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Colouring Planar Graphs With Three Colours and No Large Monochromatic Components

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We prove the existence of a function $f : \mathbb{N} \to \mathbb{N}$ such that the vertices of every planar graph with maximum degree Δ can be 3-coloured in such a way that each monochromatic component has at most $f(\Delta)$ vertices. This is best possible (the number of colours cannot be reduced and the dependence on the maximum degree cannot be avoided) and answers a question raised by Kleinberg, Motwani, Raghavan and Venkatasubramanian in 1997. Our result extends to graphs of bounded genus.

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1. Introduction

A proper vertex colouring of a graph G is an assignment of colours to the vertices of G such that every colour class is a stable set. In other words, in each colour class, connected components consist of singletons. In this paper we investigate a relaxed version of this classical version of graph colouring, where connected components in each colour class, called *monochromatic components* in the rest of the paper, have bounded size.

The famous HEX Lemma implies that in every 2-colouring of the triangular $k \times k$-grid, there is a monochromatic path on k vertices. This shows that planar graphs with maximum degree 6 cannot be 2-coloured in such a way that all monochromatic components have bounded size. On the other hand, Haxell, Szabó and Tardos [4] proved that every (not necessarily planar) graph with maximum degree at most 5 can be 2-coloured in such a way that all monochromatic components have size at most 20000. This bound was later reduced to 1908 by Berke [2].

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As for three colours, Kleinberg, Motwani, Raghavan and Venkatasubramanian [7, Theorem 4.2] constructed planar graphs that cannot be 3-coloured in such a way that each monochromatic component has bounded size. However, their examples have large maximum degree, which prompted them to ask the following question.

Question 1.1 ([7], Question 4.3). Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that every planar graph with maximum degree at most Δ has a 3-colouring in which each monochromatic component has size at most $f(\Delta)$?

A similar construction was given by Alon, Ding, Oporowski and Vertigan [1, Theorem 6.6], who also pointed out that they do not know whether examples with bounded maximum degree can be constructed. Question 1.1 was also raised more recently by Linial, Matoušek, Sheffet and Tardos [9]. Our main result is a positive answer to this question.

Theorem 1.2. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every planar graph with maximum degree Δ has a 3-colouring in which each monochromatic component has size at most $f(\Delta)$.*

This theorem will be proved in Section 3. Let us remark that we prove Theorem 1.2 with a rather large function f, namely $f(\Delta) = (15\Delta)^{32\Delta+8}$, which is almost surely far from optimal. We have attempted to make our proofs as simple as possible, and as a result we have made no effort to optimize the various bounds appearing in the paper.

In Section 4, we extend Theorem 1.2 to graphs embeddable in a fixed surface. This improves a special case of a result of Alon, Ding, Oporowski and Vertigan [1], who proved that, for every proper minor-closed class of graphs $\mathcal{G}$, there is a function $f_{\mathcal{G}} : \mathbb{N} \to \mathbb{N}$ such that every graph in $\mathcal{G}$ with maximum degree Δ can be 4-coloured in such way that each monochromatic component has size at most $f_{\mathcal{G}}(\Delta)$.

Finally, in Section 5 we conclude with some remarks and open problems.

2. Preliminaries

All graphs in this paper are finite, undirected, and simple. We let $V(G)$ and $E(G)$ denote the vertex and edge sets, respectively, of a graph G. We use the shorthand $|G|$ for the number of vertices of a graph G and denote the maximum degree of G by $\Delta(G)$. We let $\deg_G(v)$ denote the degree of vertex v in G.

The term ‘colouring’ will always refer to a vertex colouring of the graph under consideration. For simplicity, we identify colours with positive integers, and we let a k-colouring be a colouring using colours in $\{1, 2, \dots, k\}$. Note that we do not require a colouring to be proper, that is, adjacent vertices may receive the same colour. Given a colouring ϕ of G, a *monochromatic component* is a connected component of the subgraph of G induced by some colour class. A monochromatic component of colour i is also called an *i-component*. The *size* of a component is its number of vertices.

3. Proof of the main theorem

We start with a brief sketch of the proof of Theorem 1.2. We consider a decomposition of the vertex set of our planar graph G drawn in the plane into sets $O_1, O_2, \ldots, O_k$, each inducing an outerplanar graph. The set O_1 is the vertex set of the outer face of G, and for $i = 2, \ldots, k$, the set O_i is the vertex set of the outer face of the subgraph of G induced by $V(G) \setminus (\bigcup_{1 \leqslant j \leqslant i-1} O_j)$.

We colour the graph G with colours $1, 2, 3$ in such way that for each $i \in \{1, \ldots, k\}$, no vertex of O_i has colour $1 + (i \bmod 3)$. This implies that each monochromatic component is contained in the union of two consecutive sets O_i and O_{i+1}. Starting with O_k, we colour the sets O_i one after the other in decreasing order of their index i. Given a colouring of O_{i+1}, we extend this colouring to a colouring of $O_{i+1} \cup O_i$. This extension is done so as to maintain the property that in one of the two colour classes of O_i, monochromatic components are particularly small. Thus the two colours do not play symmetric roles: one is ‘small’ and the other ‘large’. The small colour of O_{i+1} then becomes the large colour of O_i, while the large colour of O_{i+1} does not appear at all in O_i.

While the above approach is natural, we found that making it work required careful handling of a number of situations. In particular, we have introduced a technical lemma, Lemma 3.1 below, whose proof might appear somewhat uninviting to the otherwise interested reader. We hope the reader will bear with us until the main part of the argument, which is provided by Theorem 3.10.

Lemma 3.1. *Let G be a connected plane graph whose vertex set is partitioned into an induced path P on at least 3 vertices, and a stable set S with a distinguished vertex r. Let d be the maximum degree of a vertex in P, and let $\Delta := \Delta(G)$. Assume further that:*

- *r is adjacent to the two endpoints of P and no other vertex of P,*
- *the outer face of G is bounded by the chordless cycle $G[V(P) \cup \{r\}]$,*
- *every vertex in S has degree at least 2,*
- *if $u \in S$ has degree exactly 2, then the two neighbours of u on P are not adjacent, and*
- *every two consecutive vertices of P have at least one common neighbour in S.*

Then there exists a 2-colouring of G in which the two endpoints of P and all the vertices in S have colour 2, each 1-component has size at most $2d + 1$, and each 2-component has size at most $(3\Delta)^{3d-4}$.

Proof. First we need to introduce a number of definitions and notations. We think of the path P as being drawn horizontally in the plane with the vertices of S above P; thus the vertices of P are ordered from left to right. This ordering induces in a natural way a linear ordering of every subset $X \subseteq V(P)$. Two vertices of such a subset X are said to be *consecutive in X* if they are consecutive in this ordering. Let x and y denote the left and right endpoint, respectively, of the path P.

For simplicity, the colour *opposite to* 1 is defined to be 2, and *vice versa.* Consider a subset $X \subseteq V(P)$ with $|X| \geqslant 2$ and call a and b the leftmost and rightmost vertices of X, respectively. If a and b are coloured, each either 1 or 2, but no vertex in $X \setminus \{a, b\}$ is coloured, then an *$\{a, b\}$-alternate colouring* of X consists of keeping the colours on a, b, and colouring the vertices of $X \setminus \{a,b\}$ (if any) as follows. We enumerate the vertices of X from left to right as $a, x_1, \ldots, x_k, b$. If $k = 1$, then x_1 is coloured with colour 2 if both a and b have colour 1; otherwise, x_1 is coloured with colour 1. If $k \geqslant 2$, then x_1 and x_k are coloured with the colour opposite to that of a and b, respectively, and for each $i \in \{2, \ldots, k-1\}$, the vertex x_i is coloured with the colour opposite to that of x_{i-1}. Let us point out some simple but useful properties of this colouring:

- no three consecutive vertices in $a, x_1, \ldots, x_k, b$ have the same colour,
- if $k \geqslant 1$ and a has colour 2, then x_1 has colour 1, and
- if $k \geqslant 1$ and b has colour 2, then x_k has colour 1.

