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ARTICLE

Many disjoint triangles in co-triangle-free graphs

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Abstract

We prove that any n-vertex graph whose complement is triangle-free contains $n^2/12 - o(n^2)$ edge-disjoint triangles. This is tight for the disjoint union of two cliques of order $n/2$. We also prove a corresponding stability theorem, that all large graphs attaining the above bound are close to being bipartite. Our results answer a question of Alon and Linial, and make progress on a conjecture of Erdős.

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1. Introduction

One of the classical results in extremal graph theory, Goodman's theorem [4], states that in every 2-colouring of the edges of the complete graph K_n the number of monochromatic triangles is at least

$$\frac{1}{4}\binom{n}{3} - o(n^3).$$

In other words, about a quarter of all possible triangles are guaranteed to be monochromatic. With this in mind, Erdős [2, 3] asked about the number of *edge-disjoint* monochromatic triangles in any 2-colouring of K_n.

To be more formal, a *triangle packing* of a graph G is a collection of edge-disjoint triangles in G. The *size* of a triangle packing is the total number of edges it contains.[1] Define $f(n)$ to be the largest number m such that every 2-colouring of the edges of K_n contains a triangle packing of size m in which each triangle is monochromatic.

As a basic example, consider $n = 6$. By the folklore fact about Ramsey numbers, any 2-colouring of K_6 contains a monochromatic triangle, and it is not hard to see that it has to contain at least two such triangles. However, they need not be edge-disjoint, as can be seen by taking a 5-cycle and replicating a vertex. So $f(6) = 3$.

In general, the obvious upper bound of $f(n) \leqslant n^2/4 - o(n^2)$ is seen to hold by considering the balanced complete bipartite graph and its complement. Erdős [2, 3] conjectured that this is tight.[2]

[1] This is obviously the number of the triangles in the packing times 3. We prefer the present scaling for technical and presentation reasons.

[2] In [2] and [3, 6] the $n^2/4 + o(n^2)$ notation is used. It is understood that the additive $o(n^2)$-term can be negative, as this is the case, for instance, in the above example. Hence we believe the expression $n^2/4 - o(n^2)$ better reflects the nature of the conjecture.

Conjecture 1.1.

$$f(n) = \frac{n^2}{4} - o(n^2).$$

To draw a parallel to Goodman's theorem, Conjecture 1.1 states that every 2-edge-colouring of K_n admits a packing with monochromatic triangles, containing about one-half of all possible edges.

In previous works, Erdős, Faudree, Gould, Jacobson and Lehel [3] proved a first non-trivial lower bound of $f(n) \geqslant (9/55)n^2 + o(n^2)$. Keevash and Sudakov [6] improved this to $f(n) \geqslant n^2/4.3 + o(n^2)$, by using the fractional relaxation of the problem. Alon and Linial (see [6]), as a step towards Conjecture 1.1, suggested considering the natural class of colourings, in which one of the colour classes is triangle-free.

At this stage it will be more convenient to break the symmetry and speak of a graph and its complement. A graph is said to be *co-triangle-free* if its complement is triangle-free. Equivalently, co-triangle-free graphs are graphs with independence number at most 2. Define $g(n)$ to be the largest number m such that every co-triangle-free graph on n vertices contains a triangle packing of size m. The same example as for $f(n)$ – the disjoint union of two cliques of order $n/2$ – shows that $g(n) \leqslant n^2/4 - o(n^2)$, and Conjecture 1.1 would imply that this is tight.

Conjecture 1.2.

$$g(n) = \frac{n^2}{4} - o(n^2).$$

Yuster [7] worked specifically on Conjecture 1.2 and proved that any potential counterexample to it must have between $0.2501n^2$ and $3n^2/8$ edges. That is, its size cannot be too close to or too far from the Mantel threshold.

Our aim in this note is to give a short proof of Conjecture 1.2.

Theorem 1.3. *We have*

$$g(n) = \frac{n^2}{4} - o(n^2).$$

Moreover, we classify the extremal graphs. An n-vertex graph is said to be ε-*far* from being bipartite if at least εn^2 edge deletions are required in order to make it bipartite.

Theorem 1.4. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that any co-triangle-free graph G of order n whose complement is ε-far from being bipartite has a triangle packing of size* $(1/4 + \delta)n^2 + o(n^2)$.

