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Dynamics of a susceptible–infected–susceptible epidemic reaction–diffusion model

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We study a susceptible–infected–susceptible reaction–diffusion model with spatially heterogeneous disease transmission and recovery rates. A basic reproduction number $\mathcal{R}_0$ is defined for the model. We first prove that there exists a unique endemic equilibrium if $\mathcal{R}_0 > 1$. We then consider the global attractivity of the disease-free equilibrium and the endemic equilibrium for two cases. If the disease transmission and recovery rates are constants or the diffusion rate of the susceptible individuals is equal to the diffusion rate of the infected individuals, we show that the disease-free equilibrium is globally attractive if $\mathcal{R}_0 \leqslant 1$, while the endemic equilibrium is globally attractive if $\mathcal{R}_0 > 1$.

Keywords: SIS epidemic reaction–diffusion model; spatial heterogeneity; disease-free equilibrium; endemic equilibrium; global attractivity

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1. Introduction

Over the past few years, several susceptible–infected-type (SI-type) epidemic reaction–diffusion models have been developed to study the impact of spatial heterogeneity of environment and movement rates of individuals on the dynamics of the models. In [3], a susceptible-infected-susceptible (SIS) reaction–diffusion model with homogeneous Neumann boundary conditions,

$$\left.\begin{aligned} S_t &= d_{\mathrm{S}}\Delta S - \frac{\beta(x)SI}{S+I} + \gamma(x)I, \\ I_t &= d_{\mathrm{I}}\Delta I + \frac{\beta(x)SI}{S+I} - \gamma(x)I, \end{aligned}\right\} \quad x \in \Omega,\ t > 0, \tag{1.1}$$

has been considered under the condition

$$\int_{\Omega} (S(x,0) + I(x,0))\,\mathrm{d}x \equiv N > 0,$$

where Ω is a bounded domain in $\mathbb{R}^m$ and N is the total number of individuals at $t = 0$.

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In [3], a basic reproduction number $\mathcal{R}_0$ is defined, and it was shown that the disease-free equilibrium is globally asymptotically stable if $\mathcal{R}_0 \leqslant 1$, while a unique endemic equilibrium exists if $\mathcal{R}_0 > 1$. Later, in [18, 20], the global attractivity of the endemic equilibrium of the model was proved for two cases: $d_{\mathrm{S}} = d_{\mathrm{I}}$ and $\gamma(x) = r\beta(x)$, although the global attractivity of the endemic equilibrium for general cases remains open. Furthermore, several results on asymptotic profiles of the equilibria (see [3, 17, 19]) have been established, which has important implications for disease control.

In addition, an SIS epidemic patch model was formulated in [2], where susceptible and infected individuals are both allowed to move between different patches. This patch model can be considered as a spatially discrete version of the model (1.1). In [14], two SIS reaction–diffusion models similar to (1.1) with Dirichlet boundary conditions were considered, and partial results on the global stability of the endemic equilibrium were obtained.

On the other hand, in [11, 22], under the assumption that recovered individuals have permanent immunity, the classic Kermack–McKendrick model (see [4]) is extended to the following reaction–diffusion model with homogeneous Neumann boundary conditions:

$$\left.\begin{aligned} S_t &= d_{\mathrm{S}}\Delta S - \beta SI, \\ I_t &= d_{\mathrm{I}}\Delta I + \beta SI - \gamma I, \end{aligned}\right\} \quad x \in \Omega,\ t > 0, \tag{1.2}$$

where β and γ are positive constants. By introducing some Lyapunov functionals (see [15] for a different proof), it is proved that the density of susceptible individuals $S(x, t)$ converges to a positive constant uniformly and the density of infected individuals $I(x, t)$ converges to zero uniformly. In sharp contrast with (1.1), this model always predicts the elimination of disease in the long run. For more studies on SI-type models, the reader is referred to the survey papers [4, 10, 21].

Note that the well-known SIS model, also due to Kermack and McKendrick (see [5]), takes the form of a system of ordinary differential equations:

$$\left.\begin{aligned} S' &= -\beta SI + \gamma I, \\ I' &= \beta SI - \gamma I, \end{aligned}\right\} \quad t > 0, \tag{1.3}$$

with initial data satisfying

$$S(0) + I(0) = N > 0,$$

where N represents the total population. A basic reproduction number can be defined as $\mathcal{R}_0 = N\beta/\gamma$. It is proved that if $\mathcal{R}_0 \leqslant 1$, the solution $(S(t), I(t))$ of (1.3) approaches the disease-free equilibrium $(N, 0)$, while if $\mathcal{R}_0 > 1$, a unique endemic equilibrium exists:

$$S^* = \frac{\gamma}{\beta}, \qquad I^* = N - \frac{\gamma}{\beta},$$

and it is globally asymptotically stable.

Our main objective is to generalize model (1.3) to an epidemic reaction–diffusion model and then study the existence of the disease-free equilibrium and the endemic equilibrium and their global attractivity. Even though our model bears a resemblance to model (1.1) by Allen *et al*. [3], there is one major difference: Allen *et al*.

consider the SIS reaction–diffusion model with frequency-dependent interaction, while we focus on mass-action-type nonlinearity. Consequently, our arguments for proving the global existence and boundedness of the model, the existence of the endemic equilibrium and the global attractivity of the endemic equilibrium are quite different.

The paper is organized as follows. In § 2, we present the model and establish the global existence and boundedness results. In § 3, we define a basic reproduction number $\mathcal{R}_0$ and prove that there exists a unique endemic equilibrium if $\mathcal{R}_0 > 1$. In § 4, we consider the global attractivity of the disease-free equilibrium and the endemic equilibrium for two cases. If the disease transmission and recovery rates are constants or the diffusion rate of the susceptible individuals is equal to the diffusion rate of the infected individuals, we show that the disease-free equilibrium is globally attractive if $\mathcal{R}_0 \leqslant 1$, while the endemic equilibrium is globally attractive if $\mathcal{R}_0 > 1$. In § 5, we conduct a concluding discussion.

2. The model

Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial\Omega$. Let $S(x,t)$ and $I(x,t)$ be the density of susceptible and infected individuals at location x and time t, respectively. We assume that the individuals randomly move in the domain Ω with diffusion rates d_{S} and d_{I} for susceptible and infected individuals, respectively. If all the infected individuals at the same location have the same rate to recover and become susceptible immediately, an SIS epidemic reaction–diffusion model can be formulated as follows:

$$\left.\begin{aligned} S_t &= d_{\mathrm{S}}\Delta S - \beta(x)SI + \gamma(x)I, \\ I_t &= d_{\mathrm{I}}\Delta I + \beta(x)SI - \gamma(x)I, \end{aligned}\right\} \quad x \in \Omega,\ t > 0, \tag{2.1}$$

where the disease transmission-rate function $\beta(x)$ describes the effective interaction between susceptible and infected individuals at location x, and the function $\gamma(x)$ represents the recovery rate of the infected individuals at location x. Both β and γ are positive Hölder-continuous functions in $\bar{\Omega}$. Furthermore, we assume that there is no flux across the boundary $\partial\Omega$, that is,

$$\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial\Omega,\ t > 0, \tag{2.2}$$

where $\partial/\partial n$ is the outward normal derivative to $\partial\Omega$. We also assume that the initial data satisfy the following hypothesis.

(H1) $S(x,0)$ and $I(x,0)$ are non-negative continuous functions in $\bar{\Omega}$, and initially the number of infected individuals is positive, i.e.

$$\int_\Omega I(x,0)\,\mathrm{d}x > 0.$$

Let

$$\int_\Omega (S(x,0) + I(x,0))\,\mathrm{d}x \equiv N$$

be the total number of individuals at $t = 0$. Adding the two equations in (2.1) and then integrating over the domain Ω, we find

$$\frac{\partial}{\partial t}\int_\Omega (S+I)\,\mathrm{d}x = 0, \quad t>0,$$

which implies that the total population size is a constant given by

$$\int_\Omega (S(x,t)+I(x,t))\,\mathrm{d}x = N. \tag{2.3}$$

We then establish the global existence and boundedness results for the model.