These properties will be used repeatedly, and sometimes implicitly, in what follows.

Let $\mathcal{F}$ be the set of bounded faces of G. For $f \in \mathcal{F}$, let ∂f denote the subgraph of G which is the boundary of f. We note that, because of our assumptions on S, every edge of P is included in the boundary of a triangular face of $\mathcal{F}$.

Let ρ denote the unique bounded face of G which includes the vertex r in its boundary. We define a rooted tree T with vertex set $\mathcal{F}$ and root ρ inductively as follows. First, let $s(\rho) = r$ and let

$$S(\rho) = (S \cap V(\partial\rho)) \setminus \{r\}.$$

Let the children of ρ in T be the faces distinct from ρ that are incident to some vertex in $S(\rho)$. Now, consider a face $f \in \mathcal{F} \setminus \{\rho\}$ with parent f^* in T. Let $s(f)$ be the unique vertex of S included in $V(\partial f) \cap V(\partial f^*)$ (the existence and uniqueness of $s(f)$ will be proved below). Let

$$S(f) = (S \cap V(\partial f)) \setminus \{s(f)\}.$$

The children of f are then all the faces $f' \neq f$ incident to a vertex of $S(f)$.

In order to show that T is well defined, we only need to prove that the vertex $s(f)$ defined above exists and is unique. The existence follows from the definition of T, since f and f^* share a vertex of $(S \cap V(\partial f^*)) \setminus \{s(f^*)\}$. Note also that for any vertex $v \in (S \cap V(\partial f^*)) \setminus \{s(f^*)\}$, the vertices v^- and v^+ just preceding and following v in a boundary walk of f^* both lie on P and any face incident to v distinct from f^* is inside the region bounded by v, v^-, v^+ and the subpath of P between v^- and v^+. It follows that $s(f)$ is unique.

The *depth* $\mathrm{dp}(f)$ of a face $f \in \mathcal{F}$ is its depth in T, the root ρ having depth 0. Observe that, because of our assumptions on S, the leaves of T are precisely the triangular faces sharing an edge with the outer face. Observe also that T can be equivalently defined as the (unique) breadth-first search tree rooted at ρ of the graph with vertex set $\mathcal{F}$ in which two vertices $f, f' \in \mathcal{F}$ are adjacent if the corresponding faces in G share a vertex of S.

A face $f \in \mathcal{F}$ is uniquely determined by, and uniquely determines, the triplet $[a,s,b]$ where $s = s(f)$ and a, b are the leftmost and rightmost neighbours of s on P included in ∂f, respectively. With a slight abuse of notation, we write $f = [a,s,b]$ to denote the face f with triplet $[a,s,b]$.

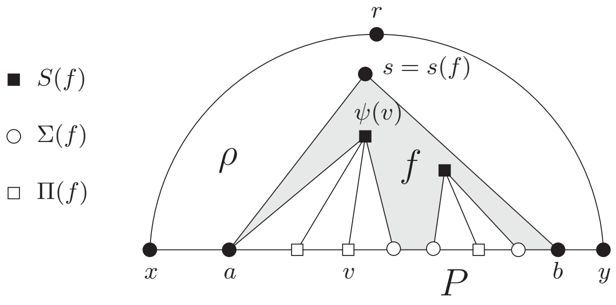

Figure 1. A face $f = [a, s, b]$ and the corresponding sets $S(f)$, $\Sigma(f)$ and $\Pi(f)$.

We define the following sets of vertices associated to a face $f = [a, s, b] \in \mathcal{F}$ (see Figure 1 for an illustration). The set

$$\Sigma(f) = (V(P) \cap V(\partial f)) \setminus \{a, b\}$$

is the set of *corners* of f, and

$$\Pi(f) = N(S(f)) \setminus (\Sigma(f) \cup \{a, b\})$$

is the set of *pivots* of f. Here, $N(X)$ denotes the set of vertices of $V(G) \setminus X$ having a neighbour in X.

Observe that the sets $\Sigma(f), \Sigma(f'), \Pi(f), \Pi(f')$ are pairwise disjoint for every two distinct faces $f, f' \in \mathcal{F}$. Moreover,

$$\bigcup_{f \in \mathcal{F}} (\Sigma(f) \cup \Pi(f)) = V(P) \setminus \{x, y\}.$$

Thus every *internal* vertex of the path P is either a corner or a pivot of some uniquely determined face f, which we denote by $f(v)$. When v is a pivot, the unique neighbour of v in S that is incident to $f(v)$ is denoted by $\psi(v)$. For each vertex $v \in S \setminus \{r\}$, similarly let $f(v)$ denote the unique face $f \in \mathcal{F}$ such that $v \in S(f)$.

Consider two faces $f = [a, s, b]$ and f' such that f' is inside the cycle formed by the edges as, bs and the path from a to b on P. Note that f is an ancestor of f' in T. The following observation describes precisely the subgraph of T induced by all the faces incident to a given internal vertex of P (see Figure 2 for an illustration).

Observation 3.2. *Let w be an internal vertex of P. Let $u_1, \ldots, u_k$ be the neighbours of w in clockwise order around w, with u_1 and u_k the left and right neighbours, respectively, of w on P. For each $i \in \{1, \ldots, k-1\}$, let f_i be the unique face in $\mathcal{F}$ with $wu_i, wu_{i+1} \in E(\partial f_i)$.*

(a) *If w is a corner of f_j for some $j \in \{1, \ldots, k-1\}$, then $f_1, f_2, \ldots, f_{k-1}$ is a path in T, with $f_j = f(w)$ being the face of smallest depth. In particular,*

$$\mathrm{dp}(f_i) - \mathrm{dp}(f(w)) \leqslant d - 2$$

for each $i \in \{1, \ldots, k-1\}$.

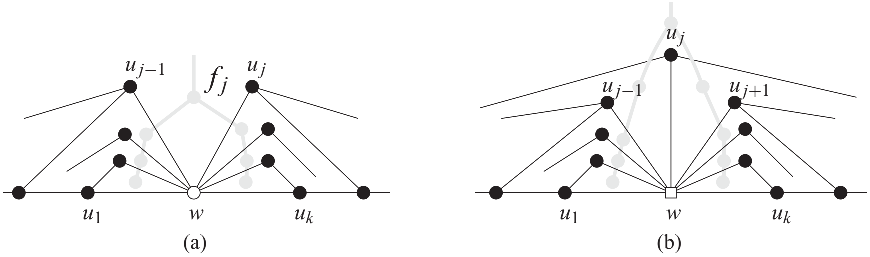

Figure 2. The two configurations in Observation 3.2. The tree T is depicted in grey.

(b) *If w is a pivot with $\psi(w) = u_j$, then $j \in \{2,\dots,k-1\}$ and $f_1,\dots,f_{j-1},f(w),f_j,\dots,f_{k-1}$ is a path in T, with $f(w)$ being the face of smallest depth. In particular,*

$$\mathrm{dp}(f_i) - \mathrm{dp}(f(w)) \leqslant d-2$$

for each $i \in \{1,\dots,k-1\}$.

An internal vertex v of P is said to be an *isolated pivot* if v is a pivot, $\deg_G(v) = 3$, and the two faces in $\mathcal{F}$ incident to v are triangular.