We say that a graph is *co-bipartite* if its complement is bipartite. Equivalently, G is co-bipartite if $V(G)$ is spanned by a disjoint union of two cliques; clearly, co-bipartite graphs are co-triangle-free. Thus Theorems 1.3 and 1.4 imply that every co-triangle-free graph on n vertices admits a triangle packing on $n^2/4 - o(n^2)$ edges, and the graphs achieving at most $n^2/4 + o(n^2)$ are essentially co-bipartite.

At the core of our proof is Lemma 3.2. It deals with the case when G is 'critical', that is, its complement $\overline{G}$ is triangle-free and not bipartite, but can be made bipartite by deleting a vertex. Lemma 3.2 states that G has a fractional triangle packing of size larger than $n(n-1)/4$. This, combined with the integer-fractional transference principle of Haxell and Rödl (Proposition 2.1), averaging over fractional packings, and a computer verification for small values of n in the spirit of [6], yields the proof of Theorem 1.3.

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To prove Theorem 1.4, in addition to the above tools, we apply a theorem of Alon, Shapira and Sudakov (Proposition 4.1) on the structure of graphs with a large edit distance to a monotone graph property.

The rest of the paper is organized as follows. In Section 2 we will collect some known facts about fractional and integer triangle packings. The proofs of the crucial Lemma 3.2 and of Theorem 1.3 are carried out in Section 3. In Section 4 we derive Theorem 1.4, and in Section 5 we discuss Conjecture 1.1 and related open questions.

2. Preliminaries

Let $\nu(G)$ denote the size of the largest triangle packing in G. In this notation,

$$g(n) = \min\{\nu(G) \colon |G| = n,\ G \text{ is co-triangle-free}\}.$$

A *fractional triangle packing* of G is a function w from $\mathcal{T}(G)$, the set of all triangles in G, to $[0, 1]$ such that every edge $e \in E(G)$ satisfies $\sum_{T\in\mathcal{T}(G)\colon e\subset T} w(T) \leqslant 1$. The *size* of a fractional packing is given by $3\sum_{T\in\mathcal{T}(G)} w(T)$. Define $\nu^*(G)$ to be the maximum size of a fractional triangle packing of G; by compactness, this is well-defined. Note that ordinary triangle packings are precisely the integer-valued fractional packings; indeed, determining $\nu^*(G)$ is the LP-relaxation of the integer linear program of finding $\nu(G)$, so that $\nu^*(G) \geqslant \nu(G)$ for every graph G. Consequently, we define the function $g^*(n)$ to be the fractional counterpart to $g(n)$,

$$g^*(n) := \min\{\nu^*(G) \colon |G| = n,\ G \text{ is co-triangle-free}\}.$$

By the above, this function satisfies $g^*(n) \geqslant g(n)$ for every n. On the other hand, as a consequence of the seminal theorem of Haxell and Rödl [5], $\nu(G) \geqslant \nu^*(G) - o(n^2)$ holds for every n-vertex graph G. Therefore we have the following result.

Proposition 2.1.

$$g(n) \geqslant g^*(n) - o(n^2).$$

By virtue of Proposition 2.1 we can work with fractional instead of integer triangle packings at virtually no loss. Hence, going forward the term 'packing', unless specified otherwise, will refer to fractional triangle packings. For an n-vertex co-triangle-free graph G, define the *packing density* of G to be

$$\eta(G) := \frac{\nu^*(G)}{n(n-1)}.$$

It is well known that packing densities are monotone under averaging (see *e.g.* Lemma 2.1 in [6]).

Lemma 2.2. *Suppose that G is a graph on n vertices and let $G_1, \ldots, G_n$ be its induced subgraphs of order $n-1$. Then*

$$\eta(G) \geqslant \frac{1}{n}\sum_{i=1}^{n} \eta(G_i).$$

Proof. Without loss of generality, assume that $V(G) = [n]$ and $V(G_i) = [n]\setminus\{i\}$. Let w_i be a packing of G_i of size $\nu^*(G_i)$. Consider

$$w = \frac{1}{n-2}\sum w_i,$$

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which is a function on $\mathcal{T}(G)$. Any given edge $\{i,j\}$ contributes 0 to $w_i + w_j$, so it receives a total weight of

$$\frac{1}{n-2}\sum_{k\neq i,j}\sum_{T=\{i,j,\ell\}\in\mathcal{T}(G_k)} w_k(T) \leqslant \frac{1}{n-2}\sum_{k\neq i,j} 1 = 1. \tag{2.1}$$