THEOREM 2.1. *Suppose that (H1) holds. Then the solution* $(S(x,t), I(x,t))$ *of problem (2.1), (2.2) exists uniquely and globally. Moreover, there exists a positive constant* M *depending on the initial data and* $\max_{x\in\bar\Omega}\{\gamma(x)/\beta(x)\}$ *such that*

$$0 < S(x,t), I(x,t) \leqslant M \quad \text{for } x\in\bar\Omega,\ t\in(0,\infty). \tag{2.4}$$

Proof. Let $(\hat S(x,t), \hat I(x,t))$ be the local solution of the following problem:

$$\left.\begin{aligned}
\hat S_t &= d_{\mathrm{S}}\Delta\hat S + \gamma(x)\hat I, \quad x\in\Omega,\ t>0,\\
\hat I_t &= d_{\mathrm{I}}\Delta\hat I + \beta(x)\hat S\hat I - \gamma(x)\hat I, \quad x\in\Omega,\ t>0,\\
\frac{\partial\hat S}{\partial n} &= \frac{\partial\hat I}{\partial n} = 0, \quad x\in\partial\Omega,\ t>0,\\
\hat S(x,0) &= S(x,0), \quad \hat I(x,0) = I(x,0), \quad x\in\bar\Omega.
\end{aligned}\right\} \tag{2.5}$$

Then $(\hat S(x,t), \hat I(x,t))$ and $(0,0)$ are a pair of coupled upper and lower solutions of (2.1), (2.2), and it follows that there exists a unique solution $(S(x,t), I(x,t))$ of (2.1), (2.2) for $x\in\bar\Omega$ and $t\in[0,T_{\max})$, where $T_{\max}$ is the maximal existence time (see [16]). Moreover, by the maximum principle, the solution is positive in $\bar\Omega\times(0,T_{\max})$. We now consider the problem for $S(x,t)$ in $\Omega\times(0,T_{\max})$:

$$\left.\begin{aligned}
S_t &= d_{\mathrm{S}}\Delta S + (\gamma(x)-\beta(x)S)I, \quad x\in\Omega,\ t\in(0,T_{\max}),\\
\frac{\partial S}{\partial n} &= 0, \quad x\in\partial\Omega,\ t\in(0,T_{\max}).
\end{aligned}\right\} \tag{2.6}$$

Choose $M_1 = \max\{\max_{x\in\bar\Omega} S(x,0), \max_{x\in\bar\Omega}\{\gamma(x)/\beta(x)\}\}$. Then, for any non-negative function $I(x,t)$, M_1 and 0 are a pair of upper and lower solutions of problem (2.6). By the comparison principle, one can see that $S(x,t)\leqslant M_1$ in $\bar\Omega\times[0,T_{\max})$. Since $\int_\Omega I(x,t)\,\mathrm{d}x\leqslant N$, in view of [1, theorem 3.1] (or see [8,9]), there exists a positive constant M_2 depending on $I(x,0)$ such that $I(x,t)\leqslant M_2$ in $\bar\Omega\times[0,T_{\max})$. Hence, it follows from the standard theory for semilinear parabolic systems that $T_{\max}=\infty$. □

3. Equilibria

We now consider the equilibria of problem (2.1), (2.2), that is, the solutions of the following semilinear elliptic system:

$$\left.\begin{aligned}
d_{\mathrm{S}}\Delta\bar S - \beta\bar S\bar I + \gamma\bar I &= 0,\\
d_{\mathrm{I}}\Delta\bar I + \beta\bar S\bar I - \gamma\bar I &= 0,
\end{aligned}\right\} \quad x\in\Omega, \tag{3.1}$$

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with boundary conditions

$$\frac{\partial \bar{S}}{\partial n} = \frac{\partial \bar{I}}{\partial n} = 0, \quad x \in \partial\Omega. \tag{3.2}$$

Here $\bar{S}(x)$ and $\bar{I}(x)$ are the densities of susceptible and infected individuals at location x, respectively. In view of (2.3), we impose an additional condition:

$$\int_\Omega (\bar{S} + \bar{I})\,\mathrm{d}x = N. \tag{3.3}$$

And we are only interested in non-negative solutions of (3.1)–(3.3). As with other epidemic models, we shall focus on the *disease-free equilibrium* (DFE) and the *endemic equilibrium* (EE). A DFE is a solution of (3.1)–(3.3) with $\bar{I}(x) = 0$ for all $x \in \bar{\Omega}$, while an EE is a solution with $\bar{I}(x) > 0$ for some $x \in \Omega$. To distinguish between these two types of equilibrium, we shall denote a DFE by $(\tilde{S}, 0)$ and an EE by (S^*, I^*). Let $|\Omega|$ be the measure of Ω. We first show that the disease-free equilibrium exists uniquely.

PROPOSITION 3.1. *Problem (3.1)–(3.3) has a unique DFE given by*

$$(\tilde{S}, 0) = \left(\frac{N}{|\Omega|}, 0\right).$$

Proof. Clearly, $(N/|\Omega|, 0)$ is a DFE. Now, for any DFE $(\tilde{S}, 0)$, by (3.1), we have that $\Delta\tilde{S} = 0$. Then, by the maximum principle and the boundary condition $\partial\tilde{S}/\partial n = 0$, $\tilde{S}$ must be a constant in $\bar{\Omega}$. It then follows from (3.3) that $\tilde{S} = N/|\Omega|$. □

We now follow the idea of [3] to linearize (3.1) around the DFE. Let $\eta(x,t) = S(x,t) - N/|\Omega|$ and $\xi(x,t) = I(x,t)$. Using (2.1) and dropping higher-order terms, we obtain the following system:

$$\eta_t = d_{\mathrm{S}}\Delta\eta - \left(\frac{N}{|\Omega|}\beta - \gamma\right)\xi, \quad x \in \Omega,\ t > 0,$$

$$\xi_t = d_{\mathrm{I}}\Delta\xi + \left(\frac{N}{|\Omega|}\beta - \gamma\right)\xi, \quad x \in \Omega,\ t > 0.$$

Let $(\eta(x,t), \xi(x,t)) = (\mathrm{e}^{-\lambda t}\phi(x), \mathrm{e}^{-\lambda t}\psi(x))$. We then derive an eigenvalue problem:

$$\left.\begin{aligned} d_{\mathrm{S}}\Delta\phi - \left(\frac{N}{|\Omega|}\beta - \gamma\right)\psi + \lambda\phi = 0, \\ d_{\mathrm{I}}\Delta\psi + \left(\frac{N}{|\Omega|}\beta - \gamma\right)\psi + \lambda\psi = 0, \end{aligned}\right\} \quad x \in \Omega, \tag{3.4}$$

with boundary conditions

$$\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n} = 0, \quad x \in \partial\Omega. \tag{3.5}$$

In view of (2.3) and proposition 3.1, we impose an additional condition

$$\int_\Omega (\phi + \psi)\,\mathrm{d}x = 0. \tag{3.6}$$

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Then, similarly to [3], there exists a least eigenvalue λ^* with its corresponding eigenvector ψ^*, where λ^* is a real number, ψ^* is strictly positive on Ω and (λ^*, ψ^*) satisfies

$$d_{\mathrm{I}}\Delta\psi^* + \left(\frac{N}{|\Omega|}\beta - \gamma\right)\psi^* + \lambda^*\psi^* = 0, \quad x \in \Omega, \quad \text{and} \quad \frac{\partial\psi^*}{\partial n} = 0, \quad x \in \partial\Omega. \tag{3.7}$$

Moreover, the eigenvalue λ^* is given by the variational formula

$$\lambda^* = \inf\left\{\int_\Omega \left(d_{\mathrm{I}}|\nabla\varphi|^2 + \left(\gamma - \frac{N}{|\Omega|}\beta\right)\varphi^2\right)\mathrm{d}x \colon \varphi \in H^1(\Omega) \text{ and } \int_\Omega \varphi^2\,\mathrm{d}x = 1\right\}.$$

We then consider the existence of the endemic equilibrium. To this end, we define a basic reproduction number $\mathcal{R}_0$. The variational formula suggests that we can define $\mathcal{R}_0$ as follows:

$$\mathcal{R}_0 = \sup\left\{\frac{(N/|\Omega|)\int_\Omega \beta\varphi^2\,\mathrm{d}x}{\int_\Omega (d_{\mathrm{I}}|\nabla\varphi|^2 + \gamma\varphi^2)\,\mathrm{d}x} \colon \varphi \in H^1(\Omega) \text{ and } \varphi \neq 0\right\}.$$

We now state a result which is similar to [3, lemmas 2.2 and 2.3].

