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CARDINALITIES OF LOCALLY COMPACT GROUPS AND THEIR STONE-ČECH COMPACTIFICATIONS

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Dedicated to Edwin Hewitt

If G is any Hausdorff topological group and βG is its Stone-Čech compactification, then $|G| \leqslant |\beta G| \leqslant 2^{2^{|G|}}$, where $|G|$ denotes the cardinality of G. It is known that if G is a discrete group then $|\beta G| = 2^{2^{|G|}}$ and if G is the additive group of real numbers with the Euclidean topology, then $|\beta G| = 2^{|G|}$. In this paper the cardinality and weight of βG, for a locally compact group G, is calculated in terms of the character and Lindelöf degree of G. The results make it possible to give a reasonably complete description of locally compact groups G for which $|\beta G| = 2^{|G|}$ or even $|\beta G| = |G|$.

0. INTRODUCTION

Compact groups and locally compact groups have been studied over the last century because of their importance and richness with the solution of Hilbert's fifth problem that a locally Euclidean group is a Lie group taking half of that period and the efforts of some of the greatest twentieth century mathematicians. The influential book [**12**] describes the Pontryagin duality and structure of locally compact Abelian groups and their centrality in harmonic analysis. The recent book [**10**] exposes the structure of compact groups. The forthcoming book [**11**] includes the beautiful structural results of Iwasawa for locally compact (not necessarily Abelian) groups. There are beautiful results on the topology of compact groups: see [**10**] and, in particular, those of Kuz'minov [**14**] which states that each compact topological group is dyadic and Shapirovsky [**15**] which implies that any compact group of weight κ can be continuously mapped onto the space $[0,1]^{\kappa}$. Structure theory proofs of Kuz'minov's Theorem and Shapirovsky's Theorem appear in [**3**] and [**4**]. In [**2**] it is proved using significant structure theory, that every locally compact group is homeomorphic to a product of $\mathbb{R}^n$, a compact connected group, a product of discrete two-point groups and a discrete group, for some nonnegative integer n.

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Weight and cardinality are, undoubtedly, among the most important cardinal functions on a topological space; we study these characteristics for locally compact topological groups and their Stone-Čech compactifications. Strong methods are needed here even for simple topological groups. It is worth mentioning that the Stone-Čech compactifications of topological groups were intensively studied after Comfort and Ross proved in [**6**] that βG is a topological group for any pseudocompact group G.

We give a complete answer as to cardinality and weight of βG for any locally compact topological group G. For the reader's benefit, our proofs do not rely on the weighty structure theorems for locally compact groups, but are presented using more elementary results.

Our general result states that, for an arbitrary locally compact group G with $\chi(G) = \kappa$ and $l(G) = \delta$, we have $|\beta G| = 2^{\kappa^\delta}$ and $w(\beta G) = \kappa^\delta$. We mentioned already that $|\beta G| = 2^{|G|}$ if G is the additive group of the reals; an easy consequence of our general theorem is that this equality holds for quite a few locally compact groups G. Since every discrete group is locally compact, an evident possibility is the equality $|\beta G| = 2^{2^{|G|}}$; we show that it holds for metrisable locally compact groups of weight at least $\mathfrak{c}$. Another result is that we can often have the equality $|\beta G| = |G|$ for a locally compact non-compact group G. It is a well-known fact that $|\beta X| \leqslant 2^{2^{|X|}}$ for any Tychonoff space X. We show that the equality is possible even if X is the space of a countably compact topological group.

1. Notation and terminology

All topological spaces and groups are assumed to be Tychonoff. Given a space X, we denote by $\tau(X)$ its topology and by βX its Stone-Čech compactification. For any $A \subset X$, let $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$; if $x \in X$ we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. The space $\mathbb{R}$ is the additive group of the reals with the natural topology, $\mathbb{I} = [0,1] \subset \mathbb{R}$ and $\mathbb{D} = \{0,1\}$ is the two-point discrete space. All cardinals are considered to carry the discrete topology if they are dealt with as topological spaces.

For spaces X and Y we denote by $C_p(X,Y)$ the subspace of the Tychonoff product Y^X which consists of all continuous maps from X to Y; we write $C_p(X)$ instead of $C_p(X, \mathbb{R})$. Besides, $C(X)$ $(C^*(X))$ is the set of all real-valued continuous (bounded) functions on the space X. If κ is an infinite cardinal then $A(\kappa)$ is the one-point compactification of the discrete space of cardinality κ. We always consider that $A(\kappa)$ $= \kappa \cup \{a\}$ where $a \notin \kappa$ is the unique non-isolated point of $A(\kappa)$. The expression $X \simeq Y$ means that the spaces X and Y are homeomorphic. If Y is a set and $P \subset Y$ then $\chi_P : Y \to \{0,1\}$ is the characteristic function of the set P defined by $\chi_P(x) = 1$ for all $x \in P$ and $\chi_P(x) = 0$ if $x \in Y \backslash P$. A set $P \subset C_p(X,Y)$ *separates the points of* X if, for any distinct $x, y \in X$, there is $f \in P$ such that $f(x) \neq f(y)$.

The *Lindelöf number (or degree)* $l(X)$ of a space X is the minimal cardinal κ such that every open cover of X has a subcover of cardinality $\leqslant \kappa$. Given a set $A \subset X$, a family $\mathcal{B} \subset \tau(A, X)$ is *a base of* A *in* X if, for any $U \in \tau(A, X)$, there is $V \in \mathcal{B}$ such that $V \subset U$. The minimal cardinality of all bases of A in X is denoted by $\chi(A, X)$; if $x \in X$, we write $\chi(x, X)$ instead of $\chi(\{x\}, X)$. The cardinal $\chi(X) = \sup\{\chi(x, X) : x \in X\}$ is called *the character of the space* X. A space X is called *zero-dimensional* if it has a base which consists of clopen ($\equiv$ closed and open) subsets of X. The rest of our notation is standard and follows [**7**].

