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The covering numbers of some Mycielski ideals may be different

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Abstract

We show that in the Silver model the inequality $\mathrm{cov}(\mathfrak{C}_2) < \mathrm{cov}(\mathfrak{P}_2)$ holds true, where $\mathfrak{C}_2$ and $\mathfrak{P}_2$ are the two-dimensional Mycielski ideals.

1 Introduction

Given any set X with at least two elements, $b \in [\omega]^\omega$ and $A \subseteq {}^\omega X$, let $\Gamma_X(A, b)$ be the infinite game of two players in which both players choose the consecutive elements of a sequence $x \in {}^\omega X$. The choice of $x(n)$ is done by the second player iff $n \in b$. The first player wins iff $x \in A$. By using such games, Mycielski [7] introduced ideals on the space ${}^\omega X$ as follows: Given a family $\mathfrak{B} = \langle b_s : s \in {}^{<\omega}2\rangle$ of infinite subsets of ω such that $b_s = b_{s^\frown 0} \dot{\cup} b_{s^\frown 1}$ (disjoint union), the **Mycielski ideal** $\mathfrak{M}_{X,\mathfrak{B}}$ is defined as the collection of all $A \subseteq {}^\omega X$ such that for all $s \in {}^{<\omega}2$ the second player has a winning strategy in the game Γ_{X,b_s}. In [7], Mycielski proved, among other things, that if $X = 2$ or $X = \omega$, then $\mathfrak{M}_{X,\mathfrak{B}}$ is a translation invariant σ-ideal that is orthogonal to $\mathcal{M} \cap \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meager and null sets, respectively (see [8, Theorem 2.2] for a list of Mycielski's results). Rosłanowski [8] introduced more ideals, denoted $\mathfrak{C}_X$ and $\mathfrak{P}_X$, that are closely related to Mycielski's $\mathfrak{M}_{X,\mathfrak{B}}$ and are also called **Mycielski ideals**:

Definition 1.1 *Let X be a set with at least two elements. Let $\mathfrak{C}_X$ be the set of all $A \subseteq {}^\omega X$ for which player II has a winning strategy in the game $\Gamma_X(A, b)$ for every $b \in [\omega]^\omega$, thus*

$$\mathfrak{C}_X = \bigcap_{\mathfrak{B}} \mathfrak{M}_{X,\mathfrak{B}}.$$

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Moreover, let

$$\mathfrak{P}_X = \{A \subseteq {}^\omega X : \forall b \in [\omega]^\omega \quad A \restriction b \neq {}^b X\}.$$

Here $A \restriction b = \{x \restriction b : x \in A\}$. Alternatively, $\mathfrak{P}_X$ is the set of those $A \subseteq {}^\omega X$ for which, for every $b \in [\omega]^\omega$, player II has a winning strategy in the game $\Gamma_X(A, b)$ that does not depend on the moves of player I. Clearly we have $\mathfrak{P}_X \subseteq \mathfrak{C}_X$.

The Mycielski ideals $\mathfrak{C}_X$ and $\mathfrak{P}_X$ have been the object of intensive research over the past decades. The main focus has been on their cardinal characteristics. Recall the following cardinals associated to any ideal $\mathfrak{I}$ on a set Y:

$$\begin{aligned}
\operatorname{add}(\mathfrak{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathfrak{I} \ \wedge \ \bigcup \mathcal{F} \notin \mathfrak{I}\},\\
\operatorname{cov}(\mathfrak{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathfrak{I} \ \wedge \ \bigcup \mathcal{F} = Y\},\\
\operatorname{non}(\mathfrak{I}) &= \min\{|Z| : Z \subseteq Y \ \wedge \ Z \notin \mathfrak{I}\},\\
\operatorname{cof}(\mathfrak{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathfrak{I} \ \wedge \ \forall Z \in \mathfrak{I} \ \exists Z' \in \mathcal{F} \ \ Z \subseteq Z'\}.
\end{aligned}$$

Trivially, if $\mathfrak{I}, \mathfrak{J}$ are ideals such that $\mathfrak{I} \subseteq \mathfrak{J}$, then $\operatorname{cov}(\mathfrak{J}) \leq \operatorname{cov}(\mathfrak{I})$.

There exists a close relation between $\mathfrak{P}_k$ and $\mathfrak{C}_k$, where $2 \leq k < \omega$, on the one hand, and the tree ideals v_k^0, u_k^0 associated to k-dimensional Silver forcing $\mathbb{SI}_k$ and forcing with k-dimensional uniform Sacks trees $\mathbb{U}_k$, respectively, on the other hand. See Definition 2.1 below for their definitions. This relation comes from the fact that a winning strategy for player II in the game $\Gamma_k(A, b)$, where b is infinite and coinfinite, can be considered as a perfect tree $u^b \subseteq {}^{<\omega}k$ with the property that for every $t \in u^b$, t is a splitnode of u^b iff $|t| \in \omega \setminus b$ and every splitnode splits into k successor nodes. Such trees are the k-dimensional uniform Sacks trees. If u^b does not depend on the moves of player II, then u^b is nothing else than a k-dimensional Silver function with domain b.

The inclusions $\mathfrak{P}_k \subseteq v_k^0$ and $\mathfrak{C}_k \subseteq u_k^0$ are pretty obvious (see [5, Lemma 4.1]); however, no equality is provable here. Moreover, no inclusion between v_k^0 and u_k^0 is provable (see [5, §4] for these results). By general knowledge, if j^0 is the tree ideal associated to some proper tree forcing P, then in the model obtained by a countable support iteration of lenght ω_2 of P over a model of CH, which will be called the P-**model** for short below, $\operatorname{cov}(j^0) = \aleph_2$ holds true. E.g., see [4, Theorem 1.2] where this is proved for the two-dimensional Sacks forcing.

We conclude from the above that in the $\mathbb{SI}_k$-model $\operatorname{cov}(\mathfrak{P}_k) = \aleph_2$ holds and in the $\mathbb{U}_k$-model $\operatorname{cov}(\mathfrak{C}_k) = \operatorname{cov}(\mathfrak{P}_k) = \aleph_2$. A main result of [10] is that the

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equality $\mathrm{cov}(\mathfrak{P}_k) = \mathrm{cov}(\mathfrak{P}_{k+1})$ holds in ZFC, for every $2 \leq k < \omega$. In [5, Theorem 7.18] it has been shown that in the $\mathbb{U}_k$-model $\mathrm{cov}(\mathfrak{C}_{k+1}) = \aleph_1$. By the results just mentioned and as by [5, Proposition 6.8] $\mathrm{cov}(\mathfrak{C}_{k+1}) \leq \mathrm{cov}(\mathfrak{C}_k)$ is always true, we conclude that in the $\mathbb{U}_2$-model we have $\mathrm{cov}(\mathfrak{C}_k) < \mathrm{cov}(\mathfrak{P}_k)$ for every $3 \leq k < \omega$, but $\mathrm{cov}(\mathfrak{C}_2) = \mathrm{cov}(\mathfrak{P}_2) = \aleph_2$. Hence the consistency of $\mathrm{cov}(\mathfrak{C}_2) < \mathrm{cov}(\mathfrak{P}_2)$ was left open.

Our main result here is that $\mathrm{cov}(\mathfrak{C}_2) < \mathrm{cov}(\mathfrak{P}_2)$ is true in the $\mathbb{SI}_2$-model. For this we have to define $\aleph_1$-many $\mathfrak{C}_2$-sets that cover ${}^\omega 2$ in the Silver model. A $\mathfrak{C}_2$-set can be coded by a family $C = \langle u^b : b \in \Omega \rangle$, where $\Omega \subseteq [\omega]^\omega$ is $\subseteq$-dense and every u^b is a strategy for player II in the game $\Gamma_2(\cdot, b)$, thus a uniform Sacks tree with $\omega \setminus b$ as its set of split-levels, as described above. Such C will be called **coding system** below. Then

$$A(C) = {}^\omega 2 \setminus \bigcup \{[u^b] : b \in \Omega\}$$

is the $\mathfrak{C}_2$-set coded by C. Clearly, sets of this type form a base of $\mathfrak{C}_2$. Clearly, no dense Ω in the ground model will remain so in a forcing extension adding reals. Indeed, given any new real x, say $x \in {}^\omega 2$, the set $\{x \restriction n : n < \omega\} \subset {}^{<\omega}2$ does not contain any infinite subset from the ground model. Therefore, one of our tasks will be to extend $\aleph_1$-many partial coding systems cofinally often during the forcing iteration.

There are two main ingredients for this construction to work. The first one consists in a careful reading of a given P_{ω_2}-name $\dot{x}$ for a new real, where P_{ω_2} is the CS-iteration of $\mathbb{SI}_2$. This process will produce (even for a name for a member of ${}^\omega\omega$) a fusion sequence $\bar{S} = \langle p_n : n < \omega \rangle$ with limit p_ω in P_{ω_2} together with a 2-splitting tree T, i.e., every node of T has at most two successor nodes, which is the tree of possibilities for $\dot{x}$ below p_ω, such that the family of refining finite maximal antichains below p_ω associated to $\bar{S}$ corresponds to the split-levels of T. For this we apply what we call the **Grigorieff dichotomy**, i.e., an idea that appears in seminal form in Grigorieff's paper [3] where it is shown that $\mathbb{SI}_2$ (as well as many more forcings of a similar type) adds a minimal real.

As a small digression let me mention that in [9] the property of some forcing that in its extension every new real is a branch through some k-ary tree in the ground model is called **k-localization property**. In [9] it is shown that the CS-iteration of $\mathbb{SI}_k$ has the k-localization property. For this, Shelah-style preservation theorems are used and it is said that"maybe some old wisdom got lost", but that it seemed impossible to prove this by classical methods. I think that this old wisdom are Grigorieff's ideas, as building on them the 2-localization property for P_{ω_2} can be shown by using a classical fusion construction. I conjecture that this can be generalized to every finite

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dimension. But then certainly some extra complexity is added, as $\mathbb{SI}_k$ is no longer minimal if $k \geq 3$.*

The second ingredient for our main result is a new idea by which it is possible to define, for any P_{ω_2}-name $\dot{x}$ for a real with fusion $\bar{S}$ and limit p_ω as above, a coding system $C = \langle u^b : b \in \Omega \rangle$ in the ground model such that

$$p_\omega \Vdash_{P_{\omega_2}} \forall b \in \Omega \quad \dot{x} \notin [u^b].$$

Moreover, the definition of C depends only on the isomorphism type of $\dot{x}$ (see Definition 4.2 below). As by CH in the ground model there are only $\aleph_1$-many such types, this will be a good start for the construction of the $\aleph_1$ $\mathfrak{C}_2$-sets we need to have in the final model. This new idea is inspired by the well-known proof that $\mathbb{SI}_2$ adds a real which splits every infinite subset of ω in the ground model.

In [6], the ideas of this work have been used to show that in the model obtained by a CS-iteration of Sacks forcing the inequality $\mathrm{cov}(\mathfrak{C}_2) < \mathrm{cov}(s^0)$ holds true. The value of $\mathrm{cov}(\mathfrak{P}_2)$ in the Sacks model is not known.

2 Basic definitions

Definition 2.1 *Let $2 \leq k < \omega$.*

*(1) Let $\mathbb{SI}_k$ denote k-**dimensional Silver forcing**, i.e., the set of all partial functions f from ω to k such that $\omega \setminus \mathrm{dom}(f)$ is infinite, ordered by extension. We shall denote $\omega \setminus \mathrm{dom}(f)$ by $\mathrm{com}(f)$. So **Silver forcing** $\mathbb{SI}$ is $\mathbb{SI}_2$. By e_f we shall denote the increasing enumeration of $\mathrm{com}(f)$.*

*Given such k-**dimensional Silver function** f and $s \in {}^{<\omega}k$, by f^s we denote the Silver function extending f such that we have $\mathrm{dom}(f^s) = \mathrm{dom}(f) \cup \{e_f(i) : i < |s|\}$ and $f^s(e_f(i)) = s(i)$ for every $i < |s|$.*

Given $f, f' \in \mathbb{SI}_k$ and $n < \omega$, we define $f' \leq^n f$ iff $f' \leq f$ and $\mathrm{com}(f)$ and $\mathrm{com}(f')$ agree on their first $n+1$ elements. It is well-known that, equipped with these orderings, $\mathbb{SI}_k$ satisfies Axiom A (see [1] for this notion).

*Given $f \in \mathbb{SI}_k$, by $[f]$ we denote its **body**, which is defined as*

$$[f] = \{x \in {}^{\omega}k : f \subseteq x\}.$$

*The k-**dimensional Silver ideal** v_k^0 is defined as the set of all $X \subseteq {}^{\omega}k$ such that*

$$\forall f \in \mathbb{SI}_k \, \exists f' \in \mathbb{SI}_k \quad (f' \leq f \wedge [f'] \cap X = \emptyset).$$

* Added in proof: This conjecture has been confirmed recently by Fabian Kaak.