Proposition 3.2. *The following statements about* λ^* *and* $\mathcal{R}_0$ *hold:*

(a) $\mathcal{R}_0 > 1$ *when* $\lambda^* < 0$, $\mathcal{R}_0 = 1$ *when* $\lambda^* = 0$ *and* $\mathcal{R}_0 < 1$ *when* $\lambda^* > 0$;

(b) *if*

$$\int_\Omega \frac{N}{|\Omega|}\beta\,\mathrm{d}x \geqslant \int_\Omega \gamma\,\mathrm{d}x,$$

then $\lambda^* \leqslant 0$ *for all* $d_{\mathrm{I}} > 0$;

(c) *if* $(N/|\Omega|)\beta - \gamma$ *changes sign on* Ω *and if*

$$\int_\Omega \frac{N}{|\Omega|}\beta\,\mathrm{d}x < \int_\Omega \gamma\,\mathrm{d}x,$$

then there exists $d_{\mathrm{I}}^* > 0$ *such that* $\lambda^* = 0$ *when* $d_{\mathrm{I}} = d_{\mathrm{I}}^*$, $\lambda^* < 0$ *when* $d_{\mathrm{I}} < d_{\mathrm{I}}^*$ *and* $\lambda^* > 0$ *when* $d_{\mathrm{I}} > d_{\mathrm{I}}^*$.

Remark 3.3. Clearly, by the variational formula, if

$$\int_\Omega \frac{N}{|\Omega|}\beta\,\mathrm{d}x > \int_\Omega \gamma\,\mathrm{d}x,$$

then $\lambda^* < 0$.

The following proposition shows that the stability of the DFE relies on the magnitude of $\mathcal{R}_0$, and it can be proved analogously to [3, lemma 2.4].

Proposition 3.4. *The DFE is stable if* $\mathcal{R}_0 < 1$, *and it is unstable if* $\mathcal{R}_0 > 1$.

We then study the existence of the endemic equilibrium. We first convert problem (3.1)–(3.3) to a more approachable problem.

LEMMA 3.5. *The pair* $(\bar{S},\bar{I})$ *is a non-negative solution of problem (3.1)–(3.3) if and only if it is a non-negative solution of the following problem:*

$$d_{\mathrm{I}}\Delta\bar{I}+\bar{I}\biggl(\frac{N}{|\Omega|}\beta-\gamma-\biggl(1-\frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\biggr)\frac{\beta}{|\Omega|}\int_{\Omega}\bar{I}\,\mathrm{d}x-\frac{d_{\mathrm{I}}\beta}{d_{\mathrm{S}}}\bar{I}\biggr)=0,\quad x\in\Omega, \tag{3.8}$$

$$\bar{S}=\frac{N}{|\Omega|}-\biggl(1-\frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\biggr)\frac{1}{|\Omega|}\int_{\Omega}\bar{I}\,\mathrm{d}x-\frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\bar{I},\quad x\in\Omega, \tag{3.9}$$

$$\frac{\partial\bar{I}}{\partial n}=0,\quad x\in\partial\Omega. \tag{3.10}$$

Proof. Through routine calculations, one can easily check that $(\bar{S},\bar{I})$ is a non-negative solution of problem (3.1)–(3.3) if and only if it solves the following problem:

$$d_{\mathrm{S}}\bar{S}+d_{\mathrm{I}}\bar{I}=K,\quad x\in\bar{\Omega}, \tag{3.11}$$

$$d_{\mathrm{I}}\Delta\bar{I}+\bar{I}(\beta\bar{S}-\gamma)=0,\quad x\in\Omega, \tag{3.12}$$

$$\frac{\partial\bar{S}}{\partial n}=\frac{\partial\bar{I}}{\partial n}=0,\quad x\in\partial\Omega, \tag{3.13}$$

$$\int_{\Omega}(\bar{S}+\bar{I})\,\mathrm{d}x=N, \tag{3.14}$$

where K is some positive constant that is independent of $x\in\Omega$. Thus, we only need to show the equivalence between problems (3.11)–(3.14) and (3.8)–(3.10). On the one hand, suppose that $(\bar{S},\bar{I})$ is a non-negative solution of (3.11)–(3.14). By (3.11), we have $\bar{S}=(K-d_{\mathrm{I}}\bar{I})/d_{\mathrm{S}}$. Substituting it into (3.14), we find

$$K=\frac{1}{|\Omega|}\biggl(d_{\mathrm{S}}N-(d_{\mathrm{S}}-d_{\mathrm{I}})\int_{\Omega}\bar{I}\,\mathrm{d}x\biggr).$$

It then follows from (3.11) that

$$\bar{S}=\frac{K-d_{\mathrm{I}}\bar{I}}{d_{\mathrm{S}}}=\frac{N}{|\Omega|}-\biggl(1-\frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\biggr)\frac{1}{|\Omega|}\int_{\Omega}\bar{I}\,\mathrm{d}x-\frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\bar{I},$$

which is (3.9). Substituting this $\bar{S}$ into (3.12), we obtain (3.8).

On the other hand, suppose that $(\bar{S},\bar{I})$ is a non-negative solution of problem (3.8)–(3.10). Taking a normal derivative of both sides of (3.9) and using (3.10), we find $\partial\bar{S}/\partial n=0$, which verifies (3.13). Furthermore, by (3.9), we have that

$$\frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\bar{I}=\frac{N}{|\Omega|}-\biggl(1-\frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\biggr)\frac{1}{|\Omega|}\int_{\Omega}\bar{I}\,\mathrm{d}x-\bar{S},$$

and substitution of this into (3.8) gives (3.12). We then integrate both sides of (3.9) over Ω to obtain (3.14). Applying the Laplace operator to both sides of (3.9), we find that $d_{\mathrm{S}}\Delta\bar{S}+d_{\mathrm{I}}\Delta\bar{I}=\Delta(d_{\mathrm{S}}\bar{S}+d_{\mathrm{I}}\bar{I})=0$. Since $\partial/\partial n(d_{\mathrm{S}}\bar{S}+d_{\mathrm{I}}\bar{I})=0$, the maximum principle implies that $d_{\mathrm{S}}\bar{S}+d_{\mathrm{I}}\bar{I}$ is a constant. In view of (3.14), this constant must be positive, which yields (3.11). □

Problem (3.8)–(3.10) is more approachable, since (3.8) and (3.10) are independent of $\bar{S}$. In addition, the following result indicates that we can actually focus on a non-local elliptic problem that involves only $\bar{I}$.

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LEMMA 3.6. *If* $\bar{I} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *is a non-negative solution of the non-local elliptic problem*

$$d_{\mathrm{I}}\Delta\bar{I} + \bar{I}\left(\frac{N}{|\Omega|}\beta - \gamma - \left(1 - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\right)\frac{\beta}{|\Omega|}\int_\Omega \bar{I}\,\mathrm{d}x - \frac{d_{\mathrm{I}}\beta}{d_{\mathrm{S}}}\bar{I}\right) = 0, \quad x \in \Omega, \tag{3.15}$$

$$\frac{\partial\bar{I}}{\partial n} = 0, \quad x \in \partial\Omega, \tag{3.16}$$

then we have that

$$\left(1 - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\right)\frac{1}{|\Omega|}\int_\Omega \bar{I}\,\mathrm{d}x + \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\bar{I}(x) \leqslant \frac{N}{|\Omega|} \quad \textit{for all } x \in \bar{\Omega}. \tag{3.17}$$

Proof. If $\bar{I}$ is trivial, then the claim holds. If $\bar{I}$ is not identically zero on $\bar{\Omega}$, we assume to the contrary that the claim is false. Since $\bar{I}$ is continuous on $\bar{\Omega}$, it attains its maximum value on $\bar{\Omega}$, say, $\bar{I}(x_0) = \max_{x\in\bar{\Omega}} \bar{I}(x) > 0$ for some $x_0 \in \bar{\Omega}$. Under the assumption, one must have that

$$\left(1 - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\right)\frac{\int_\Omega \bar{I}\,\mathrm{d}x}{|\Omega|} + \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\bar{I}(x_0) > \frac{N}{|\Omega|}. \tag{3.18}$$

