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Geostrophic adjustment with gyroscopic waves: barotropic fluid without the traditional approximation

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We study geostrophic adjustment in rotating barotropic fluid when the angular speed of rotation $\boldsymbol{\Omega}$ does not coincide in direction with the acceleration due to gravity; the traditional and hydrostatic approximations are not used. Linear adjustment results in a tendency of any localized initial state towards a geostrophically balanced steady columnar motion with columns parallel to $\boldsymbol{\Omega}$. Nonlinear adjustment is examined for small Rossby numbers Ro and aspect ratio H/L (H and L are the layer depth and the horizontal scale of motion), using multiple-time-scale perturbation theory. It is shown that an arbitrary perturbation is split in a unique way into slow and fast components evolving with characteristic time scales $(Rof)^{-1}$ and f^{-1}, respectively, where f is the Coriolis parameter. The slow component does not depend on depth and is close to geostrophic balance. On times $O(1/f\,Ro)$ the slow component is not influenced by the fast one and is described by the two-dimensional fluid dynamics equation for the geostrophic streamfunction. The fast component consists of long gyroscopic waves and is a packet of inertial oscillations modulated by an amplitude depending on coordinates and slow time. On times $O(1/f\,Ro)$ the fast component conserves its energy, but it is coupled to the slow component: its amplitude obeys an equation with coefficients depending on the geostrophic streamfunction. Under the traditional approximation, the inertial oscillations are trapped by the quasi-geostrophic component; 'non-traditional' terms in the amplitude equation provide a meridional dispersion of the packet on times $O(1/f\,Ro)$, and, therefore, an effective radiation of energy from the initial perturbation domain. Another important effect of the non-traditional terms is that on longer times $O(1/f\,Ro^2)$ a transfer of energy between the fast and the slow components becomes possible.

Key words: quasi-geostrophic flows, rotating flows, waves in rotating fluids

1. Introduction

Gyroscopic waves (GW) exist owing to rotation (e.g. LeBlond & Mysak 1978); no stratification or gravity are necessary, although both of these factors strongly affect the structure and properties of these waves. In a 'pure' form the GWs occur in a

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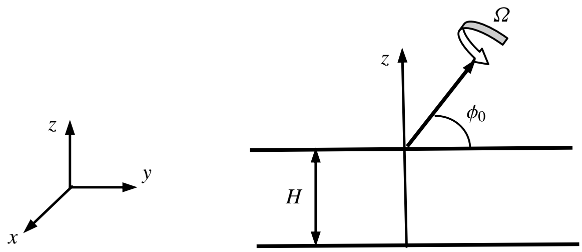

FIGURE 1. Schematic representation of the barotropic fluid layer of constant depth H rotating at the angular speed $\boldsymbol{\Omega}$.

barotropic fluid layer of constant depth, bounded by two rigid lids and rotating as a whole at a constant angular speed whose direction can be different from gravity (see figure 1).

Under the traditional approximation (TA) when the horizontal component of the Coriolis force is neglected, the GWs in the barotropic layer are sub-inertial, i.e. their frequencies σ do not exceed the vertical component of twice the angular speed of rotation $f = 2\Omega \sin\phi_0$ (see figure 1), i.e. $\sigma \leqslant f$; without the TA both sub-inertial and super-inertial GWs with $\sigma \geqslant f$ are possible (e.g. Brekhovskikh & Goncharov 1994; Kasahara 2003). In stably stratified fluid under the TA the sub-inertial GWs exist together with super-inertial internal waves only if the minimal buoyancy frequency $N_{min} < f$ (e.g. Kamenkovich 1977). In strongly stratified fluid, i.e. for $N_{min} > f$, only super-inertial internal waves are possible. However, without the TA sub-inertial waves (so-called internal inertio-gravity waves) occur even in strongly stratified fluid (Kasahara 2003; Gerkema & Shrira 2005; Gerkema *et al.* 2008). Like the GWs these waves cannot exist without rotation.

Up to now, studies of geostrophic adjustment have taken into account only surface and/or internal gravity waves (Reznik, Zeitlin & Ben Jelloul 2001; Zeitlin, Reznik & Ben Jelloul 2003, see e.g.). Our aim is to include in the analysis gyroscopic waves. In the present paper we examine the geostrophic adjustment of a barotropic fluid layer where GWs are the only possible wave motion. An example of stratified fluid where GWs co-exist with internal waves was considered by Reznik (2013*a*,*b*).

The paper is organized as follows. In § 2 the governing equations with boundary and initial conditions are presented. In § 3 linear gyroscopic waves are discussed. Linear adjustment is examined in § 4. In §§ 5–9 nonlinear adjustment of the long-wave ($L \gg H$) initial state is analysed. Non-dimensional equations and the asymptotic procedure for finding the solution are given in § 5. The lowest-order solution as a sum of a depth-independent quasi-geostrophic (QG) component and modulated inertial oscillations is represented in § 6. In § 7 the first-order solution is analysed and slow evolution of the QG flow and inertial oscillations is described. The behaviour of the QG flow on times longer than a typical geostrophic time is discussed in § 8. The role of the β-effect is considered in § 9. Section 10 contains a discussion and conclusions. The majority of the results below are obtained by direct analytic calculations. As sometimes they are rather cumbersome, in order to simplify the presentation many technical details are relegated to appendices A and B.

2. Governing equations

The equations of motion for the barotropic fluid layer represented in figure 1 can be written in the form:

$$u_t + uu_x + vu_y + wu_z - fv + f_s w = -p_x/\rho_0, \tag{2.1a}$$
$$v_t + uv_x + vv_y + wv_z + fu = -p_y/\rho_0, \tag{2.1b}$$
$$w_t + uw_x + vw_y + ww_z - f_s u = -p_z/\rho_0, \tag{2.1c}$$
$$u_x + v_y + w_z = 0. \tag{2.1d}$$

Here u, v, w are the velocity components associated with the x, y, z axes, respectively, the z-axis being directed upward parallel to gravity (origin at the upper surface); ρ_0 is the fluid density, p the deviation of pressure from the hydrostatic one, f and f_s are the vertical and horizontal components of twice the angular speed $\boldsymbol{\Omega}$, respectively:

$$f = 2\Omega \sin\phi_0, \quad f_s = 2\Omega \cos\phi_0. \tag{2.2a,b}$$

The velocity field obeys the no-flux conditions at the surface and bottom:

$$w|_{z=0,-H} = 0, \tag{2.3}$$

and the initial conditions

$$(u, v, w)_{t=0} = (u_I, v_I, w_I)(x, y, z); \quad w_I = -\int_{-H}^{z} (\partial_x u_I + \partial_y v_I)\, \mathrm{d}z. \tag{2.4a,b}$$

In a geophysical context (2.1) represent the so-called non-traditional f-plane approximation; in this case ϕ_0 is the reference latitude around which the west–east, south–north and vertical Cartesian coordinates x, y, z are introduced (see figure 1). For a 'non-traditional' β-plane, Grimshaw (1975) showed that, for dynamical consistency (angular momentum and vorticity conservation) the term f_s should be constant and the next-order term linear in y should be included only in the vertical component f, i.e.

$$f_s = 2\Omega \cos\phi_0, \quad f = 2\Omega \sin\phi_0 + \beta y, \quad \beta = 2\Omega \cos\phi_0/a, \tag{2.5a–c}$$

where a is the Earth's radius (see also Gerkema *et al.* 2008, and references therein).

The following analysis uses primarily the f-plane model with constant f, f_s given by (2.2). The influence of the β-effect is discussed at the end of the paper in § 9. The ratio $q = f_s/f = \cot\phi_0$ is assumed to be of the order of unity,

$$q = O(1), \tag{2.6}$$

which corresponds to the mid-latitudes.

3. Linear gyroscopic waves

Greenspan (1968) discusses the general characteristics of the linear waves and geostrophic flow in a homogeneous rotating fluid where the container walls are not perpendicular to the rotation vector. In §§ 3 and 4 we will examine these in our more geophysical context as a basis for the study of the nonlinear evolution of the geostrophic flow and its interactions with gyroscopic waves.

For $\beta = 0$ and in the absence of a free surface and stratification the only wave-making mechanism is the rotation, i.e. only gyroscopic waves are possible here. Linearized equations (2.1) can be reduced to one equation for the vertical velocity (e.g. Miropol'sky 2001):

$$(\partial_{tt} + f^2)w_{zz} + \nabla_h^2 w_{tt} + 2 f f_s w_{yz} + f_s^2 w_{yy} = 0, \quad w|_{z=0,-H} = 0, \tag{3.1a,b}$$

where $\nabla_h^2 = \partial_x^2 + \partial_y^2$. The wave solution

$$w = W(z)\exp[\mathrm{i}(kx + ly - \sigma t)] \tag{3.2}$$

is determined by the equations ($\kappa = \sqrt{k^2 + l^2}$):

$$(f^2 - \sigma^2)W_{zz} + 2\mathrm{i} f f_s l W_z + (\kappa^2\sigma^2 - f_s^2 l^2)W = 0, \quad W|_{z=0,-H} = 0. \tag{3.3a,b}$$

Non-trivial solutions to the eigenvalue problem (3.3) exist only for

$$\sigma \neq f \tag{3.4}$$

and have the form

$$W = \mathrm{e}^{\lambda^+ z} - \mathrm{e}^{\lambda^- z}, \quad \lambda^\pm = a \pm \mathrm{i}b, \tag{3.5a,b}$$

$$a = -\frac{\mathrm{i} f f_s l}{f^2 - \sigma^2}, \quad b = \frac{\sigma\kappa}{|f^2 - \sigma^2|}\sqrt{f^2 - \sigma^2 + \bar{f}_s^2}, \quad \bar{f}_s = f_s\frac{|l|}{\kappa}. \tag{3.6a–c}$$

The solution (3.5) satisfies the boundary conditions (3.3b) if b is real, i.e. the frequency σ cannot exceed the limit value $\sigma_0 = \sqrt{f^2 + \bar{f}_s^2}$,

$$\sigma \leqslant \sigma_0, \tag{3.7}$$

and if $\sin bH = 0$, i.e.

$$b = b_n = n\pi/H, \quad n = 1, 2, \ldots. \tag{3.8}$$

Inequality (3.7) means that the frequencies of the gyroscopic waves cannot exceed twice the angular speed $2\boldsymbol{\Omega}$; Greenspan (1968) showed that this limitation is valid for oscillations of fluid in a reservoir of arbitrary shape.