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*(2) By $\mathbb{U}_k$ we denote k-**dimensional uniform (Sacks) tree forcing**, i.e., the set of all perfect trees $u \subseteq {}^{<\omega}k$ such that there exists an infinite and coinfinite set $a^u \subset \omega$ with the property that for every node $t \in u$, t is a splitnode of u iff $|t| \in a^u$ and every splitnode of u has full splitting, i.e., it has k immediate successors. The order on $\mathbb{U}_k$ is inclusion. We shall denote $\mathbb{U}_2$ by $\mathbb{U}$.*

*If $u \in \mathbb{U}_k$, by $[u]$ we denote the **body** of u, i.e., the set of all its branches. The tree ideal associated to $\mathbb{U}_k$, denoted by u_k^0, is defined analogously as v_k^0: It contains all $X \subseteq {}^{\omega}k$ such that for every $u \in \mathbb{U}_k$ there is $u' \in \mathbb{U}_k$ with $u' \leq u$ and $[u'] \cap X = \emptyset$.*

*(3) Given any tree T, by $\operatorname{split}(T)$ we denote the set of all **splitnodes** of T, i.e., those $t \in T$ which have at least two immediate successors. By $\operatorname{split}_n(T)$ we denote the nth **splitlevel** of T, i.e., the set of those $t \in \operatorname{split}(T)$ which have precisely n proper initial segments which are splitnodes of T.*

Definition 2.2 *Let P_α be the CS-iteration of $\mathbb{SI}$ of length α, in particular, its elements are functions p with countable $\operatorname{supp}(p) := \operatorname{dom}(p) \subseteq \alpha$ such that for every $\beta \in \operatorname{supp}(p)$ we have that $p(\beta)$ is a P_β-name and the maximal condition in P_β forces $p(\beta) \in \mathbb{SI}^{\mathbf{V}^{P_\beta}}$.*

As explained in detail in [1, §5], given $\beta < \alpha$, P_α is forcing equivalent to an iteration $P_\beta * P_\alpha/\dot{G}_\beta$, where $\dot{G}_\beta$ is the canonical P_β-name for the P_β-generic filter, such that $P_\alpha/\dot{G}_\beta$ is again (the P_β-name of) a CS-support iteration of $\mathbb{SI}$. Using this identification, given $p \in P_\alpha$ and a P_β-generic filter G_β, we can partially evaluate $p[G_\beta]$ as a pair of sequences $(\underline{f}, \overline{f})$ such that $\underline{f} = \langle f_\gamma : \gamma \in \operatorname{supp}(p) \cap \beta\rangle$ is a sequence of Silver functions $f_\gamma = p(\gamma)[G_\beta]$ and $\overline{f} = \langle \dot{f}_\gamma : \gamma \in \operatorname{supp}(p) \setminus \beta\rangle$ is a condition of the CS-iteration of $\mathbb{SI}$ along $\alpha \setminus \beta$ in $\mathbf{V}[G_\beta]$.

As is well-known from proper forcing, we may assume that all $p \in P_\alpha$ are hereditarily countable (see [11, Chapter III, Definition 4.1] for this notion and proof of this fact), hence in particular, for every $\beta \in \operatorname{supp}(p)$, the P_β-name $p(\beta)$ can be evaluated as a Silver function by $\langle g_\gamma : \gamma \in \operatorname{supp}(p) \cap \beta\rangle$, where g_γ is the Silver real added by the γth iterand.

Wlog we may assume $0, 1 \in \operatorname{supp}(p)$ for every p.

*(1) For $p, q \in P_\alpha$, $F \in [\operatorname{supp}(p)]^{<\aleph_0}$ and $\eta : F \to \omega$ we call the triple (p, F, η) a **fusion condition** and we write $q \leq_{F,\eta} p$ for*

$$q \leq p \quad \wedge \quad \forall \beta \in F \quad q \restriction \beta \Vdash_{P_\beta} \quad q(\beta) \leq^{\eta(\beta)} p(\beta).$$

*(2) A **fusion sequence** in P_α is a sequence $\bar{S} = \langle (p_n, F_n, \eta_n) : n < \omega\rangle$ of*

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fusion conditions such that

(a) $F_0 = F_1 = \{0\}$, $F_2 = \{0,1\}$ *(as long as* $\alpha > 1$ *of course),* $F_n \subseteq F_{n+1}$ *and* $|F_{n+1} \setminus F_n| \leq 1$;

(b) $\eta_0(0) = 0$, $\eta_1(0) = 1$, $\forall \beta \in F_n \quad \eta_n(\beta) \leq \eta_{n+1}(\beta) \leq \eta_n(\beta) + 1$ *and there is precisely one* $\beta \in F_{n+1}$*, which we call the* ***active coordinate*** *of* F_{n+1}*, such that* $\eta_{n+1}(\beta) = \eta_n(\beta) + 1$ *in case* $\beta \in F_n$*, and* $\eta_{n+1}(\beta) = 0$ *in case* $\beta \notin F_n$*; we also call* $0 \in F_0$ *active;*

(c) $\forall \beta \in F_n \, \exists m \quad \eta_m(\beta) \geq n$;

(d) $\bigcup \{F_n : n < \omega\} = \bigcup \{\mathrm{supp}(p_n) : n < \omega\}$;

(e) $p_{n+1} \leq_{F_n, \eta_n} p_n$.

(3) Every fusion sequence $\langle (p_n, F_n, \eta_n) : n < \omega \rangle$ *determines its* ***fusion limit***

$$p_\omega = \inf \{p_n : n < \omega\}$$

in P_α*. Clearly,* $\mathrm{supp}(p_\omega) = \bigcup \{F_n : n < \omega\}$.

(4) For every $f \in \mathbb{SI}$ *and* $s \in 2^{<\omega}$ *we have defined* $f^s \in \mathbb{SI}$ *(see Definition 2.1(1)). If* (p, F, η) *is a fusion condition, for* $\sigma \in \prod_{\beta \in F} {}^{\eta(\beta)}2$ *we define* $p * \sigma \in P_\alpha$ *as follows: For every* $\beta \in F$ *let* $(p * \sigma)(\beta)$ *a* P_β*-name for a condition in* $\mathbb{SI}$ *such that*

$$p \restriction \beta \Vdash_{P_\beta} \quad (p * \sigma)(\beta) = p(\beta)^{\sigma(\beta)}.$$

For every $\beta \in \alpha \setminus F$ *let* $(p * \sigma)(\beta) = p(\beta)$.

(5) For $\beta \in \mathrm{supp}(p_\omega)$ *let* $Z(\beta) = Z(\bar{S}, \beta)$ *denote the set*

$$\{n < \omega : \beta \text{ is the active coordinate of } F_n\}.$$

Clearly $\langle Z(\beta) : \beta \in \mathrm{supp}(p_\omega) \rangle$ *is a partition of* ω *into infinite sets.*

(6) For $\beta \in \mathrm{supp}(p_\omega)$*, the increasing bijection between* $Z(\beta)$ *and* $\mathrm{com}(p_\omega(\beta))$ *will be denoted by the* P_β*-name* $\dot{c}(p_\omega, \beta)$.

Remark 2.1 *(1) The demand* $F_0 = F_1 = \{0\}$ *and* $F_2 = \{0,1\}$ *in Definition 2.2(2)(a) is only to avoid having to consider several cases in some proofs below.*

(2) If (p, F, η) *is a fusion condition then*

$$\langle p * \sigma : \sigma \in \prod_{\beta \in F} {}^{\eta(\beta)}2 \rangle$$

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is a (finite) maximal antichain below p.

(3) If $\bar{S} = \langle (p_n, F_n, \eta_n) : n < \omega \rangle$ *is a fusion sequence with limit* p_ω*, then*

$$\langle p_{n+1} * \sigma : \sigma \in \prod_{\beta \in F_n} {}^{\eta_n(\beta)+1}2 \rangle$$

is a maximal antichain below p_{n+1} *of size* 2^{n+1} *and it induces a maximal antichain* $A_{n+1}(\bar{S})$ *below* p_ω *of the same size, as*

$$(p_{n+1} * \sigma) \wedge p_\omega = p_\omega * \sigma,$$

such that every member of $A_n(\bar{S})$ *gets refined by two members of* $A_{n+1}(\bar{S})$*, where we let* $A_0(\bar{S}) = \{p_\omega\}$*. Hence we naturally enumerate* $A_n(\bar{S})$ *by* $\langle a^s : s \in {}^n2 \rangle$ *such that* $a^{s^\frown 0}$, $a^{s^\frown 1}$ *are the two members of* $A_{n+1}(\bar{S})$ *below* a^s *and, moreover, if* γ *is the active coordinate of* F_n*, thus* $n \in Z(\gamma)$*, then* $a^{s^\frown i} = p_\omega * \sigma$ *for some* $\sigma \in \prod_{\beta \in F_n} {}^{\eta_n(\beta)+1}2$ *with the last digit of* $\sigma(\gamma)$ *being* i*.*

Definition 2.3 *Let* $\dot{y}$ *be a* P_α*-name for an element of* ω *and* (p, F, η) *a fusion condition. We say that* (p, F, η) ***weakly decides*** $\dot{y}$ *if for every* $\sigma \in \prod_{\beta \in F} {}^{\eta(\beta)+1}2$*,* $p * \sigma$ *decides* $\dot{y}$*.*

The following lemma is well-known (e.g., see [5, Lemma 7.6] where it is proved for $\mathbb{U}_k$).

Lemma 2.1 *Suppose* $\Vdash_{P_\alpha} \dot{x} \in \omega^\omega$ *and* $p \in P_\alpha$*. There exists a fusion sequence* $\langle (p_n, F_n, \eta_n) : n < \omega \rangle$ *in* P_α *such that the following hold:*

(1) $p_0 = p$*;*

(2) (p_n, F_n, η_n) *weakly decides* $\dot{x} \restriction n$*, and hence, for every* $n < \omega$*,* (p_ω, F_n, η_n) *weakly decides* $\dot{x} \restriction n$*.*

3 The Grigorieff dichotomy

Definition 3.1 *Let* $\alpha \leq \omega_2$ *and suppose that* P_α *is a CS-iteration of* $\mathbb{SI}$*,* $\dot{x}$ *is a* P_α*-name such that* $\Vdash_{P_\alpha} \dot{x} \in 2^\omega \setminus \mathbf{V}$ *and* $p \in P_\alpha$*. By* $\dot{x}[p]$ *we denote the longest* $t \in {}^{<\omega}2$ *such that*

$$p \Vdash_{P_\alpha} t \subseteq \dot{x}.$$

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Following Grigorieff [3] we say that $k < \omega$ is ***indifferent to*** $p, \dot{x}$*, if* $k \in \mathrm{com}(p(0))$ *and there is no* $q \leq p$ *such that* $k \in \mathrm{com}(q(0))$ *and*

$$\dot{x}\left[q^{(k,0)}\right] \perp \dot{x}\left[q^{(k,1)}\right],$$

where

$$q^{(k,i)} = (q(0) \cup \{(k,i)\})^\frown q \restriction [1, \alpha).$$

We have the following ***Grigorieff dichotomy****: Either (G1) or (G2) holds, where*

(G1) $\forall p \exists q \leq p \forall r \leq q \forall k \quad k$ *is not indifferent to* $r, \dot{x}$*;*

(G2) $\exists p \forall q \leq p \exists r \leq q \exists k \quad k$ *is indifferent to* $r, \dot{x}$*.*

The following lemma, whose prototype is [3, Lemma 4.7], shows that if $\dot{x}$ is a name for a new real, then (G1) must hold.

Lemma 3.1 *Suppose* $\Vdash_{P_\alpha} \dot{x} \in \omega^\omega$ *and* $p \in P_\alpha$ *witnesses that (G2) of Grigorieff's dichotomy holds. Then* $p \Vdash_{P_\alpha} \dot{x} \in \mathbf{V}$.

Proof: We construct a fusion sequence $\langle (p_n, F_n, \eta_n) : n < \omega \rangle$ in P_α and families $\langle n_k : k < \omega \rangle$, $\langle j_k : k < \omega \rangle$ and $\langle \xi_n : n < \omega \rangle$ such that $p_0 \leq p$ and for every $n < \omega$ the following hold:

(1) $\langle n_k : k < \omega \rangle$ increasingly enumerates $Z(0)$ (hence $n_0 = 0$ and $n_1 = 1$);

(2) if $Z(0) \cap n = \{n_0, ..., n_{k-1}\}$, then $\{j_0 < ... < j_{k-1}\}$ is an initial segment of $\mathrm{com}(p_n(0))$ and j_ℓ is indifferent to $p_n, \dot{x}$ for every $\ell < k$;

(3) if $n = n_k$ then $\{j_0 < ... < j_k\}$ is an initial segment of $\mathrm{com}(p_n(0))$ and j_ℓ is indifferent to $p_n, \dot{x}$ for every $\ell \leq k$;

(4) $\xi_n \in {}^n 2$ and $p_n \Vdash_{P_\alpha} \dot{x} \restriction n = \xi_n$.

We present the first three steps of the recursive construction in detail to make clear the crucial arguments in a simple situation. After that we shall give the general step.

We apply (G2) and obtain $p_0 \leq p$ and $j_0 \in \mathrm{com}(p_0(0))$ such that j_0 is indifferent to $p_0, \dot{x}$. Wlog we may assume $j_0 = \min(\mathrm{com}(p_0(0)))$. Letting $\xi_0 = \emptyset$, (2), (3) and (4) hold for $n = n_0 = 0$.

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By Lemma 2.1, we can choose $q_1 \leq_{F_0,\eta_0} p_0$ (hence $j_0 \in \operatorname{com}(q_1(0))$) such that (q_1, F_0, η_0) weakly decides $\dot{x} \restriction 1$. Note that then q_1 even decides $\dot{x} \restriction 1$, say as ξ_1, as otherwise we had

$$\dot{x}\left[q_1 * \sigma_0\right] \perp \dot{x}\left[q_1 * \sigma_1\right],$$

where $\sigma_i = \langle\langle i\rangle\rangle$, which contradicts the indifference of j_0 (note that $q_1^{(j_0,i)} = q_1 * \sigma_i$). By (G2) we can find $p_1 \leq_{F_0,\eta_0} q_1$ and j_1 such that j_1 is indifferent to $p_1 * \sigma_0, \dot{x}$. Hence $j_1 > j_0$ and wlog we may assume that j_1 is the second member of $\operatorname{com}(p_1(0))$.

We claim that j_1 is indifferent even to $p_1, \dot{x}$, and hence (2), (3) and (4) hold for $n = n_1 = 1$. Indeed, otherwise we could find $q \leq_{F_1,\eta_1} p_1$ such that, letting $\sigma_{ij} = \langle\langle i, j\rangle\rangle$ (recall that by Definition 2.2(2) $F_1 = \{0\}$ and $\eta_1(0) = 1$),

$$\dot{x}\left[q * \sigma_{10}\right] \perp \dot{x}\left[q * \sigma_{11}\right],$$

say

$$\dot{x}\left[q * \sigma_{10}\right](m) \neq \dot{x}\left[q * \sigma_{11}\right](m)$$

for some m. Wlog we may assume that (q, F_1, η_1) weakly decides $\dot{x} \restriction m+1$. By the indifference of j_1 to $q * \sigma_0, \dot{x}$, as above we conclude that $q * \sigma_0$ decides $\dot{x}(m)$. But now we can choose j such that

$$\dot{x}\left[q * \sigma_{0j}\right] \perp \dot{x}\left[q * \sigma_{1j}\right],$$

contradicting that j_0 is indifferent to $p_0, \dot{x}$.