With these definitions in hand, we may now describe our colouring of the graph G. First, recall that the vertices in S must be coloured with colour 2. So it remains to colour the vertices of P. These vertices are coloured as follows. We perform a depth-first walk in T starting from its root ρ, and for each face $f \in \mathcal{F}$ encountered we colour the vertices in $\Pi(f)$ and $\Sigma(f)$. This ensures that, when considering a face $f = [a,s,b]$ distinct from the root ρ, the two vertices a and b are already coloured. Given $f = [a,s,b]$:

- if $f = \rho$, we colour both x and y with colour 2,
- if $f \neq \rho$, we perform an $\{a,b\}$-alternate colouring of $\Sigma(f) \cup \{a,b\}$,
- we colour each isolated pivot in $\Pi(f)$ with colour 2, and
- we colour each non-isolated pivot in $\Pi(f)$ with colour 1 if $\mathrm{dp}(f) \bmod 2d \in \{0,\dots,d-1\}$, and with colour 2 otherwise.

Let us consider the maximum size of monochromatic components in this colouring of G, starting with colour 1. Since all vertices in S and the two endpoints x and y of P have colour 2, each 1-component of G is a subpath of $P \setminus \{x,y\}$. We define a 1-*path* as a (not necessarily maximal) subpath of $P \setminus \{x,y\}$, every vertex of which has colour 1.

Claim 3.3. *If Q is a 1-path, then each vertex in S has at most two neighbours on Q, in which case they are consecutive vertices of Q.*

Proof. Let $w_1,\dots,w_k$ be the vertices of Q enumerated from left to right. Arguing by contradiction, suppose there exists $u \in S$ adjacent to w_i and w_j with $i+1 < j$, and choose such a triple (u,w_i,w_j) with $j-i$ minimum, and with respect to this, $\mathrm{dp}(f(u))$ maximum. The vertices w_i and w_{i+1} have a common neighbour $u' \in S$. If $u = u'$, then the triple (u,w_{i+1},w_j) is a better choice than (u,w_i,w_j), unless $j = i+2$, in which case w_{i+1} is an isolated pivot and has colour 2 (here we use the fact that there cannot be any vertex of

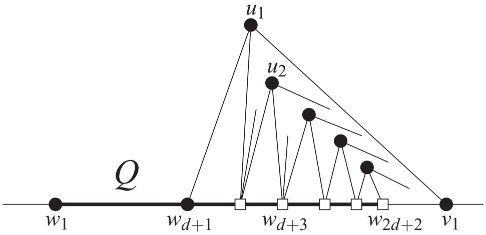

Figure 3. Illustration of the proof of Claim 3.4.

S inside the cycles uw_iw_{i+1} and $uw_{i+1}w_{i+2}$ since such a vertex would have degree exactly 2 and would be adjacent to two consecutive vertices of P). Thus we obtain a contradiction in both cases, and hence, $u \neq u'$. It follows that $\mathrm{dp}(f(u')) > \mathrm{dp}(f(u))$ since u' is inside the cycle consisting of the edges uw_i, uw_j and the subpath of Q between w_i and w_j. Now, the vertex u' cannot have degree exactly 2, and thus $u'w_\ell \in E(G)$ for some $\ell \in \{i+2,\dots,j\}$. However, the triple (u', w_i, w_ℓ) is then a better choice than (u, w_i, w_j), a contradiction (indeed, either $\ell < j$, or $\ell = j$ but $\mathrm{dp}(f(u')) > \mathrm{dp}(f(u))$). ☐

We deduce that 1-components have bounded size.

Claim 3.4. *Every* 1*-path has at most* $2d+1$ *vertices.*

Proof. Arguing by contradiction, suppose that Q is a 1-path with $2d+2$ vertices, and let $w_1,\dots,w_{2d+2}$ be its vertices enumerated from left to right. Let $u_1 \in S$ be a common neighbour of w_{d+1} and w_{d+2}. By Claim 3.3, w_{d+1} and w_{d+2} are the only neighbours of u_1 on Q, therefore u_1 has a neighbour in $V(P) \setminus V(Q)$ by our assumption on S. Let v_1 be such a neighbour at minimum distance from w_{d+2} on P. Either v_1 is on the right of Q or on the left of Q; since w_{d+1}, w_{d+2} are the two middle vertices of Q, these two cases are symmetric, and thus we may assume without loss of generality that v_1 is on the right of Q. Then $[w_{d+2}, u_1, v_1]$ is a face distinct from the root face ρ. Let z be the right neighbour of w_{2d+2} on P (thus $z \notin V(Q)$), and let A denote the z–v_1 subpath of P.

For $i = 2,\dots,d+1$, let $u_i \in S$ be a common neighbour of w_{d+i} and w_{d+i+1} (see Figure 3). By Claim 3.3, the vertices $u_1,\dots,u_k$ are all distinct, and thus each such vertex has a neighbour in $V(P) \setminus V(Q)$, which must then be on A because of the face $[w_{d+2}, u_1, v_1]$. Moreover, for each $i \in \{1,\dots,d+1\}$, we have that w_{d+i+1} is a pivot, and $u_i = \psi(w_{d+i+1})$. For each such index i, let $f_i = f(w_{d+i+1}) = f(u_i)$. It follows from Claim 3.2 that $1 \leqslant \mathrm{dp}(f_{i+1}) - \mathrm{dp}(f_i) \leqslant d-2$. This in turn implies that there exists an index $i \in \{1,\dots,d+1\}$ such that $\mathrm{dp}(f_i) \bmod 2d \in \{d,\dots,2d-1\}$. But then the pivot vertex w_{d+i+1} was coloured 2 in our colouring of G, a contradiction. ☐

We now bound the size of monochromatic components of colour 2. Therefore let K be a 2-component of G. We start by gathering a few observations about K.

Observe that if $f \in \mathcal{F}$ with $f = [a,s,b]$ then $\{a,b\}$ separates all vertices v such that $v \in S(f') \cup \Sigma(f') \cup \Pi(f')$ for some face f' that is a descendant of f in T from the remaining vertices of G. (Note that f is considered to be a descendant of itself.)

Observation 3.5. *Let* $f \in \mathcal{F}$ *with* $f = [a,s,b]$*, and let* K_f *be the set of vertices* $v \in V(K)$ *such that* $v \in S(f') \cup \Sigma(f') \cup \Pi(f')$ *for some face* f' *that is a descendant of* f *in* T*. If there are two vertices* $u, v \in V(K)$ *with* $u \in V(K_f)$ *and* $v \notin V(K_f)$*, then at least one of* a, b *is in* K*.*

Let $\mathcal{F}_K$ be the set of faces $f \in \mathcal{F}$ such that $S(f) \cup \Sigma(f) \cup \Pi(f)$ contains a vertex of K, and let T_K denote the subgraph of T induced by $\mathcal{F}_K$. Suppose that T_K is not connected. Then $\mathcal{F}_K$ contains two faces $f = [a,s,b]$ and f' such that the parent f^* of f is not in $\mathcal{F}_K$ and f' is not a descendant of f. By Observation 3.5, this implies that at least one of a, b is in K, and consequently $s \in V(K)$. Since $s \in S(f^*)$, we deduce that $f^* \in \mathcal{F}_K$, a contradiction.

Observation 3.6. T_K *is a subtree of* T*.*

Let $\widetilde{f}$ be the face in $\mathcal{F}_K$ having smallest depth in T. We see T_K as being rooted at $\widetilde{f}$. Our aim now is to bound the height of T_K.