If $x_0 \in \Omega$, we can choose a closed ball B centred at x_0 such that $B \subset \Omega$. By (3.15) and (3.18), we can make the ball so small that $d_{\mathrm{I}}\Delta\bar{I} > 0$ in B. Since $\bar{I}$ attains its maximum at an interior point x_0 of B, by the strong maximum principle, $\bar{I}$ must be a constant in B. But this is impossible, since $d_{\mathrm{I}}\Delta\bar{I} > 0$ in B. So $x_0 \in \partial\Omega$, and $I(x_0) > I(x)$ for all $x \in \Omega$. Then we can find a closed ball $\hat{B} \subset \Omega$ such that $\hat{B} \cap \bar{\Omega} = \{x_0\}$. Again, by (3.15) and (3.18), we can make the ball so small that $d_{\mathrm{I}}\Delta\bar{I} > 0$ in the interior of $\hat{B}$. It then follows from the Hopf lemma that $\partial\bar{I}/\partial n(x_0) > 0$, which is also impossible by virtue of (3.16). □

In view of lemma 3.6, if there is a non-negative solution $\bar{I}$ of the non-local elliptic problem (3.15), (3.16), then one can define

$$\bar{S} \equiv \frac{N}{|\Omega|} - \left(1 - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\right)\frac{1}{|\Omega|}\int_\Omega \bar{I}\,\mathrm{d}x - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\bar{I} \quad \text{for } x \in \bar{\Omega},$$

which is non-negative by (3.17). Then it follows that the pair $(\bar{S}, \bar{I})$ solves problem (3.8)–(3.10).

Let $Y = \{z \in C^{2,\alpha}(\bar{\Omega})\colon \partial z/\partial n = 0 \text{ on } \partial\Omega\}$. For simplicity, we introduce

$$f(\tau, \bar{I}) = \frac{N}{|\Omega|}\beta - \gamma - \left(1 - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\right)\frac{\beta}{|\Omega|}\tau - \frac{d_{\mathrm{I}}\beta}{d_{\mathrm{S}}}\bar{I},$$

and define a mapping $F\colon \mathbb{R}^+ \times Y \to C^\alpha(\bar{\Omega})$ by

$$F(\tau, \bar{I}) = d_{\mathrm{I}}\Delta\bar{I} + \bar{I}f(\tau, \bar{I}).$$

We then consider an eigenvalue problem:

$$\left.\begin{aligned} d_{\mathrm{I}}\Delta\varphi + f(\tau, 0)\varphi + \lambda\varphi = 0, \quad x \in \Omega, \\ \frac{\partial\varphi}{\partial n} = 0, \quad x \in \partial\Omega, \end{aligned}\right\} \tag{3.19}$$

and let λ_τ be the principal eigenvalue of (3.19). Note that $\lambda_0 = \lambda^*$, where λ^* is the principal eigenvalue of (3.7).

We now state a well-known result about the existence of positive solutions of an elliptic problem.

LEMMA 3.7. *Suppose that* $\tau \geqslant 0$ *and consider the problem*

$$\left.\begin{aligned} d_{\mathrm{I}}\Delta\bar{I} + \bar{I}f(\tau,\bar{I}) &= 0, \quad x\in\Omega,\\ \frac{\partial\bar{I}}{\partial n} &= 0, \quad x\in\partial\Omega. \end{aligned}\right\} \tag{3.20}$$

Then the following statements hold:

(a) *if* $\lambda_\tau \geqslant 0$*, the only non-negative solution of (3.20) is* $\bar{I}=0$*;*

(b) *if* $\lambda_\tau < 0$*, there is a unique positive solution* $\bar{I}\in Y$ *of (3.20).*

Using the implicit function theorem (cf. [6]), we then prove the following result.

LEMMA 3.8. *Suppose that* $\lambda^* < 0$ *and* $d_{\mathrm{S}} > d_{\mathrm{I}}$*. Then there exists a smooth curve* $(\tau, \bar{I}_\tau(x))$ *in* $\mathbb{R}^+\times Y$ *such that* $F(\tau,\bar{I}_\tau) = 0$*. And there is a* $T>0$ *such that* $\bar{I}_\tau(x) > 0$ *for all* $x\in\bar{\Omega}$ *and* $\tau\in[0,T)$ *and* $\bar{I}_T = 0$*. Moreover,* $\bar{I}_\tau$ *is decreasing and continuously differentiable in* τ *on* $(0,T)$*.*

Proof. Suppose that $(\tau_0,\bar{I}_{\tau_0})\in\mathbb{R}^+\times Y$ satisfies $F(\tau_0,\bar{I}_{\tau_0}) = 0$ and $\bar{I}_{\tau_0}(x) > 0$ on $\bar{\Omega}$. The Fréchet derivative of F with respect to the second variable at $(\tau_0,\bar{I}_{\tau_0})$ is $F_y(\tau_0,\bar{I}_{\tau_0})w = d_{\mathrm{I}}\Delta w + (f(\tau_0,\bar{I}_{\tau_0}) - (d_{\mathrm{I}}/d_{\mathrm{S}})\beta\bar{I}_{\tau_0})w$ for all $w\in Y$. To see that $F_y(\tau_0,\bar{I}_{\tau_0})$ is invertible, for any $h\in C^\alpha(\bar{\Omega})$, consider the following problem:

$$\left.\begin{aligned} d_{\mathrm{I}}\Delta w + \left(f(\tau_0,\bar{I}_{\tau_0}) - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\beta\bar{I}_{\tau_0}\right)w &= h, \quad x\in\Omega,\\ \frac{\partial w}{\partial n} &= 0, \quad x\in\partial\Omega. \end{aligned}\right\} \tag{3.21}$$

Let σ_{τ_0} be the principal eigenvalue of the problem

$$\left.\begin{aligned} d_{\mathrm{I}}\Delta\varphi + \left(f(\tau_0,\bar{I}_{\tau_0}) - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\beta\bar{I}_{\tau_0}\right)\varphi + \sigma\varphi &= 0, \quad x\in\Omega,\\ \frac{\partial\varphi}{\partial n} &= 0, \quad x\in\partial\Omega. \end{aligned}\right\} \tag{3.22}$$

By the Fredholm alternative, (3.21) has a unique solution for every $h\in C^\alpha(\bar{\Omega})$ if 0 is not an eigenvalue of (3.22). To show this, we note that, since $F(\tau_0,\bar{I}_{\tau_0}) = 0$, $\bar{I}_{\tau_0}$ is an eigenvector of the eigenvalue problem

$$\left.\begin{aligned} d_{\mathrm{I}}\Delta\varphi + f(\tau_0,\bar{I}_{\tau_0})\varphi + \sigma\varphi &= 0, \quad x\in\Omega,\\ \frac{\partial\varphi}{\partial n} &= 0, \quad x\in\partial\Omega, \end{aligned}\right\} \tag{3.23}$$

for the eigenvalue $\sigma = 0$. Then by the Krein–Rutman theorem, the positivity of $\bar{I}_{\tau_0}$ implies that $\sigma = 0$ is the principal eigenvalue of (3.23). Since $f(\tau_0,\bar{I}_{\tau_0}) -$

$(d_{\mathrm{I}}/d_{\mathrm{S}})\beta\bar{I}_{\tau_0} < f(\tau_0, \bar{I}_{\tau_0})$, it follows that $\sigma_{\tau_0} > 0$. So all the eigenvalues of problem (3.22) are positive, which yields the unique solvability of (3.21). The continuity of the inverse of $F_y(\tau_0, \bar{I}_{\tau_0})$ follows from the classical $C^{2,\alpha}$ estimates. Since $\lambda_0 = \lambda^* < 0$, by lemma 3.7, there exists a unique positive $\bar{I}_0 \in Y$ such that $F(0, \bar{I}_0) = 0$. Then, by the implicit function theorem, there is a unique $\bar{I}_\tau \in Y$ such that $F(\tau, \bar{I}_\tau) = 0$ for $\tau \in [0, \tau')$ with $\tau' > 0$, and this $\bar{I}_\tau$ is continuously differentiable with respect to τ.