It follows from (3.6b) and (3.8) that

$$(1 + \bar{b}_n^2)(f^2 - \sigma^2)^2 - (f^2 - \bar{f}_s^2)(f^2 - \sigma^2) - f^2\bar{f}_s^2 = 0, \quad \bar{b}_n = \frac{n\pi}{\kappa H}, \tag{3.9a,b}$$

whence we find (e.g. Brekhovskikh & Goncharov 1994; Kasahara 2003) the dispersion relation $\sigma = \sigma(k, l, n)$ consisting of the sub-inertial branches σ_n^{sub}:

$$\sigma_n^{sub} = \left\{ f^2 - \frac{1}{2(1 + \bar{b}_n^2)}\left[(f^2 - \bar{f}_s^2) + \sqrt{\sigma_0^4 + 4\bar{b}_n^2 f^2 \bar{f}_s^2}\right]\right\}^{1/2}, \tag{3.10a}$$

and the super-inertial branches $\sigma_n^{\sup}$:

$$\sigma_n^{\sup} = \left\{ f^2 + \frac{1}{2(1 + \bar{b}_n^2)}\left[-(f^2 - \bar{f}_s^2) + \sqrt{\sigma_0^4 + 4\bar{b}_n^2 f^2 \bar{f}_s^2}\right]\right\}^{1/2}. \tag{3.10b}$$

The branches are presented in figure 2.

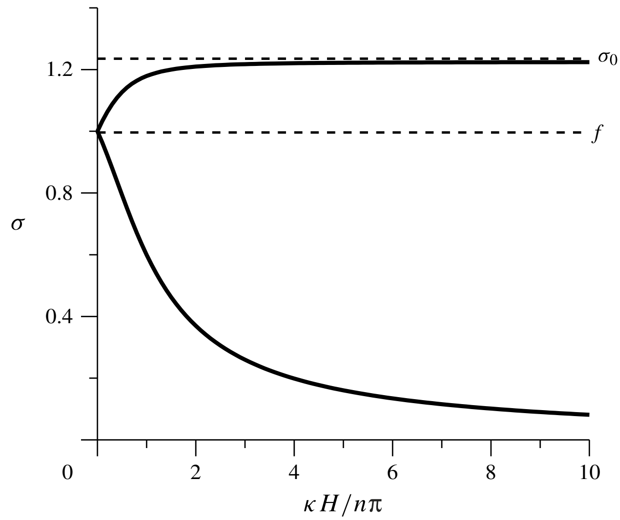

FIGURE 2. Dispersion relation for the barotropic gyroscopic waves.

We now consider in more detail the case when the horizontal scale of motion L greatly exceeds its vertical scale H. In the long-wave approximation $\kappa H \ll 1$ both the sub- and super-inertial frequencies (3.10) are close to the inertial frequency f; in this case

$$\sigma_n^{sub} = f - \frac{f_s H}{2n\pi}|l| + f\,O(\bar{b}_n^{-2}), \quad \sigma_n^{\sup} = f + \frac{f_s H}{2n\pi}|l| + f\,O(\bar{b}_n^{-2}). \tag{3.11a,b}$$

The long-wave asymptotics (3.11) are universal in the sense that they remain valid in stratified fluid too (Gerkema & Shrira 2005). We emphasize that the gyroscopic waves are close to inertial oscillations if $L \gg H$; this is not the case for the surface and internal gravity waves which are nearly inertial if $L \gg L_R$ where L_R is the Rossby scale. Reznik (2013*a*) showed that this property of GWs is valid also in stratified fluid. Usually $L_R \gg H$, therefore the presence of GWs results in the existence of inertial oscillations with shorter horizontal scales $L \leqslant L_R$.

Using the scales L and H, and f^{-1} as the time scale, we write (3.1) in the non-dimensional form:

$$(\partial_{tt} + 1)w_{zz} + 2\delta q w_{yz} + \delta^2(\nabla_h^2 w_{tt} + q^2 w_{yy}) = 0, \quad w|_{z=0,-1} = 0, \tag{3.12a,b}$$

where $\delta = H/L \ll 1$ and $q = f_s/f = \cot\phi_0$. The smallness of δ allows a solution to (3.12) to be sought in the following asymptotic form:

$$w = w_0(x, y, z, t, T_1, \ldots) + \delta w_1(x, y, z, t, T_1, \ldots) + \cdots, \tag{3.13}$$

where $T_n = \delta^n t,\ \ n = 1, 2, \ldots$ are the slow times.

Substitution of (3.13) into (3.12) gives in the lowest order:

$$(\partial_{tt} + 1)w_{0zz} = 0, \quad w|_{z=0,-1} = 0, \tag{3.14a,b}$$

whence we have:

$$w_0 = W_0(x, y, z, T_1, \ldots)e^{-it} + \text{c.c.}; \tag{3.15}$$

c.c. denotes complex-conjugate value. Thus, the lowest-order solution is inertial oscillations modulated by the arbitrary amplitude $W_0(x, y, z, T_1, \ldots)$, which depends on the coordinates and slow time. Obviously, the existence of the approximate solution (3.15) is related to the fact that all the modes σ_n^{sub}, σ_n^{sup}, independently of their number n, degenerate into inertial oscillations $\propto e^{-ift}$ in the long-wave limit.

The amplitude W_0 is determined from the first-order equation:

$$(\partial_{tt} + 1)w_{1zz} = -2\partial_{tT_1} w_{0zz} - 2q w_{0yz}. \tag{3.16}$$

The correction w_1 is bounded in the 'fast' time t if the right-hand side of (3.16) is zero whence one obtains:

$$\partial_{T_1} W_{0zz} + \mathrm{i}q W_{0yz} = 0. \tag{3.17}$$

Equation (3.17) should be solved under the boundary conditions

$$W_0|_{z=0,-1} = 0, \tag{3.18}$$

and the initial condition

$$W_0|_{T_1=0} = W_I = \tfrac{1}{2} w_I(x, y, z). \tag{3.19}$$

Representing the derivative W_{0z} as the Fourier series

$$W_{0z} = \sum_{n=-\infty}^{n=\infty} \hat{W}_n(x, y, T_1) e^{\mathrm{i}2n\pi z}, \tag{3.20}$$

and substituting (3.20) into (3.17) one obtains:

$$\hat{W}_{nT_1} + \frac{q}{2n\pi} \hat{W}_{ny} = 0, \tag{3.21}$$

i.e. the Fourier amplitudes in (3.20) have the form:

$$\hat{W}_n = \hat{W}_n \left(x, y - \frac{q}{2n\pi} T_1 \right). \tag{3.22}$$

Finally, by virtue of (3.20) and (3.22) the amplitude W_0 is given by the formula:

$$W_0 = \frac{\mathrm{i}}{2\pi} \sum_{n=-\infty}^{n=\infty} \frac{1}{n} \hat{W}_n \left(x, y - \frac{q}{2n\pi} T_1 \right) \left(1 - e^{\mathrm{i}2n\pi z} \right). \tag{3.23}$$

The solution (3.23) describes an along-meridional (along the y-axis) dispersive spreading of the perturbation: each vertical mode with number n travels along the y-axis at the group velocity $q/2n\pi$ that, obviously, agrees with the asymptotics (3.11). The group velocity does not depend on the horizontal wavenumbers k, l, therefore the modes $\hat{W}_n(1 - e^{\mathrm{i}2n\pi z})/n$ in the series (3.23) uniformly translate one after another conserving their shapes (see figure 3). With increasing n the group velocity decreases, i.e. at a fixed point x, y the velocity field has a tendency to become more and more small-scale in the vertical direction. Under the TA ($q = 0$) the meridional dispersion at time $T_1 \sim 1$ disappears; in this case the inertial oscillations disperse in all allowable directions on the longer time $T_2 \sim 1$.

By virtue of (3.5), (3.8) the horizontal velocities u, v and pressure p in the GW always depend on the vertical coordinate z, therefore the assumption that motion does not depend on depth (which is frequently used in the barotropic models under the TA) is equivalent to the filtering of GWs. In what follows we assume that the initial fields u_I, v_I in (2.4) depend on all coordinates.

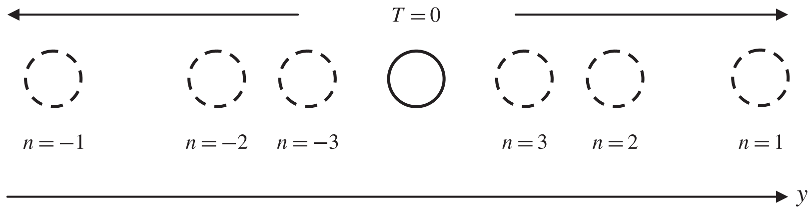

FIGURE 3. Schematic representation of dispersion spreading of a horizontally localized initial field (solid circle); n denotes the number of corresponding vertical modes (dashed circles).