Claim 3.7. T_K *has height at most* $3d - 5$*.*

Proof. Let f_1 be a leaf of T_K farthest from $\widetilde{f}$. We may assume that $f_1 \neq \widetilde{f}$, since otherwise T_K has height 0 and the claim trivially holds. Let A_K be a set of ancestors of f_1 in T_K, f_1 included. Thus A_K induces a path in T_K with endpoints f_1 and $\widetilde{f}$. Starting with f_1, we define inductively a sequence $f_1, f_2, \ldots, f_t$ of faces, with $f_i = [a_i, s_i, b_i]$ and $f_i \in A_K$ for each $i \in \{1, \ldots, t\}$, as follows. For $i \geqslant 2$, if f_{i-1} is distinct from $\widetilde{f}$, then by Observation 3.5, at least one of a_{i-1}, b_{i-1} is in K. Let h_{i-1} denote such a vertex, and let $f_i = f(h_{i-1})$. If $f_{i-1} = \widetilde{f}$ then f_i is not defined, and $f_{i-1} = f_t$ becomes the last face in the sequence.

Let $i \in \{2, \ldots, t\}$. By definition of T_K, the face f_i is in T_K. By Observation 3.2, f_i is an ancestor of f_{i-1}, which implies inductively that $f_i \in A_K$ (since $f_1 \in A_K$). Moreover, $\mathrm{dp}(f_i) < \mathrm{dp}(f_{i-1})$ since $f_i \neq f_{i-1}$. Since $f_i = f(h_{i-1})$ and f_{i-1} is incident to h_{i-1}, Observation 3.2 also implies that $\mathrm{dp}(f_{i-1}) \leqslant \mathrm{dp}(f_i) + d - 2$.

Let $i \in \{2, \ldots, t-1\}$. The vertex h_i has to be connected to h_{i-1} by a path in K. It follows from the definition of an $\{a_i, b_i\}$-alternate colouring that h_{i-1} cannot be a corner of f_i. (In fact, this is the key property of an $\{a_i, b_i\}$-alternate colouring.) Therefore, h_{i-1} is a pivot of f_i. Since

$$S(f_{i-1}) \cup \Sigma(f_{i-1}) \cup \Pi(f_{i-1}) \neq \varnothing,$$

the face f_{i-1} is not triangular, and hence h_{i-1} is not an isolated pivot. It follows that $\mathrm{dp}(f_i) \bmod 2d \in \{d, \ldots, 2d-1\}$.

Write $\mathrm{dp}(f_2) = 2kd + \ell_2$, with $d \leqslant \ell_2 \leqslant 2d - 1$, and for each $i \in \{3, \ldots, t-1\}$, let $\ell_i = \mathrm{dp}(f_i) - 2kd$. Since

$$\mathrm{dp}(f_i) < \mathrm{dp}(f_{i-1}) \leqslant \mathrm{dp}(f_i) + d - 2$$

for each $i \in \{2, \ldots, t-1\}$, and

$$\mathrm{dp}(f_i) \bmod 2d \in \{d, \ldots, 2d-1\},$$

we have

$$d \leqslant \ell_{t-1} < \ell_{t-2} < \cdots < \ell_2 \leqslant 2d-1.$$

In particular, $\ell_2 - \ell_{t-1} \leqslant d-1$. Now, the height of T_K is precisely

$$\begin{aligned}\mathrm{dp}(f_1) - \mathrm{dp}(f_t) &= \sum_{i=2}^{t}(\mathrm{dp}(f_{i-1}) - \mathrm{dp}(f_i)) \\ &= \mathrm{dp}(f_1) - \mathrm{dp}(f_2) + \ell_2 - \ell_{t-1} + \mathrm{dp}(f_{t-1}) - \mathrm{dp}(f_t).\end{aligned}$$

By Observation 3.2,

$$\mathrm{dp}(f_{t-1}) - \mathrm{dp}(f_t) \leqslant d-2 \quad \text{and} \quad \mathrm{dp}(f_1) - \mathrm{dp}(f_2) \leqslant d-2.$$

Using that $\ell_2 - \ell_{t-1} \leqslant d-1$, we obtain that the height of T_k is at most $3d-5$. □

Consider a face $f = [a, s, b]$ of T_K. By the definition of an $\{a, b\}$-alternate colouring, there are at most two consecutive vertices with colour 2 in $\Sigma(f)$. Therefore, using Observation 3.5, $\Sigma(f)$ contains at most 2 vertices of K and $S(f)$ contains at most 3 vertices of K. It follows that $|K \cap \Pi(f)| \leqslant 3(\Delta - 2)$, which implies that $S(f) \cup \Sigma(f) \cup \Pi(f)$ contains at most $3\Delta - 1$ vertices of K. Also, we deduce that f has at most 3Δ children in T_K. Using Claim 3.7, we then obtain

$$|T_K| \leqslant \sum_{i=0}^{3d-5}(3\Delta)^i = \frac{(3\Delta)^{3d-4}}{3\Delta - 1}$$

and hence

$$|K| \leqslant (3\Delta - 1)|T_K| \leqslant (3\Delta)^{3d-4},$$

as desired. This concludes the proof of Lemma 3.1. □

At the expense of a slightly larger bound on the size of monochromatic components, we may relax the requirements in Lemma 3.1 as follows.

Lemma 3.8. *Let G be a connected plane graph whose vertex set is partitioned into a chordless cycle C and a stable set S such that the cycle C bounds a face of G. Let d be the maximum degree of a vertex in C, and let $\Delta := \Delta(G)$. Then there exists a 2-colouring of G in which each vertex in S has colour 2, each 1-component has size at most $2d + 5$, and each 2-component has size at most $d(6\Delta)^{3d+2}$.*

Proof. We may assume without loss of generality that C bounds the outer face of G. Let S^* be the set of vertices $v \in S$ such that either $\deg_G(v) \leqslant 1$, or $\deg_G(v) = 2$ and the two neighbours of v are adjacent. Let $G^* = G \setminus S^*$, and remove from S the vertices in S^*.

(We will treat the vertices in S^* at the very end.) We construct a new graph G' from G^* in two steps as follows.

Step 1. Take a maximal stable set Z of the vertices $\{v \in V(C), \deg_{G^*}(v) = 2\}$ (Z might be empty), and for each vertex $v \in Z$ add a vertex s_v in S adjacent to v and its two neighbours in C.

Note that this can be done so that the embedding stays planar and C still bounds the outer face of the graph. By our choice of Z, after Step 1 every vertex of C has degree at least 3 (and thus has at least one neighbour in S).

Step 2. For each pair of consecutive vertices u, v in C in anticlockwise order having no common neighbour in S, do the following. Let f be the inner face incident to uv, and let s be the unique neighbour of u in S that is incident to f. Add an edge between s and v.

Again, this can be done so that the embedding stays planar and C still bounds the outer face of the graph. (How the degrees of vertices increased will be considered later.) Let G' be the graph obtained after Step 2. Note that G' is a supergraph of G^*.

Let x, y be two arbitrarily chosen consecutive vertices of C, and let P denote the x–y path in C that avoids the edge xy. Subdivide the edge xy by adding a vertex r between x and y, and add r to S. Observe that the graph G'' obtained after this operation together with the set S satisfy the assumptions of Lemma 3.1. Indeed, P is an induced path in G'' with endpoints x and y, and $r \in S$ is only adjacent to x, y, while all other vertices in S are inside the cycle induced by $V(P) \cup \{r\}$. Moreover, every vertex in S has degree at least 2; if $u \in S$ has degree exactly 2, then the two neighbours of u on P are not adjacent, and every two consecutive vertices of P have at least one common neighbour in S.