Thus w is a packing of G of size

$$\frac{1}{n-2}\sum_{i=1}^{n}\nu^*(G_i),$$

which implies

$$n(n-1)\eta(G) = \nu^*(G) \geqslant \frac{1}{n-2}\sum_{i=1}^{n}\nu^*(G_i) = (n-1)\sum_{i=1}^{n}\eta(G_i),$$

and the desired inequality follows. □

Corollary 2.3. *With the above notation,*

$$\eta(G) \geqslant \min_{1\leqslant i\leqslant n}\eta(G_i).$$

We shall need the following straightforward bound on packings of co-bipartite graphs.

Lemma 2.4. *For any co-bipartite G of order $n \geqslant 6$, we have*

$$\nu^*(G) \geqslant \frac{n(n-2)}{4}.$$

Proof. G contains two disjoint cliques of sizes a and $n-a$ for some $0 \leqslant a \leqslant n/2$. Since each clique of order $m \geqslant 3$ admits a packing of size $\binom{m}{2}$, by convexity of the binomial coefficients we have

$$\nu^*(G) \geqslant \binom{a}{2} + \binom{n-a}{2} \geqslant 2\binom{n/2}{2} = \frac{n(n-2)}{4}.$$ □

A *fractional triangle decomposition* of G is a packing in which $\sum_{T\in\mathcal{T}(G):\, e\subset T} w(T) = 1$ holds for every edge $e \in E(G)$. Fractional decompositions are packings of the largest possible size $e(G)$.

Lemma 2.5. *Suppose that G is a graph on n vertices, and let $G_1, \ldots, G_n$ be its induced subgraphs of order $n-1$. If each G_i has a fractional triangle decomposition, then so does G.*

Proof. Assuming $V(G) = [n]$, and $V(G_i) = [n]\backslash\{i\}$, define w as in the proof of Lemma 2.2. We obtain (2.1) with equality in place of the inequality. Thus w is a fractional decomposition of G. □

Let K_n^{-k} denote the graph obtained from K_n by removing a k-edge matching.

Lemma 2.6. *For all integers $n \geqslant 7$ and $0 \leqslant k \leqslant \lfloor n/2 \rfloor$, the graph K_n^{-k} has a fractional triangle decomposition.*

Proof. It is easy to check by hand that this holds for $n = 7$. The rest follows by induction, applying Lemma 2.5. □

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3. Proof of Theorem 1.3

Theorem 1.3 follows readily from the following stability result.

Lemma 3.1. *Suppose that G is co-triangle-free with $|G| \geqslant 26$ and $\eta(G) \leqslant 1/4$. Then G is co-bipartite.*

The reason for the threshold of 26 is that for $|G| \leqslant 25$ the 'natural enemy' of bipartite graphs in our problem, namely the blow-up of the 5-cycle, achieves $\eta \leqslant 1/4$. This, however, happens only for small n: at $n = 25$ the 5-blow-up of C_5 attains precisely $\eta = 1/4$, and for larger n, as Lemma 3.1 claims, only co-bipartite graphs achieve packing densities of at most $1/4$.

Let us first show that Theorem 1.3 is indeed implied by Lemma 3.1.

Proof of Theorem 1.3. The complement of $K_{n/2,n/2}$ certifies that $g(n) \leqslant n^2/4 - o(n^2)$. To see the other direction, suppose for a contradiction that $g(n) \leqslant n^2/4 - \Omega(n^2)$. Then, by Proposition 2.1, we have

$$g^*(n) \leqslant \frac{n^2}{4} - \Omega(n^2).$$

This means that there exists $\varepsilon > 0$ such that for large n there is a co-triangle-free G with n vertices and $\eta(G) < 1/4 - \varepsilon$. By Lemma 3.1, G is co-bipartite. However, in this case, by Lemma 2.4,

$$\nu^*(G) \geqslant \frac{n(n-2)}{4},$$

so $\eta(G) \geqslant 1/4 - O(1/n)$, contradicting $\eta(G) < 1/4 - \varepsilon$. Hence

$$g(n) = \frac{n^2}{4} - o(n^2). \qquad \square$$

The proof of Lemma 3.1 is carried out by induction on n. For both the induction base ($n = 26$) and the step we require the following crucial lemma. Call a co-triangle-free graph G *critical* if G is not co-bipartite but contains a vertex whose removal will make it co-bipartite.