In order to find p_2 as desired we first choose $q_2 \leq_{F_1,\eta_1} p_1$ such that (q_2, F_2, η_2) weakly decides $\dot{x}(1)$. Again we claim that q_2 even decides $\dot{x}(1)$. Otherwise, as each $\sigma \in \prod_{\beta \in F_2} {}^{\eta_2(\beta)+1}2$ is of the form $\sigma_{ijk} = \langle\langle i, j\rangle, \langle k\rangle\rangle$, we find distinct $\sigma = \sigma_{ijk}$ and $\sigma' = \sigma_{i'j'k'}$ such that $q_2 * \sigma$ and $q_2 * \sigma'$ force different values to $\dot{x}(1)$, say x and x'. The most interesting case is $\langle i, j\rangle = \langle i', j'\rangle$ and hence $k \neq k'$. By indifference of j_1 we have that, letting $\sigma'' = \langle\langle i, 1-j\rangle, k\rangle$, $q_2 * \sigma''$ decides $\dot{x}(1)$ as x. Now we can define $r \leq_{F_0,\eta_0} q_2 * \langle i\rangle$ such that $r * \langle 1-j\rangle$ decides $\dot{x}(1)$ as x while $r * \langle j\rangle$ decides $\dot{x}(1)$ as x', which contradicts the indifference of j_1. Indeed, let r such that $r(0) = (q_2 * \langle i\rangle)(0)$ (which is the same thing as $q_2(0)^{\langle i\rangle}$) and

$$r \restriction [1, \alpha) = \begin{cases} q_2(1)^{\langle k\rangle} {}^\frown q_2 \restriction [2, \alpha), & \text{if } r(0)^{\langle i, 1-j\rangle} \text{ (belongs to the generic } G(0)) \\ q_2(1)^{\langle k'\rangle} {}^\frown q_2 \restriction [2, \alpha), & \text{otherwise.} \end{cases}$$

The other cases, i.e., $\langle i, j\rangle \neq \langle i', j'\rangle$, are similar.

Now we can let $p_2 = q_2$ and ξ_2 is what q_2 decides about $\dot{x} \restriction 2$. Then clearly (2) and (4) hold for $n = 2$ and (3) does not apply.

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Now we consider the general recursive step, where we have to apply essentially the same arguments we have seen above. Suppose that $(p_0, F_0, \eta_0), \dots,$ (p_n, F_n, η_n), $\xi_0, \dots, \xi_n$ for $n \geq 2$ have been constructed for some $n < \omega$ together with some initial segment $\langle j_\ell : \ell \leq k \rangle$ of $\mathrm{com}(p_n(0))$, where k is such that $Z(0) \cap (n+1) = \{n_0 < ... < n_k\}$ such that

$$p_n \Vdash_{P_\alpha} \dot{x} \restriction n = \xi_n$$

and every j_ℓ $(\ell \leq k)$ is indifferent to $p_n, \dot{x}$.

We have to distinguish two cases:

Case 1: $n+1 \in Z(0)$, thus $n+1 = n_{k+1}$ and clearly $n > k$, $\eta_n(0) = k$ and $\eta_{n+1}(0) = k+1$.

Fix some $\tau \in \prod_{\beta \in F_n} {}^{\eta_n(\beta)+1}2$. By (G2) and Lemma 2.1 we can find $p_{n+1} \leq_{F_n, \eta_n} p_n$ and j_{k+1} such that j_{k+1} is indifferent to $p_{n+1} * \tau, \dot{x}$ and p_{n+1} weakly decides $\dot{x}(n)$. Then clearly $j_{k+1} > j_k$. Wlog we may assume that $\{j_0, ..., j_{k+1}\}$ is an initial segment of $\mathrm{com}(p_{n+1}(0))$.

At first we show that p_{n+1} even decides $\dot{x}(n)$ (and hence $\dot{x} \restriction n+1$). Then a similar argument together with the one above showing indifference of j_1 to $p_1, \dot{x}$ will prove that j_{k+1} is even indifferent to $p_{n+1}, \dot{x}$.

For $\sigma \in \prod_{\beta \in F_n} {}^{\eta_n(\beta)+1}2$ let x_σ such that

$$p_{n+1} * \sigma \Vdash_{P_\alpha} \dot{x}(n) = x_\sigma.$$

For $\sigma, \sigma' \in \prod_{\beta \in F_n} {}^{\eta_n(\beta)+1}2$ let $s = s(\sigma, \sigma')$ be the size of the set

$$\{\ell < \eta_n(0) + 1 : \sigma(0)(\ell) \neq \sigma'(0)(\ell)\}.$$

We shall prove $x_\sigma = x_{\sigma'}$ by induction on s. Suppose first that $s(\sigma, \sigma') = 0$ (thus $\sigma(0) = \sigma'(0)$) and $\sigma \neq \sigma'$. Let σ'_1 be defined as follows: If $(\sigma(0))(0) = i$, σ'_1 equals σ' except for $(\sigma'_1(0))(0) = 1 - i$.

As j_0 is indifferent to p_{n+1} we must have $x_{\sigma'} = x_{\sigma'_1}$. Hence if we had $x_\sigma \neq x_{\sigma'}$, then also $x_\sigma \neq x_{\sigma'_1}$. But this leads to a contradiction to the indifference of j_0, as follows: Define $r \in P_\alpha$ such that

$$r(0) = p_{n+1}(0)^{\sigma(0) \restriction \eta_n(0)+1 \setminus \{0\}}$$

(hence $j_0 = \min(\mathrm{com}(r(0)))$ and $r(0)$ does not change if σ is replaced by σ') and, for $j < 2$,

$$r(0)^{(j_0, j)} \Vdash_{\mathbb{SI}} \quad r \restriction [1, \alpha) = p_{n+1} * \tau_j,$$

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where $\tau_i = \sigma \restriction F_n \setminus \{0\}$ and $\tau_{1-i} = \sigma' \restriction F_n \setminus \{0\}$. Clearly $r \leq p_n$, but

$$\dot{x}\left[r^{(j_0,i)}\right](n) = x_\sigma \neq x_{\sigma'} = \dot{x}\left[r^{(j_0,1-i)}\right](n),$$

a contradiction.

Now suppose $s(\sigma, \sigma') = m+1$ and for $s \leq m$ the claim is true. Let $\ell^* \leq \eta_n(0)$ (hence $\ell^* \leq k$) be maximal such that $(\sigma(0))(\ell^*) \neq (\sigma'(0))(\ell^*)$. Now define σ'_1 as follows : If $(\sigma(0))(\ell^*) = i$ let σ'_1 be equal to σ' except for $(\sigma'_1(0))(\ell^*) = i$.

By the indifference of j_{ℓ^*} we must have $x_{\sigma'} = x_{\sigma'_1}$. By the inductive hypothesis we have $x_\sigma = x_{\sigma'_1}$. Hence we conclude $x_\sigma = x_{\sigma'_1} = x_{\sigma'}$ and thus p_{n+1} decides $\dot{x} \restriction n+1$ as claimed. Denote this decision by ξ_{n+1}. Then clearly (2), (3) and (4) hold for $n+1$.

Now let us sketch why j_{k+1} is indifferent to $p_{n+1}, \dot{x}$. Otherwise we could find $q \leq_{F_{n+1},\eta_{n+1}}$, $\tau' \in \prod_{\beta \in F_n} {}^{\eta_n(\beta)+1}2$ and $m < \omega$ such that $\dot{x}\left[q * \tau' {}^\frown 0\right](m) \neq \dot{x}\left[q * \tau' {}^\frown 1\right](m)$ and q weakly decides $\dot{x} \restriction m+1$. Similarly as we showed above that p_{n+1} decides $\dot{x}(n)$, applying indifference of j_{k+1} to $p_{n+1} * \tau$, we can show that q even decides $\dot{x}(m)$, which is a contradiction.

<u>Case 2:</u> $n+1 \notin Z(0)$.

Choose $p_{n+1} \leq_{F_n,\eta_n} p_n$ such that p_{n+1} weakly decides $\dot{x}(n)$. Very similarly as in Case 1 we can show that p_{n+1} even decides $\dot{x}(n)$, hence also $\dot{x} \restriction n+1$, and we denote this decision by ξ_{n+1}. Then clearly (2) and (4) hold for $n+1$ and (3) does not apply. □

By Lemma 3.1 we conclude that if $\dot{x}$ is a P_α-name for a new real, then (G1) of the Grigorieff dichotomy must hold. As Theorem 3.1 below will show, this enables us to find a very precise form of continuous reading of $\dot{x}$. The prototype of this result is [3, Lemma 4.6].

The following definition was introduced in [5] (see Definition 7.8):

Definition 3.2 *Let $\dot{x}$ be a P_α-name for an element of ω^ω. We say that a fusion condition (p, F, η)* ***splits*** *$\dot{x}$ at $\gamma \in F$ if for every*

$$\sigma \in \prod_{\beta \in F \setminus \{\gamma\}} {}^{\eta(\beta)+1}2 \quad \text{and} \quad s_0, s_1 \in {}^{(\eta(\gamma)+1)}2$$

with $s_0 \restriction \eta(\gamma) = s_1 \restriction \eta(\gamma)$ and $s_0(\eta(\gamma)) < s_1(\eta(\gamma))$,

$$\dot{x}\left[p * (\sigma \cup \{(\gamma, s_0)\})\right] \perp \dot{x}\left[p * (\sigma \cup \{(\gamma, s_1)\})\right].$$

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Theorem 3.1 *Suppose* $p \in P_\alpha$ *and*

$$p \Vdash_{P_\alpha} \dot{x} \in \omega^\omega \setminus \bigcup_{\beta<\alpha} \mathbf{V}^{P_\beta}.$$

There exists a fusion sequence $\bar{S} = \langle (p_n, F_n, \eta_n) : n < \omega \rangle$ *below* p *such that for every* n*, if* γ *is the active coordinate of* F_n *then* (p_n, F_n, η_n) *splits* $\dot{x}$ *at* γ*.*

Moreover, letting $A_n(\bar{S})$ *and* $\langle a^s : s \in {}^n2 \rangle$ *be defined as in Remark 2.1 and letting* t_s *the longest common initial segment of* $\dot{x}\left[a^{s^\frown 0}\right]$ *and* $\dot{x}\left[a^{s^\frown 1}\right]$*, the following hold:*

(0) $\dot{x}\left[a^{s^\frown 0}\right](|t_s|) \perp \dot{x}\left[a^{s^\frown 1}\right](|t_s|)$*;*

(1) if $|s| < |s'|$ *then* $|t_s| < |t_{s'}|$*;*

(2) if γ *is the active coordinate of* F_n *and* $s, s' \in {}^n2$ *are such that*

$$s \restriction n \setminus \bigcup \{Z(\beta) : \beta \geq \gamma\} \neq s' \restriction n \setminus \bigcup \{Z(\beta) : \beta \geq \gamma\}$$

we have $|t_s| \neq |t_{s'}|$*. (For the definition of* $Z(\beta)$ *see Definition 2.2(5).)*

Proof: Our recursive construction will guarantee that the following demand holds for every n:

$(*)_n$ If γ is the active coordinate of F_n then

$$p_n \restriction \gamma \Vdash_{P_\gamma} \forall r \leq p_n \restriction [\gamma, \alpha) \, \forall k \quad k \text{ is not indifferent to } r, \dot{x}.$$

For $(*)_n$ to make sense we use the well-known fact that the quotient forcing $P_\alpha / \dot{G}_\gamma$ is again a CS-support iteration of $\mathbb{SI}$ (see [1, §5]).

We have to find p_0 such that (p_0, F_0, η_0) splits $\dot{x}$ at 0. For this we first apply Lemma 3.1 and the assumption and conclude that (G2) of Grigorieff's dichotomy fails below p, hence (G1) holds below p and we obtain q as in (G1). Let $k = \min(\mathrm{com}(q(0)))$. As k is not indifferent to $q, \dot{x}$ we find $p_0 \leq q$ as desired. By (G1) we also know that at every later stage $n \in Z(0)$ we will have $(*)_n$.

Now suppose we have gotten $(p_0, F_0, \eta_0), ..., (p_n, F_n, \eta_n)$ as desired such that $(*)_m$ holds for every $m \leq n$.

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Let γ be the active coordinate of F_{n+1}. We perform a recursion along the lexicopraphic order of the set

$$\Sigma = \left\{ s \restriction n+1 \setminus \bigcup \{Z(\beta) : \beta \geq \gamma\} : s \in {}^{n+1}2 \right\}.$$

Note that its members correspond to functions

$$\sigma \in \prod_{\beta \in F_{n+1} \cap \gamma} {}^{\eta_{n+1}(\beta)+1}2 \tag{3}$$

which are then ordered accordingly, say by $\prec$.

We have two subcases according to whether $n+1 = \min(Z(\gamma))$ or not. In the second case we shall apply $(*)_m$ for some $m < n+1$ in $Z(\gamma)$ and perform the same resursion as will be done in the first case after some preliminary step. Hence we treat only the first case.