4. Geostrophic mode and linear adjustment

In the linear approximation (2.1) take the form:

$$u_t - fv + f_s w = -p_x/\rho_0, \quad v_t + fu = -p_y/\rho_0, \qquad (4.1a,b)$$
$$w_t - f_s u = -p_z/\rho_0, \quad u_x + v_y + w_z = 0. \qquad (4.1c,d)$$

Besides the gyroscopic waves with frequency $\sigma > 0$ considered in § 3, there exists another eigenfunction of the system (4.1): the so-called geostrophic mode which does not depend on time (Greenspan 1968). This mode is related to a special invariant of the system (4.1) with the boundary conditions (2.3).

To derive the invariant we introduce the new variables:

$$x' = x, \quad y' = y - qz, \quad z' = z. \qquad (4.2)$$

The coordinates (4.2) are not orthogonal; the planes $y' = y - qz = \text{const}$ are parallel to the angular speed $\boldsymbol{\Omega}$ (see figure 4). We emphasize that only the coordinates are transformed, the velocity components are determined by the geometry of the boundaries as before. In the coordinates (4.2), (4.1) are written as

$$u_t - fv + f_s w = -p_{x'}/\rho_0, \quad v_t + fu = -p_{y'}/\rho_0, \qquad (4.3a,b)$$
$$w_t - f_s u = -(p_{z'} - qp_{y'})/\rho_0, \quad u_{x'} + v_{y'} + w_{z'} - qw_{y'} = 0. \qquad (4.3c,d)$$

Excluding p from (4.3*a,b*) and using the continuity equation (4.3*d*) one obtains the following equation for the vertical vorticity:

$$(v_{x'} - u_{y'})_t = fw_{z'}. \qquad (4.4)$$

The transformation (4.2) does not change the no-flux condition (2.3), i.e.

$$w|_{z'=0,-H} = 0, \qquad (4.5)$$

therefore the following conservation integral is obtained from (4.4):

$$\int_{-H}^{0} (v_{x'} - u_{y'})\, \mathrm{d}z' = \bar{\Omega}_I^{(z')}(x', y'). \qquad (4.6)$$

The right-hand side of (4.6) is determined by the initial conditions (2.4):

$$\bar{\Omega}_I^{(z')}(x', y') = \int_{-H}^{0} \Omega_I^{(z)}(x', y' + qz', z')\, \mathrm{d}z', \qquad (4.7)$$

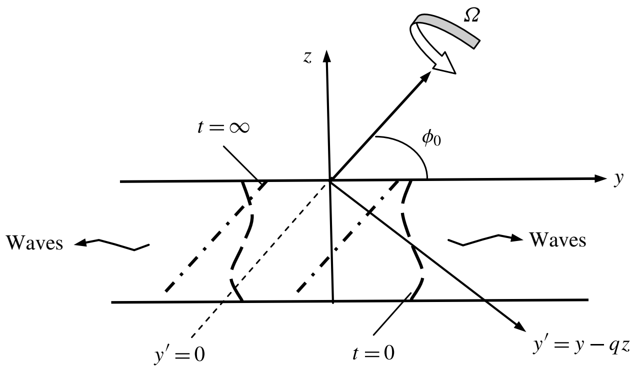

FIGURE 4. The coordinates (4.2) and schematic representation of linear geostrophic adjustment of an initial perturbation (thick long-dashed lines) to a z'-independent vortex state (thick dot-dashed lines) oriented along $\boldsymbol{\Omega}$.

where $\Omega_I^{(z)}$ is the initial vertical vorticity:

$$\Omega_I^{(z)} = \Omega_I^{(z)}(x, y, z) = \partial_x v_I - \partial_y u_I. \tag{4.8}$$

The conservation law (4.6) expresses a conservation of z'-averaged circulation along the so-called geostrophic contours (Greenspan 1968). The compact form of the integral is related to the fact that in our simple geometry any closed contour lying in one of the boundary planes $z' = 0, -H$, is a geostrophic contour.

The gyroscopic wave (3.2) is a solution to the system (4.3) with the boundary condition (4.5), therefore the invariant (4.6) also exists for the wave. One can readily see, however, that any linear invariant should be zero for any wave solution harmonically depending on time; otherwise the invariant harmonically depends on time, i.e. it is not an invariant. In our case this general principle implies that for the gyroscopic wave $\boldsymbol{u}^{(w)}(\boldsymbol{r}, t)$, $p^{(w)}(\boldsymbol{r}, t)$

$$\int_{-H}^{0} (v_{x'}^{(w)} - u_{y'}^{(w)})\, \mathrm{d}z' = 0, \tag{4.9}$$

where the bold $\boldsymbol{u}$ and $\boldsymbol{r}$ denote the velocity and radius vectors, the superscript (w) the wave solution.

This property of waves allows the solution to the linear problem (4.1), (2.3), (2.4) to be represented as a sum of a stationary component $\bar{\boldsymbol{u}}(\boldsymbol{r})$, $\bar{p}(\boldsymbol{r})$ with non-zero conservation integral (4.6) and a wave component $\tilde{\boldsymbol{u}}(\boldsymbol{r}, t)$, $\tilde{p}(\boldsymbol{r}, t)$ with the zero invariant:

$$(\boldsymbol{u}, p) = (\bar{\boldsymbol{u}}, \bar{p}) + (\tilde{\boldsymbol{u}}, \tilde{p}). \tag{4.10}$$

The stationary component obeys the equations:

$$-f\bar{v} + f_s\bar{w} = -\bar{p}_{x'}/\rho_0, \quad f\bar{u} = -\bar{p}_{y'}/\rho_0, \tag{4.11a,b}$$

$$f_s\bar{u} = (\bar{p}_{z'} - q\bar{p}_{y'})/\rho_0, \quad \bar{u}_{x'} + \bar{v}_{y'} + \bar{w}_{z'} - q\bar{w}_{y'} = 0. \tag{4.11c,d}$$

From (4.11b,c) one obtains $\bar{p}_{z'} = 0$, and from (4.11a,b,d), $\bar{w}_{z'} = 0$. By virtue of the boundary conditions (4.5) we have $\bar{w} = 0$.

Thus the stationary solution is a geostrophic mode (Greenspan 1968) which does not depend on the depth z on the planes parallel to the angular speed $\boldsymbol{\Omega}$:

$$\bar{u}=-\frac{1}{f\rho_0}\bar{p}_{y'} \quad \bar{v}=\frac{1}{f\rho_0}\bar{p}_{x'}, \quad \bar{w}=0, \quad \bar{p}_{z'}=0. \tag{4.12}$$

The geostrophic mode is characterized by a columnar motion, the column axes being directed along the rotation speed $\boldsymbol{\Omega}$ so that the motion is parallel to the rigid boundaries and the vertical velocity is zero (see figure 4). Geostrophic pressure $\bar{p}$ is found from (4.6) and (4.12):

$$\nabla_h^2\bar{p}=\frac{f\rho_0}{H}\bar{\Omega}_I^{(z')}(x',y'). \tag{4.13}$$

The wave component of solution obeys (4.1):

$$\tilde{u}_t-f\tilde{v}+f_s\tilde{w}=-\tilde{p}_x/\rho_0, \quad \tilde{v}_t+f\tilde{u}=-\tilde{p}_y/\rho_0, \tag{4.14a,b}$$
$$\tilde{w}_t-f_s\tilde{u}=-\tilde{p}_z/\rho_0, \quad \tilde{u}_x+\tilde{v}_y+\tilde{w}_z=0, \tag{4.14c,d}$$

with the boundary conditions (2.3) and the initial conditions

$$(\tilde{u}_I,\tilde{v}_I)=(u_I-\bar{u},v_I-\bar{v}). \tag{4.15}$$

In addition, the conservation integral (4.6) for the wave component is zero, i.e.

$$\int_{-H}^{0}(\tilde{v}_{x'}-\tilde{u}_{y'})\,\mathrm{d}z'=0. \tag{4.16}$$

Solution to the problem (4.14)–(4.16), (2.3) is a superposition of the gyroscopic waves considered in § 3. The waves are dispersive, therefore for localized initial conditions (when $u_I,v_I\to 0$ $r=\sqrt{x^2+y^2}\to\infty$) the wave solution $\tilde{\boldsymbol{u}},\tilde{p}$ decays with increasing time at a fixed point in space and the full solution (4.10) tends to the geostrophic mode (4.12), (4.13). In other words, any localized initial state tends with time to a geostrophically balanced localized vortex with axis parallel to $\boldsymbol{\Omega}$ (see figure 4). This tendency to columnar motion seems to be very persistent and is observed, for example, in laboratory experiments with turbulence in rotating tanks (see e.g. Davidson, Staplehurst & Dalziel 2006; Staplehurst, Davidson & Dalziel 2008, and references therein).

The typical time T_w of the wave adjustment can be defined as $T_w=L/c_g$ where L is the typical horizontal scale of the initial perturbation and c_g is the typical group velocity of radiated waves. It readily follows from the dispersion relations (3.10) and (3.11) that for the large and moderate scales $L\geqslant H$ the group velocity $c_g=O(fH)$ and for the small scales $L\ll H$ in the super-inertial (sub-inertial) range $c_g=O(fL^3/H^2)$ ($c_g=O(fL^2/H)$), i.e.

$$T_w=\frac{L}{H}f^{-1} \quad \text{for } \delta=\frac{H}{L}\leqslant 1, \tag{4.17a}$$
$$T_w\geqslant\frac{H}{L}f^{-1} \quad \text{for } \delta=\frac{H}{L}\gg 1. \tag{4.17b}$$

Thus the typical time of the wave adjustment is of the order of the inertial time f^{-1} for perturbations of moderate scales with $L\sim H$ and greatly exceeds this time in the large-scale ($L\gg H$) and short-scale ($L\ll H$) domains.

