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CERTAIN LOCALLY NILPOTENT VARIETIES OF GROUPS

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Let $c \geqslant 0$, $d \geqslant 2$ be integers and $\mathcal{N}_c^{(d)}$ be the variety of groups in which every d-generator subgroup is nilpotent of class at most c. N.D. Gupta asked for what values of c and d is it true that $\mathcal{N}_c^{(d)}$ is locally nilpotent? We prove that if $c \leqslant 2^d + 2^{d-1} - 3$ then the variety $\mathcal{N}_c^{(d)}$ is locally nilpotent and we reduce the question of Gupta about the periodic groups in $\mathcal{N}_c^{(d)}$ to the prime power exponent groups in this variety.

1. INTRODUCTION AND RESULTS

Let $c \geqslant 0$, $d \geqslant 2$ be integers and $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c. We denote by $\mathcal{N}_c^{(d)}$ the variety of groups in which every d-generator subgroup is in $\mathcal{N}_c$. In [**2**], Gupta posed the following question:

For what values of c and d it is true that $\mathcal{N}_c^{(d)}$ is locally nilpotent?

Then he proved that for $c \leqslant (d^2 + 2d - 3)/4$, the variety $\mathcal{N}_c^{(d)}$ is locally nilpotent. In [**1**], Endimioni improved the latter result where he proved that for $c \leqslant 2^d - 2$, the variety $\mathcal{N}_c^{(d)}$ is locally nilpotent. Here we improve the number $2^d - 2$ to $2^d + 2^{d-1} - 3$. In fact we prove:

THEOREM 1.1.

(1) *For $c \leqslant 2^d + 2^{d-1} - 3$, the variety $\mathcal{N}_c^{(d)}$ is locally nilpotent.*

(2) *For $c \leqslant 2^d + 2^{d-1} + 2^{d-2} - 3$, every p-group in the variety $\mathcal{N}_c^{(d)}$ is locally nilpotent, where $p \in \{2, 3, 5\}$.*

Note that the variety $\mathcal{N}_c^{(2)}$ is contained in the variety of c-Engel groups and it is yet unknown whether every c-Engel group is locally nilpotent, even, so far there is no published example of a non-locally nilpotent group in the variety $\mathcal{N}_c^{(2)}$. In the last section of this paper, where we consider the problem of locally nilpotency of the variety $\mathcal{N}_c^{(2)}$, we study periodic groups in this variety. Note that since every two generator subgroup of a group in $\mathcal{N}_c^{(2)}$ is nilpotent, every periodic group in $\mathcal{N}_c^{(2)}$ is a direct product of p-groups (p prime). We reduce the question of Gupta for periodic groups in $\mathcal{N}_c^{(d)}$ to the locally nilpotency of p-groups of finite exponent in this variety where the exponent depends only on the numbers p and c. In fact we prove that

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Theorem 1.2. *Let p be a prime, $c > 1$ an integer and $r = r(c,p)$ be the integer such that $p^{r-1} < c - 1 \leqslant p^r$.*

(1) *every p-group in $\mathcal{N}_c^{(2)}$ is locally nilpotent.*

(2) *if p is odd, every p-group of exponent dividing p^r in $\mathcal{N}_c^{(2)}$ is locally nilpotent, and if $p = 2$, every 2-group of exponent dividing 2^{r+1} in $\mathcal{N}_c^{(2)}$ is locally nilpotent.*

2. Groups in the variety $\mathcal{N}_c^{(d)}$

Let F_∞ be the free group of infinite countable rank on the set $\{x_1, x_2, \dots\}$, we define inductively the following words in F_∞:

$$
\begin{aligned}
W_1 &= W_1(x_1,x_2) = [x_1,x_2,x_1,x_2],\\
W_n &= W_n(x_1,x_2,\dots,x_{n+1}) = [W_{n-1},x_{n+1},W_{n-1},x_{n+1}] \quad n > 1;\\
V_1 &= V_1(x_1,x_2,x_3) = \big[[x_2,x_1,x_1,x_1,x_1],x_3,[x_2,x_1,x_1,x_1,x_1],x_3\big],\\
V_n &= V_n(x_1,x_2,x_3,\dots,x_{n+2}) = [V_{n-1},x_{n+2},V_{n-1},x_{n+2}] \quad n > 1.
\end{aligned}
$$

For a group G and a subgroup H of G, we denote by $HP(G)$ the Hirsch-Plotkin radical of G and H^G the normal closure of H in G. We use the following result due to Heineken (see Lemma 8 of [**3**] and see Lemma 2 of [**5**] for the left-normed version).