To show that $\bar{I}_\tau$ is decreasing with respect to τ, we may consider $0 < \tau_1 < \tau_2 < \tau'$. Since $d_{\mathrm{S}} > d_{\mathrm{I}}$, we have that $F(\tau_1, \bar{I}_{\tau_2}) > 0$, and hence $\bar{I}_{\tau_2}$ is a lower solution of the equation $F(\tau_1, \bar{I}) = 0$. On the other hand, we can choose a sufficiently large number as an upper solution. Then the method of upper/lower solutions and the uniqueness of the positive solution of $F(\tau_2, \tilde{I}) = 0$ imply $\bar{I}_{\tau_1} > \bar{I}_{\tau_2}$.

The curve $(\tau, \bar{I}_\tau)$ with $\bar{I}_\tau > 0$ continues as long as $\lambda_\tau < 0$. By the variational formula, λ_τ is increasing with respect to τ and $\lambda_\tau > 0$ for large τ. Thus, by lemma 3.7, there is no positive solution of $F(\tau, \bar{I}) = 0$ if τ is large. Let $[0, T)$ be the maximal interval of existence of τ such that $\bar{I}_\tau > 0$. Then $\bar{I}_T = 0$. □

An analogous result in the case $d_{\mathrm{S}} < d_{\mathrm{I}}$ can also be proved.

LEMMA 3.9. *Suppose that $\lambda^* < 0$ and $d_{\mathrm{S}} < d_{\mathrm{I}}$. Then there exists a smooth curve $(\tau, \bar{I}_\tau(x))$ in $\mathbb{R}^+ \times Y$ such that $F(\tau, \bar{I}_\tau) = 0$ with $\bar{I}_\tau(x) > 0$ for all $x \in \bar{\Omega}$ and $\tau \in (0, \infty)$. Moreover, $\bar{I}_\tau$ is increasing and continuously differentiable in τ on $(0, \infty)$, and it satisfies the following estimate:*

$$\int_\Omega \bar{I}_\tau \,\mathrm{d}x \leqslant \frac{d_{\mathrm{S}}}{d_{\mathrm{I}}} N + \left(1 - \frac{d_{\mathrm{S}}}{d_{\mathrm{I}}}\right)\tau.$$

Proof. The existence and continuity of the curve $(\tau, \bar{I}_\tau)$ follow from a similar argument as in the proof of lemma 3.8. Since $d_{\mathrm{S}} < d_{\mathrm{I}}$, one can see that $\bar{I}_\tau$ is increasing with respect to τ, and thus the curve continues as $\tau \to \infty$. It then remains to show the estimate. For any $\tau > 0$, one can check that

$$\hat{I} = \frac{d_{\mathrm{S}} N}{d_{\mathrm{I}} |\Omega|} + \left(1 - \frac{d_{\mathrm{S}}}{d_{\mathrm{I}}}\right)\frac{\tau}{|\Omega|}$$

is an upper solution of $F(\tau, \bar{I}) = 0$. On the other hand, $\check{I} = \bar{I}_{\check{\tau}}$ with $\check{\tau} < \tau$ is a lower solution of $F(\tau, \bar{I}) = 0$. Then the method of upper/lower solutions implies that

$$\bar{I}_\tau \leqslant \frac{d_{\mathrm{S}} N}{d_{\mathrm{I}} |\Omega|} + \left(1 - \frac{d_{\mathrm{S}}}{d_{\mathrm{I}}}\right)\frac{\tau}{|\Omega|},$$

which, upon integration over Ω, yields the estimate. □

We are now in a position to prove the existence of the endemic equilibrium.

THEOREM 3.10. *If $\mathcal{R}_0 > 1$, then there exists a unique EE.*

Proof. By lemmas 3.5 and 3.6, it suffices to show that problem (3.15), (3.16) has a unique positive solution. The case $d_{\mathrm{S}} = d_{\mathrm{I}}$ follows directly from lemma 3.7. We then consider the case $d_{\mathrm{S}} > d_{\mathrm{I}}$. By lemma 3.8, there exists a smooth curve

$(\tau, \bar{I}_\tau)$ for $\tau \in [0, T)$ with $F(\tau, \bar{I}_\tau) = 0$. By the definition of F, $\bar{I}_\tau$ is a solution of problem (3.15), (3.16) if $\tau = \int_\Omega \bar{I}_\tau \, \mathrm{d}x$. Since

$$0 < \int_\Omega \bar{I}_0 \, \mathrm{d}x \quad \text{and} \quad T > \int_\Omega \bar{I}_T \, \mathrm{d}x = 0,$$

the continuity and monotonicity of $\bar{I}_\tau$ in τ implies that there exists a unique $\tau_0 \in [0, T)$ such that $\tau_0 = \int_\Omega \bar{I}_{\tau_0} \, \mathrm{d}x$. Hence, problem (3.15), (3.16) has a unique positive solution.

We now consider the case $d_\mathrm{S} < d_\mathrm{I}$. By lemma 3.9, there exists a smooth curve $(\tau, \bar{I}_\tau)$ with $F(\tau, \bar{I}_\tau) = 0$. Since $0 < \int_\Omega \bar{I}_0 \, \mathrm{d}x$, by the continuity and monotonicity in $\bar{I}_\tau$, and using the estimate of $\int_\Omega \bar{I}_\tau \, \mathrm{d}x$ in lemma 3.9, we can see that there exists a unique $\tau_0 > 0$ such that $\tau_0 = \int_\Omega \bar{I}_{\tau_0} \, \mathrm{d}x$. □

We then discuss the non-existence of the endemic equilibrium.

THEOREM 3.11. *If $d_\mathrm{S} \geqslant d_\mathrm{I}$, then the EE does not exist when $\mathcal{R}_0 \leqslant 1$; if $d_\mathrm{S} < d_\mathrm{I}$, then the EE does not exist when $\mathcal{R}_0 \leqslant d_\mathrm{S}/d_\mathrm{I}$. Furthermore, if $d_\mathrm{S} < d_\mathrm{I}$ and $\gamma(x) = r\beta(x)$ with r a positive constant, then the EE does not exist when $\mathcal{R}_0 \leqslant 1$.*

Proof. The case $d_\mathrm{S} = d_\mathrm{I}$ follows directly from lemma 3.7. We then consider the case $d_\mathrm{S} > d_\mathrm{I}$ with $\mathcal{R}_0 \leqslant 1$. Assume to the contrary that an EE, (S^*, I^*), exists. Then there is a $\tau^* > 0$ such that $\tau^* = \int_\Omega I^* \, \mathrm{d}x$ and $F(\tau^*, I^*) = 0$, and it follows from lemma 3.7 that $\lambda_{\tau^*} < 0$. Since $f(\tau, 0)$ is decreasing in τ when $d_\mathrm{S} > d_\mathrm{I}$, by the variational formula, $\lambda^* = \lambda_0 \leqslant \lambda_{\tau^*} < 0$. This implies $\mathcal{R}_0 > 1$ by proposition 3.2, which is a contradiction.