Lemma 3.2. *Every critical graph G with $|G| = n \geqslant 18$ satisfies*

$$\nu^*(G) \geqslant \frac{n^2 - 17}{4} > \frac{n(n-1)}{4}.$$

In particular,

$$\eta(G) > \frac{1}{4}.$$

Before giving the proof of Lemma 3.2, let us show how it implies Lemma 3.1.

Proof of Lemma 3.1. We proceed by induction on n. The statement for $n = 26$ has been computer-verified via the following algorithm (the program and the execution logs are provided in the Supplementary material). Our code is a modification of the code from the paper of Keevash and Sudakov [6], tailored to meet the specific requirements of our proof.

Initialization. Create the list L_n of all triangle-free graphs on $n = 6$ vertices, and calculate ν^* for their complements.

Iteration. For each $n \geqslant 7$, go through all one-vertex triangle-free extensions of the graphs in L_{n-1}, and select from them the graphs H with $\eta(\overline{H}) \leqslant 1/4$, to form the list L_n. By Corollary 2.3, any other triangle-free graph G of order n must have $\eta(\overline{G}) > 1/4$. If L_n is empty, the algorithm terminates. Otherwise move to the next iteration step.

At $n = 17$, before proceeding with the iteration, delete from L_{17} all bipartite graphs (be aware that this is a one-off action, which is carried out only at $n = 17$). After that, perform the iteration step for $n = 18$, and continue as previously. By Lemma 3.2 and Corollary 2.3, for $n \geqslant 18$ every co-triangle-free n-vertex graph G with $\chi(\overline{G}) > 2$ and $\eta(G) \leqslant 1/4$ is a one-vertex extension of an $(n-1)$-vertex graph with the same properties. Therefore, for each $n \geqslant 18$ the list L_n will contain precisely all triangle-free, non-bipartite n-vertex graphs H satisfying $\eta(\overline{H}) \leqslant 1/4$.

Termination. The algorithm terminates if, for some n, the list L_n is empty.

Outcome. The program run terminates at $n = 26$, when L_{26} turns out to be empty. In fact, at $n = 25$ the single graph in L_n, up to isomorphism, is the 5-blow-up of C_5, and it has no valid extensions to $n = 26$. This completes the proof of the induction base.

To see that Lemma 3.2 also implies the induction step for Lemma 3.1, let G be as in Lemma 3.1, with $|G| = n \geqslant 27$, and let $G_1, \ldots, G_n$ be the induced subgraphs of G of order $n-1$. By Corollary 2.3, we have $\eta(G_i) \leqslant 1/4$ for some i, and note that G_i is co-triangle-free. By the induction hypothesis, G_i is co-bipartite. If G is co-bipartite, we are done. Otherwise G is critical, so by Lemma 3.2 we have $\eta(G) > 1/4$, a contradiction. □

Remark 3.3. Strictly speaking, the proof of Lemma 3.1 uses Lemma 3.2 only for $n \geqslant 27$. The latter was stated and proved for $n \geqslant 18$ for the purpose of accelerating the computer search needed to prove Lemma 3.1 for $n = 26$.

Proof of Lemma 3.2. Suppose that $n \geqslant 18$ and G is a critical graph on n vertices. Then there exists a vertex $v \in G$ such that $G' := \overline{G \backslash \{v\}}$ is bipartite. Let $U \cup W$ be a bipartition of $V(G')$, *i.e.* $G' = G'[U, W]$, and note that the graphs $G[U]$ and $G[W]$ are complete. Note also that we can assume

$$\min\{|U|, |W|\} \geqslant 7,$$

as otherwise $G[U] \cup G[W]$ would contain a packing of size more than $n(n-1)/4$ and we would be done. Define

$$\begin{aligned} A &:= N_G(v) \cap U, \\ B &:= N_G(v) \cap W, \\ X &:= U \backslash A = N_{\overline{G}}(v) \cap U, \quad \text{and} \\ Y &:= W \backslash B = N_{\overline{G}}(v) \cap W. \end{aligned}$$

Note that X and Y are non-empty, since if, for instance, $X = \emptyset$ then $G[U \cup \{v\}]$ and $G[W]$ are complete, so G would be co-bipartite, a contradiction. Moreover, since for every $(x', y') \in X \times Y$ we have $\{x', y'\} \subseteq X \cup Y \subseteq N_{\overline{G}}(v)$, we must have $\{x', y'\} \in E(G)$, as $\overline{G}$ is triangle-free. Hence $G[X, Y]$ is complete bipartite.