In the first case, as for the construction of p_0, (working in $\mathbf{V}[\dot{G}_\gamma]$) we first have to apply our assumption together with (G1) to extend $p_n \restriction [\gamma, \alpha)$ to some $p^1 \in P_\alpha/\dot{G}_\gamma$ such that $p^1 \leq_{F_{n+1}\setminus\gamma+1, \eta_{n+1}\restriction F_{n+1}\setminus\gamma+1} p_n \restriction [\gamma, \alpha)$ and

$$p_n \restriction \gamma \Vdash_{P_\gamma} \forall r \leq p^1 \, \forall k \quad k \text{ is not indifferent to } r, \dot{x}. \tag{4}$$

Such p^1 is obtained as the last element of a $\leq_{F_{n+1}\setminus\gamma+1, \eta_{n+1}\restriction F_{n+1}\setminus\gamma+1}$-decreasing chain considering each

$$\tau \in \prod_{\beta \in F_{n+1} \setminus \gamma+1} {}^{\eta_{n+1}(\beta)+1}2. \tag{5}$$

As pedantically, p^1 is a only name for a condition in the quotient forcing (hence $(p_n \restriction \gamma, p^1)$ is not a member of P_α), by properness we obtain $p^0 \leq_{F_n \cap \gamma, \eta_n \restriction \gamma} p_n \restriction \gamma$ which decides $\operatorname{supp}(p^1)$, i.e., turns p^1 into a countable sequence of names so that (p^0, p^1) will be a member of P_α. (See [1, §5] for more details about this.)

Now we start our recursion below (p^0, p^1). Let σ be the first sequence as in (3). We have to construct $q \leq_{F_n, \eta_n} (p^0, p^1)$ such that for every pair of

$$u_0, u_1 \in {}^{\eta_{n+1}(\gamma)+1}2$$

and every τ as in (5), if $u_0 \restriction \eta_{n+1}(\gamma) = u_1 \restriction \eta_{n+1}(\gamma)$ and $u_0(\eta_{n+1}(\gamma)) < u_1(\eta_{n+1}(\gamma))$ then

$$\dot{x}\left[q * (\sigma \cup \{(\gamma, u_0)\} \cup \tau)\right] \perp \dot{x}\left[q * (\sigma \cup \{(\gamma, u_1)\} \cup \tau)\right],$$

and, letting

$$t(q, \sigma, (u_0, u_1), \tau)$$

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denote the longest common initial segment of these two incompatible nodes, we have

$$|t_s| < |t(q, \sigma, (u_0, u_1), \tau)| \tag{6}$$

for every $s \in {}^n2$. Note that $t(q, \sigma, (u_0, u_1), \tau) = t_s$ for some $s \in {}^{n+1}2$.

This is easy to achieve. Simply build a finite $\leq_{F_n,\eta_n}$-descending chain of conditions $r \leq (p^0, p^1)$ taking care of every (u_0, u_1) and τ as above. More precisely, if we have obtained r and have to consider (u_0, u_1) and τ, by (4) we know

$$r \restriction \gamma \Vdash_{P_\gamma} \forall r' \leq r \restriction [\gamma, \alpha) \, \forall k \quad k \text{ is not indifferent to } r', \dot{x}.$$

Choose G_γ a P_γ-generic filter with $r \restriction \gamma * \sigma \in G_\gamma$. Work in $\mathbf{V}[G_\gamma]$. Let $u := u_0 \restriction \eta_{n+1}(\gamma) = u_1 \restriction \eta_{n+1}(\gamma)$. Choose $r' \leq (r(\gamma)^u, r \restriction [\gamma+1, \alpha) * \tau)$ deciding an initial segment of $\dot{x}$, say ξ, that is longer than all the t_s for $s \in {}^n2$, and let $k = \min(\mathrm{com}(r'(\gamma)))$. By (4) we can find $r'' \leq r'$ such that $k \in \mathrm{com}(r''(\gamma))$ and

$$\dot{x}[(r''(\gamma) \cup \{(k,0)\}, r'' \restriction [\gamma+1, \alpha)] \perp \dot{x}[(r''(\gamma) \cup \{(k,1)\}, r'' \restriction [\gamma+1, \alpha)].$$

Then let r_1 be such that $r_1(\gamma)$ is $r(\gamma)$ except that $r_1(\gamma)^u = r''(\gamma)$ and

$$r_1(\gamma)^u \Vdash_{P_\alpha/\dot{G}_{\gamma+1}} r_1 \restriction [\gamma+1, \alpha) * \tau = r'' \restriction [\gamma+1, \alpha).$$

Finally choose $r_0 \leq_{F_n \cap \gamma, \eta_n} r \restriction \gamma$ such that $r_0 * \sigma \in G_\gamma$ forces all this, decides ξ and also decides $\mathrm{supp}(r_1)$ (see the above remark how to get (p^0, p^1)). Then $r = (r_0, r_1)$ is the next condition in our finite chain. Let q be its last element. We denote $q = (q \restriction \gamma, q \restriction [\gamma, \alpha))$ by (q_σ, q^σ).

Now suppose we have already dealt with an initial segment of σ's as in (3), built a $\leq_{F_n,\eta_n}$-descending chain of conditions below (p^0, p^1) with last element q, and σ' is the next sequence we have to consider. We essentially repeat the above recursion below q, but this time deciding long enough initial segments of $\dot{x}$ so that

$$|t((q_\sigma, q^\sigma), \sigma, (u_0, u_1), \tau)| < |t((q_{\sigma'}, q^{\sigma'}), \sigma', (u_0', u_1'), \tau')|$$

will hold for every $\sigma \prec \sigma'$ and all $(u_0, u_1), (u_0', u_1'), \tau, \tau'$ as above. Suppose now we have done this for every σ. We define p_{n+1} as the last condition we have obtained. Then by construction we have

$$\dot{x}\,[p_{n+1} * (\sigma \cup \{(\gamma, u_0)\}) \cup \tau] \perp \dot{x}\,[p_{n+1} * (\sigma \cup \{(\gamma, u_1)\} \cup \tau)]$$

for every σ, (u_0, u_1) and τ, and if $\sigma \prec \sigma'$ then

$$|t(p_{n+1}, \sigma, (u_0, u_1), \tau)| < |t(p_{n+1}, \sigma', (u_0', u_1'), \tau')|$$

for any $(u_0, u_1), (u_0', u_1'), \tau, \tau'$. Hence the theorem is proved. □

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Remark 3.1 *Suppose that* $\dot{x}$*, the fusion sequence* $\bar{S} = \langle (p_n, F_n, \eta_n) : n < \omega \rangle$ *and* $\langle t_s : s \in {}^{<\omega}2 \rangle$ *are as in Theorem 3.1 and let* p_ω *be the fusion limit of* $\bar{S}$*. Moreover, associated with these we have the refining finite antichains* $A_n(\bar{S}) = \langle a^s : s \in {}^n2 \rangle$ *as explained in Remark 2.1(3).*

(1) Let $T = T(\dot{x}, \bar{S})$ *be the tree generated by* $\bar{t} = \langle t_s : s \in {}^n2 \rangle$*. By construction,* $T \subseteq {}^{<\omega}\omega$ *is a 2-ary tree such that*

$$p_\omega \Vdash_{P_\alpha} \dot{x} \in [T]$$

and $\mathrm{split}(T) = \{t_s : s \in {}^{<\omega}2\}$*. Moreover*

$$T = \{t \in T : \neg\, p_\omega \Vdash_{P_\alpha} \neg t \subseteq \dot{x}\}$$

is the ***tree of possibilities for*** $\dot{x}$ ***below*** p_ω*.*

(2) If we step into $\mathbf{V}[G_\gamma]$ *for some* $\gamma < \omega_2$*, where* G_γ *is* P_γ*-generic containing* $p_\omega \restriction \gamma$*, then* G_γ *evaluates* $\dot{x}$ *partially. More precisely, the generic reals* $\langle g_\beta : \beta \in \mathrm{supp}(p_\omega) \cap \gamma \rangle$ *determine a possibly partial function* $H_\gamma = H_\gamma(\bar{S}) : \omega \to 2$ *with*

$$\mathrm{dom}(H_\gamma) = \bigcup \{Z(\beta) : \beta \in \mathrm{supp}(p_\omega) \cap \gamma\}$$

such that for every $n < \omega$*, only those* $a^s(\bar{S})$ *in* $A_n(\bar{S})$ *are compatible with* $p_\omega\,[G_\gamma]$ *for which* s *is compatible with* H_γ*, and hence*

$$A^\gamma_n(\bar{S})\,[G_\gamma] := \{a^s(\bar{S})\,[G_\gamma] : s \in {}^n2 \wedge \forall i \in \mathrm{dom}(s) \cap \mathrm{dom}(H_\gamma)\ s(i) = H_\gamma(i)\}$$

is a maximal antichain in the quotient forcing P_{ω_2}/G_γ *below* $p_\omega\,[G_\gamma]$*. More explicitly, given* $\beta \in \mathrm{supp}(p_\omega) \cap \gamma$ *and* $i \in Z(\beta)$*, we have*

$$H_\gamma(i) = g_\beta \circ \dot{c}(p_\omega, \beta)[G_\beta](i).$$

(See Definition 2.2(6).)

Note that for any $a^s(\bar{S})\,[G_\gamma], a^{s'}(\bar{S})\,[G_\gamma] \in A^\gamma_n(\bar{S})\,[G_\gamma]$ *we have*

$$a^s(\bar{S})\,[G_\gamma] \restriction \gamma = a^{s'}(\bar{S})\,[G_\gamma] \restriction \gamma.$$

Clearly, H_γ *is a total function iff* $\mathrm{supp}(p_\omega) \subseteq \gamma$*, otherwise* H_γ *is a Silver function, and if* $\beta < \gamma$*,* $H_\gamma \restriction Z(\beta)$ *is completely determined by* g_β*, hence belongs to* $\mathbf{V}^{P_{\beta+1}}$*.*

Corollary 3.1 *[9, Corollary 2.6] The CS-iteration of* $\mathbb{SI}$ *has the 2-localization property.*

As outlined in the introduction, [9, Corollary 2.6] proves the k-localization property for the CS-iteration of $\mathbb{SI}_k$, as well as for $\mathbb{U}_k$ and k-dimensional Sacks forcing $\mathbb{S}_k$, for every $2 \leq k < \omega$.

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4 $\mathfrak{C}_2$-sets covering new reals

In this section we prove that, given any P_{ω_2}-name $\dot{x}$ for a new real, where P_{ω_2} is the CS-iteration of $\mathbb{SI}$, it is possible to define a coding system $\langle u^b : b \in \Omega \rangle$ in the ground model such that $\Vdash_{P_{\omega_2}} \forall b \in \Omega \quad \dot{x} \notin [u^b]$. Let me first explain the core idea in the simple case where we replace P_{ω_2} by $\mathbb{SI}$ and $\dot{x}$ by the $\mathbb{SI}$-name $\dot{g}$ for the Silver real.

From now on let Ω denote the set of all infinite and coinfinite subsets of ω. For $b \in \Omega$ we define $u^b \in \mathbb{U}$ with $a^{u^b} = \omega \setminus b$ (see Definition 2.1(2)) by recursion on levels as follows: Suppose we have $n \in b$ and $t \in u^b \cap {}^n 2$. For any finite partial function s from ω to 2 we let

$$i_s = |\{j \in \operatorname{dom}(s) : s(j) = 1\}| \bmod 2.$$

We stipulate that $t^\frown i_t$ is the (only) successor node of t in u^b. Hence u^b is defined and thus also the coding system $C = \langle u^b : b \in \Omega \rangle$. Now we can prove:

$$(*) \qquad \Vdash_{\mathbb{SI}} \dot{g} \in A(C).$$

Recall

$$A(C) = {}^\omega 2 \setminus \bigcup \{[u^b] : b \in \Omega\}.$$

For this, let $f \in \mathbb{SI}$ and $b \in \Omega$. Let $m = \min(\operatorname{com}(f))$ and $n = \min(b \setminus m+1)$. Wlog we may assume that $n+1 \setminus \{m\} \subseteq \operatorname{dom}(f)$. Define

$$k = \begin{cases} 0, & \text{if } i_{f \restriction n \setminus \{m\}} = 1 - f(n) \\ 1, & \text{otherwise} \end{cases}$$

and $f' = f \cup \{(m,k)\}$. Then clearly $f' \Vdash_{\mathbb{SI}} \dot{g} \notin \left[u^b\right]$, as $f' \restriction n+1 \notin u^b$. This proves $(*)$. In the general case, the correct definition of C is more complex. It is given in the following definition. Then Lemma 4.1 will generalize the above argument.

Definition 4.1 *(1) Let Y be a nonempty set of ordinals and $\bar{Z} = \langle Z(\beta) : \beta \in Y \rangle$ a partition of ω into infinite sets. Let*

$$F_n = \{\beta \in Y : Z(\beta) \cap n+1 \neq \emptyset\}.$$

Let $T \subseteq {}^{<\omega}2$ be a perfect tree such that $\bar{t} = \langle t_s : s \in {}^{<\omega}2 \rangle$ enumerates canonically $\operatorname{split}(T)$*, hence $\{t_s : s \in {}^n 2\} = \operatorname{split}_n(T)$, such that the following are satisfied:*

(i) $|t_s| < |t_{s'}|$ *whenever* $|s| < |s'|$;

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(ii) *if* $n \in Z(\gamma)$ *and* $s, s' \in {}^{n}2$ *are such that*

$$s \restriction n \setminus \bigcup \{Z(\beta) : \beta \geq \gamma\} \neq s' \restriction n \setminus \bigcup \{Z(\beta) : \beta \geq \gamma\},$$

we have $|t_s| \neq |t_{s'}|$.