We now consider the case $d_\mathrm{S} < d_\mathrm{I}$ with $\mathcal{R}_0 \leqslant d_\mathrm{S}/d_\mathrm{I}$. Assume to the contrary that an EE (S^*, I^*) exists. Let $\tau^* = \int_\Omega I^* \, \mathrm{d}x$. Then $F(\tau^*, I^*) = 0$, and this implies $\lambda_{\tau^*} < 0$ by lemma 3.7. On the other hand, by lemma 3.6, one can see that, for all $x \in \Omega$,

$$\left(1 - \frac{d_\mathrm{I}}{d_\mathrm{S}}\right) \frac{1}{|\Omega|} \int_\Omega I^* \, \mathrm{d}x + \frac{d_\mathrm{I}}{d_\mathrm{S}} I^*(x) \leqslant \frac{N}{|\Omega|}.$$

Integrating the above inequality over Ω then gives

$$\tau^* = \int_\Omega I^* \, \mathrm{d}x \leqslant N.$$

Since $f(\tau, 0)$ is increasing in τ when $d_\mathrm{S} < d_\mathrm{I}$, $\lambda_N \leqslant \lambda_{\tau^*} < 0$. Note that λ_N is the principal eigenvalue of the following problem:

$$\begin{aligned} d_\mathrm{I} \Delta \varphi + \left(\frac{d_\mathrm{I} N}{d_\mathrm{S} |\Omega|} \beta - \gamma\right) \varphi + \lambda \varphi = 0, \quad & x \in \Omega, \\ \frac{\partial \varphi}{\partial n} = 0, \quad & x \in \partial\Omega. \end{aligned}$$

Similarly to $\mathcal{R}_0$, one can define $\mathcal{R}_0'$ as follows:

$$\mathcal{R}_0' = \sup \left\{ \left(\frac{d_\mathrm{I} N}{d_\mathrm{S} |\Omega|} \int_\Omega \beta \varphi^2 \, \mathrm{d}x \right) \left(\int_\Omega (d_\mathrm{I} |\nabla \varphi|^2 + \gamma \varphi^2) \, \mathrm{d}x \right)^{-1} : \varphi \in H^1(\Omega) \text{ and } \varphi \neq 0 \right\}.$$

Then $\lambda_N < 0$ if and only if $\mathcal{R}_0' > 1$. Since $\mathcal{R}_0' = d_{\mathrm{I}}/d_{\mathrm{S}}\mathcal{R}_0$, $\lambda_N < 0$ implies $\mathcal{R}_0 > d_{\mathrm{S}}/d_{\mathrm{I}}$, which is a contradiction.

We then consider the case $d_{\mathrm{S}} < d_{\mathrm{I}}$ with $\mathcal{R}_0 \leqslant 1$ for $\gamma(x) = r\beta(x)$. Assume to the contrary that an EE (S^*, I^*) exists. Proceeding as in the proof of lemma 3.6, one can see that, for all $x \in \Omega$,

$$\left(1 - \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}}\right)\frac{1}{|\Omega|}\int_\Omega I^* \,\mathrm{d}x + \frac{d_{\mathrm{I}}}{d_{\mathrm{S}}} I^*(x) \leqslant \frac{N}{|\Omega|} - r.$$

Integrating the above inequality over Ω then gives

$$\int_\Omega I^* \,\mathrm{d}x \leqslant N - r|\Omega|,$$

and it follows that I^* is a lower solution of the problem:

$$\left.\begin{aligned} d_{\mathrm{S}}\Delta\tilde{I} + \tilde{I}\left(\frac{N}{|\Omega|}\beta - \gamma - \beta\tilde{I}\right) &= 0, \quad x \in \Omega, \\ \frac{\partial\tilde{I}}{\partial n} &= 0, \quad x \in \partial\Omega. \end{aligned}\right\} \tag{3.24}$$

On the other hand, it is easy to see that $M = N/|\Omega|$ is an upper solution of problem (3.24). Then, by the upper/lower solution argument, there exists a unique positive solution of (3.24). However, since $\mathcal{R}_0 \leqslant 1$, $\lambda^* \geqslant 0$ by proposition 3.2. It then follows from the variational formula of λ^* and $\gamma(x) = r\beta(x)$ that $\gamma \geqslant N\beta/|\Omega|$. Thus, problem (3.24) has no positive solution. This leads to a contradiction. □

4. Global attractivity

In this section, we consider the global attractivity of the disease-free equilibrium and the endemic equilibrium. As for most SI models, one may expect that the DFE is globally attractive when $\mathcal{R}_0 \leqslant 1$, while the EE is globally attractive when $\mathcal{R}_0 > 1$. However, it is generally difficult to establish such results for reaction–diffusion models with variable coefficients. For the model (1.1), global attractivity analysis is conducted only for the case $d_{\mathrm{S}} = d_{\mathrm{I}}$ and the case $\gamma(x) = r\beta(x)$ (the second case is equivalent to the case that β and γ are constants) (see [18, 20]). Here, for our model (2.1)–(2.3), we are also able to establish such results for these two cases.

4.1. The case of constant coefficients

We first consider the case that the coefficients β and γ are positive constants. In this case, we can see that the DFE equals $(\tilde{S}, 0) = (N/|\Omega|, 0)$ and the EE equals $(S^*, I^*) = (\gamma/\beta, N/|\Omega| - \gamma/\beta)$ if it exists. To conduct our discussion, we shall mainly rely on the LaSalle invariance principle for nonlinear dynamical systems (see [13]). Let $X = L^p(\Omega)$ with $p > m$. We define a closed linear operator A with dense domain $D(A)$ given by

$$Au = -\Delta u, \qquad D(A) = \left\{u \in W^{2,p}(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}.$$

Then $-A$ generates an analytic semigroup e^{-tA} on X. Let X_α $(0 \leqslant \alpha \leqslant 1)$ be the fractional power space of X with respect to A. Since the embedding $X_\alpha \subset C^{1,\mu}(\bar{\Omega})$ is compact if $1+\mu < 2\alpha - m/p$, we choose α close to 1 and p large such that X_α compactly embedded into $C^{1,\mu}(\bar{\Omega})$. Let $P \subset X_\alpha$ be the cone of all non-negative functions of X_α with non-empty interior. We introduce

$$D = \left\{(u,v) \in X_\alpha \times X_\alpha \colon \int_\Omega (u+v)\,\mathrm{d}x = N \text{ and } u, v \in P\right\}.$$

Then D is a closed subset of $X_\alpha \times X_\alpha$, and the solution (S, I) of problem (2.1)–(2.3) induces a nonlinear dynamical system $\{\Phi(t),\ t \in \mathbb{R}^+\}$ on D given by

$$\Phi(t)(S_0, I_0) := (S(t), I(t)), \quad t \in \mathbb{R}^+,$$

where (S, I) is the solution of (2.1)–(2.3) with the initial condition $(S_0, I_0) \in D$.

Theorem 4.1. *If β and γ are positive constants, then the following statements hold:*

(a) *if $\mathcal{R}_0 \leqslant 1$, then the DFE is globally attractive;*

(b) *if $\mathcal{R}_0 > 1$, then the EE is globally attractive.*

Proof. If β and γ are constants, then $\mathcal{R}_0 = N\beta/(|\Omega|\gamma)$. Suppose that $\mathcal{R}_0 \leqslant 1$, i.e. $\gamma/\beta - N/|\Omega| \geqslant 0$. Define a continuously differentiable real-valued function $V \colon D \to \mathbb{R}$ by

$$V(S, I) = \frac{1}{2}\int_\Omega (S - \tilde{S})^2\,\mathrm{d}x + B\int_\Omega I\,\mathrm{d}x$$

for all $(S, I) \in D$ with B a non-negative constant to be determined. We can check that, for all $(S, I) \in D \cap (D(A) \times D(A))$,

$$\begin{aligned}
\dot{V}(S, I) &= \limsup_{t \to 0^+} \frac{V(\Phi(t)(S, I)) - V(S, I)}{t} \\
&= \int_\Omega ((S - \tilde{S})(d_{\mathrm{S}}\Delta S + I(-\beta S + \gamma)))\,\mathrm{d}x + B\int_\Omega (d_{\mathrm{I}}\Delta I + I(\beta S - \gamma))\,\mathrm{d}x \\
&= -d_{\mathrm{S}}\int_\Omega |\nabla S|^2\,\mathrm{d}x - \int_\Omega I(\beta S - \gamma)(S - \tilde{S} - B)\,\mathrm{d}x \\
&= -d_{\mathrm{S}}\int_\Omega |\nabla S|^2\,\mathrm{d}x - \int_\Omega I(\beta S - \gamma)\left(S - \frac{\gamma}{\beta}\right)\mathrm{d}x \\
&\leqslant 0,
\end{aligned}$$

where $B = \gamma/\beta - \tilde{S} = \gamma/\beta - N/|\Omega|$. Since V is continuously differentiable and $D \cap (D(A) \times D(A))$ is dense in D, we find that

$$\dot{V}(S, I) = -d_{\mathrm{S}}\int_\Omega |\nabla S|^2\,\mathrm{d}x - \beta\int_\Omega I\left(S - \frac{\gamma}{\beta}\right)^2\mathrm{d}x \leqslant 0 \quad \text{for all } (S, I) \in D.$$

Thus, V is a Lyapunov functional on D.