Nonlinear adjustment at small Rossby number $Ro = U/fL \ll 1$ (U is the horizontal velocity scale) results in a slow (as compared to the inertial time f^{-1}) evolution of the geostrophic component on the advective time $T_a = O(1/Rof)$. Scenario of the adjustment depends on the relationship between the advective time T_a and the wave adjustment time T_w. In the case $T_w \ll T_a$ the group velocity c_g greatly exceeds the flow velocity U, i.e. the waves rapidly run away from the initial perturbation and do not interact effectively with the geostrophic mode. The residual flow left behind, after all the waves have been propagated away, slowly changes on the advective time and is close to geostrophic balance. This scenario is realized for perturbations with moderate scale $L \sim H$ since in this case $T_w = O(f^{-1}) \ll T_a = O(Ro^{-1}f^{-1})$. For large-and small-scale perturbations the time $T_w \gg f^{-1}$, therefore in these scale domains the waves can effectively interact with the geostrophic mode if $T_w \geqslant T_a$ and, therefore, $c_g \leqslant U$. The moderate and small-scale cases will be considered elsewhere; in the rest of paper we examine the nonlinear evolution of large-scale perturbations with $L \gg H$ assuming the advective time T_a and the wave time T_w from (4.17*a*) to be of the same order. This assumption means that the group velocity c_g is of the order of the flow velocity U:

$$c_g = O(fH) \sim U. \tag{4.18}$$

5. Non-dimensional equations and asymptotic procedure

We now write the system (2.1) in the coordinates (4.2) and then in non-dimensional form using the scales L, H, f^{-1}, U and the scales of vertical velocity $W = (H/L)U$ and of pressure $P = \rho_0 fUL$ (the primes are omitted):

$$u_t + Ro(uu_x + vu_y + wu_z - \delta qwu_y) - v + \delta qw = -p_x, \tag{5.1a}$$
$$v_t + Ro(uv_x + vv_y + wv_z - \delta qwv_y) + u = -p_y, \tag{5.1b}$$
$$\delta^2 w_t + \delta^2 Ro(uw_x + vw_y + ww_z - \delta qww_y) - \delta qu = -p_z + \delta qp_y, \tag{5.1c}$$
$$u_x + v_y + w_z - \delta qw_y = 0; \tag{5.1d}$$

in the boundary and initial conditions (2.3), (2.4) the depth H is replaced by 1, and x, y, z in (2.3), (2.4*a*) are replaced by the variables (4.2). In terms of the small Rossby number $Ro = U/fL$ and parameter $\delta = H/L$ the condition (4.18) means that

$$Ro = \delta \ll 1. \tag{5.2}$$

Solution to the problem (5.1), (2.3), (2.4) is represented in an asymptotic form analogous to (3.13):

$$(u, v, w, p) = (u_0, v_0, w_0, p_0)(x, y, z, t, T_1, \ldots) + \delta(u_1, v_1, w_1, p_1) + \cdots . \tag{5.3}$$

Substitution of (5.3) into (5.1), (2.3), and (2.4) gives at the first three orders: for δ^0

$$u_{0t} - v_0 = -p_{0x}, \quad v_{0t} + u_0 = -p_{0y}, \quad p_{0z} = 0, \quad u_{0x} + v_{0y} + w_{0z} = 0, \tag{5.4 a–d}$$
$$w_0|_{z=0,-1} = 0, \quad (u_0, v_0)_{t=0} = (u_I, v_I)(x, y, z); \tag{5.5a,b}$$

for δ^1

$$u_{1t} - v_1 = -u_{0T_1} - N_u^{(0)} - qw_0 - p_{1x}, \tag{5.6a}$$
$$v_{1t} + u_1 = -v_{0T_1} - N_v^{(0)} - p_{1y}, \tag{5.6b}$$

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$$p_{1z}=q(u_0+p_{0y}),\quad u_{1x}+v_{1y}+w_{1z}-qw_{0y}=0, \tag{5.6c,d}$$

$$w_1|_{z=0,-1}=0,\quad (u_1,\,v_1)_{t=0}=0, \tag{5.7a,b}$$

$$N_u^{(0)}=(u_0^2)_x+(u_0v_0)_y+(w_0u_0)_z,\quad N_v^{(0)}=(u_0v_0)_x+(v_0^2)_y+(w_0v_0)_z; \tag{5.8a,b}$$

for δ^2

$$u_{2t}-v_2=-u_{0T_2}-u_{1T_1}-N_u^{(1)}-qw_1-p_{2x}, \tag{5.9a}$$
$$v_{2t}+u_2=-v_{0T_2}-v_{1T_1}-N_v^{(1)}-p_{2y}, \tag{5.9b}$$

$$p_{2z}=q(u_1+p_{1y})-w_{0t},\quad u_{2x}+v_{2y}+w_{2z}-qw_{1y}=0, \tag{5.9c,d}$$

$$w_2|_{z=0,-1}=0,\quad (u_2,\,v_2)_{t=0}=0, \tag{5.10a,b}$$

$$N_u^{(1)}=u_0u_{1x}+u_1u_{0x}+v_0u_{1y}+v_1u_{0y}+w_0u_{1z}+w_1u_{0z}-qw_0u_{0y}, \tag{5.11a}$$
$$N_v^{(1)}=u_0v_{1x}+u_1v_{0x}+v_0v_{1y}+v_1v_{0y}+w_0v_{1z}+w_1v_{0z}-qw_0v_{0y}. \tag{5.11b}$$

For $n\geqslant 1$ the nth-order set of equations constitutes a linear system for the nth-order fields with right-hand sides depending on the fields of previous orders. Analysis of the nth-order system consists of two steps (e.g. Reznik *et al.* 2001; Zeitlin *et al.* 2003). First, the dependence on the slow times of the previous order fields is determined from conditions of absence of secular terms in the right-hand sides. Second, the dependence on the coordinates and fast time t of the nth-order solution is determined from the corresponding system free of secular terms. Below we sequentially analyse the systems (5.4)–(5.11); cumbersome calculations are given in corresponding appendices A and B.

6. The lowest-order solution: slow QG flow and inertial oscillations

Here and below we use the representation of a physical field a as the sum of the depth-averaged component and the component with zero mean over depth:

$$a=\bar{a}(x,y)+\hat{a}(x,y,z),\quad \bar{a}=\int_{-1}^{0}a\,\mathrm{d}z\quad \int_{-1}^{0}\hat{a}\,\mathrm{d}z=0. \tag{6.1a–c}$$

Let us write the horizontal velocities u_0, v_0 in the form (6.1); in this case one obtains from (5.4), (5.5) the following equations:

$$\bar{u}_{0t}-\bar{v}_0=-p_{0x},\quad \bar{v}_{0t}+\bar{u}_0=-p_{0y},\quad \bar{u}_{0x}+\bar{v}_{0y}=0; \tag{6.2a–c}$$

$$(\bar{u}_0,\,\bar{v}_0)_{t=0}=(\bar{u}_I,\,\bar{v}_I)(x,y)=\int_{-1}^{0}(u_I,\,v_I)\,\mathrm{d}z; \tag{6.2d}$$

$$\hat{u}_{0t}-\hat{v}_0=0,\quad \hat{v}_{0t}+\hat{u}_0=0,\quad \hat{u}_{0x}+\hat{v}_{0y}+w_{0z}=0; \tag{6.3a–c}$$

$$w_0|_{z=0,-1}=0,\quad (\hat{u}_0,\,\hat{v}_0)_{t=0}=(u_I,\,v_I)-(\bar{u}_I,\,\bar{v}_I). \tag{6.3d,e}$$

It readily follows from (6.2*a–c*) that:

$$\bar{\zeta}_0=\bar{v}_{0x}-\bar{u}_{0y}=\bar{\zeta}_0(x,y,T_1,\ldots),\quad \bar{u}_0=-\bar{\psi}_{0y},\quad \bar{v}_0=\bar{\psi}_{0x}, \tag{6.4a–c}$$

where $\bar{\zeta}_0$ is the depth-averaged vorticity and $\bar{\psi}_0$ the streamfunction which can be introduced in view of (6.2*c*). It is seen from (6.4) that the depth-averaged zero-order horizontal velocities $\bar{u}_0$, $\bar{v}_0$ do not depend on the fast time t, and, therefore, by virtue of (6.2*a,b*) they satisfy geostrophic relations, i.e. one can set

$$\bar{\psi}_0 = p_0. \tag{6.4d}$$

System (6.3) is also readily solved:

$$\hat{u}_0 + \mathrm{i}\hat{v}_0 = A_0(x, y, z, T_1, \ldots)\mathrm{e}^{-\mathrm{i}t}, \quad \hat{u}_0 = \tfrac{1}{2}A_0\mathrm{e}^{-\mathrm{i}t} + \text{c.c.}, \quad \hat{v}_0 = -\tfrac{1}{2}\mathrm{i}A_0\mathrm{e}^{-\mathrm{i}t} + \text{c.c.}, \tag{6.5a–c}$$

$$w_0 = -\tfrac{1}{2}\mathrm{e}^{-\mathrm{i}t}\int_{-1}^{z} \mathrm{s}(A_0)\,\mathrm{d}z + \text{c.c.} \tag{6.5d}$$

Here the operator s is

$$\mathrm{s} = \partial_x - \mathrm{i}\partial_y, \tag{6.6}$$

and the amplitude A_0 obeys the conditions

$$\int_{-1}^{0} A_0\,\mathrm{d}z = 0, \quad A_0(x, y, z, 0) = \hat{u}_I + \mathrm{i}\hat{v}_I. \tag{6.7a,b}$$

Thus, the zero-order solution is the sum of a depth-independent slow geostrophic component and fast ageostrophic inertial oscillations modulated by amplitude which depends on the coordinates and slow times. Equations to determine the geostrophic streamfunction $\bar{\psi}_0$ and the amplitude A_0 follow from analysis of the next-order approximations.