Lemma 2.1. *Let G be a group and g an element in G such that $[g,x,g,x] = 1$ for all $x \in G$. Then the normal closure of $\langle g\rangle$ in $\langle g\rangle^G$ is Abelian. In particular, $g \in HP(G)$.*

Lemma 2.2. *Let G be a group satisfying the law $W_n = 1$ for some integer $n \geqslant 1$. Then, G has a normal series $1 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 = G$ in which each factor G_i/G_{i+1} is locally nilpotent $(i = 1,2,\dots,n-1)$.*

Proof: We argue by induction on n. If $n = 1$, then Lemma 2.1 yields that $x_1 \in HP(G)$ for all $x_1 \in G$ and so G is locally nilpotent. Now suppose that the lemma is true for n and G satisfies the law $W_{n+1} = 1$. By Lemma 2.1, we have $W_n(x_1,\dots,x_{n+1}) \in HP(G)$ for all $x_1,\dots,x_{n+1} \in G$ and so $G/HP(G)$ satisfies the law $W_n = 1$. Thus by the induction hypothesis $G/HP(G)$ has a normal series of length n with locally nilpotent factors. This completes the proof. ∎

Lemma 2.3. *Let G be a p-group satisfying the law $V_n = 1$ for some integer $n \geqslant 1$ where $p \in \{2,3,5\}$. Thus G has a normal series $1 = G_{n+1} \lhd G_n \lhd \cdots \lhd G_1 = G$ in which each factor G_i/G_{i+1} is locally nilpotent $(i = 1,2,\dots,n)$.*

Proof: We argue by induction on n. If $n = 1$, then by Lemma 2.1 $[x_2,x_1,x_1,x_1,x_1] \in HP(G)$ for all $x_1, x_2 \in G$. Thus $G/HP(G)$ is a 4-Engel group and since every 4-Engel p-group is locally nilpotent where $p \in \{2,3,5\}$ (see Traustason [**9**] and Vaughan-Lee [**10**]), $G/HP(G)$ is locally nilpotent. Now suppose that G satisfies the law $V_{n+1} = 1$. By

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Lemma 2.1, $V_n(x_1,\dots,x_{n+2}) \in HP(G)$ for all $x_1,\dots,x_{n+2} \in G$, so $G/HP(G)$ satisfies the law $V_n = 1$. Thus by the induction hypothesis, $G/HP(G)$ has a normal series of length $n+1$ with locally nilpotent factors. This completes the proof. □

We use in the sequel the following special case of the well-known fact due to Plotkin [**6**] that every Engel radical group is locally nilpotent. (see also Lemma 2.2 of [**1**])

Lemma 2.4. *Let H be a normal subgroup of an Engel group G. If H and G/H are locally nilpotent, then G is locally nilpotent.*

Proof of Theorem 1.1: One can see that W_n and V_n are, respectively, in the $(2^{n+1}+2^n-2)$th term and $(2^{n+2}+2^{n+1}+2^n-2)$th term of the lower central series of F_∞, for all integers $n \geqslant 1$.

(1) every group G in the variety $\mathcal{N}_c^{(d)}$ satisfies the law $W_{d-1} = 1$ and so it follows from Lemmas 2.2 and 2.4, that G is locally nilpotent.

(2) if $d = 2$ then $c \leqslant 4$ and so every group in the variety $\mathcal{N}_c^{(d)}$ is 4-Engel. But as it is mentioned in the proof of Lemma 2.3, every 4-Engel p-group is locally nilpotent, where $p \in \{2,3,5\}$. Now assume that $d \geqslant 3$, then every group G in the variety $\mathcal{N}_c^{(d)}$ satisfies the law $V_{d-2} = 1$ and so it follows from Lemmas 2.3 and 2.4, that G is locally nilpotent. □

3. p-groups in the variety $\mathcal{N}_c^{(2)}$

Lemma 3.1. *Let $c \geqslant 1$ be an integer, p a prime number and G a finite c-Engel p-group. Suppose that $x, y \in G$ such that $x^{p^n} = y^{p^n} = 1$ for some integer $n > 0$. Let r be the integer such that $p^{r-1} < c \leqslant p^r$. Then:*