2. Locally compact groups and free unions of Cantor cubes

It turns out that analysis of weights and cardinalities of βG for a locally compact group G can be reduced to the situation when G is topologically a free union of copies of the Cantor cube $\mathbb{D}^\kappa$ for some infinite cardinal κ. Then a purely topological reasoning makes it possible to obtain a complete computation of these cardinal functions.

To simplify our notation, we consider each cardinal ν to be a discrete topological space of cardinality ν. In particular, the expression $G \simeq G' \times \nu$ means that G is homeomorphic to a space $G \times D$ where D is a discrete space of cardinality ν.

Proposition 2.1. *Any locally compact group G has an open σ-compact subgroup G'; if G is not σ-compact then, for any such G', the space of G is homeomorphic to $G' \times \delta$ where $\delta = l(G)$.*

Proof: Take any symmetric open neighbourhood U of the identity e of G such that $\overline{U}$ is compact. It is a standard fact (see for example [**12**, Theorems 5.7 and 5.13]) that the set $G' = \{U^n : n \in \mathbb{N}\}$ is a clopen σ-compact subgroup of G. It is evident that $G \simeq G' \times \nu$ where ν is the number of cosets of G' If the group G is not σ-compact then ν has to be infinite (even uncountable). Choose a point x_H in each coset H of the group G'; the set $D = \{x_H : H \in G/G'\}$ is closed and discrete and therefore $\nu = |D| \leqslant l(G) = \delta$. On the other hand, each coset is homeomorphic to G' so the group G is a union of ν-many Lindelöf subspaces; as a consequence, $\delta = l(G) \leqslant l(G') \cdot \nu = \omega \cdot \nu = \nu$. ∎

Proposition 2.2. *Given a non-metrisable locally compact group G, there exists an open σ-compact subgroup G' of the group G and a compact subgroup K of the same group G such that $K \subset G'$, $|K| = |G'|$, the group G'/K is second countable and $\chi(K) = \chi(G)$.*

Proof: Apply Proposition 2.1 to find an open σ-compact subgroup G' of the group G. It is straightforward that the group G' cannot be metrisable. There exists a normal compact subgroup K of the group G' such that the quotient group G'/K is second countable [**12**, Theorem 8.7]. Observe that the quotient map $\pi : G' \to G'/K$ is open and perfect [**12,** Theorems 5.17 and 5.18] which implies $\chi(K, G') = \chi(G'/K) = \omega$. Thus $\chi(e, G') \leqslant \chi(e, K) \cdot \chi(K, G') = \chi(K) \cdot \omega = \chi(K)$, that is, $\chi(K) = \chi(G')$. It is easy to see that the character of any open subgroup of G coincides with the character of G, so $\chi(K) = \chi(G') = \chi(G)$. The group K cannot be discrete or even metrisable because $\chi(K) = \chi(G) > \omega$. Every second countable space has cardinality $\leqslant \mathfrak{c}$ so $|G'| = |K| \cdot |G'/K| \leqslant |K| \cdot \mathfrak{c} = |K|$ because $|K| \geqslant \mathfrak{c}$ for any non-discrete compact group [**15**]. This proves that $|K| = |G'|$. ∎

THEOREM 2.3. *Let G be a non-compact locally compact group with $\chi(G) = \kappa$ and $l(G) = \delta$. Then $w(\beta G) = w\big(\beta(\mathbb{D}^\kappa \times \delta)\big)$ and $|\beta G| = \big|\beta(\mathbb{D}^\kappa \times \delta)\big|$.*

PROOF: Consider first the case of a non-metrisable group G. Apply Proposition 2.2 to find an open σ-compact subgroup G' of the group G and a compact subgroup K of the group G' such that G'/K is second countable, $|K| = |G'|$ and $\chi(K) = \chi(G)$. Fix a countable dense set $S \subset G'/K$; since the quotient map $\pi : G' \to G'/K$ is open, the set $\pi^{-1}(S)$ is dense in G'. It is clear that $\pi^{-1}(S)$ is a countable union of cosets of K and therefore $K \times \omega$ maps continuously onto a dense subspace of G'.

The subgroup G' is open in G so $G \simeq G' \times \delta'$ for some $\delta' \leqslant \delta$ (it is even possible that $\delta' = 1$); thus the space $(K \times \omega) \times \delta \simeq K \times \delta$ maps continuously onto a dense subspace of G. This implies that $\beta(K \times \delta)$ maps continuously onto βG; therefore $|\beta G| \leqslant \big|\beta(K \times \delta)\big|$ and $w(\beta G) \leqslant w\big(\beta(K \times \delta)\big)$.

We have $w(K) = \chi(K) = \kappa$ [**5,** Theorem 3.12] and therefore $\mathbb{D}^\kappa$ maps continuously onto the space K [**14**]; an immediate consequence is that $\mathbb{D}^\kappa \times \delta$ maps continuously onto $K \times \delta$ and thus $\beta(\mathbb{D}^\kappa \times \delta)$ maps continuously onto $\beta(K \times \delta)$. This gives us inequalities $|\beta G| \leqslant \big|\beta(K \times \delta)\big| \leqslant \big|\beta(\mathbb{D}^\kappa \times \delta)\big|$ and $w(\beta G) \leqslant w\big(\beta(K \times \delta)\big) \leqslant w\big(\beta(\mathbb{D}^\kappa \times \delta)\big)$.