Such a tree T *will be called a* ***coding tree*** *and we write* $T = T(\bar{Z})$ *to indicate with respect to which* $\bar{Z}$ *property (ii) is satisfied.*

Given such T *we define a coding system* $C = C(T) = \langle u^b : b \in \Omega(T)\rangle$ *as follows: For* $\beta \in Y$ *let*

$$L(\beta) = \{|t_s| : |s| \in Z(\beta)\}.$$

Clearly, by (i) the $L(\beta)$ *are infinite and pairwise disjoint. We define* $\Omega(T)$ *as follows:*

$$\Omega(T) = \Omega \cap \{b : [\exists \beta \in Y\ b \subseteq L(\beta)] \vee [b \subseteq \bigcup_{\beta \in Y} L(\beta) \wedge \forall \beta \in Y\ |b \cap L(\beta))| \leq 1] \vee [b \cap \bigcup_{\beta \in Y} L(\beta) = \emptyset]\}.$$

Clearly $\Omega(T)$ *is dense in* $[\omega]^\omega$ *and every* $b \in \Omega(T)$ *is coinfinite.*

We define $C = \langle u^b : b \in \Omega(T)\rangle$ *such that for* $b \in \Omega(T)$ *we have* $u^b \in \mathbb{U}$ *with* $a^{u^b} = \omega \setminus b$ *defined according to the three types of members of* $\Omega(T)$ *as follows:*

(I) *Suppose* $b \subseteq L(\beta)$ *for* $\beta \in Y$. *We determine whether* $t \in {}^{n+1}2$ *belongs to* u^b *by recursion on* $n \in b$. *Suppose* $t \in u^b \cap {}^{n}2$. *Now we require the following:*

- *if* $t \notin T$ *let* $t^\frown 0 \in u^b$;
- *if* $t \in T \setminus \operatorname{split}(T)$ *let* $t^\frown i \in u^b$ *such that* $t^\frown i \notin T$;
- *if* $t \in \operatorname{split}(T)$, *thus* $t = t_s$ *for some* $s \in {}^{<\omega}2$, *letting*

$$i(t, \beta) := |\{j \in |s| \cap Z(\beta) : s(j) = 1\}| \bmod 2,$$

we stipulate that

$$t^\frown t_{s^\frown i(t,\beta)}(|t|) \in u^b.$$

(II) *Suppose* $\forall \beta \in Y\ |b \cap L(\beta)| \leq 1$ *and* $b \subseteq \bigcup_{\beta \in Y} L(\beta)$. *We require the following:*

- *if* $t \notin T$ *or* $t \in \operatorname{split}(T)$ *let* $t^\frown 0 \in u^b$;

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- *if* $t \in T \setminus \operatorname{split}(T)$ *let* $t^\frown i \in u^b$ *such that* $t^\frown i \notin T$*;*

(III) Suppose $b \cap \bigcup_{\beta \in Y} L(\beta) = \emptyset$*. Hence for no* $t \in \operatorname{split}(T)$ *do we have* $|t| \in b$*. We define* u^b *as in case II.*

(2) Suppose that the objects as in (1) are given and Y' *is another countable set of ordinals that is order isomorphic to* Y*. Let* $\pi : Y \to Y'$ *be the isomorphism. Let* $\bar{Z}' = \langle Z'(\beta) : \beta \in Y' \rangle$ *be the partition of* ω *induced by* π*, i.e.,* $Z'(\pi(\beta)) = Z(\beta)$ *for every* $\beta \in Y$*. Similarly, we obtain the induced finite sets* $F'_n \subseteq Y'$ *such that* $Y' = \bigcup \{F'_n : n < \omega\}$*.*

Then clearly (1)(ii) holds for $F'_n, Z'(\beta)$ *as well and* $L(\beta) = L'(\pi(\beta))$*, where*

$$L'(\pi(\beta)) = \{|t_s| : |s| \in Z'(\pi(\beta))\}.$$

Moreover, $\Omega(T(\bar{Z})) = \Omega(T(\bar{Z}'))$ *and we obtain the same trees* u^b *if we replace* $Z(\beta)$ *by* $Z'(\pi(\beta))$ *and* $L(\beta)$ *by* $L'(\pi(\beta))$*.*

Lemma 4.1 *Suppose that* P_α *is a CS-iteration of* $\mathbb{SI}$*,* $p \in P_\alpha$ *and*

$$p \Vdash_{P_\alpha} \dot{x} \in \omega^\omega \setminus \bigcup_{\beta < \alpha} \mathbf{V}^{P_\beta}.$$

Suppose also that $\bar{S} = \langle (p_n, F_n, \eta_n) : n < \omega \rangle$ *is a fusion sequence below* p *with limit* p_ω *for* $\dot{x}$ *as in Theorem 3.1. Hence we have the associated partition* $\bar{Z} = \langle Z(\beta) : \beta \in \operatorname{supp}(p_\omega) \rangle$ *and perfect tree* $T = T(\dot{x}, \bar{S})$ *with splitnodes* $\bar{t} = \langle t_s : s \in {}^n2 \rangle$*. Clearly* $T = T(\bar{Z})$ *is a coding tree as in Definition 4.1. If the coding system* $C = C(T) = \langle u^b : b \in \Omega(T) \rangle$ *is defined as there we have*

$$p_\omega \Vdash_{P_\alpha} \forall b \in \Omega(T) \cap \mathbf{V} \quad \dot{x} \notin [u^b].$$

Proof: Suppose that $q \leq p_\omega$ and $b \in \Omega(T) \cap \mathbf{V}$ are arbitrary. We have to find $q' \leq q$ such that

$$q' \Vdash_{P_\alpha} \dot{x} \notin \left[u^b\right].$$

Let the antichains $A_n(\bar{S}) = \langle a^s : s \in {}^n2 \rangle$ be defined as in Remark 2.1(3).

Define

$$R^q = \{s \in {}^{<\omega}2 : \neg\, q \perp a^s\}$$

and let

$$T^q = \{t \in T : \neg\, q \Vdash_{P_\alpha} \neg\, t \subseteq \dot{x}\}$$

be the tree of possibilities for $\dot{x}$ below q. Clearly, R^q and T^q are trees and T^q is generated by $\{t_s : s \in R^q\}$. Also note that if $t \in T^q$ and $t = t_s$ for some s, then $s \in R^q$, as otherwise let $s_0 \in R^q$ be maximal with $s_0 \subseteq s$. Hence every $s' \in R^q$ that is not an initial segment of s_0 either extends $s_0 {}^\frown 1 - s(|s_0|)$ or is incompatible with s_0. As the $t_{s'}$ for $s' \in {}^{<\omega}2$ are the splitnodes of T we conclude that $t \notin T^q$, a contradiction.

Note that wlog we may assume that for every $l \in b$ and for every $t \in T^q$ of length l there is $s \in R^q$ such that $t = t_s$. Indeed, otherwise we know by the remark we just made that t is not a splitnode of T but there is $q' \leq q$ forcing $\dot{x} \restriction l = t$ and hence by definition of u^b we have

$$q' \Vdash_{P_\alpha} \dot{x} \restriction l + 1 \notin u^b,$$

and we are done.

Now the proof proceeds along the three cases of Definition 4.1:

<u>Case I:</u>

We consider $b \subseteq L(\beta)$ for some $\beta \in \operatorname{supp}(p_\omega)$.

Choose G_β a P_β-generic filter containing $q \restriction \beta$. Hence $q(\beta)\,[G_\beta] \leq^{\mathbb{SI}} p_\omega(\beta)\,[G_\beta]$ and there exist k and $v \in {}^k2$ such that

$$\operatorname{stem}(q(\beta)\,[G_\beta]) = \operatorname{stem}(p_\omega(\beta)\,[G_\beta]^v).$$

Let $m_0 = |\operatorname{stem}(q(\beta)\,[G_\beta])|$. By the bookkeeping we used for the fusion by which we obtained p_ω we know the step at which coordinate β was active for the kth time, say it was step n_0. We choose $l \in b$ larger than $\max\{|t_s| : s \in {}^{n_0}2\}$. By property (i) of a coding tree (see Definition 4.1) and by our observation above (that every $t \in T^q$ of length l is of the form t_s for some $s \in R^q$), there is a unique $n_1 \in Z(\beta)$ such that for every $t \in T^q$ of length l there exists $s \in {}^{n_1}2$ with $t = t_s$.

Moreover, in $\mathbf{V}[G_\beta]$ the tree of possibilities for $\dot{x}$ below q (with respect to the forcing P_α/G_β) has been further restricted to $T^q[G_\beta]$ defined as follows: Letting

$$R^q[G_\beta] = \left\{s \in {}^{<\omega}2 : a^s(\bar{S})[G_\beta] \restriction \beta \in G_\beta \wedge \neg\, a^s(\bar{S}) \perp_{P_\alpha/G_\beta} q\right\},$$

$T^q[G_\beta]$ is the tree generated by the set of all t_s with $s \in R^q[G_\beta]$.

Given $s \in R^q[G_\beta] \cap {}^{n_1}2$ and letting $m_1 = \dot{c}(p_\omega, \beta)[G_\beta](n_1)$ (see Definition 2.2(6)), the value of $\dot{x}(|t_s|)$ is determined by $\dot{g}_\beta$ as $t_{s^\frown \dot{g}_\beta(m_1)}(|t_s|)$, more precisely, we have

$$q \wedge a^s(\bar{S}) \Vdash_{P_{\omega_2}/G_\beta} \dot{x}(|t_s|) = t_{s^\frown \dot{g}_\beta(m_1)}(|t_s|).$$

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In $\mathbf{V}$, we can find $\hat{q} \leq q$ in P_α as follows: $\hat{q} \restriction \beta \leq q \restriction \beta$ forces all the facts we have noticed above after we fixed G_β, and $\hat{q} \restriction \beta$ also decides $k, v, m_0, n_0, l, n_1, m_1$ as above. Moreover,

$$\hat{q} \restriction \beta \Vdash_{P_\beta} \operatorname{com}(\hat{q}(\beta)) \cap m_1 + 1 = \{m_0\},$$

$\hat{q} \restriction \beta$ decides $\hat{q}(\beta) \restriction m_1+1$, thus $\dot{g}_\beta \restriction (m_1+1 \setminus \{m_0\})$, say as $\langle g(0), ..., g(m_0-1), g(m_0+1), ..., g(m_1)\rangle$, and we let $\hat{q} \restriction [\beta+1, \omega_2) = q \restriction [\beta+1, \omega_2)$.

Now we can find $s_0, s_1 \in R^{\hat{q}} \cap {}^{n_1}2$ such that $s_i(n_0) = i$ and $s_0(j) = s_1(j)$ for every $j \in Z(\beta) \cap n_1 \setminus \{n_0\}$, and hence

$$i(t_{s_0 {}^\frown g(m_1)}, \beta) \neq i(t_{s_1 {}^\frown g(m_1)}, \beta).$$

As remarked above, we know that both t_{s_0} and t_{s_1} are splitnodes of T of length l. Now choose j such that $i(t_{s_j {}^\frown g(m_1)}, \beta) \neq g(m_1)$ and a common extension q' of $\hat{q}$ and a^{s_j}. We conclude

$$q' \Vdash_{P_\alpha} t_{s_j {}^\frown g(m_1)} \restriction |t_{s_j}| + 1 \subset \dot{x}$$

and hence $q' \Vdash_{P_\alpha} \dot{x} \notin \left[u^b\right]$, by definition of u^b.

<u>Case II:</u>

We have $b \subseteq \bigcup_{\beta \in \operatorname{supp}(p_\omega)} L(\beta)$ and $\forall \beta \in \operatorname{supp}(p_\omega)\ |b \cap L(\beta)| \leq 1$. In this case we shall apply property (ii) of a coding tree (see Definition 4.1). As

$$\{n \in Z(0) : c(p_\omega, 0)(n) \in \operatorname{com}(q(0))\}$$

is infinite (see Definition 2.2(6)), certainly we can find a large enough $\beta \in \operatorname{supp}(p_\omega)$ such that there are $l \in b \cap L(\beta)$, $n \in Z(\beta)$ and $s \in {}^n2$, $s_0, s_1 \in {}^n2 \cap R^q$ such that $|t_s| = l$ and

$$s_0 \restriction n \setminus \bigcup \{Z(\gamma) : \gamma \geq \beta\} \neq s_1 \restriction n \setminus \bigcup \{Z(\gamma) : \gamma \geq \beta\}.$$

By property (ii) of a coding tree we know that $|t_{s_0}| \neq |t_{s_1}|$. Hence we can pick $i < 2$ such that $|t_{s_i}| \neq l$. We can find q' extending both q and a_{s_i} which decides $\dot{x} \restriction l$, say as t. By property (i) of a coding tree we know that t is not a splitting node of T, and hence

$$q' \Vdash_{P_\alpha} \dot{x} \restriction l + 1 \notin u^b$$

by definition of u^b.

<u>Case III:</u>

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We have $b \cap \bigcup_{\beta \in \mathrm{supp}(p_\omega)} L(\beta) = \emptyset$ and we know that no $t \in \mathrm{split}(T)$ has $|t| \in b$. Hence by definition of u^b we conclude

$$p_\omega \Vdash_{P_\alpha} \dot{x} \notin [u^b].$$

□

Definition 4.2 *Suppose we are given two P_α-names for reals, $\dot{x}$ and $\dot{x}'$, together with conditions p, p', fusion sequences $\bar{S}$, $\bar{S}'$ below p, p', respectively, fusion limits p_ω, p'_ω and associated partitions of ω, $\bar{Z}$ and $\bar{Z}'$, respectively, as in Lemma 4.1, such that*

$$p \Vdash_{P_\alpha} \dot{x} \in \omega^\omega \setminus \bigcup_{\beta<\alpha} \mathbf{V}^{P_\beta} \text{ and } p' \Vdash_{P_{\alpha'}} \dot{x}' \in \omega^\omega \setminus \bigcup_{\beta<\alpha'} \mathbf{V}^{P_\beta}.$$

We call $\dot{x}$ and $\dot{x}'$ ***isomorphic*** *if there is an order isomorphism*

$$\pi : \mathrm{supp}(p_\omega) \to \mathrm{supp}(p'_\omega)$$

such that $Z(\beta) = Z'(\pi(\beta))$ and $T(\dot{x}, \bar{S}) = T(\dot{x}', \bar{S}')$ with the same sequence $\bar{t} = \langle t_s : s \in {}^{<\omega}2 \rangle$ of splitnodes. Then Definition 4.1(2) applies and we know that $\Omega(T(\dot{x}, \bar{S})) = \Omega(T(\dot{x}', \bar{S}'))$ and $C(T(\dot{x}, \bar{S})) = C(T(\dot{x}', \bar{S}'))$, i.e., isomorphic names are associated with the same coding system. Clearly, under CH there are only $\aleph_1$-many isomorphism types of names.