The degree of each vertex of P increased by at most two during Steps 1 and 2, while the degree of each vertex in S can at worst be doubled at Step 2 (we might add an edge sv for every neighbour u of s). It follows that G'' has maximum degree at most 2Δ and vertices of P have degree at most $d+2$. By Lemma 3.1, G'' has a 2-colouring such that 1-components have size at most $2d+5$, 2-components have size at most $(6\Delta)^{3d+2}$, and x, y, r have colour 2 (in particular, x and y are in the same 2-component).

We now add back the edge xy and the vertices of S^*, which we connect to their original neighbours in G, and colour them with colour 2. By the definition of S^* and the remark above, this does not connect different 2-components of G''. Since each vertex of C had at most d neighbours in S^*, in the resulting graph G''' the size of each 2-component is at most $d(6\Delta)^{3d+2}$, while 1-components still have size at most $2d+5$ since they remain unchanged. Since the graph G is a subgraph of G''', these two bounds obviously hold for this colouring restricted to G. □

Let $g_1 : \mathbb{N} \to \mathbb{N}$ and $g_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ denote the bounds on the sizes of 1- and 2-components, respectively, appearing in Lemma 3.8; namely $g_1(d) := 2d+5$ and $g_2(d,\Delta) := d(6\Delta)^{3d+2}$.

For a plane graph G, we denote by $O(G)$ the set of vertices lying on the boundary of the outer face of G, and by $O_2(G)$ the set of vertices not in $O(G)$ that are adjacent to a vertex in $O(G)$. A plane graph is *near-triangulated* if all its faces are triangular, except possibly for the outer face. Note that if G is near-triangulated, then $O_2(G)$ is precisely the set of vertices on the outer face of $G \setminus O(G)$.

We will use the following simple observation.

Observation 3.9. *Let $\ell \geqslant 1$ be an integer. Suppose we have a colouring of a graph with maximum degree at most $\Delta \geqslant 1$ in which every i-component has size at most k, for some colour class i. Then, if we recolour at most ℓ vertices of the graph, in the new colouring every i-component has size at most $\ell \Delta k + \ell \leqslant 2\ell\Delta k$.*

We now use Lemma 3.8 to prove the following result by induction.

Theorem 3.10. *Every connected near-triangulated plane graph G with maximum degree at most $\Delta \geqslant 1$ has a 3-colouring such that:*

(i) *no vertex of $O(G)$ is coloured with colour 3,*
(ii) *no vertex of $O_2(G)$ is coloured with colour 1,*
(iii) *each 1-component intersecting $O(G)$ has size at most $f_1(\Delta) = 16\Delta^2 g_1(\Delta)$,*
(iv) *each 2-component intersecting $O(G) \cup O_2(G)$ has size at most*

$$f_2(\Delta) = 16\Delta^2 f_1(\Delta)\, g_2(\Delta, \Delta f_1(\Delta)),$$

and

(v) *each monochromatic component has size at most $6\Delta^2 f_2(\Delta)$.*

Proof. We prove the theorem by induction on $|G|$. The proof is split into five cases, depending on the structure of the outerplanar graph J induced by $O(G)$. In fact, to make induction work, we will need to prove additional properties in some of the cases. Instead of giving the exact statement that we prove by induction (which would be lengthy), we describe at the beginning of each case below the extra properties we wish to guarantee in that case (if any).

Case 0. $|G| = 1$.

This is the base case of the induction, which trivially holds. Let us now consider the inductive case $|G| > 1$.

Case 1. G has a vertex of degree one.

Let v be such a vertex. Since G is near-triangulated, v and its neighbour u both lie on the boundary of the outer face. We can colour $G \setminus v$ by induction and assign to v a colour (1 or 2) different from that of u. This does not affect the sizes of existing monochromatic components, and the newly created monochromatic component has size 1. Thus the resulting colouring of G satisfies conditions (i)–(v). In the rest of the proof we assume that G has minimum degree at least two.

Case 2. The outerplanar graph J is a chordless cycle.

In this case we show a strengthened version of (iii) and (iv) where a multiplicative factor of $16\Delta^2$ is saved in the bounds, as well as a better bound for 3-components intersecting

$O_2(G)$:

(a) each 1-component intersecting $O(G)$ has size at most $g_1(\Delta)$,
(b) each 2-component intersecting $O(G) \cup O_2(G)$ has size at most $f_1(\Delta)\, g_2(\Delta, \Delta f_1(\Delta))$, and
(c) each 3-component intersecting $O_2(G)$ has size at most $f_2(\Delta)$.

Since G is near-triangulated and J is a chordless cycle, the graph $H = G \setminus O(G)$ is connected and near-triangulated, or it is empty. If H is empty then $G = J$ is a cycle, and G can trivially be 2-coloured in such a way that monochromatic components have size at most 2. We may thus suppose that H is not empty. Observe that $O(H) = O_2(G)$. By induction, H has a 3-colouring such that:

(i′) no vertex of $O(H)$ is coloured with colour 1,
(ii′) no vertex of $O_2(H)$ is coloured with colour 2,
(iii′) every 2-component intersecting $O(H)$ has size at most $f_1(\Delta)$,
(iv′) every 3-component intersecting $O(H) \cup O_2(H)$ has size at most $f_2(\Delta)$, and
(v′) every monochromatic component has size at most $6\Delta^2 f_2(\Delta)$.

Our aim now is to extend this colouring of H to one of G by colouring the vertices of $O(G)$ using colours 1 and 2. Let G' be the graph obtained from G by removing all vertices of H coloured with colour 3 and all monochromatic components of H that are disjoint from $O(H)$, and contracting each 2-component of H intersecting $O(H)$ into a single vertex. Note that G' is a plane graph as in Lemma 3.8, with S the set of contracted 2-components.

Observe that vertices of G' in $O(G') = O(G)$ still have degree at most Δ, and that vertices in S have degree at most $\Delta \cdot f_1(\Delta)$ by property (iii′) of H. We colour G' using Lemma 3.8. In this colouring, 1-components of G' have size at most $g_1(\Delta)$, while 2-components of G' have size at most $g_2(\Delta, \Delta f_1(\Delta))$.

The colouring of G' induces a colouring of the vertices of $O(G)$ that extends the colouring of H we previously obtained to the graph G. In this colouring of G, since no vertex of $O(H)$ is coloured with colour 1 by property (i′) of H, each 1-component intersecting $O(G)$ has size at most $g_1(\Delta)$ by the previous paragraph, which proves (a). Also, each 2-component of G intersecting $O(G) \cup O_2(G)$ corresponds to a 2-component of G' of size at most $g_2(\Delta, \Delta f_1(\Delta))$. Hence, each such 2-component of G has size at most $f_1(\Delta)\, g_2(\Delta, \Delta f_1(\Delta))$, showing (b). Moreover, 3-components intersecting $O_2(G) = O(H)$ have size at most $f_2(\Delta)$ by (iv′), which proves (c). Finally, every monochromatic component of G not considered above has size at most $6\Delta^2 f_2(\Delta)$ by (v′), which concludes this case.

Case 3. All bounded faces of J are triangular.

Let uv be an arbitrarily chosen edge of J lying on the boundary of its outer face. Let $\phi(u)$ and $\phi(v)$ be colours for u and v, respectively, arbitrarily chosen among 1 and 2. We show that G has a 3-colouring satisfying (i)–(v) and the following three extra properties:

(1) each monochromatic component of G intersecting $O(G)$ is contained in $O(G)$ and has size at most 2Δ,
(2) all vertices in $O_2(G)$ have colour 3, and
(3) u and v are coloured with colours $\phi(u)$ and $\phi(v)$, respectively, and moreover no neighbour of u in $V(G) \setminus \{v\}$ is coloured with colour $\phi(u)$.