First suppose that $|Y|$ is even (the case when $|X|$ is even is symmetric). Let $x \in X$ be an arbitrary vertex. In the complete graph $G[Y]$ select a matching M_Y on $|Y|$ vertices, and note that $\mathcal{Y} := \{y_1y_2x\colon y_1y_2 \in M_Y\}$ is a triangle packing in G containing $|V(M_Y)| = |Y|$ edges from $G[U, W]$. Next, let $y \in Y$ be an arbitrary vertex, and in the complete graph $G[X \backslash x]$ select a matching M_X on at least $|X| - 2$ vertices, so that $\mathcal{X} := \{x_1x_2y\colon x_1x_2 \in M_X\}$ is a triangle packing in G with $|V(M_X)| \geqslant |X| - 2$ edges from $G[U, W]$. By construction, $\mathcal{X}$ and $\mathcal{Y}$ are edge-disjoint and $\mathcal{X} \cup \mathcal{Y}$ contains at least $|X| + |Y| - 2$ edges from $G[U, W]$.

If both $|X|$ and $|Y|$ are odd, we select $x \in X$ arbitrarily and M_Y to be a matching on $|Y| - 1$ vertices. In the second step we select y to be the sole vertex in $Y \backslash M_Y$ and M_X to be a matching on $|X| - 1$ vertices in $X \backslash \{x\}$. We obtain two edge-disjoint triangle packings in G, $\mathcal{X}$ and $\mathcal{Y}$, containing together $|X| + |Y| - 2$ edges from $G[U, W]$.

Similarly, in the complete graphs $G[A]$ and $G[B]$ we select matchings M_A and M_B, with at least $|A|-1$ and $|B|-1$ vertices respectively, to define triangle packings $\mathcal{A}:=\{a_1a_2v\colon a_1a_2\in M_A\}$ and $\mathcal{B}:=\{b_1b_2v\colon b_1b_2\in M_B\}$. Note that $\mathcal{A}$ contains at least $|A|-1$ edges from $G[v,U]$, $\mathcal{B}$ contains at least $|B|-1$ edges from $G[v,W]$, and $\mathcal{A},\mathcal{B},\mathcal{X}$ and $\mathcal{Y}$ are edge-disjoint.

Therefore $\mathcal{A}\cup\mathcal{B}\cup\mathcal{X}\cup\mathcal{Y}$ is a triangle packing of G containing at least

$$|A|-1+|B|-1+|X|+|Y|-2=|U|+|W|-4=n-5$$

edges that are not in $G[U]$ or $G[W]$. The edges of $G[U]$ and $G[W]$ that are not part of $\mathcal{A}\cup\mathcal{B}\cup\mathcal{X}\cup\mathcal{Y}$ form on each of U and W a complete graph with a matching removed. Since $\min\{|U|,|W|\}\geqslant 7$, by Lemma 2.6 those are fractionally decomposable into triangles. Hence

$$\begin{aligned}\nu^*(G)&\geqslant\binom{|U|}{2}+\binom{|W|}{2}+(n-5)\\&\geqslant 2\binom{(n-1)/2}{2}+n-5\\&=\frac{(n-1)(n-3)+4n-20}{4}\\&=\frac{n^2-17}{4}\\&>\frac{n^2-n}{4}.\end{aligned}$$

In particular,

$$\eta(G)=\frac{\nu^*(G)}{n(n-1)}>\frac{1}{4}.$$ □

Put

$$\gamma_n:=\min\{\eta(H)\colon |H|=n,\alpha(H)\leqslant 2,\chi(\overline{H})>2\}.$$

Lemma 3.1 implies $\gamma_n>1/4$ for all $n\geqslant 26$. In the next section we will need the following quantitative form of this statement.