5 Extending the coding systems

As our goal is to prove $\mathrm{cov}(\mathfrak{C}_2) = \aleph_1$ in the iterated Silver model, we need to construct a family of $\aleph_1$-many coding systems coding $\mathfrak{C}_2$-sets covering 2^ω. For this, Lemma 4.1 is a good start, as any two isomorphic names of reals give rise to the same coding system as we have noticed at the end of Definition 4.2.

However, whenever by some forcing new reals are added then no dense $\Theta \subseteq [\omega]^\omega$ in $\mathbf{V}$ will be dense in the extension (see the argument in the introduction). Hence the coding system $C(T(\bar{Z})) = \langle u^b : b \in \Omega(T(\bar{Z})) \rangle$ defined in Definition 4.1 where $T(\bar{Z}) = T(\dot{x}, \bar{S})$ as in Lemma 4.1 has to be extended cofinally often during the iteration so that it will code a $\mathfrak{C}_2$-set A in the final model. This extension has to be done in such a way that Lemma 4.1 will remain true for every name belonging to the equivalence class $\mathcal{K}$ of $\dot{x}$, i.e., the evaluation of every $\dot{x}' \in \mathcal{K}$ belongs to A.

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As is well-known, new reals appear only in intermediate models $\mathbf{V}^{P_\gamma}$ where $\gamma > 0$ and $\mathrm{cf}(\gamma)$ is countable (i.e., finite and hence 1, or countably infinite). Hence at stages $\gamma \leq \omega_2$ of uncountable cofinality we can take unions of the coding systems we already constructed. Hence we actively extend the coding systems only at steps $\gamma > 0$ of countable cofinality. In order to make sure that no old real is a branch through some new tree we are adding to the system at that stage, we make use of the γ-th Silver real $\dot{g}_\gamma$. Hence this extension will be completed only in the model $\mathbf{V}^{P_{\gamma+1}}$.

We let $\Omega^0(T(\bar{Z})) = \Omega(T(\bar{Z}))$ and $C^0(T(\bar{Z})) = C(T(\bar{Z}))$. So both belong to $\mathbf{V}$. Now we define $\Omega^\gamma(T)$ and the coding system $C^\gamma(T(\bar{Z})) = \langle u^{\dot{b}} : \dot{b} \in \Omega^\gamma(T)\rangle$ for every $0 < \gamma < \omega_2$ such that $\Omega^\gamma(T) \in \mathbf{V}^{P_\gamma}$ always and $C^\gamma(T(\bar{Z})) \in \mathbf{V}^{P_\gamma}$ if $\mathrm{cf}(\gamma) = \omega_1$ and $C^\gamma(T(\bar{Z})) \in \mathbf{V}^{P_{\gamma+1}}$ if $\mathrm{cf}(\gamma)$ is countable.

Suppose we have already extended $C(T(\bar{Z}))$ to a coding system $C^\beta(T(\bar{Z})) = \langle u^{\dot{b}} : \dot{b} \in \Omega^\beta(T)\rangle$ in $\mathbf{V}^{P_{\beta+1}}$ for every $\beta < \gamma$, for some $0 < \gamma < \omega_2$. If γ is a limit of uncountable cofinality we let

$$\Omega^\gamma(T(\bar{Z})) = \bigcup\{\Omega^\beta(T(\bar{Z})) : \beta < \gamma\}$$

and

$$C^\gamma(T(\bar{Z})) = \bigcup\{C^\beta(T(\bar{Z})) : \beta < \gamma\}.$$

If γ has countable cofinality, let

$$\Omega^\gamma(T(\bar{Z})) = \{\dot{b} \in \Omega(T(\bar{Z}))^{\mathbf{V}^{P_\gamma}} : \neg\, \exists c \in [\omega]^\omega \cap \bigcup_{\beta<\gamma} \mathbf{V}^{P_\beta} \quad c \subseteq \dot{b}\}.$$

(For the definition of $\Omega(T(\bar{Z}))$ see Definition 4.1(1).) Clearly, $\Omega^\gamma(T(\bar{Z}))$ is dense in $[\omega]^\omega \cap \mathbf{V}^{P_\gamma}$. In this case, in $\mathbf{V}^{P_{\gamma+1}}$ we shall define a coding system $\langle u^{\dot{b}} : \dot{b} \in \Omega^\gamma(T(\bar{Z}))\rangle$ such that for every $\dot{x}' \in \mathcal{K}$ (with associated isomorphism π', fusion sequence $\bar{S}'$ and limit p'_ω), its evaluation in the final model is not a branch through any of the $u^{\dot{b}}$.

As in Definition 4.1, the definition of $u^{\dot{b}}$ will depend on the three types of members of $\Omega^\gamma(T(\bar{Z}))$. But now, the two first ones of these (i.e., $\dot{b} \subset L(\beta)$ for some $\beta \in \mathrm{supp}(p_\omega)$ or $\dot{b} \subseteq \bigcup_{\beta\in\mathrm{supp}(p_\omega)} L(\beta)$ and $\forall \beta \in \mathrm{supp}(p_\omega)\ |\dot{b} \cap L(\beta)| \leq 1$) split into two subcases taking care of how $\mathrm{supp}(p'_\omega)$ and $\pi'(\beta)$ are positioned with respect to γ. For this, disjoint infinite subsets $\dot{b}_0, \dot{b}_1$ of $\dot{b}$ are chosen and then, on levels from $\dot{b}_0$, $u^{\dot{b}}$ is defined in $\mathbf{V}^{P_\gamma}$, whereas on levels from $\dot{b}_1$ we need the Silver real $\dot{g}_\gamma$ to define it.

For our main result we shall need to know that in the Silver model, the ground model reals are a $\mathfrak{P}_2$-set. This seems to be proved by [2, Corollary 4.4] by using the Sacks property of $\mathbb{SI}$. Let us give a direct proof here:

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Lemma 5.1 *Suppose that $f_0 \in \mathbb{SI}$ and $\dot{b}$ is a $\mathbb{SI}$-name such that $f_0 \Vdash_{\mathbb{SI}} \dot{b} \in [\omega]^\omega$. There exist $\mathbb{SI}$-names $\dot{c}$, $\dot{y}$ and a condition $f \in \mathbb{SI}$ such that $f \leq f_0$ and*

$$f \Vdash_{\mathbb{SI}} \dot{c} \in [\dot{b}]^\omega \wedge \dot{y} \in {}^{\dot{c}}2 \setminus \{x \restriction \dot{c} : x \in {}^\omega 2 \cap \mathbf{V}\}.$$

Proof: By a simple fusion we can find $f \in \mathbb{SI}$ and a one-to-one family $\langle k_s : s \in {}^{<\omega}2\rangle$ in ω such that $f \leq f_0$ and

$$f^s \Vdash_{\mathbb{SI}} k_s \in \dot{b}$$

for every $s \in {}^{<\omega}2$. Letting $\dot{g}$ be the canonical name for the Silver real and defining

$$\dot{c} := \{k_s : f^s \subseteq \dot{g}\},$$

clearly $f \Vdash_{\mathbb{SI}} \dot{c} \in [\dot{b}]^\omega$. Now we define $\dot{y}$ on $\dot{c}$ such that

$$\dot{y}(k_s) = 0 \quad \text{iff} \quad k_{s^\frown i} \in \dot{c} \ \wedge \ k_{s^\frown i} < k_{s^\frown 1-i}.$$

Given any $f' \leq f$ and $x \in {}^\omega 2 \cap \mathbf{V}$, let s be maximal with $f' \leq f^s$ and let $j = \min(\mathrm{com}(f'))$. Hence $f' \Vdash_{\mathbb{SI}} k_s \in \dot{c}$, and $\dot{y}(k_s)$ depends on $\dot{g}(j)$, thus has not yet been decided by f'. Hence we can find $f'' \leq f'$ forcing $\dot{y}(k_s) \neq x(k_s)$. Note that the lemma is nontrivial only if $\dot{b}$ is forced to be outside $\mathbf{V}$. $\square$

Corollary 5.1 *(1) $\mathbf{V}^{\mathbb{SI}} \models {}^\omega 2 \cap \mathbf{V} \in \mathfrak{P}_2$.*

(2) If P_{ω_2} is the CS iteration of $\mathbb{SI}$ of length ω_2 then $\mathbf{V}^{P_{\omega_2}} \models {}^\omega 2 \cap \mathbf{V} \in \mathfrak{P}_2$.

Proof: (1) In $\mathbf{V}^{\mathbb{SI}}$, let $\Theta \subseteq {}^\omega[\omega]$ be dense such that for every $b \in \Theta$ we have $[b]^\omega \cap \mathbf{V} = \emptyset$. By Lemma 5.1, for every $b \in \Theta$ find a Silver function f^b with $\mathrm{dom}(f^b) = b$ such that for no $x \in {}^\omega 2 \cap \mathbf{V}$, $x \restriction b = f^b$. Then $C = \langle f^b : b \in \Theta\rangle$ codes the $\mathfrak{P}_2$-set

$$A(C) = {}^\omega 2 \setminus \bigcup \{[f^b] : b \in \Theta\}$$

which has the property ${}^\omega 2 \cap \mathbf{V} \subseteq A(C)$.

(2) For every $\gamma < \omega_2$, in $\mathbf{V}^{P_{\gamma+1}}$ we can choose a dense $\Omega^{\gamma+1} \subseteq [\omega]^\omega$ such that for every $b \in \Omega^{\gamma+1}$, $[b]^\omega \cap \mathbf{V}^{P_\gamma} = \emptyset$, and then, as in (1), define a coding system $C^{\gamma+1} = \langle f^b : b \in \Omega^{\gamma+1}\rangle$ such that ${}^\omega 2 \cap \mathbb{V}^{P_\gamma} \subseteq A(C^{\gamma+1})$. In $\mathbf{V}^{P_{\omega_2}}$ we have the coding system $C = \langle f^b : b \in \Omega^{\gamma+1}, \gamma < \omega_2\rangle$ defining the $\mathfrak{P}_2$-set $A(C)$ which covers ${}^\omega 2 \cap \mathbf{V}$. $\square$

Remark 5.1 *In [2, Proposition 4.13] it is shown that Corollary 5.1(1) is false for Laver as well as for Miller forcing.*

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Now the precise crucial definition of how we extend our coding systems along the iteration is as follows:

Definition 5.1 *Suppose that in* $\mathbf{V}$ *we are given some coding tree* $T(\bar{Z})$*. Recall that in particular* $\bar{Z} = \langle Z(\beta) : \beta \in Y \rangle$ *is a partition of* ω *into infinite sets,* $\bar{t} = \langle t_s : s \in {}^{<\omega}2 \rangle$ *canonically enumerates* $\operatorname{split}(T(\bar{Z}))$ *and* $L(\beta) = \{|t_s| : |s| \in Z(\beta)\}$*. Let* $\dot{x}$ *with fusion* $\bar{S}$ *and limit* p_ω *be associated to* $T(\bar{Z})$*, so in particular* $\operatorname{supp}(p_\omega) = Y$ *and* $T(\bar{Z}) = T(\dot{x}, \bar{S})$*. Let* $A_n(\bar{S}) = \langle a^s = a^s(\bar{S}) : s \in {}^n2 \rangle$ *for* $n < \omega$ *be the associated refining maximal antichains below* p_ω *(see Remark 2.1(3)). Let* $\pi = \mathrm{id}_Y$*.*

Let $\mathcal{K}$ *be the isomorphism class of names associated to* $T(\bar{Z})$*. For* $\dot{x}' \in \mathcal{K}$ *let* $\bar{S}'$ *be the associated fusion,* p'_ω *its limit,* $\pi' : Y \to \operatorname{supp}(p'_\omega)$ *the isomorphism and* $A_n(\bar{S}') = \langle a^s(\bar{S}') : s \in {}^n2 \rangle$ *the associated maximal antichains below* p'_ω*. Moreover, let* $\dot{g}_\gamma \in \mathbf{V}^{P_{\gamma+1}}$ *be the Silver real added by the iterand* $\dot{Q}_\gamma$ *of the iteration.*

Given any $b \subseteq \omega$*, define* $Z(b) = \{|s| : |t_s| \in b\}$*. Note that it may happen that there are* s *with* $|s| \in Z(b)$ *but* $|t_s| \notin b$ *(see property (ii) of a coding tree).*

For every $\gamma < \omega_2$*, in* $\mathbf{V}^{P_\gamma}$ *we have already defined* $\Omega^\gamma(T(\bar{Z}))$ *at the beginning of this section. Now let us define the coding system*

$$C^\gamma(T(\bar{Z})) = \langle u^{\dot{b}} : \dot{b} \in \Omega^\gamma(T(\bar{Z})) \rangle$$

in $\mathbf{V}^{P_{\gamma+1}}$*. For* $\dot{b} \in \Omega^\gamma(T(\bar{Z}))$*,* $u^{\dot{b}}$ *will be defined according to the three types of members of* $\Omega^\gamma(T(\bar{Z}))$*. Always,* $u^{\dot{b}}$ *will be defined as a uniform tree with* $a^{u^{\dot{b}}} = \omega \setminus \dot{b}$*.*

(I) Suppose $\dot{b} \in \Omega^\gamma(T(\bar{Z}))$ *is such that* $\dot{b} \subseteq L(\beta)$ *for some* $\beta \in Y$*. In* $\mathbf{V}^{P_\gamma}$*, let* $\dot{b} = \dot{b}_0 \cup \dot{b}_1$ *be a partition into two infinite sets. Levels of* $u^{\dot{b}}$ *in* $\dot{b}_0$ *are defined in* $\mathbf{V}^{P_\gamma}$ *while levels in* $\dot{b}_1$ *are defined in* $\mathbf{V}^{P_{\gamma+1}}$*. More precisely, for every* $l \in \dot{b}_0$ *and* $t \in {}^l2$*, at first we define a next digit* $d(t) < 2$ *(in* $\mathbf{V}^{P_\gamma}$*). Then, in* $\mathbf{V}^{P_{\gamma+1}}$*, for every* $l \in \dot{b}_1$ *and* $t \in {}^l2$ *a next digit* $d(t)$ *will be defined. Then, in* $\mathbf{V}^{P_{\gamma+1}}$*,* $u^{\dot{b}}$ *will be the non-empty uniform tree such that for every* $t \in u^{\dot{b}}$*, if* $|t| \in \dot{b}$*, then its next digit is* $d(t)$*.*

(0) *(On levels from* $\dot{b}_0$ *we make sure that we can deal with* $\dot{x}' \in \mathcal{K}$ *with associated isomorphism* π' *such that* $\gamma \leq \pi'(\beta)$*.) On levels in* $\dot{b}_0$ *we define* $u^{\dot{b}}$ *in* $\mathbf{V}^{P_\gamma}$ *analogously as in Definition 4.1(1)(I): Suppose* $l \in \dot{b}_0$ *and* $t \in {}^l2$*. Then* $d(t)$ *is determined as follows:*

- *if* $t \notin T(\bar{Z})$ *let* $d(t) = 0$*;*

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- *if* $t \in T(\bar{Z}) \setminus \operatorname{split}(T(\bar{Z}))$ *let* $d(t)$ *be such that* $t^\frown d(t) \notin T(\bar{Z})$;
- *if* $t \in \operatorname{split}(T(\bar{Z}))$, *thus* $t = t_s$ *for some* $s \in {}^{<\omega}2$, *letting*

$$i(t, \beta) := |\{j \in |s| \cap Z(\beta) : s(j) = 1\}| \bmod 2,$$

we stipulate that

$$d(t) = t_{s^\frown i(t,\beta)}(|t|).$$

(1) *(On levels from* $\dot{b}_1$, *we take care of those* $\dot{x}'$ *with* $\pi'(\beta) < \gamma$.*) In* $\mathbf{V}^{P_{\gamma+1}}$ *we can choose a real* $\dot{h} : Z(\dot{b}_1) \to 2$ *such that* $\dot{h} \notin \mathbf{V}^{P_\gamma}$, *e.g., if* $\dot{\rho} : Z(\dot{b}_1) \to \omega$ *is a bijection in* $\mathbf{V}^{P_\gamma}$, *let* $\dot{h} = \dot{g}_\gamma \circ \dot{\rho}$.