On the other hand, a theorem of Shapirovsky [**15**] says that the space K maps continuously onto $\mathbb{I}^\kappa$. Therefore $K \times \delta$ maps continuously onto $\mathbb{I}^\kappa \times \delta$ and thus $\beta(K \times \delta)$ maps continuously onto the space $\beta(\mathbb{I}^\kappa \times \delta)$ which gives us inequalities $\big|\beta(K \times \delta)\big| \geqslant \big|\beta(\mathbb{I}^\kappa \times \delta)\big| \geqslant \big|\beta(\mathbb{D}^\kappa \times \delta)\big|$ and $w\big(\beta(K \times \delta)\big) \geqslant w\big(\beta(\mathbb{I}^\kappa \times \delta)\big) \geqslant w\big(\beta(\mathbb{D}^\kappa \times \delta)\big)$.

If G is not σ-compact then $G \simeq G' \times \delta$ by Proposition 2.1; an immediate consequence is that $K \times \delta$ is a closed subspace of G. If the group G is σ-compact and there are infiniteley many cosets of G' in G then again $K \times \delta = K \times \omega$ is a closed subspace of G. If the number of cosets of G' is finite then G' can not be compact because G is not compact. The map π being perfect, the second countable group G'/K is not compact and therefore there is an infinite closed discrete $D \subset G'/K$. It is evident that the space $\pi^{-1}(D)$ is a closed subspace of G' homeomorphic to $K \times \omega = K \times \delta$. This shows that, in all cases, $K \times \delta$ is a closed subspace of G so $|\beta G| \geqslant \big|\beta(K \times \delta)\big| \geqslant \big|\beta(\mathbb{D}^\kappa \times \delta)\big|$ and $w(\beta G) \geqslant w\big(\beta(K \times \delta)\big) \geqslant w\big(\beta(\mathbb{D}^\kappa \times \delta)\big)$.

Now assume that G is metrisable and hence $\chi(G) = \omega$. If G is discrete then $|G| = l(G) = \delta \geqslant \omega$ and hence G is homeomorphic to the space $\mathbb{D}^1 \times \delta$. Since

$\chi(G) = 1$, the result follows. If G is non-discrete, fix an open σ-compact subgroup G' of the group G. If G/G' is infinite, take any open neighbourhood U of the identity e such that $U \subset G'$ and $K = \overline{U}$ is compact. If G/G' is finite then the second countable group G' is not compact so we have a neighbourhood U of the identity e such that $K = \overline{U}$ is compact and the family $\{a \cdot U : a \in A\}$ is discrete for some infinite $A \subset G'$ (recall that a family $\mathcal{F}$ of subsets of G' is called *discrete* if any point of G' has a neighbourhood which intersects at most one element of $\mathcal{F}$).

Observe that, if G is not σ-compact then $G \simeq G' \times \delta$ by Proposition 2.1. Since $K \subset G'$, this implies that $K \times \delta$ is a closed subspace of G. If the group G is σ-compact and G/G' is infinite then again $K \times \delta = K \times \omega$ is a closed subspace of G because $G \simeq G' \times \omega$. Finally, if G/G' is finite then, by our choice of U, there is an infinite $A \subset G'$ such that the family $\{a \cdot U : a \in A\}$ is discrete. The family $\mathcal{F} = \{a \cdot \overline{U} : a \in A\}$ is also discrete and $\cup\mathcal{F}$ is homeomorphic to $K \times \omega = K \times \delta$. Thus, in all cases, $K \times \delta$ can be considered a closed subspace of G.

Since G' is separable the space $K \times \omega$ maps continuously onto a dense subspace of G'. The subgroup G' is open in G so $G \simeq G' \times \delta'$ for some $\delta' \leqslant \delta$; thus the space $(K \times \omega) \times \delta \simeq K \times \delta$ maps continuously onto a dense subspace of G. This implies that $\beta(K \times \delta)$ maps continuously onto βG; therefore $|\beta G| \leqslant |\beta(K \times \delta)|$ and $w(\beta G) \leqslant w(\beta(K \times \delta))$. We have $w(K) = \omega$ and therefore $\mathbb{D}^\omega$ maps continuously onto the space K; an immediate consequence is that $\mathbb{D}^\omega \times \delta$ maps continuously onto $K \times \delta$ and thus $\beta(\mathbb{D}^\omega \times \delta)$ maps continuously onto $\beta(K \times \delta)$. This gives us inequalities $|\beta G| \leqslant |\beta(K \times \delta)| \leqslant |\beta(\mathbb{D}^\omega \times \delta)|$ and $w(\beta G) \leqslant w(\beta(K \times \delta)) \leqslant w(\beta(\mathbb{D}^\omega \times \delta))$.