7. The first-order solution and slow evolution of QG flow and inertial oscillations

7.1. *Derivation of slow evolution equations*

Using (5.6*c*) and (6.4*b,d*) the pressure p_1 is represented in the form (6.1):

$$p_1 = \bar{p}_1 + \hat{p}_1, \tag{7.1}$$

where the depth-averaged pressure $\bar{p}_1 = \bar{p}_1(x, y, t)$ is still unknown and the deviation $\hat{p}_1$ is

$$\hat{p}_1 = q\left(\int_{-1}^{z} \hat{u}_0\,\mathrm{d}z + \int_{-1}^{0} z\hat{u}_0\,\mathrm{d}z\right). \tag{7.2}$$

Representing all fields in (5.6) in the form (6.1) one obtains from (5.6)–(5.8) the equations for the depth-averaged and zero-mean components:

$$\bar{u}_{1t} - \bar{v}_1 = -\bar{u}_{0T_1} - \bar{N}_u^{(0)} - q\bar{w}_0 - \bar{p}_{1x}, \tag{7.3a}$$

$$\bar{v}_{1t} + \bar{u}_1 = -\bar{v}_{0T_1} - \bar{N}_v^{(0)} - \bar{p}_{1y}, \tag{7.3b}$$

$$\bar{u}_{1x} + \bar{v}_{1y} - q\bar{w}_{0y} = 0; \tag{7.3c}$$

$$\hat{u}_{1t} - \hat{v}_1 = -\hat{u}_{0T_1} - \hat{N}_u^{(0)} - q\hat{w}_0 - \hat{p}_{1x}, \tag{7.4a}$$

$$\hat{v}_{1t} + \hat{u}_1 = -\hat{v}_{0T_1} - \hat{N}_v^{(0)} - \hat{p}_{1y}, \tag{7.4b}$$

$$\hat{u}_{1x} + \hat{v}_{1y} + w_{1z} - q\hat{w}_{0y} = 0. \tag{7.4c}$$

Here we have:

$$\bar{N}_u^{(0)} = (\overline{u_0^2})_x + (\overline{u_0 v_0})_y, \quad \bar{N}_v^{(0)} = (\overline{u_0 v_0})_x + (\overline{v_0^2})_y, \tag{7.5a,b}$$

$$\hat{N}_u^{(0)} = (u_0^2 - \overline{u_0^2})_x + (u_0 v_0 - \overline{u_0 v_0})_y + (w_0 u_0)_z, \tag{7.6a}$$

$$\hat{N}_v^{(0)} = (u_0 v_0 - \overline{u_0 v_0})_x + (v_0^2 - \overline{v_0^2})_y + (w_0 v_0)_z. \tag{7.6b}$$

Elimination of $\bar{p}_1$ from (7.3*a*,*b*) taking into account (7.3*c*) gives the first-order vorticity equation:

$$\bar{\zeta}_{1t} = -\bar{\zeta}_{0T_1} - (\partial_x \bar{N}_v^{(0)} - \partial_y \bar{N}_u^{(0)}), \quad \zeta_1 = v_{1x} - u_{1y}. \tag{7.7a,b}$$

One can show (see appendix A) that:

$$\partial_x \bar{N}_v^{(0)} - \partial_y \bar{N}_u^{(0)} = \mathrm{J}(\bar{\psi}_0, \bar{\zeta}_0) - \tfrac{1}{4}\left[\mathrm{e}^{-2\mathrm{i}t}\mathrm{L}\left(\overline{A_0^2}\right) + \mathrm{c.c.}\right], \tag{7.8}$$

where J is the Jacobian and

$$\mathrm{L} = \mathrm{is}^2 = 2\partial_{xy} + \mathrm{i}(\partial_{xx} - \partial_{yy}). \tag{7.9}$$

It readily follows from (7.7*a*), (6.4) and (7.8) that the vorticity $\bar{\zeta}_1$ is bounded as $t \to \infty$ only if

$$\bar{\zeta}_{0T_1} + \mathrm{J}(\bar{\psi}_0, \bar{\zeta}_0) = 0, \quad \bar{\zeta}_0 = \nabla_h^2 \bar{\psi}_0. \tag{7.10a,b}$$

Using (7.10*a*) and (7.8) one obtains $\bar{\zeta}_1$ from (7.7*a*):

$$\bar{\zeta}_1 = \Pi_1(x, y, T_1, \ldots) + \tfrac{1}{8}\mathrm{i}\left[\mathrm{e}^{-2\mathrm{i}t}\mathrm{L}\left(\overline{A_0^2}\right) - \mathrm{c.c.}\right], \tag{7.11}$$

where Π_1 is a still unknown function of the horizontal coordinates and slow times.

To determine the deviations $\hat{u}_1$, $\hat{v}_1$ we write (7.4*a*,*b*) as one complex equation:

$$\hat{U}_{1t} + \mathrm{i}\hat{U}_1 = -[\hat{U}_{0T_1} + \hat{N}_u^{(0)} + \mathrm{i}\hat{N}_v^{(0)} + \hat{p}_{1x} + \mathrm{i}\hat{p}_{1y} + q\hat{w}_0], \tag{7.12}$$

where

$$\hat{U}_0 = \hat{u}_0 + \mathrm{i}\hat{v}_0, \quad \hat{U}_1 = \hat{u}_1 + \mathrm{i}\hat{v}_1. \tag{7.13a,b}$$

Obviously, the complex velocity $\hat{U}_1$ is bounded if secular terms proportional to $\mathrm{e}^{-\mathrm{i}t}$ are absent from the right-hand side of (7.12). It follows from (6.5*d*) that

$$\bar{w}_0 = \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}t}\int_{-1}^{0} z\mathrm{s}(A_0)\,\mathrm{d}z + \mathrm{c.c.}, \quad \hat{w}_0 = -\tfrac{1}{2}\mathrm{e}^{-\mathrm{i}t}\left[\int_{-1}^{z}\mathrm{s}(A_0)\,\mathrm{d}z + \int_{-1}^{0} z\mathrm{s}(A_0)\,\mathrm{d}z\right] + \mathrm{c.c.} \tag{7.14a,b}$$

Using (7.2) and (7.14*b*) one finds:

$$\hat{p}_{1x} + \mathrm{i}\hat{p}_{1y} + q\hat{w}_0 = \mathrm{e}^{-\mathrm{i}t}\mathrm{i}q\left(\int_{-1}^{z} A_0\mathrm{d}z + \int_{-1}^{0} zA_0\,\mathrm{d}z\right)_y. \tag{7.15}$$

It is shown in appendix A that:

$$\hat{N}_u^{(0)} + \mathrm{i}\hat{N}_v^{(0)} = \mathrm{e}^{-\mathrm{i}t}[\mathrm{J}(\bar{\psi}_0, A_0) + \tfrac{1}{2}\mathrm{i}\nabla_h^2\bar{\psi}_0 A_0] + NR; \tag{7.16}$$

here and below NR denotes non-resonant terms. Now using (7.15), (7.16), and (6.5*a*) one concludes that the resonant terms on the right-hand side of (7.12) are absent under the condition

$$A_{0T_1} + \mathrm{J}(\bar{\psi}_0, A_0) + \tfrac{1}{2}\mathrm{i}\nabla_h^2\bar{\psi}_0 A_0 + \mathrm{i}q\left(\int_{-1}^{z} A_0\mathrm{d}z + \int_{-1}^{0} zA_0\,\mathrm{d}z\right)_y = 0. \tag{7.17}$$

7.2. *Discussion of the slow evolution*

Equations (7.10) and (7.17) describe the slow evolution of the zero-order fields on times $t \sim 1/\delta$ ($T_1 \sim 1$). Equation (7.10) expresses conservation of the quasi-geostrophic potential vorticity $\bar{\zeta}_0 = \nabla_h^2 \bar{\psi}_0$ and coincides here with the two-dimensional fluid dynamics equation. Importantly, the slow QG component does not depend on the fast ageostrophic inertial oscillations and is determined only by (7.10) and the initial conditions (6.2*d*).

Evolution of the amplitude of inertial oscillations is determined by equation (7.17), and depends strongly on the QG streamfunction $\bar{\psi}_0$. At the same time one can readily show that

$$\frac{\partial}{\partial T_1} \int \mathrm{d}x\, \mathrm{d}y \overline{|A_0|^2} = 0, \quad \frac{\partial}{\partial T_1} \int \mathrm{d}x\, \mathrm{d}y (\nabla \bar{\psi}_0)^2 = 0, \tag{7.18}$$

i.e. on times $t \sim 1/\delta$ both the fast and the slow components conserve their total energies. In the next section we show that an energy transfer between the inertial oscillations and QG flow is possible on times $t \sim 1/\delta^2$.

Under the TA, i.e. for $q = 0$, (7.17) is substantially simplified, especially if the QG streamfunction $\bar{\psi}_0$ is axisymmetric and, therefore, it does not depend on time as follows from (7.10), i.e. $\bar{\psi}_0 = \bar{\psi}_0(r)$. In this case the solution to (7.17) has the form:

$$A_0 = \exp\left(-\tfrac{1}{2}\mathrm{i}\nabla_h^2 \bar{\psi}_0 T_1\right) A_I \left(r, \theta - \frac{\bar{\psi}_0'}{r} T_1\right), \tag{7.19}$$

where the prime means differentiation with respect to r and $A_I(r, \theta)$ is the initial amplitude (6.7*b*) written in polar coordinates. The exponential factor in (7.19) shifts the inertial frequency f to the so-called effective inertial frequency $f + \bar{\zeta}_0/2$ (Kunze 1985). The factor $A_I(\ldots)$ describes advection of the inertial oscillations by the QG flow, the radial gradients of the amplitude becoming sharp due to the differential rotation. In accordance with (7.19) the inertial oscillations are trapped by the QG vortex, (7.19) displaying no asymmetry between cyclonic and anticyclonic vortices in their 'trapping ability' (cf. Kunze 1985). This lack of asymmetry is related to the lack of dispersion of the long gyroscopic waves here: under the TA the dispersion becomes significant on longer times $T_2 \sim 1$.