Now suppose $l \in \dot{b}_1$ *and* $t \in u^{\dot{b}} \cap {}^l 2$. *Define* $d(t) < 2$ *as follows:*

- *if* $t \notin T(\bar{Z})$ *let* $d(t) = 0$;
- *if* $t \in T(\bar{Z}) \setminus \operatorname{split}(T(\bar{Z}))$ *let* $d(t)$ *such that* $t^\frown d(t) \notin T(\bar{Z})$;
- *if* $t \in \operatorname{split}(T(\bar{Z}))$, *hence* $t = t_s$ *for some* s, *and* $|s| = i \in Z(\dot{b}_1)$ *for some* i, *we let*

$$d(t) = t_{s^\frown h(i)}(|t|).$$

(II) Suppose $\dot{b} \in \Omega^\gamma(T(\bar{Z}))$ *is such that* $\forall \beta \in Y \ |\dot{b} \cap L(\beta)| \leq 1$ *and* $\dot{b} \subseteq \bigcup \{L(\beta) : \beta \in Y\}$. *For* $b_* \subseteq \dot{b}$ *let*

$$Y(b_*) = \{\beta \in Y : b_* \cap L(\beta) \neq \emptyset\}.$$

Clearly, $Y(\dot{b}) \in \mathbf{V}^{P_\gamma}$ *is infinite. Note that for any* $\dot{x}' \in \mathcal{K}$ *we have*

$$Y(b_*) = \{\beta \in Y : b_* \cap L'(\pi'(\beta)) \neq \emptyset\}.$$

(See Definition 4.1(2).) Let $\dot{\delta}$ *be the largest accumulation point of* $Y(\dot{b})$. *It is easy to choose* $\dot{b}_0$ *and* $\dot{b}_1$ *in* $\mathbf{V}^{P_\gamma}$ *such that* $\dot{b}_0$ *and* $\dot{b}_1$ *are disjoint infinite subsets of* $\dot{b}$ *with* $\sup(Y(\dot{b}_0)) = \sup(Y(\dot{b}_1)) = \dot{\delta}$ *and* $\dot{\delta} \notin Y(\dot{b}_0 \cup \dot{b}_1)$. *As in Case I, on the levels in* $\dot{b}_0$, $u^{\dot{b}}$ *will be defined in* $\mathbf{V}^{P_\gamma}$, *while its levels in* $\dot{b}_1$ *will be defined in* $\mathbf{V}^{P_{\gamma+1}}$, *by defining a next-digit-function* d *on* $\bigcup_{l \in \dot{b}_0} {}^l 2$, $\bigcup_{l \in \dot{b}_1} {}^l 2$, $\bigcup_{l \in \dot{b} \setminus (\dot{b}_0 \cup \dot{b}_1)} {}^l 2$, *respectively, and then letting* $u^{\dot{b}}$ *be the non-empty tree such that for every* $t \in u^{\dot{b}}$ *with* $|t| \in \dot{b}$, *its next digit is* $d(t)$.

(0) *(On levels from* $\dot{b}_0$ *we take care of* $\dot{x}' \in \mathcal{K}$ *such that we have* $\sup(\pi[Y(\dot{b}_0)]) \geq \gamma + 1$, *and hence* $\sup(\pi[Y(\dot{b}_0)]) > \gamma + 1$.*) We define* $u^{\dot{b}}$ *on levels from* $\dot{b}_0$ *as in Definition 4.1(II): Given* $t \in {}^l 2$ *with* $l \in \dot{b}_0$, $d(t)$ *is determined such that*

- *if* $t \notin T(\bar{Z})$ *or* $t \in \operatorname{split}(T(\bar{Z}))$ *then* $d(t) = 0$;

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- *if* $t \in T(\bar{Z}) \setminus \text{split}(T(\bar{Z}))$, $d(t) < 2$ *is defined such that* $t^\frown d(t) \notin T(\bar{Z})$.

(1) *(On levels from* $\dot{b}_1$ *we take care of* $\dot{x}' \in \mathcal{K}$ *such that we have* $\sup(\pi'[Y(\dot{b}_0)]) \leq \gamma$.*) Then* $Z(\dot{b}_1) \in \mathbf{V}^{P_\gamma}$ *(see the beginning of the present definition for its definition). We choose* $\dot{h} : Z(\dot{b}_1) \to 2$ *in* $\mathbf{V}^{P_{\gamma+1}}$ *such that for no* $x : Z(\dot{b}_1) \to 2$ *in* $\mathbf{V}^{P_\gamma}$ *we have* $x = \dot{h}$. *Now suppose* $l \in \dot{b}_1$ *and* $t \in {}^l2$. *Then* $d(t)$ *is defined as follows:*

- *if* $t \notin T(\bar{Z})$ *let* $d(t) = 0$*;*
- *if* $t \in T(\bar{Z}) \setminus \text{split}(T(\bar{Z}))$ *let* $d(t) < 2$ *such that* $t^\frown d(t) \notin T(\bar{Z})$*;*
- *if* $t \in \text{split}(T(\bar{Z}))$*, thus* $t = t_s$ *for some* $s \in {}^{<\omega}2$ *(and* $|s| \in Z(\dot{b}_1)$*), we let*

$$d(t) = t_{s^\frown h(|s|)}(|t|).$$

(2) *For* $t \in u^{\dot{b}}$ *with* $|t| \in \dot{b} \setminus (\dot{b}_0 \cup \dot{b}_1)$*, its next digit in* $u^{\dot{b}}$ *is irrelevant, we require that it is* 0.

(III) Suppose $\dot{b} \in \Omega^\gamma(T(\bar{Z}))$ *is such that* $\dot{b} \cap \bigcup\{L(\beta) : \beta \in Y\} = \emptyset$. *Hence no splitnode of* $T(\bar{Z})$ *has its length in* $\dot{b}$. *We define* $u^{\dot{b}}$ *in* $\mathbf{V}^{P_\gamma}$ *such that for every* $l \in \dot{b}$ *and* $t \in {}^l2 \cap u^{\dot{b}}$ *the following hold:*

- *if* $t \notin T(\bar{Z})$ $t^\frown 0 \in u^{\dot{b}}$*;*
- *if* $t \in T(\bar{Z})$ *let* $t^\frown i \in u^{\dot{b}}$ *such that* $t^\frown i \notin T(\bar{Z})$.

This completes the definition of $C^\gamma(T(\bar{Z})) = \langle u^{\dot{b}} : \dot{b} \in \Omega^\gamma(T(\bar{Z}))\rangle$.

Finally, we define a coding system $C(T(\bar{Z}))$ *in* $\mathbf{V}^{P_{\omega_2}}$ *as*

$$C(T(\bar{Z})) = \langle u^{\dot{b}} : \dot{b} \in \bigcup\{\Omega^\gamma(T(\bar{Z})) : \gamma < \omega_2\}\rangle.$$

The following main lemma shows that the coding system constructed in Definition 5.1 has the desired property.

Lemma 5.2 *If* $T(\bar{Z})$*,* $\mathcal{K}$ *and* $C(T(\bar{Z}))$ *are as in Definition 5.1,* $\dot{x}' \in \mathcal{K}$ *is arbitrary and* $\bar{S}'$ *is the associated fusion with limit* p'_ω*, then*

$$p'_\omega \Vdash_{P_{\omega_2}} \forall b \in \bigcup\{\Omega^\gamma(T(\bar{Z})) : \gamma < \omega_2\} \quad \dot{x}' \notin [u^b].$$

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Proof: Let $q \leq p'_\omega$, $\gamma < \omega_2$ and $\dot{b}$ be arbitrary such that

$$q \Vdash_{P_{\omega_2}} \dot{b} \in \Omega^\gamma(T(\bar{Z})).$$

We have to find $q' \leq q$ such that $q' \Vdash_{P_{\omega_2}} \ \dot{x}' \notin \left[u^{\dot{b}}\right]$.

Wlog we may assume that q forces that t is a split node of $T(\bar{Z})$ whenever $|t| \in \dot{b}$ and t is a possible initial segment of $\dot{x}'$ below condition q, as otherwise, by the definition of $u^{\dot{b}}$, some $q' \leq q$ clearly forces that $\dot{x}' \restriction |t| + 1 \notin u^{\dot{b}}$. It follows that we can ignore Case III.

We need to consider the different cases in Definition 5.1.

Case I:

Wlog we may assume that $q \Vdash_{P_{\omega_2}} \dot{b} \subseteq L(\beta)$, for some $\beta \in Y$, hence we have $\pi'(\beta) \in \operatorname{supp}(p'_\omega)$ and in $\mathbf{V}^{P_\gamma}$ we have fixed the partition $\dot{b} = \dot{b}_0 \cup \dot{b}_1$ for the definition of $u^{\dot{b}}$.

Let $G_{\gamma+1}$ be a $P_{\gamma+1}$-generic filter containing $q \restriction \gamma + 1$. Moreover, we let $b = \dot{b}[G_\gamma]$ and $b_i = \dot{b}_i[G_\gamma]$ for $i < 2$.

Subcase I(0): We have $\pi'(\beta) \geq \gamma$. We can proceed essentially as in the first case of the proof of Lemma 4.1 with b_0 in place of b, as $\dot{g}_{\pi'(\beta)}$ is generic over $\mathbf{V}[G_\gamma]$, where we have defined the levels from b_0 of u^b.

In $\mathbf{V}[G_\gamma]$, we obtain the partial evaluations $T^q[G_\gamma]$ and $R^q[G_\gamma]$ of the trees $T^q = T^q(\dot{x}')$ and $R^q = R^q(\dot{x}')$ defined accordingly for $\dot{x}'$ and q as in the proof of Lemma 4.1, i.e.:

$$R^q[G_\gamma] = \{s \in {}^{<\omega}2 : a^s(\bar{S}')[G_\gamma] \restriction \gamma \in G_\gamma \wedge \neg\, q[G_\gamma] \perp_{P_{\omega_2}/G_\gamma} a^s(\bar{S}')[G_\gamma]\}$$

and $T^q[G_\gamma]$ is the subtree of T^q generated by all $t_s \in T^q$ where $s \in R^q[G_\gamma]$.

Wlog we may assume that in $\mathbf{V}[G_\gamma]$ there are $k, v, m_0, n_0, l, n_1, m_1$ such that $q \restriction \pi'(\beta)$ forces (with respect to the forcing P_{ω_2}/G_γ) the following:

- $v \in {}^k2$, $\operatorname{stem}(q(\pi'(\beta))) = \operatorname{stem}(p'_\omega(\pi'(\beta))^v)$ and $m_0 = |\operatorname{stem}(q(\pi'(\beta)))|$;
- at step n_0 of the fusion $\bar{S}'$ that produced p'_ω, $\pi'(\beta)$ was active for the k-th time;
- $l \in b_0$ is bigger than $\max\{|t_s| : s \in {}^{n_0}2\}$;
- $n_1 \in Z(\beta)$ is such that $\forall\, t \in T^q[G_\gamma] \cap {}^l2\, \exists\, s \in R^q[G_\gamma] \cap {}^{n_1}2 \quad t = t_s$;

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- $\dot{c}(p'_\omega, \pi'(\beta))(n_1) = m_1$, and
- for every $s \in R^q\,[G_\gamma]$ with $|s| = n_1$, we have

$$q\,[G_\gamma] \wedge a^s(\bar{S}')\,[G_\gamma] \Vdash_{P_{\omega_2}/G_\gamma} \dot{x}'(|t_s|) = t_{s^\frown \dot{g}_{\pi'(\beta)}(m_1)}(|t_s|).$$

Analogously as in the proof of Lemma 4.1, in $\mathbf{V}$ we find $\hat{q} \leq q$ in P_{ω_2} such that $\hat{q} \restriction \pi'(\beta)$ forces all these facts and $\hat{q} \restriction \pi'(\beta)$ also decides $k, v, m_0, n_0, l, n_1, m_1$ as above. Moreover,

$$\hat{q} \restriction \pi'(\beta) \Vdash_{P_{\pi'(\beta)}} \operatorname{com}(\hat{q}(\pi'(\beta))) \cap m_1 + 1 = \{m_0\}$$

and $\hat{q} \restriction \pi'(\beta)$ decides $\hat{q}(\pi'(\beta)) \restriction m_1 + 1$, thus $\dot{g}_{\pi'(\beta)} \restriction (m_1 + 1 \setminus \{m_0\})$, say as $\langle g(0), ..., g(m_0 - 1), g(m_0 + 1), ..., g(m_1)\rangle$.