The metrisable compact space $K = \overline{U}$ cannot have isolated points because the group G is non-discrete. Applying an easier version of the theorem of Shapirovsky **[15]** we conclude that the space K maps continuously onto $\mathbb{I}^\omega$. Therefore $K \times \delta$ maps continuously onto $\mathbb{I}^\omega \times \delta$ and thus $\beta(K \times \delta)$ maps continuously onto the space $\beta(\mathbb{I}^\omega \times \delta)$ which gives us inequalities $|\beta(K \times \delta)| \geqslant |\beta(\mathbb{I}^\omega \times \delta)| \geqslant |\beta(\mathbb{D}^\omega \times \delta)|$ and $w(\beta(K \times \delta)) \geqslant w(\beta(\mathbb{I}^\omega \times \delta)) \geqslant w(\beta(\mathbb{D}^\omega \times \delta))$. We already saw that $K \times \delta$ is a closed subspace of the group G so we have $|\beta G| \geqslant |\beta(K \times \delta)| \geqslant |\beta(\mathbb{D}^\omega \times \delta)|$ and $w(\beta G) \geqslant w(\beta(K \times \delta)) \geqslant w(\beta(\mathbb{D}^\omega \times \delta))$. ∎

Recall that, given a space X, a set $A \subset C^*(X)$ is *uniformly dense* in $C^*(X)$ if, for any $f \in C^*(X)$ and any $\varepsilon > 0$, there is $g \in A$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$. We shall also need the usual sup-metric on $C^*(X)$ defined by $d(f,g) = \sup\{|f(x)-g(x)| : x \in X\}$ for any $f,g \in C^*(X)$. It is easy to see that A is uniformly dense in $C^*(X)$ if and only if A is a dense set of the metric space $(C^*(X), d)$.

DEFINITION 2.4: Given a space X, let $\mathrm{ud}(X) = \min\{|A| : A \subset C^*(X)$ and A is uniformly dense in $C^*(X)\}$.

The following two simple facts are well-known as a folklore but we give their proofs here for the sake of completeness.

PROPOSITION 2.5. *If X is any compact space X, then $w(X) = \mathrm{ud}\,(X)$.*

PROOF: If A is uniformly dense in $C^*(X) = C(X)$ then it is also dense in $C_p(X)$; therefore $w(X) = d\big(C_p(X)\big) \leqslant |A|$ ([**1**, Theorem I.1.5]) and hence $w(X) \leqslant \mathrm{ud}\,(X)$. On the other hand, there exists a dense $A \subset C_p(X)$ with $|A| = w(X)$. If $R(A)$ is the rational algebra generated by A then $|A| = \big|R(A)\big|$ and $R(A)$ is uniformly dense in $C(X)$ by the Stone-Weierstrass theorem [**7**, Theorem 3.2.21]. This shows that $\mathrm{ud}\,(X) \leqslant w(X)$ so the equality $\mathrm{ud}\,(X) = w(X)$ is established. ∎

PROPOSITION 2.6. *For any Tychonoff space X, we have $\mathrm{ud}\,(X) = w(\beta X)$.*

PROOF: Each $f \in C^*(X)$ has a continuous extension $u(f) : \beta X \to \mathbb{R}$. Denote by π the restriction map from $C(\beta X)$ to $C^*(X)$. It is clear that $\pi\big(u(f)\big) = f$ for any $f \in C^*(X)$. Apply Proposition 2.5 to find a set $A \subset C(\beta X)$ which is uniformly dense in $C(\beta X)$ and $|A| = w(\beta X)$. Given any $f \in C^*(X)$ and $\varepsilon > 0$, there is $g \in A$ such that $\big|u(f)(x) - g(x)\big| < \varepsilon$ for all $x \in \beta X$. This implies $\big|\pi(g)(x) - f(x)\big| = \big|g(x) - f(x)\big| < \varepsilon$ for all $x \in X$. This shows that $\pi(A)$ is uniformly dense in $C^*(X)$ and therefore $\mathrm{ud}\,(X) \leqslant |A| = w(\beta X)$.

Now take any set $A \subset C^*(X)$ which is uniformly dense in $C^*(X)$ and $|A| = \mathrm{ud}\,(X)$. The set $B = u(A) = \big\{u(f) : f \in A\big\}$ is uniformly dense in $C(\beta X)$. Indeed, take any $f \in C(\beta X)$ and $\varepsilon > 0$. There exists $g \in A$ such that $\big|g(x) - \pi(f)(x)\big| < \varepsilon/2$ for all $x \in X$. Since $\pi(f)(x) = f(x)$ for any $x \in X$, we have $\big|g(x) - f(x)\big| < \varepsilon/2$ for all $x \in X$. If $g' = u(g)$ then the function $h = |g' - f|$ is continuous on βX and $h(X) \subset \big[0, \varepsilon/2\big)$. Since X is dense in βX we have $h(\beta X) \subset \big[0, (\varepsilon/2)\big] \subset [0, \varepsilon)$ which shows that $\big|g'(y) - f(y)\big| < \varepsilon$ for all $y \in \beta X$. Thus the set B is uniformly dense in $C(\beta X)$ and therefore $w(\beta X) \leqslant |B| = \mathrm{ud}\,(X)$. ∎

THEOREM 2.7. *For any infinite cardinals κ and δ, we have $w\big(\beta(\mathbb{D}^\kappa \times \delta)\big) = \kappa^\delta$.*

PROOF: By Proposition 2.6, it suffices to prove that $\mathrm{ud}\,(\mathbb{D}^\kappa \times \delta) = \kappa^\delta$. Since $w(\mathbb{D}^\kappa) = \mathrm{ud}\,(\mathbb{D}^\kappa) = \kappa$ (Proposition 2.5), we can find a set $F = \{f_\gamma : \gamma < \kappa\} \subset C(\mathbb{D}^\kappa)$ such that F is uniformly dense in $C(\mathbb{D}^\kappa)$. The set $P_\delta(F)$ of all possible functions from δ to F has cardinality κ^δ. Given any $\varphi \in P_\delta(F)$, let $g_\varphi(x, \beta) = \varphi(\beta)(x)$ for any $(x, \beta) \in \mathbb{D}^\kappa \times \delta$. Then $g_\varphi : \mathbb{D}^\kappa \times \delta \to \mathbb{R}$ for each $\varphi \in P_\delta(F)$ and the set $G = \big\{g_\varphi : \varphi \in P_\delta(F)\big\}$ is uniformly dense in $C^*(\mathbb{D}^\kappa \times \delta)$.