Let $E := \{(S, I) \in D \colon \dot{V}(S, I) = 0\}$ and M be the largest positively invariant subset of E. It follows from theorem 2.1 and standard arguments (see [12, 13]) that the orbit $\{(S(t), I(t)), t > 0\}$ is pre-compact in D. So by the LaSalle invariance principle, we have that

$$\lim_{t\to\infty} \operatorname{dist}(\Phi(t)(S_0, I_0), M) = 0.$$

In view of $\dot{V}$, $\int_\Omega |\nabla S|^2 \, \mathrm{d}x = 0$ implies that S is a constant, and $\int_\Omega I(S - \gamma/\beta)^2 \, \mathrm{d}x = 0$ implies that either $I = 0$ or $S = \gamma/\beta$. If $S = \gamma/\beta$, then

$$\int_\Omega I \, \mathrm{d}x = N - \frac{|\Omega|\gamma}{\beta} \leqslant 0,$$

which yields $I = 0$. Thus, we must have $I = 0$, and so $E = \{(\tilde{S}, 0)\}$. Hence, $M = \{(\tilde{S}, 0)\}$, and it follows that the DFE is globally attractive.

Now suppose that $\mathcal{R}_0 > 1$. Then we have that $\beta N/|\Omega| - \gamma > 0$ and the EE equal to

$$(S^*, I^*) = \left(\frac{\gamma}{\beta}, \frac{N}{|\Omega|} - \frac{\gamma}{\beta}\right)$$

exists. Define another continuously differentiable real-valued function $W\colon D \to \mathbb{R}$ by

$$W(S, I) = \frac{1}{2}\int_\Omega ((S - S^*) + (I - I^*))^2 \, \mathrm{d}x + \tfrac{1}{2}B\int_\Omega (S - S^*)^2 \, \mathrm{d}x$$

for all $(S, I) \in D$ with B a positive constant to be determined. We can check that, for all $(S, I) \in D \cap (D(A) \times D(A))$,

$$\begin{aligned}
\dot{W}(S, I) &= \limsup_{t\to 0^+} \frac{W(\Phi(t)(S, I)) - W(S, I)}{t} \\
&= \int_\Omega ((S - S^*) + (I - I^*))(d_{\mathrm{S}}\Delta S + d_{\mathrm{I}}\Delta I) \, \mathrm{d}x \\
&\quad + B\int_\Omega (S - S^*)(d_{\mathrm{S}}\Delta S + I(-\beta S + \gamma)) \, \mathrm{d}x \\
&= -\int_\Omega (d_{\mathrm{S}}|\nabla S|^2 + (d_{\mathrm{S}} + d_{\mathrm{I}})\nabla S \cdot \nabla I + d_{\mathrm{I}}|\nabla I|^2) \, \mathrm{d}x \\
&\quad - Bd_{\mathrm{S}}\int_\Omega |\nabla S|^2 \, \mathrm{d}x - B\beta\int_\Omega I(S - S^*)^2 \, \mathrm{d}x \\
&= -\int_\Omega (|B_1\nabla S + \sqrt{d_{\mathrm{I}}}\nabla I|^2) \, \mathrm{d}x - d_{\mathrm{S}}\int_\Omega |\nabla S|^2 \, \mathrm{d}x - B\beta\int_\Omega I(S - S^*)^2 \, \mathrm{d}x \\
&\leqslant 0,
\end{aligned}$$

where $B_1 = (d_{\mathrm{S}} + d_{\mathrm{I}})/(2\sqrt{d_{\mathrm{I}}})$ and $B = B_1^2/d_{\mathrm{S}}$. Since W is continuously differentiable and $D \cap (D(A) \times D(A))$ is dense in D, we find that $\dot{W}(S, I) \leqslant 0$ for all $(S, I) \in D$. Thus, W is a Lyapunov functional on D.

Let $E' := \{(S, I) \in D\colon \dot{W}(S, I) = 0\}$ and M' be the largest positively invariant subset of E'. Then, by the LaSalle invariance principle, we have that

$$\lim_{t\to\infty} \operatorname{dist}(\Phi(t)(S_0, I_0), M') = 0.$$

In view of $\dot{W}$, one can see that

$$E' = \{(S^*, I^*)\} \cup \{(\tilde{S}, 0)\}.$$

It follows that either $(S(t), I(t)) \to (S^*, I^*)$ or $(S(t), I(t)) \to (\tilde{S}, 0)$ as $t \to \infty$. Assume that $(S(t), I(t)) \to (\tilde{S}, 0)$. Since $\beta\tilde{S} - \gamma > 0$, we can choose $\varepsilon > 0$ so small that $\beta\tilde{S} - \varepsilon\beta - \gamma > 0$. For this ε, there exists a $T > 0$ such that $S(x, t) > \tilde{S} - \varepsilon$ for all $(x, t) \in \bar{\Omega} \times [T, \infty)$. By $(2.1)_2$, the following inequality holds:

$$I_t - d_{\mathrm{I}}\Delta I \geqslant \beta(\tilde{S} - \varepsilon)I - \gamma I \quad \text{for } (x, t) \in \Omega \times [T, \infty). \tag{4.1}$$

We now consider a related problem:

$$\left.\begin{aligned} J_t = d_{\mathrm{I}}\Delta J + \beta(\tilde{S} - \varepsilon)J - \gamma J, &\quad x \in \Omega,\ t \in (T, \infty),\\ \frac{\partial J}{\partial n} = 0, &\quad x \in \partial\Omega,\ t \in (T, \infty),\\ J(x, T) = \min_{x\in\bar{\Omega}} I(x, T) \equiv I_m(T), &\quad x \in \bar{\Omega}. \end{aligned}\right\} \tag{4.2}$$

Then the comparison principle yields that $I \geqslant J$ on $\bar{\Omega} \times [T, \infty)$, and it is easy to check that $J = I_m(T)\exp((\beta\tilde{S} - \varepsilon\beta - \gamma)t)$. Since $J \to \infty$ as $t \to \infty$, so does I, which contradicts the fact that I is bounded. Hence, we must have that

$$(S(t), I(t)) \to (S^*, I^*) \quad \text{as } t \to \infty.$$

□

4.2. The case $d_{\mathrm{S}} = d_{\mathrm{I}}$

We then consider the case $d_{\mathrm{S}} = d_{\mathrm{I}} \equiv d$. In this case, the EE exists if and only if $\mathcal{R}_0 > 1$. Adding up the two equations in (2.1), we have that $(S + I)_t = d\Delta(S + I)$, and it follows from condition (2.3) that $S(x, t) + I(x, t) \to N/|\Omega|$ uniformly for $x \in \bar{\Omega}$. Similarly to the case of constant coefficients, a result about the global attractivity of the equilibria can be established.

THEOREM 4.2. *If $d_{\mathrm{S}} = d_{\mathrm{I}} = d$, then the following statements hold:*

(a) *if $\mathcal{R}_0 \leqslant 1$, then the DFE is globally attractive;*

(b) *if $\mathcal{R}_0 > 1$, then the EE is globally attractive.*

Proof. Suppose that $\mathcal{R}_0 < 1$. Let $\varepsilon > 0$ be given. Since $S(x, t) + I(x, t) \to N/|\Omega|$ as $t \to \infty$, there exists a $T > 0$ such that $S(x, t) \leqslant N/|\Omega| + \varepsilon - I(t)$ for all $t > T$. Then by (2.1), (2.2), I satisfies the following:

$$\left.\begin{aligned} I_t - d\Delta I \leqslant I\biggl(\biggl(\frac{N}{|\Omega|} + \varepsilon\biggr)\beta - \gamma - \beta I\biggr), &\quad x \in \Omega,\ t \in (T, \infty),\\ \frac{\partial I}{\partial n} = 0, &\quad x \in \partial\Omega,\ t \in (T, \infty). \end{aligned}\right\} \tag{4.3}$$

Let $\tilde{I}$ be the solution of a related problem:

$$\left.\begin{aligned} \tilde{I}_t - d\Delta\tilde{I} = \tilde{I}\biggl(\biggl(\frac{N}{|\Omega|} + \varepsilon\biggr)\beta - \gamma - \beta\tilde{I}\biggr), &\quad x \in \Omega,\ t \in (T, \infty),\\ \frac{\partial\tilde{I}}{\partial n} = 0, &\quad x \in \partial\Omega,\ t \in (T, \infty),\\ \tilde{I}(x, T) = I(x, T). & \end{aligned}\right\} \tag{4.4}$$