For each bounded face f of J, let H_f be the subgraph of G induced by the vertices lying in the proper interior of f. As in Case 2, these graphs H_f are either connected and near-triangulated, or are empty. Using induction, for each bounded face f of J such that H_f is not empty, we colour H_f with colours 1, 2, 3 in such a way that:

(i′) no vertex of $O(H_f)$ is coloured with colour 1,
(ii′) no vertex of $O_2(H_f)$ is coloured with colour 2,
(iii′) every 2-component intersecting $O(H_f)$ has size at most $f_1(\Delta)$,
(iv′) every 3-component intersecting $O(H_f) \cup O_2(H_f)$ has size at most $f_2(\Delta)$, and
(v′) every monochromatic component has size at most $6\Delta^2 f_2(\Delta)$,

and we recolour with colour 3 the at most 3Δ vertices of $O(H_f)$ (*i.e.*, the vertices of H_f that are adjacent to some vertex in the boundary of f). Next, we colour u and v with colours $\phi(u)$ and $\phi(v)$, respectively, and colour the remaining vertices of J according to the parity of their distances to $\{u,v\}$ in J: we use colour $\phi(u)$ if the distance is even, and the colour opposite to $\phi(u)$ if it is odd. (As before, the colour opposite to 1 is 2, and *vice versa.*)

Clearly, the resulting colouring of G satisfies (2) and (3). Also, each monochromatic component K of G that includes a vertex of some graph H_f is contained in H_f. The bounds on the size of K are then guaranteed by (i′)–(v′), except possibly in the case where K is a 3-component intersecting $O(H_f)$. In that case, since we recoloured with colour 3 at most 3Δ vertices of H_f, using (iv′) and Observation 3.9 (with $\ell = 3\Delta$) we obtain that K has at most $6\Delta^2 f_2(\Delta)$ vertices, as desired. Hence, properties (ii)–(v) are satisfied for monochromatic components of G avoiding $O(G)$. Since the remaining monochromatic components are contained in J, and since we only used colours 1 and 2 when colouring that graph, it only remains to establish property (1).

Consider thus a monochromatic component K of J. First suppose that K contains v. Here there are two possibilities: either $\phi(u) = \phi(v)$, in which case $V(K) = \{u,v\}$, or $\phi(u) \neq \phi(v)$, in which case all vertices in $V(K) \setminus \{v\}$ are neighbours of u or v. Note that $|V(K) \setminus \{v\}| \leqslant 2\Delta - 2$ in the latter case since $uv \in E(G)$. Hence $|K| \leqslant 2\Delta$ holds in both cases.

Now assume that K avoids the vertex v. Then by the definition of our colouring, all vertices in K are at the same distance i from $\{u,v\}$ in J. If $i = 0$ then $V(K) = \{u\}$, and (1) trivially holds, so assume $i > 0$. Let X be the set of vertices of J at distance $i-1$ from $\{u,v\}$ and having a neighbour in K. If $|X| \geqslant 3$, then considering the union of three shortest paths from u to three distinct vertices in X together with the connected subgraph K, we deduce that J contains $K_{2,3}$ as a minor. However, this contradicts the fact that J is outerplanar. Hence, we must have $|X| \leqslant 2$, and therefore K has at most $|X| \cdot \Delta \leqslant 2\Delta$ vertices, showing (1). This concludes Case 3.

Before proceeding with the final case we need to introduce some terminology. First, note that each bounded face of J is bounded by a cycle of J (since J is outerplanar), and that each vertex of J is in the boundary of at least one bounded face of J (since G has minimum degree at least 2). In particular, every such vertex is contained in a cycle of J. These basic properties will be used implicitly in what follows.

Since neither Case 2 nor Case 3 applies, J has at least two bounded faces, and at least one of them is not triangular. For a bounded face f of J, let G_f denote the subgraph of G induced by the union of the vertices in the boundary of f and the vertices lying in the proper interior of f.

We define a rooted tree $\mathcal{T}$ whose vertices are the bounded faces of J. First, choose arbitrarily a bounded face of J and make it the root of $\mathcal{T}$. The tree $\mathcal{T}$ is then defined inductively as follows. If f is a vertex of $\mathcal{T}$ then its children in $\mathcal{T}$ are the bounded faces f' of J that are distinct from the parent of f in $\mathcal{T}$ (if f is not the root), and such that the boundaries of f and f' intersect in a non-empty set $X_{f'}$ of vertices which separates $G_f \setminus X_{f'}$ from $G_{f'} \setminus X_{f'}$ in G. The set $X_{f'}$ is then said to be the *attachment* of the face f'. Observe that, since J is outerplanar, $X_{f'}$ consists either of a single vertex or of two adjacent vertices. For definiteness, we let the attachment of the root of $\mathcal{T}$ be the empty set.

A bounded face f of J is *bad* if f is triangular and $|X_f| = 2$, otherwise f is *good*. Observe that, in particular, the root of $\mathcal{T}$ is good.

Case 4. None of the previous cases apply.

Let f be a good face maximizing its depth in $\mathcal{T}$. Thus all strict descendants of f in $\mathcal{T}$ are bad (if any). Let $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k$ be the trees resulting from the removal of f in $\mathcal{T}$, where $\mathcal{T}_0$ contains the parent of f if f is not the root and is otherwise empty, and each $\mathcal{T}_i$ ($i \in \{1, \ldots, k\}$) contains a different child of f in $\mathcal{T}$. (Note that possibly $k = 0$ if f is not the root.) Let X_0 denote the attachment of f, and for each $i \in \{1, \ldots, k\}$ let X_i denote the attachment of the unique child of f contained in $\mathcal{T}_i$. By the choice of f, each X_i ($i \in \{1, \ldots, k\}$) consists of two adjacent vertices u_i, v_i of the boundary of f, and at least one of them, say v_i, is not in X_0.

For each $i \in \{0, \ldots, k\}$, let

$$G_i := G[\cup_{f' \in V(\mathcal{T}_i)} V(G_{f'})].$$

Notice that, for each $i \in \{1, \ldots, k\}$, all bounded faces of G_i are triangular. Let us also recall once again that G_0 is the empty graph if f is the root of $\mathcal{T}$ (in which case X_0 is empty as well).

We proceed in three steps.

Step 1. We start by colouring G_0 using induction (if G_0 is not empty), and G_f using Case 2 of the induction, so the resulting colouring of G_f also satisfies (a)–(c).

Step 2. We recolour three sets of vertices of G_f. First, recolour in G_f the vertices of X_0 to match their colour in G_0. Next, recolour the at most two vertices in $O(G_f) \setminus X_0$ having a neighbour in X_0 with a colour (1 or 2) distinct from the colour of their unique neighbour in X_0. Note that the latter can be done precisely because f is good. (Indeed, either $|X_0| \leqslant 1$, or $|X_0| = 2$ and the cycle bounding the outer face of G_f has length at least 4.) Finally, recolour with colour 3 all vertices in $G_f \setminus O(G_f)$ having a neighbour in X_0 (note that there are at most $2\Delta - 4$ such vertices).

Step 3. For each $i \in \{1, \ldots, k\}$, colour G_i using Case 3 of the induction, choosing respectively u_i and v_i as u and v in (3), and $\phi(u_i)$ and $\phi(v_i)$ as the colours of u_i and v_i after Step 2 above. Recall that $v_i \notin X_0$.

We claim that the colouring of G obtained by taking the union of colourings of $G_f, G_0, \dots, G_k$ satisfies (i)–(v). First we remark that, because of the recolouring of X_0 at Step 2 and the use of property (3) in Step 3, the colourings of $G_f, G_0, \dots, G_k$ coincide on the pairwise intersection of the vertex sets of these graphs, so the union of these colourings is well defined.