Indeed, take any $f \in C^*(\mathbb{D}^\kappa \times \delta)$ and any $\varepsilon > 0$; for each $\beta < \delta$, the function $f \mid \big(\mathbb{D}^\kappa \times \{\beta\}\big)$ is continuous on $\mathbb{D}^\kappa \times \{\beta\}$ which is a copy of $\mathbb{D}^\kappa$. Therefore there is $g_\beta \in F$ such that $\big|g_\beta(x) - f(x, \beta)\big| < \varepsilon$ for all $x \in \mathbb{D}^\kappa$. Now, if $\varphi(\beta) = g_\beta$ then $g_\varphi \in G$ and $\big|g_\varphi(x, \beta) - f(x, \beta)\big| < \varepsilon$ for all $x \in \mathbb{D}^\kappa$ and $\beta < \delta$; this proves that G is uniformly dense in $C^*(\mathbb{D}^\kappa \times \delta)$. Consequently, we have $\mathrm{ud}\,(\mathbb{D}^\kappa \times \delta) \leqslant |G| \leqslant \kappa^\delta$.

To show that $\mathrm{ud}(\mathbb{D}^\kappa \times \delta) \geqslant \kappa^\delta$ observe that $\mathbb{D}^\kappa \times \delta$ is normal so $\mathrm{ud}(\mathbb{D}^\kappa \times \delta) \geqslant \mathrm{ud}(Y)$ for any closed $Y \subset \mathbb{D}^\kappa \times \delta$. The one-point compactification $A(\kappa)$ of the discrete space κ, embeds in $\mathbb{D}^\kappa$ so let us take $Y = A(\kappa) \times \delta$. We shall prove that $\mathrm{ud}(Y) \geqslant \kappa^\delta$; take any function $\varphi : \delta \to \kappa$ and consider the set $P_\varphi = \{(\varphi(\beta), \beta) : \beta < \delta\} \subset A(\kappa) \times \delta$. It is evident that P_φ is a clopen subset of Y so $h_\varphi = \chi_{P_\varphi}$ is a continuous function on Y. Since $\varphi \mapsto h_\varphi$ is a bijection, the set $G = \{h_\varphi : \varphi$ is a function from δ to $\kappa\}$ has cardinality κ^δ and $d(g, g') = 1$ for any distinct $g, g' \in G$.

Finally, assume that $A \subset C(Y)$ is uniformly dense in $C^*(Y)$ and $|A| < \kappa^\delta$. For each $g \in G$ fix a function $p_g \in A$ such that $d(p_g, g) < 1/3$. Since $|G| = \kappa^\delta$ and $|A| < \kappa^\delta$, there exist distinct $g, g' \in G$ such that $p_g = p_{g'}$. Then $1 = d(g, g') \leqslant d(g, p_g) + d(p_g, g') < 1/3 + 1/3 = 2/3 < 1$ which is a contradiction. Thus it is impossible that $|A| < \kappa^\delta$ so $\mathrm{ud}(\mathbb{D}^\kappa \times \delta) \geqslant \mathrm{ud}(Y) \geqslant \kappa^\delta$ and our proof is complete. ∎

The following easy fact is also a folklore of C_p-theory.

FACT 2.8. If X is a zero-dimensional space then $C_p(X, \mathbb{D})$ is dense in $\mathbb{D}^X$.

PROOF: Given disjoint finite sets $K, L \subset X$, let $O(K, L) = \{f \in \mathbb{D}^X : f(K) \subset \{0\}$ and $f(L) \subset \{1\}\}$; it is evident that the family $\mathcal{B} = \{O(K, L) : K, L$ are disjoint finite subsets of $X\}$ is a base of the space $\mathbb{D}^X$. For an arbitrary $U = O(K, L) \in \mathcal{B}$ we can find a disjoint family $\{W_x : x \in K \cup L\}$ of clopen subsets of X such that $x \in W_x$ for each $x \in K \cup L$. If $W = \cup\{W_x : x \in L\}$ then W is a clopen subset of X so $\chi_W \in C_p(X, \mathbb{D}) \cap O(K, L)$ which proves that $C_p(X, \mathbb{D})$ is dense in $\mathbb{D}^X$. ∎

FACT 2.9. If K is a zero-dimensional compact space and $A \subset C_p(K, \mathbb{D})$ separates the points of K then the ring $R(A)$ generated by the set A coincides with $C_p(K, \mathbb{D})$.

PROOF: It is evident that $R(A)$ also separates the points of K. Besides, given any $f, g \in R(A)$, the functions $1 - f$, $\max\{f, g\} = f + g + f \cdot g$, and $\min\{f, g\} = f \cdot g$ also belong to $R(A)$. This makes it possible to apply [**1**, Lemma IV.3.2] to conclude that $R(A) = C_p(K, \mathbb{D})$. ∎

LEMMA 2.10. *For any infinite cardinals κ and δ, the space $(A(\kappa))^\omega \times \delta$ maps continuously onto a dense subspace of $\mathbb{D}^{\kappa^\delta}$.*