Now we can find $s_0, s_1 \in R^{\hat{q}} \cap {}^{n_1}2$ such that $s_i(n_0) = i$ and $s_0(j) = s_1(j)$ for every $j \in Z(\beta) \cap n_1 \setminus \{n_0\}$, and hence

$$i(t_{s_0{}^\frown g(m_1)}, \beta) \neq i(t_{s_1{}^\frown g(m_1)}, \beta).$$

We know that both t_{s_0} and t_{s_1} are splitnodes of $T(\bar{Z})$ of length l. We can choose j such that $i(t_{s_j{}^\frown g(m_1)}, \beta) \neq g(m_1)$ and a common extension q' of $\hat{q}$ and $a^{s_j}(\bar{S}')$. We conclude

$$q' \Vdash_{P_{\omega_2}} t_{s_j{}^\frown g(m_1)} \restriction |t_{s_j}| + 1 \subset \dot{x}'$$

and hence $q' \Vdash_{P_{\omega_2}} \dot{x}' \notin \left[u^{\dot{b}}\right]$, by the definition of $u^{\dot{b}}$.

Subcase I(1): We have $\pi'(\beta) < \gamma$. In the intermediate model $\mathbf{V}[G_\gamma]$ we have the restricted tree of possibilities for $\dot{x}'$, $T^q(\dot{x}')[G_\gamma]$ (which is analogously defined as in Subcase I(0) above). We know that every

$$t \in T^q(\dot{x}')[G_\gamma] \cap \bigcup \left\{ {}^l 2 : l \in b_1 \right\}$$

is a splitnode of $T(\bar{Z})$, hence $t = t_s$ for some

$$s \in \bigcup \left\{ {}^i 2 : i \in Z(b_1) \right\}.$$

Now we have that if $l \in L(\beta)$, $t \in T^q(\dot{x}')[G_\gamma]$, $|t| = l$ and $t = t_s$, then t_s is no longer a splitnode of $T^q(\dot{x}')[G_\gamma]$, and its successive digit is

$$t_{s^\frown g_{\beta'} \circ \dot{c}(p'_\omega, \beta')[G_\gamma](|s|)}(l).$$

If in addition $l \in b_1$, we have $|s| \in Z(b_1)$. Note that by property (i) of a coding tree, $|s|$ does not depend on t (but on l of course).

Hence in $\mathbf{V}[G_\gamma]$ we can define a function $F = F(\dot{x}') : Z(b_1) \to 2$ as follows: Given $i \in Z(b_1)$, we let

$$F(i) = g_{\beta'} \circ \dot{c}(p'_\omega, \beta')[G_\gamma](i).$$

For the definition of u^b on levels in b_1 in the present case we applied some new function $h = \dot{h}[G_{\gamma+1}] \in \mathbf{V}\,[G_{\gamma+1}] \setminus \mathbf{V}\,[G_\gamma]$, $h : Z(b_1) \to 2$. Hence clearly we have

$$(*) \qquad \exists^\infty i \in Z(b_1) \quad h(i) \neq F(i).$$

By construction we can find s with $|s| = i$, $i \in Z(b_1)$, $h(i) \neq F(i)$ and $t_s \in T^q(\dot{x}')[G_\gamma]$. Letting $l = |t_s|$, hence $l \in b_1$, we get that

$$t_{s ^\frown F(i)}(l) \neq t_{s ^\frown h(i)}(l).$$

Hence, as $t_{s^\frown F(i)} \in T^q(\dot{x}')[G_\gamma]$, there exists $q' \leq q$ in P_{ω_2} (with $q' \restriction \gamma \in G_\gamma$) such that

$$q' \Vdash_{P_{\omega_2}} \dot{x}' \restriction l+1 = t_{s^\frown F(i)} \restriction l+1,$$

and therefore, by the definition of $u^{\dot{b}}$ on levels in $\dot{b}_1$, we conclude

$$q' \Vdash_{P_{\omega_2}} \dot{x}' \notin [u^{\dot{b}}].$$

Note that the proof here, in particular the correct choice of s that leads to a contradiction, is quite subtle. On the one hand we have that for every $t \in T^q(\dot{x}')[G_\gamma]$ such that $t = t_{s'}$ for some s' of same length as s we have

$$t_{s' ^\frown F(i)}(|t_{s'}|) \neq t_{s' ^\frown h(i)}(|t_{s'}|).$$

Note that this does not imply that, letting $l = |t_s|$ and $l' = |t_{s'}|$,

$$t_{s ^\frown F(i)}(l) = t_{s' ^\frown F(i)}(l')$$

(which is generally false). On the other hand, such other s' might be useless for our purpose as not necessarily $|t_{s'}| \in b_1$ (see the remark after the the definition of $Z(b)$ at the beginning of Definition 5.1).

Case II:

Let $\dot{b} \in \Omega^\gamma(T(\bar{Z}))$, $\dot{\delta}$ and $\dot{b}_0, \dot{b}_1 \subseteq \dot{b}$ be as there, and let $G_{\gamma+1}$ be a $P_{\gamma+1}$-generic filter containing $q \restriction \gamma + 1$. Let b, δ, b_0, b_1 be the evaluations of $\dot{b}_0, \dot{b}_1, \dot{b}$ by G_γ, respectively.

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Subcase II(0): We have $\sup(\pi'[Y(b_0)]) > \gamma$. We proceed essentially as in the second case in the proof of Lemma 4.1. We argue in $\mathbf{V}[G_\gamma]$. The set

$$\{\beta \in Y(b_0) : \pi'(\beta) > \gamma\}$$

is infinite, so we can find a large enough $\beta \in Y(b_0)$, $l \in b_0 \cap L(\beta)$, $n \in Z(\beta)$ and $s \in {}^n2$, $s_0, s_1 \in {}^n2 \cap R^q(\dot{x}')[G_\gamma]$ such that $|t_s| = l$ and

$$s_0 \restriction n \cap \{\nu \in Y : \pi'(\nu) \in (\gamma, \pi'(\beta))\} \neq s_1 \restriction n \cap \{\nu \in Y : \pi'(\nu) \in (\gamma, \pi'(\beta))\}.$$

As in the proof of Lemma 4.1 we can apply properties (i) and (ii) of the coding tree $T(\bar{Z})$ to find $q' \leq q$, $j < 2$ and $t \in {}^l2$ compatible with t_{s_j} such that t is not a splitnode of $T(\bar{Z})$ and

$$q' \Vdash_{P_{\omega_2}} \dot{x}' \restriction l = t,$$

and hence

$$q' \Vdash_{P_{\omega_2}} \dot{x}' \restriction l + 1 \notin [u^{\dot{b}}].$$

Subcase II(1): This is similar to Subcase I(1). We have $\sup(\pi'[Y(b_0)]) \leq \gamma$. In the intermediate model $\mathbf{V}[G_\gamma]$ we have the restricted tree of possibilities for $\dot{x}'$, $T^q(\dot{x}')[G_\gamma]$ (which is analogously defined as in Subcase I(0) above). We know that every $t \in T^q(\dot{x}')[G_\gamma] \cap \bigcup \left\{ {}^l2 : l \in b_1 \right\}$ is a splitnode of $T(\bar{Z})$, hence $t = t_s$ for some $s \in \bigcup \{ {}^i2 : i \in Z(b_1)\}$.

Now we have that if $\beta \in \mathrm{dom}(p_\omega)$, $\beta' = \pi'(\beta) < \gamma$, $l \in L(\beta)$, $t \in T^q(\dot{x}')[G_\gamma]$, $|t| = l$ and $t = t_s$, then t_s is no longer a splitnode of $T^q(\dot{x}')[G_\gamma]$, and its successive digit is

$$t_{s ^\frown g_{\beta'} \circ \dot{c}(p'_\omega, \beta')[G_\gamma](|s|)}(l).$$

If in addition $l \in b_1$, we have $|s| \in Z(b_1)$.

By what we noticed so far, in $\mathbf{V}[G_\gamma]$ we can define a function $F = F(\dot{x}') : Z(b_1) \to 2$ as follows: Given $i \in Z(b_1)$, letting $\beta' \in \mathrm{supp}(p'_\omega)$ with $i \in Z(\beta')$, we let

$$F(i) = g_{\beta'} \circ \dot{c}(p'_\omega, \beta')[G_\gamma](i).$$

As $h \neq F$ ($h = \dot{h}[G_{\gamma+1}]$ where $\dot{h}$ is from the Definition 5.1(II)(1)) there is $i \in Z(b_1)$ such that $h(i) \neq F(i)$. By construction we can find s and β' with $|s| = i$ and $i \in Z(\beta')$ such that $|t_s| \in b_1$ and $t_s \in T^q(\dot{x}')[G_\gamma]$. We get that

$$t_{s ^\frown F(i)}(|t_s|) \neq t_{s ^\frown h(i)}(|t_s|).$$

Hence, letting $l := |t_s|$, as $t_{s ^\frown F(i)} \in T^q(\dot{x}')[G_\gamma]$ there exists $q' \leq q$ in P_{ω_2} (with $q' \restriction \gamma \in G_\gamma$) such that

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$$q' \Vdash_{P_{\omega_2}} \dot{x}' \restriction l+1 = t_{s^\frown F(i)} \restriction l+1,$$

and therefore, by the definition of $u^{\dot{b}}$ on levels in $\dot{b}_1$, we conclude

$$q' \Vdash_{P_{\omega_2}} \dot{x}' \notin [u^{\dot{b}}].$$

<u>Case III:</u>

In this case we have that no splitnode of $T(\bar{Z})$ has its length in $\dot{b}$, and therefore, as we have noticed already,

$$p'_\omega \Vdash_{P_{\omega_2}} \dot{x}' \notin [u^b]$$

follows immediately by the definition of u^b.

We have completed the proof of Lemma 5.2. □

6 Conclusion

By Lemma 5.2 we obtain our main result:

Theorem 6.1 *If* $\mathbf{V} \models \mathrm{ZFC} + \mathrm{CH}$ *and* P_{ω_2} *is the CS-iteration of Silver forcing* $\mathbb{SI}$ *of length* ω_2*, then*

$$\mathbf{V}^{P_{\omega_2}} \models \quad \mathrm{cov}(\mathfrak{C}_2) < \mathrm{cov}(\mathfrak{P}_2).$$

Proof: Given any P_{ω_2}-name $\dot{x}$ and $p \in P_{\omega_2}$ such that

$$p \Vdash_{P_{\omega_2}} \dot{x} \in {}^\omega 2 \setminus \mathbf{V},$$

by Theorem 3.1 we obtain a fusion sequence $\bar{S}$ with limit $p_\omega \leq p$ and a tree $T = T(\dot{x}, \bar{S})$ with properties (0), (1) and (2), which is the tree of possibilities of $\dot{x}$ below p_ω. Let $\mathcal{K}$ be the isomorphism type of $\dot{x}$ as defined in Definition 4.2. Moreover, every $\dot{x}' \in \mathcal{K}$ produces the same tree.

In Definition 4.1 we have defined a coding system $C^0(T) = \langle u^b : b \in \Omega^0 \rangle$ depending only on $\mathcal{K}$ such that, by Lemma 4.1, in $\mathbf{V}$

$$p_\omega \Vdash_{P_\alpha} \forall b \in \Omega^0 \quad \dot{x} \notin \left[u^b\right].$$

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In Definition 5.1, for every $0 < \gamma < \omega_2$ of countable cofinality we have defined a coding system $C^\gamma(T) = \langle u^b : b \in \Omega^\gamma(T)\rangle$ in $\mathbf{V}^{P_{\gamma+1}}$, where $\Omega^\gamma(T) \subseteq [\omega]^\omega$ is dense in $\mathbf{V}^{P_\gamma}$ with $\Omega^\gamma(T) \cap \bigcup_{\beta<\gamma} \mathbf{V}^{P_\beta} = \emptyset$, such that by Lemma 5.2 we have

$$p_\omega \Vdash_{P_{\omega_2}} \forall b \in \bigcup \{\Omega^\gamma(T) : 0 < \gamma < \omega_2\} \quad \dot{x} \notin [u^b].$$

For $\gamma < \omega_2$ of uncountable cofinality, $\Omega^\gamma(T)$ and $C^\gamma(T)$ were defined by taking the union of the $\Omega^\beta(T)$, $C^\beta(T)$, respectively, for $\beta < \gamma$. Hence in $\mathbf{V}^{P_{\omega_2}}$ we have the $\mathfrak{C}_2$-set $A(C)$, where $C = \langle u^b : b \in \Omega^\gamma, \gamma < \omega_2\rangle$, such that

$$p_\omega \Vdash_{P_{\omega_2}} \dot{x} \in A(C).$$

As $A(C)$ only depends on T, and as by CH in the ground model there are only $\aleph_1$-many isomorphism types $\mathcal{K}$, we have proved that in $\mathbf{V}^{P_{\omega_2}}$, ${}^\omega 2 \setminus \mathbf{V}$ can be covered by $\aleph_1$-many $\mathfrak{C}_2$-sets. By Corollary 5.1(2) we know that ${}^\omega 2 \cap \mathbf{V}$ is even in $\mathfrak{P}_2$. Hence $\text{cov}(\mathfrak{C}_2) = \aleph_1$ holds in $\mathbf{V}^{P_{\omega_2}}$.

As we have explained in the introduction, $\mathbf{V}^{P_{\omega_2}} \models \text{cov}(v^0) = \aleph_2$, where v^0 is the Silver ideal. As $\mathfrak{P}_2 \subseteq v^0$, we have $\mathbf{V}^{P_{\omega_2}} \models \text{cov}(\mathfrak{P}_2) = \aleph_2$. □

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