(a) *if p is odd and $n > r$ then $[x^{p^{n-1}}, y^{p^{n-1}}] = 1$.*

(b) *if $p = 2$ and $n > r+1$ then $[x^{2^{n-1}}, y^{2^{n-1}}] = 1$.*

Proof: Suppose that $K \leqslant H$ are two normal subgroups of G such that H/K is elementary Abelian and a is an arbitrary element of G. Put $t = aK$ and $V = H/K$. Since $[V,_c t] = 1$, we have that $[V,_{p^r} t] = 1$ and $0 = (t-1)^{p^r} = t^{p^r} - 1$ in $\mathrm{End}(V)$. Thus $[H, a^{p^r}] \leqslant K$ for all $a \in G$. Now let $N = \langle x^{p^r}, y^{p^r}\rangle$. Then $[H,N] \leqslant K$ and since K, H are normal in G; $[H,M] \leqslant K$ where $M = N^G$ the normal closure of N in G. Thus M is a normal subgroup of G centralising every elementary Abelian normal section of G. By a result of Shalev [**8**]; if p is odd then M is powerful and if $p = 2$ then M^2 as well as all subgroups of M^2 which are normal in G, are powerful. Suppose that p is odd. Since M is generated by $\{(x^g)^{p^r}, (y^g)^{p^r} \mid g \in G\}$ and M is powerful, by Corollary 1.9 of [**4**], $M^{p^{n-r}}$ is generated by $\{(x^g)^{p^n}, (y^g)^{p^n} \mid g \in G\}$ and so $M^{p^{n-r}} = 1$. On the other hand $M^{p^{n-r-1}}$ is powerful by Corollary 1.2 of [**4**]. Thus $[M^{p^{n-r-1}}, M^{p^{n-r-1}}] \leqslant (M^{p^{n-r-1}})^p$. Now by Theorem 1.3 of [**4**], we have $(M^{p^{n-r-1}})^p = M^{p^{n-r}} = 1$. Thus $M^{p^{n-r-1}}$ is Abelian and part (a) has been proved. Now assume that $p = 2$. As mentioned above, since the subgroup $\langle x^{2^{r+1}}, y^{2^{r+1}}\rangle^G$ of M is normal in G, it is also powerful and the rest of the proof is similar to the previous case. ∎

LEMMA 3.2. *Let $c > 1$ be an integer, p a prime number and G a p-group in the variety $\mathcal{N}_c^{(2)}$ and let r be the integer satisfying $p^{r-1} < c-1 \leqslant p^r$.*

(a) *if p is odd then G^{p^r} is locally nilpotent.*

(b) *if $p = 2$ then $G^{2^{r+1}}$ is locally nilpotent.*

PROOF: Note that G is a c-Engel group. Suppose p is odd. First we prove that if x is an element of G such that $x^{p^n} = 1$ for some $n > r$, then $[a, x^{p^{n-1}}, x^{p^{n-1}}] = 1$ for all $a \in G$. Let $y = (x^{-1})^a$, then it is enough to show that $[y^{p^{n-1}}, x^{p^{n-1}}] = 1$. Since $\langle x, a\rangle \in \mathcal{N}_c$ then $\langle y, x\rangle \in \mathcal{N}_{c-1}$; therefore by Lemma 3.1, we have that $[y^{p^{n-1}}, x^{p^{n-1}}] = 1$. Now let z be an arbitrary element of G such that $z^{p^n} = 1$ for some positive integer n. We prove by induction on n that $z^{p^r} \in HP(G)$ and so it completes the proof for the case p odd. If $n \leqslant r$, then $z^{p^r} = 1 \in HP(G)$. Assume that $n > r$ then by the induction hypothesis, $z^{p^{r+1}} \in HP(G)$. Now by the first part of the proof, $[a, z^{p^r}, z^{p^r}] = 1 \bmod HP(G)$, for all $a \in G$. So the normal closure of $z^{p^r}HP(G)$ in $G/HP(G)$ is Abelian. Therefore $\langle z^{p^r}\rangle^G$ is (locally nilpotent)-by-Abelian, hence Lemma 2.4 implies that $z^{p^r} \in HP(G)$.

The case $p = 2$ is similar. ∎

PROOF OF THEOREM 1.2: Suppose that p is odd and every p-group of exponent dividing p^r in $\mathcal{N}_c^{(2)}$ is locally nilpotent. Let G be a p-group in $\mathcal{N}_c^{(2)}$, then by Lemma 3.2(a), G^{p^r} is locally nilpotent. By assumption, G/G^{p^r} is locally nilpotent and since G is a c-Engel group, it follows from Lemma 2.4 that G is locally nilpotent. The case $p = 2$ is similar and the converse is obvious. ∎

Now we use this result to some special cases. In fact we prove:

PROPOSITION 3.3. *Every 2-group or 3-group in the variety $\mathcal{N}_5^{(2)}$ is locally nilpotent.*

PROOF: By Theorem 1.2, we must prove that every 2-group of exponent dividing 8 and every 3-group of exponent dividing 9 in $\mathcal{N}_5^{(2)}$ is locally nilpotent. Suppose that G is a 2-group of exponent 8 in $\mathcal{N}_5^{(2)}$. Let $x, y \in G$, then $\langle x, y\rangle \in \mathcal{N}_5$ and is of exponent 8. It is easy to see that $[x^4, y, x^4, y] = 1$. So by Lemma 2.1, $x^4 \in HP(G)$ for all $x \in G$. Therefore $G/HP(G)$ is of exponent dividing 4. By a famous result of Sanov (see 14.2.4 of [**7**]), $G/HP(G)$ is locally nilpotent and so by Lemma 2.4, G is locally nilpotent.

Now suppose that G is a 3-group of exponent 9 in the variety $\mathcal{N}_5^{(2)}$. Let $x, y \in G$, it is easy to see that $[x^3, y, x^3, y] = 1$. The rest of the proof is similar to the previous case, but we may use this well-known result that every group of exponent 3 is nilpotent (see 12.3.5 and 12.3.6 of [**7**]). ∎

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