PROOF: Given any $\beta < \delta$, let $\pi_\beta : (A(\kappa))^\delta \to A(\kappa)$ be the natural projection onto the β-th factor. For any $\xi < \kappa$, let $f_\xi = \chi_{\{\xi\}}$ be the characteristic function of the set $\{\xi\}$ in the space $A(\kappa)$. Let u denote the function which is identically zero on $A(\kappa)^\delta$. Then $A_\beta = \{f_\xi \circ \pi_\beta : \xi < \kappa\} \cup \{u\}$ is a subspace of $C_p\big((A(\kappa))^\delta, \mathbb{D}\big)$ homeomorphic to $A(\kappa)$. Indeed, take any family $\mathcal{U} \subset \tau\big(C_p((A(\kappa))^\delta, \mathbb{D})\big)$ with $A_\beta \subset \cup\mathcal{U}$. There is $U \in \mathcal{U}$ such that $u \in U$; the set U being open in $C_p\big((A(\kappa))^\delta, \mathbb{D}\big)$ there is a finite $P \subset (A(\kappa))^\delta$ such that $\big\{f \in C_p\big((A(\kappa))^\delta, \mathbb{D}\big) : f(P) = \{0\}\big\} \subset U$. The set $Q = \pi_\beta(P) \subset A(\kappa)$ is finite; if $\xi \in \kappa \backslash Q$ then $f_\xi(Q) = \{0\}$ and therefore $(f_\xi \circ \pi_\beta)(P) = \{0\}$. This shows that the set $A_\beta \backslash U \subset \{f_\xi \circ \pi_\beta : \xi \in Q\}$ is finite. It turns out that any neighbourhood of the point u contains all but finitely many points of the space A_β. Since $|A_\beta| = \kappa$, we have $A_\beta \simeq A(\kappa)$ for each $\beta < \delta$.

Our next step is to verify that the set $A = \cup\{A_\beta : \beta < \delta\}$ separates the points of the space $\left(A(\kappa)\right)^\delta$. If $x, y \in \left(A(\kappa)\right)^\delta$ are distinct points then $x' = \pi_\beta(x)$ and $y' = \pi_\beta(y)$ are distinct for some $\beta < \delta$. Therefore one of the points x', y', say x', is distinct from a, that is, $x' = \xi$ for some $\xi < \kappa$ and hence f_ξ separates x' and y'. Consequently, the function $f_\xi \circ \pi_\beta$ separates the points x and y.

Fix a homeomorphism $h_\beta : A(\kappa) \to A_\beta$ for each $\beta < \delta$. Given any point $z = (x, \beta) \in A(\kappa) \times \delta$ let $h(z) = h_\beta(x) \in A$. It is immediate that the map $h : A(\kappa) \times \delta \to A$ is continuous and onto. Let $P(A) = \{f_1 \cdot \ldots \cdot f_n : n \in \mathbb{N},\ f_i \in A \text{ for all } i \leqslant n\}$ and $S(A) = \{g_1 + \ldots + g_m : m \in \mathbb{N},\ g_i \in P(A) \text{ for all } i \leqslant m\}$. It is clear that $S(A)$ is a ring in $C_p\left(\left(A(\kappa)\right)^\delta, \mathbb{D}\right)$ which contains A; it is immediate that any ring that contains A has to contain $S(A)$ so $S(A) = R(A)$.

Next we show that $R(A) = S(A)$ can be represented as a countable union of continuous images of finite powers of A. Indeed, given $m \in \mathbb{N}$ and $n_1, \ldots, n_m \in \mathbb{N}$, the space $P(n_1, \ldots, n_m) = A^{n_1} \times \ldots \times A^{n_m}$ is a finite power of A. For each $i \in \mathbb{N}$, define a map $p_i : A^i \to C_p\left(\left(A(\kappa)\right)^\delta, \mathbb{D}\right)$ as follows: $p_1(f) = f$ for all $f \in A = A^1$ and $p_n(f_1, \ldots, f_n) = f_1 \cdot \ldots \cdot f_n$ for each $(f_1, \ldots, f_n) \in A^n$ for all $n > 1$.

Next we define a map $\varphi = \varphi_{n_1, \ldots, n_m} : P(n_1, \ldots, n_m) \to C_p\left(\left(A(\kappa)\right)^\delta, \mathbb{D}\right)$ as follows: for each $f = (f^1, \ldots, f^m) \in P(n_1, \ldots, n_m)$, where $f^i = (f^i_1, \ldots, f^i_{n_i}) \in A^{n_i}$ for each natural number $i \leqslant m$, let $\varphi(f) = p_{n_1}(f^1) + \ldots + p_{n_m}(f^m) \in C_p\left(\left(A(\kappa)\right)^\delta, \mathbb{D}\right)$. The map $\varphi_{n_1, \ldots, n_m}$ is continuous for any $(n_1, \ldots, n_m)$. If we denote by $Q(n_1, \ldots, n_m)$ the image of the set $P(n_1, \ldots, n_m)$ under the map $\varphi_{n_1, \ldots, n_m}$ then we obtain the equality $R(A) = \cup\{Q(n_1, \ldots, n_m) : m \in \mathbb{N} \text{ and } n_1, \ldots, n_m \in \mathbb{N}\}$.

Thus we have shown that there is a sequence $\{A^{n_i} : i \in \mathbb{N}\}$ of finite powers of A and a sequence $\{g_i : i \in \mathbb{N}\}$ of continuous maps such that $g_i : A^{n_i} \to R(A)$ for all $i \in \mathbb{N}$ and $R(A) = \cup\{g_i(A_{n_i}) : i \in \mathbb{N}\}$.