It is seen from (7.19) that the magnitude $|A_0|$ behaves exactly like a passive scalar in the steady QG flow. For $q = 0$ the same is valid for any $\bar{\psi}_0$ since by virtue of (7.17) we have:

$$|A_0|_{T_1} + \mathrm{J}(\bar{\psi}_0, |A_0|) = 0. \tag{7.20}$$

Equation (7.20) means that under the TA the inertial oscillations are trapped by the QG velocity field $\bar{u}_0$, $\bar{v}_0$.

At $q \neq 0$ the 'non-traditional' term in (7.17) changes the situation. In the absence of the slow component, i.e. for $\bar{\psi}_0 = 0$, (7.17) is similar to (3.17) considered above in § 3 (one can readily see this by differentiating (7.17) with respect to z, setting $\bar{\psi}_0 = 0$ and applying the operator (6.6) to the resulting equation). Therefore, the non-traditional term in (7.17) produces a tendency for the meridional (along the y-axis) propagation of the inertial oscillations. To analyse the general case $\bar{\psi}_0 \neq 0$, $q \neq 0$ we represent the solution to (7.17) in a form analogous to (3.20):

$$A_0 = \sum_{n=-\infty}^{n=\infty} \hat{A}_n(x, y, T_1, \ldots)\mathrm{e}^{\mathrm{i}2n\pi z}. \tag{7.21}$$

The equation for the Fourier amplitude $\hat{A}_n$ is written as

$$\hat{A}_{nT_1} + \mathrm{J}(\bar{\psi}_0, \hat{A}_n) + \tfrac{1}{2}\mathrm{i}\nabla_h^2\bar{\psi}_0\hat{A}_n + \frac{q}{2n\pi}\hat{A}_{ny} = 0. \tag{7.22}$$

The equation for the module $|\hat{A}_n|$ analogous to (7.20) simply follows from (7.22):

$$|\hat{A}_n|_{T_1} + \mathrm{J}\left(\bar{\psi}_n, |\hat{A}_n|\right) = 0. \tag{7.23}$$

Here $\bar{\psi}_n$ is the sum of $\bar{\psi}_0$ and a superimposed constant meridional flow $(q/2n\pi)x$:

$$\bar{\psi}_n = \bar{\psi}_0 + \frac{q}{2n\pi}x. \tag{7.24}$$

Thus, the field $|\hat{A}_n|$ behaves as a passive scalar in the velocity field $\bar{u}_n = -\bar{\psi}_{ny}$, $\bar{v}_n = \bar{\psi}_{nx}$. Let the QG component $\bar{\psi}_0$ contain intense vortices with closed streamlines; in this case the streamline field (7.24) consists of the closed streamlines related to the vortices, and unclosed ones, each of the unclosed streamlines tending to the straight line $\bar{\psi}_n = (q/2n\pi)x = \text{const}$ as $y \to \pm\infty$. If the QG flow is time-independent then the module $|\hat{A}_n|$ is trapped in the domains with the closed streamlines and travels away from the initial disturbance location along the unclosed streamlines. The 'propagation ability' depends on the mutual strength of the field $\bar{\psi}_0$ and the superimposed flow $(q/2n\pi)x$ and decreases with increasing n. In the case of time-dependent QG flow the situation is more complicated since Lagrangian trajectories do not coincide with the streamlines. However, one can assume that the time-dependent $\bar{\psi}_0$, at least, does not reduce the 'propagation ability' of the inertial oscillations since in this case the Lagrangian trajectory can escape from the closed streamlines (e.g. Aref 1984).

An analogue of (7.17) in stratified fluid was derived by Young & Ben Jelloul (1997), and analysed by Balmforth, Llewellyn Smith & Young (1998), Balmforth & Young (1999), Klein & Llewellyn-Smith (2001) and Klein, Llewellyn-Smith & Lapeyre (2004): the QG flow in these works was assumed to be prescribed. In the context of geostrophic adjustment an analogue of (7.17) was derived by Reznik *et al.* (2001) (for barotropic shallow water with a free surface) and by Zeitlin *et al.* (2003) (for stratified fluid). In all these works, the TA was used and the inertial oscillations were long gravity (surface or internal) waves with horizontal scales greatly exceeding the corresponding Rossby scales.

In the theory presented here, no special constraints on the initial states (2.4) are required: for example, decay of the motion at infinity is unnecessary. For horizontally periodic flows, a more rigorous theory has been developed (see e.g. Wingate *et al.* 2011, and references therein). It is shown that the spatially periodic motion, too, is split into the slow QG and fast wave components. The main result of these studies is that the fast–fast interactions (i.e. interactions between the fast waves) do not contribute to the slow component at least on times $O(1/f\,Ro)$ (longer times were not considered).

8. Long-term evolution of QG flow

We now proceed to the problem (5.9), (5.10) and (5.11). We are interested in corrections to (7.10) for the slow QG component, therefore we integrate (5.9*a,b,d*) over the depth to derive the equations for $\bar{u}_2, \bar{v}_2, \bar{p}_2$:

$$\bar{u}_{2t} - \bar{v}_2 = -\bar{u}_{0T_2} - \bar{u}_{1T_1} - \bar{N}_u^{(1)} - q\bar{w}_1 - \bar{p}_{2x}, \tag{8.1a}$$

$$\bar{v}_{2t} + \bar{u}_2 = -\bar{v}_{0T_2} - \bar{v}_{1T_1} - \bar{N}_v^{(1)} - \bar{p}_{2y}, \tag{8.1b}$$

$$\bar{u}_{2x} + \bar{v}_{2y} - q\bar{w}_{1y} = 0. \tag{8.1c}$$

Like the preceding section we use the vorticity equation following from (8.1*a,b*) (cf. (7.7)):

$$\bar{\zeta}_{2t} = -\bar{\zeta}_{1T_1} - \bar{\zeta}_{0T_2} - (\partial_x \bar{N}_v^{(1)} - \partial_y \bar{N}_u^{(1)}), \quad \zeta_2 = v_{2x} - u_{2y}. \tag{8.2a,b}$$

Averaging (8.2*a*) over the fast time t gives:

$$\bar{\zeta}_{0T_2} + \langle \bar{\zeta}_1 \rangle_{T_1} + \langle \partial_x \bar{N}_v^{(1)} - \partial_y \bar{N}_u^{(1)} \rangle = 0, \tag{8.3}$$

where the average is defined as

$$\langle a \rangle = \lim \frac{1}{T_0} \int_0^{T_0} a \, \mathrm{d}t \quad \text{as } T_0 \to \infty. \tag{8.4}$$

Using (7.3*c*), (7.7*b*) and (7.14*a*) one finds:

$$\langle \bar{\zeta}_1 \rangle = \nabla_h^2 \bar{\psi}_1, \quad \langle \bar{u}_1 \rangle = -\bar{\psi}_{1y}, \quad \langle \bar{v}_1 \rangle = \bar{\psi}_{1y}. \tag{8.5a–c}$$

The third term in (8.3) is calculated in appendix B:

$$\langle \partial_x \bar{N}_v^{(1)} - \partial_y \bar{N}_u^{(1)} \rangle = \mathrm{J}(\bar{\psi}_0, \langle \bar{\zeta}_1 \rangle) + \mathrm{J}(\bar{\psi}_1, \bar{\zeta}_0) + \mathrm{G}(\bar{\psi}_0, A_0) + q\mathrm{H}(A_0) \tag{8.6a}$$

where

$$\mathrm{G}(\bar{\psi}_0, A_0) = \tfrac{1}{2} \left(\mathrm{M}(\bar{\psi}_0) \overline{|A_0|^2} \right)_{xy} - \tfrac{1}{2} \mathrm{M} \left(\bar{\psi}_{0xy} \overline{|A_0|^2} \right), \tag{8.6b}$$

$$\mathrm{H}(A_0) = -\tfrac{1}{4} \mathrm{i s} \left[\int_{-1}^{0} \mathrm{d}z A_0 \int_{-1}^{z} \mathrm{s}^*(A_0^*) \, \mathrm{d}z \right]_y + \text{c.c.}, \tag{8.6c}$$

and the operator $\mathrm{M} = \partial_{xx} - \partial_{yy}$.

Equation (8.3) together with (8.5) and (8.6) describe the next-order correction to equation (7.10*a*) which should be taken into account when studying the slow evolution of the QG component for times much longer than $1/\delta$. Combining (7.10*a*) with (8.3) (see also Reznik *et al.* 2001) one derives the 'refined' QG equation valid on times of the order of $1/\delta^2$ (the subscripts are omitted):

$$\bar{\zeta}_T + \mathrm{J}(\bar{\psi}, \bar{\zeta}) + \delta[\mathrm{G}(\bar{\psi}, A) + q\mathrm{H}(A)] = 0, \tag{8.7}$$

where $\bar{\zeta} = \nabla_h^2 \bar{\psi}$ and $T = \delta t$.

If all fields decay at infinity then we have

$$\int \bar{\psi} \mathrm{G}(\bar{\psi}, A) \, \mathrm{d}x \, \mathrm{d}y = 0, \tag{8.8}$$

therefore the energy $\bar{E}$ of the QG component changes in time as

$$\partial_T \bar{E} = \delta q \int \bar{\psi} \mathrm{H}(A) \, \mathrm{d}x \, \mathrm{d}y, \quad \bar{E} = \tfrac{1}{2} \int (\nabla_h \bar{\psi})^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{8.9a,b}$$

For $q \neq 0$ the right-hand side of (8.9*a*), is, generally, non-zero whence the important conclusion follows that without the TA a transfer of energy between the QG component and inertial oscillations can exist.