Corollary 3.4. *There exist an absolute constant n_0 such that, for all $n\geqslant n_0$, we have*

$$\gamma_n\geqslant\frac{n^2-17}{4n(n-1)}.$$

Proof. We may assume that $n\geqslant 27$. By Lemma 3.2 and Corollary 2.3,

$$\gamma_n\geqslant\min\left\{\frac{n^2-17}{4n(n-1)},\gamma_{n-1}\right\}.$$

Expanding this recursively gives

$$\gamma_n\geqslant\min\left\{\frac{n^2-17}{4n(n-1)},\ldots,\frac{27^2-17}{4\cdot 27(27-1)},\gamma_{26}\right\}.$$

All of these numbers are strictly above $1/4$, the sequence

$$\frac{n^2-17}{4n(n-1)}$$

is monotone decreasing from $n=33$ and converges to $1/4$. Therefore, for large values of n the above minimum will be attained by the first term. □

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4. Proof of Theorem 1.4

For an n-vertex graph G let $\Delta_{\text{bip}}(G)$ denote the *edit distance* of G to the set of bipartite graphs, *i.e.* the minimum number of edge deletions needed to turn G into a bipartite graph. Let $E_{\text{bip}}(G) := \Delta_{\text{bip}}(G)/n^2$ be the corresponding density. So, G being ε-far from being bipartite is equivalent to $E_{\text{bip}}(G) \geqslant \varepsilon$.

In order to prove Theorem 1.4, we need the following deep theorem of Alon, Shapira and Sudakov on monotone graph properties [1, Theorem 1.2], which we state here for the property of being bipartite.

Proposition 4.1 ([1]). *For every $\varepsilon > 0$ there is $m(\varepsilon)$ with the following property. Let G be any graph and suppose we randomly pick a subset M on m vertices from $V(G)$. Let G' denote the graph induced by G on M. Then*

$$\mathbb{P}[|E_{\text{bip}}(G') - E_{\text{bip}}(G)| > \varepsilon] < \varepsilon.$$

It is implicit in [1] that m tends to infinity when ε goes to 0 (in fact it is not hard to see that this is the only way for Proposition 4.1 to be true). Thus, applying Proposition 4.1 with parameter $\varepsilon/2$ to graphs G with $E_{\text{bip}}(G) \geqslant \varepsilon$, we obtain the following statement.

Corollary 4.2. *For every $\varepsilon > 0$ there exists $m = m(\varepsilon)$, with $m \to \infty$ as $\varepsilon \to 0$, as follows. Suppose that $|G| =: n \geqslant m$, and G is ε-far from being bipartite. Then at least $(1 - \varepsilon/2)\binom{n}{m}$ m-vertex induced subgraphs of G are not bipartite.*

Proof of Theorem 1.4. Suppose that $\varepsilon > 0$ and let $m = m(\varepsilon)$ be as in Corollary 4.2. Let n_0 be the constant from Corollary 3.4. Choosing ε to be sufficiently small, by Corollary 4.2 we may assume that $m > \max\{n_0, 100\}$.

Then by Corollary 3.4 we have

$$\min\{\eta(H) \colon |H| = m, \alpha(H) \leqslant 2, \chi(\overline{H}) > 2\} \geqslant \frac{m^2 - 17}{4m(m-1)} = \frac{1}{4} + \frac{m - 17}{4m(m-1)}, \tag{4.1}$$

and for co-bipartite graphs H of order m, by Lemma 2.4, we have

$$\eta(H) \geqslant \frac{m(m-2)}{4m(m-1)} = \frac{1}{4} - \frac{m}{4m(m-1)}. \tag{4.2}$$

Suppose now that G is co-triangle-free with $|G| = n \geqslant m$, and $\overline{G}$ is ε-far from being bipartite. Applying Lemma 2.2 iteratively gives

$$\eta(G) \geqslant \frac{1}{\binom{n}{m}} \sum_{M \in \binom{V(G)}{m}} \eta(G[M]).$$

Combining this with Corollary 4.2, (4.1) and (4.2), we obtain

$$\begin{aligned}\eta(G) &\geqslant \frac{1}{\binom{n}{m}} \sum_{M \in \binom{V(G)}{m}} \eta(G[M]) \\ &= \frac{1}{\binom{n}{m}} \left(\sum_{M \colon \chi(\overline{G}[M]) > 2} \eta(G[M]) + \sum_{M \colon \chi(\overline{G}[M]) \leqslant 2} \eta(G[M]) \right)\end{aligned}$$