Recall that the space $A(\kappa) \times \delta$ maps continuously onto A; note that we have $(A(\kappa) \times \delta)^m \simeq \left(A(\kappa)\right)^m \times \delta$ and therefore $\left(A(\kappa)\right)^m \times \delta$ maps continuously onto A^m for each $m \in \mathbb{N}$. Since $\left(A(\kappa)\right)^\omega$ can be mapped onto each $\left(A(\kappa)\right)^m$ for each natural number m, the space $\left(A(\kappa)\right)^\omega \times \delta$ maps continuously onto $\left(A(\kappa)\right)^m \times \delta$ and hence it also maps continuously onto A^m. Besides, it follows from $\left(A(\kappa)\right)^\omega \times \delta \simeq \left(\left(A(\kappa)\right)^\omega \times \delta\right) \times \omega$ that $\left(A(\kappa)\right)^\omega \times \delta$ maps continuously onto $R = \bigoplus\{A^{n_i} : i \in \mathbb{N}\}$ and R maps continuously onto $R(A)$. Finally, observe that $R(A)$ coincides with $C_p\left(\left(A(\kappa)\right)^\delta, \mathbb{D}\right)$ by Fact 2.9; as a consequence, $\left(A(\kappa)\right)^{\omega} \times \delta$ maps continuously onto $C_p\Big(\left(A(\kappa)\right)^{\delta}, \mathbb{D}\Big)$ which is dense in $\mathbb{D}^{(A(\kappa))^{\delta}}$ by Fact 2.8. Since $\mathbb{D}^{(A(\kappa))^{\delta}}$ is homeomorphic to $\mathbb{D}^{\kappa^{\delta}}$, the proof of our lemma is complete. ∎

THEOREM 2.11. *For any infinite cardinals κ and δ, we have $\left|\beta(\mathbb{D}^{\kappa} \times \delta)\right| = 2^{\kappa^{\delta}}$.*

PROOF: Since $|X| \leqslant 2^{w(X)}$ for an arbitrary regular space X [**9**, Theorem 3.1], for $X = \mathbb{D}^{\kappa} \times \delta$, we have $\left|\beta(\mathbb{D}^{\kappa} \times \delta)\right| \leqslant 2^{w(X)} \leqslant 2^{\kappa^{\delta}}$ by Theorem 2.7 so we must prove only that $\left|\beta(\mathbb{D}^{\kappa} \times \delta)\right| \geqslant 2^{\kappa^{\delta}}$.

Since $w\Big(\left(A(\kappa)\right)^{\omega}\Big) = \kappa$, the compact space $K = \left(A(\kappa)\right)^{\omega}$ embeds in $\mathbb{D}^{\kappa}$ and therefore $K \times \delta$ is a closed subspace of $\mathbb{D}^{\kappa} \times \delta$. Since the space $\mathbb{D}^{\kappa} \times \delta$ is normal, the closure of the set $K \times \delta$ in the space $\beta(\mathbb{D}^{\kappa} \times \delta)$ is homeomorphic to $\beta(K \times \delta)$; thus $\left|\beta(\mathbb{D}^{\kappa} \times \delta)\right| \geqslant \left|\beta(K \times \delta)\right|$. The space $K \times \delta$ maps continuously onto a dense subspace of $\mathbb{D}^{\kappa^{\delta}}$ by Lemma 2.10. Therefore the space $\beta(K \times \delta)$ maps continuously onto $\mathbb{D}^{\kappa^{\delta}}$ and, in particular, $\left|\beta(K \times \delta)\right| \geqslant |\mathbb{D}^{\kappa^{\delta}}| = 2^{\kappa^{\delta}}$. As a consequence, $\left|\beta(\mathbb{D}^{\kappa} \times \delta)\right| \geqslant \left|\beta(K \times \delta)\right| \geqslant 2^{\kappa^{\delta}}$. ∎

COROLLARY 2.12. *Let κ and δ be infinite cardinals. If G is a locally compact non-compact group with $\chi(G) = \kappa$ and $l(G) = \delta$ then $|\beta G| = 2^{\kappa^{\delta}}$ and $w(\beta G) = \kappa^{\delta}$.*

PROOF: Apply Theorems 2.3, 2.7 and 2.11. ∎

COROLLARY 2.13. *Let κ and δ be infinite cardinals such that $\kappa^{\delta} = \kappa$. If G is a locally compact group with $\chi(G) = \kappa$ and $l(G) = \delta$ then $|\beta G| = |G|$ and $w(\beta G) = w(G)$.*

PROOF: Since the case of a compact G is trivial, we can assume that G is not compact. Obviously, the cardinal κ is uncountable so Proposition 2.2 is applicable to find an open σ-compact subgroup G' of the group G such that there is a compact subgroup $K \subset G'$ with $\chi(K) = \kappa$ and hence $|K| = 2^{\kappa}$ [**5**, Theorem 3.9]. Thus $|G| \geqslant |K| = 2^{\kappa}$ and $|G| \leqslant |\beta G| = 2^{\kappa^{\delta}} = 2^{\kappa}$. This shows that $|G| = |\beta G| = 2^{\kappa}$. Besides, $w(\beta G) = \kappa^{\delta} = \kappa$ by Corollary 2.12 and $w(G) \geqslant w(K) = \chi(K) = \kappa$ so $w(\beta G) = w(G) = \kappa$. ∎

COROLLARY 2.14. *Suppose that G is a locally compact metrisable group with $l(G) \geqslant \mathfrak{c}$. Then $|\beta G| = 2^{2^{|G|}}$.*

PROOF: Take an open σ-compact subgroup $G' \subset G$. Being metrisable, the group G' is second countable so $|G'| \leqslant \mathfrak{c}$. The group G is not σ-compact because $\delta = l(G) > \omega$ so Propositon 2.1 can be applied to conclude that $G \simeq G' \times \delta$ and hence the group G is a union of δ-many copies of G'. Consequently, $\delta \leqslant |G| = \delta \cdot |G'| \leqslant \delta \cdot \mathfrak{c} = \delta$. Therefore $|G| = \delta$ while $|\beta G| = 2^{\omega^{\delta}} = 2^{2^{\delta}} = 2^{2^{|G|}}$ by Corollary 2.12. ∎