To understand why the Coriolis force (which does not do any work) participates in the energy transfer we represent all the fields in the form (6.1) and derive from the system (5.1) the energy equations for the depth-averaged and zero-mean components:

$$\frac{\partial \bar{E}}{\partial t} = -\delta q \left\langle (\bar{u} + \bar{p}_y)\bar{w} \right\rangle_{x,y,z} - Q_N, \quad \bar{E} = \tfrac{1}{2} \left\langle \bar{u}^2 + \bar{v}^2 \right\rangle_{x,y,z}; \tag{8.10a,b}$$

$$\frac{\partial \hat{E}}{\partial t} = \delta q \left\langle (\bar{u} + \bar{p}_y)\bar{w} \right\rangle_{x,y,z} + Q_N, \quad \hat{E} = \tfrac{1}{2} \left\langle \hat{u}^2 + \hat{v}^2 + \delta^2 w^2 \right\rangle_{x,y,z}; \tag{8.11a,b}$$

$$Q_N = -Ro \left\langle \bar{u}\bar{N}_u + \bar{v}\bar{N}_v \right\rangle_{x,y,z}. \tag{8.12}$$

Here $\bar{N}_u$, $\bar{N}_v$ are the depth-averaged nonlinear terms in the brackets in (5.1*a,b*), $\langle\rangle_{x,y,z}$ denotes integration in all three coordinates. The parameters δ and Ro in (8.10)–(8.12) can be arbitrary. We see that the 'non-traditional' term $\delta q \left\langle (\bar{u} + \bar{p}_y)\bar{w} \right\rangle_{x,y,z}$ provides a redistribution of energy between the vertically averaged (QG in our case) flow and the zero-mean (ageostrophic in our case) components; Q_N is the energy flux due to nonlinear terms. We note that the redistribution of energy by non-zero f_s plays an important role in the dynamics of Ekman flows (see Gerkema *et al.* 2008, and references therein).

9. Influence of the β-effect

To understand the role of the β-effect we examine the problem discussed in §§ 5–8 with variable parameter f given by (2.5*b,c*). In doing this one replaces the linear term $-v$ in (5.1*a*) by $-(1 + \delta\bar{\beta}y + \delta^2\bar{\beta}qz)v$ and the term u in (5.1*b*) by $(1 + \delta\bar{\beta}y + \delta^2\bar{\beta}qz)u$. Here $\bar{\beta} = (L/a)/\delta$ and we assume that $\bar{\beta} = O(1)$. Analysis of the modified system (5.1) is very similar to that in §§ 5–8 and only the modified slow evolution equations are given here:

$$\bar{\zeta}_T + \mathrm{J}(\bar{\psi}, \bar{\zeta}) + \delta[\mathrm{G}(\bar{\psi}, A) + q\mathrm{H}(A)] + \bar{\beta}\bar{\psi}_x = 0, \tag{9.1}$$

$$A_T + \mathrm{J}(\bar{\psi}, A) + \mathrm{i}\left(\tfrac{1}{2}\nabla_h^2\bar{\psi} + \bar{\beta}y\right)A + \mathrm{i}q\left(\int_{-1}^{z} A\,\mathrm{d}z + \int_{-1}^{0} zA\,\mathrm{d}z\right)_y = 0. \tag{9.2}$$

As seen from (9.1), the β-effect does not affect the nonlinear interaction between the QG flow and inertial oscillations and results only in giving rise to Rossby waves which contribute to the distortion of the slow component. As for the inertial oscillations, the β-term appears in the third term of the modified equation (9.2); this makes the effective inertial frequency equal to $f + \bar{\zeta}/2$ as before (see § 7.2) but with the variable parameter f given by (2.5*b,c*). The results on the slow evolution of inertial oscillations represented in § 7.2 remain valid.

10. Summary and discussion

We have examined geostrophic adjustment in a rotating barotropic fluid layer of constant depth bounded by two rigid lids. The angular speed of rotation $\boldsymbol{\Omega}$ does not coincide in direction with the gravity; the traditional and hydrostatic approximations are not used. The only possible wave motions in our model are the gyroscopic waves due to rotation. In the linear approximation the adjustment causes any localized initial state to tend to a geostrophically balanced columnar motion parallel to the layer boundaries, the columns being parallel to $\boldsymbol{\Omega}$.

Using multiple-time-scale perturbation theory we studied the nonlinear adjustment at small Rossby numbers and aspect ratio H/L. Similarly to the geostrophic adjustment with gravity waves (cf. Reznik *et al.* 2001; Zeitlin *et al.* 2003), in our case an arbitrary perturbation is split in a unique way into slow and fast components evolving with characteristic time scales $(Rof)^{-1}$ and f^{-1}, respectively. The slow component is close to geostrophic balance and does not depend on depth. On times $t \sim (f\,Ro)^{-1}$ the slow component is not influenced by the fast one and is described by the two-dimensional fluid dynamics equation for the geostrophic streamfunction.

The fast component is a packet of inertial oscillations modulated by amplitude depending on coordinates and the slow time. The depth-integrated horizontal flow induced by the inertial oscillations is zero in the leading order. The inertial oscillations are long gyroscopic waves with horizontal scale L exceeding the depth layer H. We note that in geostrophic adjustment with gravity waves (surface or internal) the inertial oscillations arise only if the dominating scale of the initial perturbation exceeds the corresponding Rossby scale L_R (cf. Reznik *et al.* 2001; Zeitlin *et al.* 2003). The slow QG component in this case obeys the so-called frontal dynamics equation. If $L_R \gg H$ (as in the atmosphere and the ocean) then in the presence of gyroscopic waves 'shorter' inertial oscillations with scales $L \leqslant L_R$ are possible. The significant vertical velocities of the near-inertial oscillations observed by van Haren & Millot (2005) in the practically barotropic deep Western Mediterranean Sea can be related to this property of gyroscopic waves. We note that some other regions of the deep ocean are also characterized by a very weak stratification, as for example, the Canada Basin in the Arctic Ocean (Timmermans, Melling & Rainville 2007), or the Pacific Ocean near 179° E (Gerkema *et al.* 2008).

On times $t \sim (f\,Ro)^{-1}$ the fast component conserves its energy but it is coupled to the slow component: its amplitude obeys an equation with coefficients depending on the geostrophic streamfunction. In accordance with this equation, under the TA the inertial oscillations are trapped by the QG component; dispersion of the inertial oscillations packet occurs on much longer times $t \sim (f\,Ro^2)^{-1}$. Without the TA the 'non-traditional' terms in the amplitude equation provide much faster meridional dispersion of the packet on times $t \sim (f\,Ro)^{-1}$, and, therefore, cause an effective radiation of energy from the initial perturbation domain. Another important effect of the non-traditional terms is that on the longer times $t \sim (f\,Ro^2)^{-1}$ the slow component ceases to be independent of the fast one, and a transfer of energy between the components becomes possible. Preceding studies of the geostrophic adjustment use the TA and in these works the fast ageostrophic component does not affect the slow QG component even on times much longer than the typical geostrophic time $t \sim (f\,Ro)^{-1}$ (cf. Reznik *et al.* 2001; Zeitlin *et al.* 2003). Therefore, it was unclear in what way the slow QG flow could interact (if at all) with the fast ageostrophic motions. The presence and properties of this interaction are of importance for understanding of the still vague mechanism of dissipation of the atmospheric and oceanic mesoscale motions. Our example shows that the 'non-traditional' terms in equations of motion can play a certain role in this mechanism. It would be useful to estimate the efficiency of these terms numerically using a non-hydrostatic model without the TA.

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Appendix A. Some details of the first-order solution

Using (6.1) and (6.2*c*), (6.3*c*) we find:

$$\bar{N}_u^{(0)} = \bar{u}_0\bar{u}_{0x} + \bar{v}_0\bar{u}_{0y} + (\overline{\hat{u}_0^2})_x + (\overline{\hat{u}_0\hat{v}_0})_y, \tag{A 1a}$$

$$\bar{N}_v^{(0)} = \bar{u}_0\bar{v}_{0x} + \bar{v}_0\bar{v}_{0y} + (\overline{\hat{u}_0\hat{v}_0})_x + (\overline{\hat{v}_0^2})_y; \tag{A 1b}$$

$$\hat{N}_u^{(0)} = \bar{u}_0\hat{u}_{0x} + \bar{v}_0\hat{u}_{0y} + \hat{u}_0\bar{u}_{0x} + \hat{v}_0\bar{u}_{0y} + (\hat{u}_0^2 - \overline{\hat{u}_0^2})_x + (\hat{u}_0\hat{v}_0 - \overline{\hat{u}_0\hat{v}_0})_y + (w_0\hat{u}_0)_z, \tag{A 2a}$$

$$\hat{N}_v^{(0)} = \bar{u}_0\hat{v}_{0x} + \bar{v}_0\hat{v}_{0y} + \hat{u}_0\bar{v}_{0x} + \hat{v}_0\bar{v}_{0y} + (\hat{u}_0\hat{v}_0 - \overline{\hat{u}_0\hat{v}_0})_x + (\hat{v}_0^2 - \overline{\hat{v}_0^2})_y + (w_0\hat{v}_0)_z. \tag{A 2b}$$

The formula (7.8) follows from (A 1*a,b*) and (6.5*b,c*), (6.2*c*), and (6.4).