After Steps 2 and 3, no vertex in X_0 is coloured with a colour that is used for some of its neighbours in $G_f \setminus X_0$ (after Step 3, this follows from properties (2) and (3)). This implies that every monochromatic component of G intersecting $V(G_0)$ is contained in $V(G_0)$, and therefore satisfies (i)–(v) by induction. Similarly, monochromatic components intersecting $V(G_i)$ but avoiding X_i for some $i \in \{1, \dots, k\}$ satisfy (i)–(v) by induction. Hence, it only remains to consider monochromatic components of G avoiding G_0 (and in particular, avoiding X_0) and intersecting $V(G_f)$.

Since we recoloured at most two vertices of $O(G_f) \setminus X_0$, it follows from Observation 3.9 (with $\ell = 2$) that the size of 1-components after Step 2 of the graph G_f intersecting $O(G_f) \setminus X_0$ is at most 4Δ times the maximum size of 1-components of G_f intersecting $O(G_f)$ before we recoloured vertices of G_f, which was at most $g_1(\Delta)$ by property (a) from Case 2. Now, a 1-component K of the graph G which intersects $O(G_f) \setminus X_0$ is the union of a *single* 1-component K' of G_f after Step 2 intersecting $O(G_f) \setminus X_0$ with at most $2|K'|$ 1-components from the graphs $G_1, \dots, G_k$ (since every vertex of $V(K') \cap O(G_f)$ lies in at most two such graphs). It follows from (1) in Case 3 and the observation above that $|K| \leqslant 2|K'| \cdot 2\Delta \leqslant 16\Delta^2 g_1(\Delta)$. This proves (iii).

Similarly, using property (b) from Case 2, we deduce that after Step 2, 2-components of G_f intersecting $(O(G_f) \setminus X_0) \cup O_2(G_f)$ have size at most $4\Delta \cdot f_1(\Delta) g_2(\Delta, \Delta f_1(\Delta))$. Applying the same reasoning as for 1-components of G intersecting $O(G_f) \setminus X_0$ above, we deduce that 2-components of G intersecting $(O(G_f) \setminus X_0) \cup O_2(G_f)$ have size at most $16\Delta^2 \cdot f_1(\Delta) g_2(\Delta, \Delta f_1(\Delta))$, which proves (iv).

Finally, using property (c) from Case 2 and the fact that at most $2\Delta - 4 \leqslant 2\Delta$ vertices of $O_2(G_f)$ have been recoloured with colour 3 in Step 2, we have that 3-components of G intersecting $O_2(G_f)$ have size at most $4\Delta^2 f_2(\Delta) \leqslant 6\Delta^2 f_2(\Delta)$ by Observation 3.9 (with $\ell = 2\Delta$). Therefore, (v) also holds. □

From Theorem 3.10 we easily derive our main theorem, Theorem 1.2, with an explicit bound.

Corollary 3.11. *Every plane graph G with maximum degree $\Delta \geqslant 1$ can be 3-coloured in such a way that:*

(i) *each monochromatic component has size at most* $(15\Delta)^{32\Delta+8}$,
(ii) *only colours 1 and 2 are used for vertices on the outer face,*
(iii) *each 1-component intersecting $O(G)$ is included in $O(G)$ and has size at most* $6^4\,\Delta^3$.

Proof. If G is not connected we can colour each component of G separately, so we may suppose that G is connected. We may further assume that $\Delta \geqslant 3$ since otherwise G is properly 3-colourable. If G is not near-triangulated, we do the following for every bounded face f of G. Let $x_1, x_2, \dots, x_k$ be a boundary walk of f (note that a vertex appears at least twice in the walk if and only if it is a cut-vertex of G). We add a cycle $u_1, u_2, \dots, u_k$ of length k inside f and link each vertex u_i to x_i and x_{i-1} (indices are taken modulo k). Next, for each $i \in \{1, \dots, \lceil k/2 \rceil - 1\}$ we add the edges $u_i u_{k-i}$ and $u_i u_{k-i+1}$ (if they are not already present). The graph obtained is near-triangulated and every new vertex has degree at most 6. For every original vertex v of G, we added at most two edges incident to v in between every two consecutive original edges in the cyclic ordering of the edges around v. Hence the maximum degree of the new graph is at most $\max(6, 3\Delta) \leqslant 3\Delta$ and the result follows from Theorem 3.10 (with Δ replaced by 3Δ). ☐

4. Extension to surfaces of higher genus

In this section we extend our main result to graphs embeddable in a fixed surface. In this paper, a *surface* is a non-null compact connected 2-manifold without boundary. Recall that the *Euler genus* of a surface Σ is $2 - \chi(\Sigma)$, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. We refer the reader to the monograph by Mohar and Thomassen [11] for basic terminology and results about graphs embedded in surfaces.

Let $f(\Delta) = (15\Delta)^{32\Delta+8}$ be the bound on the size of monochromatic components in Corollary 3.11.

Theorem 4.1. *Every graph G with maximum degree $\Delta \geqslant 1$ embedded in a surface Σ of Euler genus g can be 3-coloured in such a way that each monochromatic component has size at most $(5\Delta)^{2^g-1} f(\Delta)^{2^g}$.*

Proof. The proof proceeds by induction on g. If $g = 0$ then G is planar, and the result follows from Corollary 3.11. Assume now that $g > 0$.

We may suppose that some cycle of G is not contractible (as a closed curve on the surface), since otherwise G can be embedded in the plane. Let C be a shortest non-contractible cycle of G. If C has a chord e, then at least one of the two cycles obtained from C using the edge e is not contractible, as follows from the so-called 3-Path Property (see [11, p. 110]). However, this contradicts the minimality of C. Thus the cycle C is induced.

Each connected component of $G' := G \setminus V(C)$ can be embedded in a surface of Euler genus strictly less than g (see [11, Chapter 4.2]). Thus, applying induction on each connected component of G', we deduce that G' can be 3-coloured in such a way that each monochromatic component has size at most $s = (5\Delta)^{2^{g-1}-1} f(\Delta)^{2^{g-1}}$.

Let $t := |C|$. We extend the colouring of G' obtained above to a colouring of G by colouring the vertices of C as follows. We divide them into k circular intervals $I_1, \dots, I_k$ (where the circular ordering is of course given by C), each of length $s+1$, except I_1 whose length is t if $t \leqslant s$, and $s + 1 + (t \pmod{s+1}) \leqslant 2s + 1$ if $t > s$. We colour all vertices in I_1 with colour 1, and for each $i \in \{2, \dots, k\}$, we colour vertices in I_i with colour 2 if i is even, and colour 3 if i is odd.

If some monochromatic component K of G' has a neighbour u in some interval I_i and another neighbour v in an interval I_j with $i \neq j$ that are coloured the same as K, then one can find a path P from u to v having all its internal vertices in K, and thus being internally disjoint from C. Recall that $|K| \leqslant s$, and that by our colouring of the intervals, there are at least $s+1$ vertices between u and v on both sections of the cycle C. Hence, the two cycles obtained by shortcutting C using the path P are shorter than C. However, by the 3-Path Property, at least one of them is not contractible, contradicting our choice of C.