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THEOREM 2.15. *Under the Continuum Hypothesis, for a metrisable locally compact group G, we have $|\beta G| = 2^{2^{|G|}}$ if and only if either G is discrete or it is not separable.*

PROOF: For any discrete space D we have $|\beta D| = 2^{2^{|D|}}$ so, if the group G is discrete then $|\beta G| = 2^{2^{|G|}}$. If G is not separable then $l(G) = w(G) \geqslant \omega_1 = \mathfrak{c}$ so Corollary 2.14 is applicable to conclude that again $|\beta G| = 2^{2^{|G|}}$.

Now, if G is a separable (and hence second countable) non-discrete locally compact metrisable group then $|G| = \mathfrak{c}$ and $l(G) = \chi(G) = \omega$ so $|\beta G| = 2^{\mathfrak{c}} = 2^{|G|}$ by Corollary 2.12. ∎

COROLLARY 2.16. *Let G be any locally compact non-discrete σ-compact group. Then $|\beta G| \leqslant 2^{|G|}$.*

PROOF: If G is metrisable then $l(G) = w(G) = \chi(G) = \omega$ so $|G| = \mathfrak{c}$. Corollary 2.12 implies that $|\beta G| = 2^{\omega^\omega} = 2^{\mathfrak{c}} = 2^{|G|}$. If G is not metrisable then $\kappa = \chi(G) > \omega$; by Proposition 2.2 we can find a compact subgroup of the group G such that G/K is second countable and $\chi(K) = \chi(G) > \omega$. Since K is a non-discrete compact group, we have $|K| = 2^{\kappa}$ [**5**, Theorem 3.9] and therefore $|G| = 2^{\kappa} \cdot \mathfrak{c} \geqslant 2^{\kappa}$. On the other hand, $|\beta G| = 2^{\kappa^\omega}$ by Corollary 2.12. Since $\kappa^\omega \leqslant 2^{\kappa}$, we have $|\beta G| \leqslant 2^{2^{\kappa}} \leqslant 2^{|G|}$. ∎

EXAMPLE 2.17. Any countably infinite discrete group G is σ-compact and locally compact while we have $|\beta G| = 2^{2^{|G|}}$. This shows that being non-discrete is essential in Corollary 2.16.

EXAMPLE 2.18. Theorem 2.15 is not true in ZFC for all metrisable locally compact groups. To see this, take any discrete group D of cardinality ω_1 and let $G = \mathbb{R} \times D$. Since $\chi(G) = \omega$ and $l(G) = \omega_1$, we have $|G| = \mathfrak{c}$ and $|\beta G| = 2^{2^{\omega_1}}$ by Corollary 2.12 so, depending on what 2^{ω_1} is, we can have $|\beta G| = 2^{2^{|G|}}$ or $|\beta G| = 2^{|G|}$.

LEMMA 2.19. *If K is a compact space and and we have a set $A \subset K$ with $|A| \leqslant \mathfrak{c}$ then there is a countably compact set $A' \subset K$ such that $A \subset A'$ and $|A'| \leqslant \mathfrak{c}$.*

PROOF: This is a standard fact proved by transfinite induction of ω_1 steps which consist in adding an accumulation point for every countable subset of the set we have at the current step (see [**7**, Example 3.10.19]). ∎

EXAMPLE 2.20. If we do not require local compactness of G, then we can have $|\beta G| = 2^{2^{|G|}}$ for a countably compact group G.

PROOF: Let $H = \mathbb{D}^{2^{\mathfrak{c}}}$; the group H has a dense subspace S with $|S| = \mathfrak{c}$ [**13**]. Take a countably compact subspace $T_0 \supset S$ with $|T_0| = \mathfrak{c}$ which exists by Lemma 2.19. Let G_0 be the group generated by the set T_0 in H. Assume that $\alpha < \omega_1$ and we have countably compact sets T_β, $\beta < \alpha$ and groups G_β, $\beta < \alpha$ such that $|T_\beta| = |G_\beta| = \mathfrak{c}$ and $T_\beta \subset G_\beta \subset T_{\beta'}$ whenever $\beta < \beta' < \alpha$.

The set $G'_\alpha = \cup\{G_\beta : \beta < \alpha\}$ has cardinality $\leqslant \mathfrak{c}$ so there is a countably compact set $T_\alpha \subset H$ such that $|T_\alpha| = \mathfrak{c}$ and $G'_\alpha \subset T_\alpha$ (Lemma 2.19). Letting G_α to be the subgroup of H generated by T_α we complete our transfinite construction which gives us families $\{T_\alpha : \alpha < \omega_1\}$ and $\{G_\alpha : \alpha < \omega_1\}$. It is evident that $G = \cup\{G_\alpha : \alpha < \omega_1\}$ is a dense subgroup of H; the group G is countably compact because $G = \cup\{T_\alpha : \alpha < \omega_1\}$ so every countable subset of G is contained in some countably compact T_α.

It is clear that $|G| = \mathfrak{c}$; besides, $\beta G = H$ because for any pseudocompact dense subspace Z of any product Π of second countable compact spaces we have $\beta Z = \Pi$ (see [**1**, Lemma I.2.6]). Finally, we have $|\beta G| = |H| = 2^{2^{\mathfrak{c}}} = 2^{2^{|G|}}$. $\square$

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