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$$\geqslant \left(1-\frac{\varepsilon}{2}\right)\left(\frac{1}{4}+\frac{m-17}{4m(m-1)}\right)+\frac{\varepsilon}{2}\left(\frac{1}{4}-\frac{m}{4m(m-1)}\right)$$
$$> \frac{1}{4}+\frac{m-17-\varepsilon m+8\varepsilon}{4m(m-1)}$$
$$> \frac{1}{4}+\frac{1}{8m}.$$

By the definition of η and Proposition 2.1,

$$\nu(G) > \left(\frac{1}{4}+\frac{1}{8m}\right)n^2+o(n^2).$$

Hence the desired statement holds with $\delta := 1/(8m)$. □

5. Discussion

One suggestion is to use the same approach in order to tackle Conjecture 1.1. Indeed, extending the definition of η to arbitrary graphs G via

$$\eta(G) := \frac{\nu^*(G)+\nu^*(\overline{G})}{n(n-1)},$$

the results of Section 2 transfer straightforwardly. That said, for general graphs $\eta(G) \leqslant 1/4$ *does not* imply that either G or $\overline{G}$ is bipartite. Take $K_{n/2,n/2}$, for instance, and add any number $\ell \leqslant n/8$ of edges to it. Then the largest monochromatic triangle packing in the resulting colouring $G \cup \overline{G}$ has size at most

$$2\binom{n/2}{2}+2\ell \leqslant \frac{n^2-2n}{4}+\frac{n}{4}=\frac{n(n-1)}{4}.$$

We suspect, however, that this is essentially the only obstruction to having $\eta(G) > 1/4$. In light of Theorem 1.3, the following strengthening of Conjecture 1.1 appears plausible.

Conjecture 5.1. *Suppose that $|G| = n \geqslant 26$ and $\eta(G) \leqslant 1/4$. Then either G or $\overline{G}$ can be made bipartite by removing at most $n/8$ edges.*

The main challenge in proving Conjecture 5.1 is to bridge the gap between computer simulations for small n and stability arguments for larger n. At present this seems much harder for general graphs than in the triangle-free case.

Further open problems

In common with several predecessor papers [3, 6], we would like to draw the reader's attention to a related conjecture of Jacobson, which states that, for every n-vertex graph G, one of G and $\overline{G}$ will have a triangle packing with at least $n^2/20 - o(n^2)$ triangles, which is tight for the C_5-blow-up. To prove this conjecture one would need a new idea, since the averaging approach *à la* Lemma 2.2 is unlikely to work.

The works [6] and [7] also discussed packings with monochromatic k-cliques instead of triangles. It would be interesting to study this systematically for arbitrary fixed graphs H and number of colours.

Question 5.2. For $c \geqslant 2$ and a fixed graph H, how many edge-disjoint monochromatic copies of H are guaranteed to exist in a c-colouring of the edges of K_n?

Specifically, it would be interesting to extend Theorem 1.3 to arbitrary graphs H.

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Question 5.3. How many edge-disjoint copies of H are guaranteed to exist in an n-vertex graph whose complement is H-free?

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Supplementary material

To view supplementary material for this article, please visit https://doi.org/10.1017/S096354832000036X.

References

[1] Alon, N., Shapira, A. and Sudakov, B. (2009) Additive approximation for edge-deletion problems. *Ann. of Math.* **170** 371–411.
[2] Erdős, P. (1997) Some recent problems and results in graph theory. *Discrete Math.* **164** 81–85.
[3] Erdős, P., Faudree, R. J., Gould, R. J., Jacobson, M. S. and Lehel, J. (1997) Edge disjoint monochromatic triangles in 2-colored graphs. *Discrete Math.* **164** 81–85.
[4] Goodman, A. W. (1959) On sets of acquaintances and strangers at any party. *Amer. Math. Monthly* **66** 778–783.
[5] Haxell, P. and Rödl, V. (2001) Integer and fractional packings in dense graphs. *Combinatorica* **21** 13–38.
[6] Keevash, P. and Sudakov, B. (2004) Packing triangles in a graph and its complement. *J. Graph Theory* **47** 203–216.
[7] Yuster, R. (2007) Packing cliques in graphs with independence number 2. *Combin. Probab. Comput.* **16** 805–817.

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