It follows that each monochromatic component of G' has neighbours in at most one interval I_i in the graph G. Using Observation 3.9 (with $\ell = 2s+1$), we deduce that monochromatic components of G have size at most

$$2(2s+1)\Delta \cdot s \leqslant 5s^2\Delta \leqslant 5\Delta \cdot (5\Delta)^{2^g-2} \cdot f(\Delta)^{2^g} = (5\Delta)^{2^g-1} f(\Delta)^{2^g},$$

as desired. □

We note that using the cutting technique introduced recently by Kawarabayashi and Thomassen [6] together with the stronger property from Corollary 3.11 that one colour can be omitted on the outer face, it is possible to obtain a bound that is linear in the genus (instead of doubly exponential). We only sketch the proof in the remainder of this section (we preferred to present the full details of the simple and self-contained proof of Theorem 4.1, at the expense of a worst bound).

Kawarabayashi and Thomassen [6, Theorem 1] proved that any graph G embedded on some surface of Euler genus g with sufficiently large face-width (say, more than $10t$, for some constant t) has a partition of its vertex set in three parts H, A, B, such that A has size at most $10tg$, B consists of the disjoint union of paths that are local geodesics[†] and are pairwise at distance at least t in G, and H induces a planar graph having a plane embedding such that the only vertices of H having a neighbour in $A \cup B$ lie on the outer face of H.

Recall that by Corollary 3.11 every plane graph of maximum degree Δ can be coloured with colours $1, 2, 3$ so that no vertex of the outer face is coloured 3 and each monochromatic component has size at most $f(\Delta) = (15\Delta)^{32\Delta+8}$. We now prove by induction on g that for every graph G of Euler genus g and maximum degree Δ there is a set of at most $10(f(\Delta)+2)g$ vertices in G whose removal yields a graph that has a 3-colouring where each monochromatic component has size at most $\Delta f(\Delta)+1$. Using Observation 3.9, this will directly imply that G has a 3-colouring in which every monochromatic component has size at most

$$f(\Delta) + 20\Delta(f(\Delta)+2)(\Delta f(\Delta)+1)\,g,$$

a bound that is linear in g.

If $g = 0$ the graph is planar, and we can apply Corollary 3.11. If the face-width is at most $10(f(\Delta)+2)g$, we remove the vertices intersecting a noose of length at most $10(f(\Delta)+2)$, and apply induction on the resulting graph (since each of its components can be embedded in a surface of Euler genus at most $g-1$). If the face-width is more than $10(f(\Delta)+2)g$ we apply the result of Kawarabayashi and Thomassen. Let H, A, B

[†] In the sense that each subpath with at most t vertices is a shortest path in G.

be the corresponding partition of G (with H having its specific plane embedding). The set A is the set of vertices we remove from G. We now colour H using Corollary 3.11, avoiding colour 3 on its outer face. Recall that each component of B is a path. For each such path P, choose arbitrarily an endpoint v of P and colour all the vertices of P with colour 3, except v and the vertices whose distance to v in P is a multiple of $f(\Delta)+2$. The latter vertices are coloured with colour 2. It can easily be checked that monochromatic components of colour 1 have size at most $f(\Delta)$ and monochromatic components of colour 3 have size at most $f(\Delta)+1$. Note that every monochromatic component of colour 2 in H has at most one neighbour coloured 2 in B, since otherwise two paths of B, or two vertices that are at distance $f(\Delta)+2$ on some path of B, would be at distance at most $f(\Delta)+1$ in G. Hence every monochromatic component of colour 2 has size at most $\Delta f(\Delta)+1$ in G, as desired.

5. Conclusion

We proved that planar graphs with maximum degree Δ can be 3-coloured in such a way that each monochromatic component has size at most $f(\Delta)=(15\Delta)^{32\Delta+8}$. It is thus natural to look for lower bounds on the best possible value for $f(\Delta)$. The examples constructed in [7] and [1] give a lower bound of $\Omega(\Delta^{1/3})$ (see also a related construction in [9]). We remark that this bound can be slightly improved as follows. Let $k \geqslant 3$ and let G_k be the graph obtained from a path P on k vertices $v_1,\dots,v_k$ by adding, for each $i \in \{2,\dots,k\}$, a path P_i on $k(2k-3)$ new vertices, and making all of them adjacent to v_{i-1} and v_i. Note that this graph is planar and has maximum degree $\Delta = 2k(2k-3)+2$. Consider any 3-colouring of G_k. We now prove that there is a monochromatic component of size at least $k=\Omega(\sqrt{\Delta})$. If the path P itself is not monochromatic, then there exists $j \in \{1,\dots,k-1\}$ such that v_j and v_{j+1} have distinct colours, say 1 and 2. If colour 1 or colour 2 appears $k-1$ times in P_j then we have a monochromatic star on k vertices. Otherwise there is a subpath of P_j with k vertices, all of which are coloured with colour 3.

As mentioned in Section 1, Alon, Ding, Oporowski, and Vertigan [1] proved that for every proper minor-closed class of graphs $\mathcal{G}$ there is a function $f_{\mathcal{G}}$ such that every graph in $\mathcal{G}$ with maximum degree Δ can be 4-coloured in such way that every monochromatic component has size at most $f_{\mathcal{G}}(\Delta)$. On the other hand, for every t, there are graphs with no K_t-minors that cannot be coloured with $t-2$ colours such that all monochromatic components have bounded size. So in this case again, the assumption that the size depends on Δ cannot be dropped. We ask whether Theorem 1.2 holds not only for graphs of bounded genus, but more generally for all proper minor-closed classes of graphs.

Question 5.1. Is it true that for each proper minor-closed class of graphs $\mathcal{G}$ there is a function $f_{\mathcal{G}}:\mathbb{N}\to\mathbb{N}$ such that every graph in $\mathcal{G}$ with maximum degree Δ can be 3-coloured in such way that each monochromatic component has size at most $f_{\mathcal{G}}(\Delta)$?

Note that the example of graphs with no K_t-minors that cannot be coloured with $t-2$ colours in such a way that all monochromatic components have bounded size shows that the famous Hadwiger conjecture, stating that graphs with no K_t-minor have a proper colouring with $t-1$ colours, is best possible even if we only ask the sizes of monochromatic components to be bounded by a function of t (instead of being of size 1). On the other hand, Kawarabayashi and Mohar [5] proved the existence of a function f such that every K_t-minor-free graph has a colouring with $\lceil \frac{31}{2}t \rceil$ colours in which each monochromatic component has size at most $f(t)$. This bound was recently reduced to $\lceil \frac{7}{2}t - \frac{3}{2} \rceil$ colours by Wood [13]. This is in contrast with the best known bound of $O(t\sqrt{\log t})$ colours for the Hadwiger conjecture (see [8, 12]).

A well-known result of Grötzsch [3] asserts that triangle-free planar graphs are 3-colourable. A natural question is whether there exists a constant c such that every triangle-free planar graph can be 2-coloured such that every monochromatic component has size at most c. The following construction shows that the answer is negative. Fix an integer $k \geqslant 2$ and consider a path $x_1, \ldots, x_k$. For each $i \in \{1, \ldots, k\}$, add a set S_i of $2k-3$ vertices which are adjacent to x_i only, and finally add a vertex u adjacent to all vertices in $\bigcup_{1 \leqslant i \leqslant k} S_i$. This graph G_k is planar and triangle-free. Take a 2-colouring of G_k and assume that the path $x_1, \ldots, x_k$ is not monochromatic. Then some vertex x_i has a colour distinct from that of u. Since u and x_i have $2k-3$ common neighbours, one of u and x_i has $k-1$ neighbours of its colours, and then lies in a monochromatic component of size k. It follows that in every 2-colouring of G_k there is a monochromatic component of size at least k. Note that this construction has unbounded maximum degree. Hence, the following natural question remains open.

Question 5.2. Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that every triangle-free planar graph with maximum degree Δ can be 2-coloured in such a way that each monochromatic component has size at most $f(\Delta)$?

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