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Then the comparison principle yields that $I(x,t) \leqslant \tilde{I}(x,t)$ on $\bar{\Omega} \times (T,\infty)$. Let $\lambda(\varepsilon)$ be the principal eigenvalue of $d\Delta\varphi + \varphi((N/|\Omega| + \varepsilon)\beta - \gamma) + \lambda\varphi = 0$, subject to the homogeneous Neumann boundary condition. Proceeding as in [6,7], one can see that problem (4.4) has a unique positive and globally attractive equilibrium if $\lambda(\varepsilon) < 0$, while it has no positive equilibrium and all solutions decay to 0 if $\lambda(\varepsilon) \geqslant 0$. Since $\mathcal{R}_0 < 1$, we have that $\lambda(0) = \lambda^* > 0$, which implies that $\lambda(\varepsilon) > 0$ if ε is small. Hence, $I(x,t) \to 0$ uniformly for $x \in \bar{\Omega}$. It then follows from $S(x,t) + I(x,t) \to N/|\Omega|$ that $S(x,t) \to N/|\Omega|$ uniformly for $x \in \bar{\Omega}$.

If $\mathcal{R}_0 = 1$, i.e. $\lambda^* = 0$, then $\lambda(\varepsilon) < 0$ and $\tilde{I}(x,t)$ converges to $\tilde{I}^*(x)$, where $\tilde{I}^*$ is the corresponding positive equilibrium. On the other hand, $\tilde{I}^* \to 0$ as $\varepsilon \to 0$ since $\lambda^* = 0$. Hence, $I(x,t) \to 0$ and $S(x,t) \to N/|\Omega|$ uniformly for $x \in \bar{\Omega}$.

Now suppose that $\mathcal{R}_0 > 1$, i.e. $\lambda^* < 0$. Since $S(x,t) + I(x,t) \to N/|\Omega|$, there exists a $T > 0$ such that $N/|\Omega| - \varepsilon - I(x,t) \leqslant S(x,t) \leqslant N/|\Omega| + \varepsilon - I(x,t)$ for all $t > T$. Then by (2.1), I satisfies the following inequality:

$$I\biggl(\biggl(\frac{N}{|\Omega|} - \varepsilon\biggr)\beta - \gamma - \beta I\biggr) \leqslant I_t - d\Delta I \leqslant I\biggl(\biggl(\frac{N}{|\Omega|} + \varepsilon\biggr)\beta - \gamma - \beta I\biggr) \tag{4.5}$$

for $(x,t) \in \Omega \times (T,\infty)$. Let $\check{I}$ and $\hat{I}$ solve the following two related problems, respectively:

$$\left.\begin{aligned} \check{I}_t - d\Delta\check{I} &= \check{I}\biggl(\biggl(\frac{N}{|\Omega|} - \varepsilon\biggr)\beta - \gamma - \beta\check{I}\biggr), && x \in \Omega,\ t \in (T,\infty), \\ \frac{\partial\check{I}}{\partial n} &= 0, && x \in \partial\Omega,\ t \in (T,\infty), \\ \check{I}(x,T) &= I(x,T) && \end{aligned}\right\} \tag{4.6}$$

and

$$\left.\begin{aligned} \hat{I}_t - d\Delta\hat{I} &= \hat{I}\biggl(\biggl(\frac{N}{|\Omega|} + \varepsilon\biggr)\beta - \gamma - \beta\hat{I}\biggr), && x \in \Omega,\ t \in (T,\infty), \\ \frac{\partial\hat{I}}{\partial n} &= 0, && x \in \partial\Omega,\ t \in (T,\infty), \\ \hat{I}(x,T) &= I(x,T). && \end{aligned}\right\} \tag{4.7}$$

By the comparison principle, we find that $\check{I}(x,t) \leqslant I(x,t) \leqslant \hat{I}(x,t)$ on $\bar{\Omega} \times (T,\infty)$. If ε is small, we have that $\lambda(\pm\varepsilon) < 0$, and it then follows that $\check{I}(x,t) \to \check{I}^*(x)$ and $\hat{I}(x,t) \to \hat{I}^*(x)$, where $\check{I}^*$ and $\hat{I}^*$ are the corresponding positive equilibria, respectively. Letting $\varepsilon \to 0$ yields that $I(x,t) \to \bar{I}(x)$ uniformly for $x \in \bar{\Omega}$, where $\bar{I}$ is the positive solution of the problem

$$\left.\begin{aligned} d\Delta\bar{I} + \bar{I}\biggl(\frac{N}{|\Omega|}\beta - \gamma - \beta\bar{I}\biggr) &= 0, \quad x \in \Omega, \\ \frac{\partial\bar{I}}{\partial n} &= 0, \quad x \in \partial\Omega. \end{aligned}\right\} \tag{4.8}$$

Again by $S(x,t) + I(x,t) \to N/|\Omega|$, we have that $\lim_{t\to\infty} S(x,t) = N/|\Omega| - \bar{I}(x)$ uniformly for $x \in \bar{\Omega}$. Let $\bar{S} = N/|\bar{\Omega}| - \bar{I}$. Since $S(x,t)$ is non-negative, so is $\bar{S}$.

And it is easy to see that $(\bar{S}, \bar{I})$ satisfies (3.1), (3.2). The uniqueness of the EE then implies that $(S^*, I^*) = (\bar{S}, \bar{I})$. □

5. Discussion

In this paper, we proposed an SIS reaction–diffusion population model and established the global existence and boundedness results. We then considered the disease-free equilibrium and the endemic equilibrium of the model. For this purpose, we defined a basic reproduction number $\mathcal{R}_0$ and proved that a unique endemic equilibrium exists if $\mathcal{R}_0 > 1$. We then conducted analysis on the global attractivity of the disease-free equilibrium and the endemic equilibrium of the model for two cases.

We now briefly discuss some implications of these results on the model dynamics and disease control. First, it is interesting to compare our model with the model (1.2) in [11, 15, 22], which is a generalization of the classical Kermack–McKendrick model. In the model (1.2), after the infected individuals recovered (or died), they no longer became infected. It is then predicted that the disease becomes extinct in the long run. However, in our model, after the infected individuals recovered, they become susceptible immediately, and consequently the disease may not become extinct. To be more specific, similar to the definition in [3], we call the domain Ω a high-risk domain if

$$\int_\Omega \frac{N\beta}{|\Omega|}\,\mathrm{d}x > \int_\Omega \gamma\,\mathrm{d}x$$

and a low-risk domain if

$$\int_\Omega \frac{N\beta}{|\Omega|}\,\mathrm{d}x < \int_\Omega \gamma\,\mathrm{d}x.$$

By remark 3.3, the basic reproduction number $\mathcal{R}_0$ is greater than 1 if Ω is a high-risk domain, and hence an epidemic equilibrium always exists by theorem 3.10. Then the disease should persist in the long run. This has been shown for two cases:

- if the disease transmission and recovery rates are constants;
- if the diffusion rate of the susceptible individuals is equal to the diffusion rate of the infected individuals.

In a low-risk domain, there exists a threshold value d_{I}^* for the diffusion rate of the infected individuals. If $d_{\mathrm{I}} > d_{\mathrm{I}}^*$, the basic reproduction number $\mathcal{R}_0$ is less than 1, and it is expected that the disease will die out, while if $d_{\mathrm{I}} < d_{\mathrm{I}}^*$, $\mathcal{R}_0$ is greater than 1, and then the disease would persist. Moreover, as for disease control, the variational formula suggests that decreasing the disease transmission rate β or increasing the recovery rate γ would lessen the likelihood of the persistence of the disease, which is consistent with the expectation.

In [3] for model (1.1), it has been demonstrated that when the endemic equilibrium exists, its I component approaches zero as the mobility of susceptible individuals approaches zero. Such a result has important implications for disease control. For model (2.1), however, due to the non-locality of (3.8), the discussion seems to become more complicated, and establishing a similar result will be left as future work.

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