Substitution of the expressions (6.4*b,c*) for $\bar{u}_0$, $\bar{v}_0$ and (6.5*b,c,d*) for $\hat{u}_0$, $\hat{v}_0$, $\hat{w}_0$ into (A 2*a,b*) gives after some algebra:

$$\hat{N}_u^{(0)} + \mathrm{i}\hat{N}_v^{(0)} = \mathrm{e}^{-\mathrm{i}t}M^{(r)} + M^{(0)} + \mathrm{e}^{\mathrm{i}t}M^{(1)} + \mathrm{e}^{-2\mathrm{i}t}M^{(2)}, \tag{A 3}$$

$$M^{(r)} = \mathrm{J}(\bar{\psi}_0, A_0) + \tfrac{1}{2}\mathrm{i}\nabla_h^2\bar{\psi}_0 A_0, \tag{A 4a}$$

$$M^{(0)} = \tfrac{1}{2}\mathrm{s}^*\left(|A_0|^2 - \overline{|A_0|^2}\right) - \tfrac{1}{2}\left[A_0\int_{-1}^{z}\mathrm{s}^*(A_0^*)\,\mathrm{d}z\right]_z, \tag{A 4b}$$

$$M^{(1)} = -\tfrac{1}{2}\mathrm{L}^*(\bar{\psi}_0)A_0^*, \tag{A 4c}$$

$$M^{(2)} = \tfrac{1}{2}\mathrm{s}\left(A_0^2 - \overline{A_0^2}\right) - \tfrac{1}{2}\left[A_0\int_{-1}^{z}\mathrm{s}(A_0)\,\mathrm{d}z\right]_z. \tag{A 4d}$$

Equation (7.16) readily follows from (A 3), (A 4).

Using (7.17), (7.15), and (A 3) one obtains from (7.12) the following expression for the velocity $\hat{U}_1$:

$$\hat{U}_1 = A_1\mathrm{e}^{-\mathrm{i}t} + \mathrm{i}M^{(0)} + \tfrac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}t}M^{(1)} - \mathrm{i}\mathrm{e}^{-2\mathrm{i}t}M^{(2)}, \tag{A 5}$$

where A_1 is a still unknown amplitude depending on the coordinates and slow times.

Appendix B. Some details of the second-order solution

To obtain (8.6) we represent $N_u^{(1)}$, $N_v^{(1)}$ in the form

$$N_u^{(1)} = 2(u_0u_1)_x + (u_0v_1 + u_1v_0)_y + (w_0u_1 + w_1u_0)_z - q(w_0u_0)_y, \tag{B 1a}$$

$$N_v^{(1)} = (u_0v_1 + u_1v_0)_x + 2(v_0v_1)_y + (w_0v_1 + w_1v_0)_z - q(w_0v_0)_y, \tag{B 1b}$$

whence using (6.2*c*), (7.3*c*) we have:

$$\partial_x\bar{N}_v^{(1)} - \partial_y\bar{N}_u^{(1)} = 2(\overline{v_0v_1} - \overline{u_0u_1})_{xy} + (\partial_{xx} - \partial_{yy})(\overline{u_0v_1} + \overline{u_1v_0}) + q\left[(\overline{u_0w_0})_y - (\overline{v_0w_0})_x\right]_y, \tag{B 2}$$

and

$$\left\langle\partial_x\bar{N}_v^{(1)} - \partial_y\bar{N}_u^{(1)}\right\rangle = \mathrm{J}(\bar{\psi}_0, \langle\bar{\zeta}_1\rangle) + \mathrm{J}(\bar{\psi}_1, \bar{\zeta}_0) + 2\left\langle\overline{\hat{v}_0\hat{v}_1} - \overline{\hat{u}_0\hat{u}_1}\right\rangle_{xy} + (\partial_{xx} - \partial_{yy})\left\langle\overline{\hat{u}_0\hat{v}_1} + \overline{\hat{u}_1\hat{v}_0}\right\rangle + q\left[(\overline{\hat{u}_0\hat{w}_0})_y - (\overline{\hat{v}_0\hat{w}_0})_x\right]_y. \tag{B 3}$$

From (6.3*a,b*), (7.4*a*) one obtains

$$\hat{v}_0 = \hat{u}_{0t}, \quad \hat{u}_0 = -\hat{v}_{0t}, \quad \hat{v}_1 = \hat{u}_{1t} + \hat{u}_{0T_1} + \hat{N}_u^{(0)} + q\hat{w}_0 + \hat{p}_{1x}, \tag{B 4a–c}$$

therefore

$$\left\langle \hat{v}_0\hat{v}_1 - \hat{u}_0\hat{u}_1 \right\rangle = \left\langle \hat{v}_0(\hat{u}_{0T_1} + \hat{N}_u^{(0)} + q\hat{w}_0 + \hat{p}_{1x}) \right\rangle, \tag{B 5a}$$

$$\left\langle \hat{u}_0\hat{v}_1 + \hat{u}_1\hat{v}_0 \right\rangle = \left\langle \hat{u}_0(\hat{u}_{0T_1} + \hat{N}_u^{(0)} + q\hat{w}_0 + \hat{p}_{1x}) \right\rangle. \tag{B 5b}$$

It follows from (6.5*b,c*) that:

$$\left\langle \hat{v}_0\hat{u}_{0T_1} \right\rangle = \tfrac{1}{4}\mathrm{i}A_0^*A_{0T_1} + \mathrm{c.c.}, \quad \left\langle \hat{u}_0\hat{u}_{0T_1} \right\rangle = \tfrac{1}{4}A_0^*A_{0T_1} + \mathrm{c.c.} \tag{B 6a,b}$$

Using (6.5*b,c*), (A 3), (A 4) one obtains:

$$\left\langle \hat{v}_0\hat{N}_u^{(0)} \right\rangle = \tfrac{1}{4}\mathrm{i}A_0^* \left[\mathrm{J}(\bar{\psi}_0, A_0) + \tfrac{1}{2}\mathrm{i}\nabla_h^2\bar{\psi}_0A_0 - \tfrac{1}{2}\mathrm{L}(\bar{\psi}_0)A_0\right] + \mathrm{c.c.}, \tag{B 6c}$$

$$\left\langle \hat{u}_0\hat{N}_u^{(0)} \right\rangle = \frac{1}{4}A_0^* \left[\mathrm{J}(\bar{\psi}_0, A_0) + \tfrac{1}{2}\mathrm{i}\nabla_h^2\bar{\psi}_0A_0 - \tfrac{1}{2}\mathrm{L}(\bar{\psi}_0)A_0\right] + \mathrm{c.c.} \tag{B 6d}$$

From (6.5*b,c*), (7.14*b*), (7.2) we find:

$$\left\langle \hat{v}_0\hat{w}_0 \right\rangle = -\frac{\mathrm{i}}{4}A_0^* \left[\int_{-1}^{z} \mathrm{s}(A_0)\mathrm{d}z + \int_{-1}^{0} z\mathrm{s}(A_0)\,\mathrm{d}z\right] + \mathrm{c.c.}, \tag{B 6e}$$

$$\left\langle \hat{u}_0\hat{w}_0 \right\rangle = -\frac{1}{4}A_0^* \left[\int_{-1}^{z} \mathrm{s}(A_0)\,\mathrm{d}z + \int_{-1}^{0} z\mathrm{s}(A_0)\,\mathrm{d}z\right] + \mathrm{c.c.}; \tag{B 6f}$$

$$\left\langle (\hat{u}_0\hat{w}_0)_y - (\hat{v}_0\hat{w}_0)_x \right\rangle = -\frac{\mathrm{i}}{4}\mathrm{s}\left[A_0\left(\int_{-1}^{z} \mathrm{s}^*(A_0^*)\,\mathrm{d}z + \int_{-1}^{0} z\mathrm{s}^*(A_0^*)\,\mathrm{d}z\right)\right] + \mathrm{c.c.} \tag{B 6g}$$

$$\left\langle \hat{v}_0\hat{p}_{1x} \right\rangle = \frac{\mathrm{i}}{4}qA_0^*\left(\int_{-1}^{z} A_{0x}\,\mathrm{d}z + \int_{-1}^{0} zA_{0x}\,\mathrm{d}z\right) + \mathrm{c.c.}, \tag{B 6h}$$

$$\left\langle \hat{u}_0\hat{p}_{1x} \right\rangle = \frac{q}{4}A_0^*\left(\int_{-1}^{z} A_{0x}\,\mathrm{d}z + \int_{-1}^{0} zA_{0x}\,\mathrm{d}z\right) + \mathrm{c.c.} \tag{B 6k}$$

Using (B 5) and (B 6) and taking into account (7.17) one obtains:

$$\left\langle \hat{v}_0\hat{v}_1 - \hat{u}_0\hat{u}_1 \right\rangle = \tfrac{1}{4}(\bar{\psi}_{0xx} - \bar{\psi}_{0yy})\,|A_0|^2, \tag{B 7a}$$

$$\left\langle \hat{u}_0\hat{v}_1 + \hat{u}_1\hat{v}_0 \right\rangle = -\tfrac{1}{2}\bar{\psi}_{0xy}\,|A_0|^2. \tag{B 7b}$$

Equation (8.6) follow from (B 3), (B 7) and (B 6*g*).

REFERENCES

AREF, H. 1984 Stirring by chaotic advection. *J. Fluid Mech.* **143**, 1–21.

BALMFORTH, N. J., LLEWELLYN SMITH, S. G. & YOUNG, W. R. 1998 Enhanced dispersion of near-inertial waves in an idealized geostrophic flow. *J. Mar. Res.* **56**, 1–40.

BALMFORTH, N. J. & YOUNG, W. R. 1999 Radiative damping of near-inertial oscillations in the mixed layer. *J. Mar. Res.* **57**, 561–584.

BREKHOVSKIKH, L. M. & GONCHAROV, V. 1994 In *Mechanics of Continua and Wave Dynamics* Springer.

DAVIDSON, P. A., STAPLEHURST, P. J. & DALZIEL, S. B. 2006 On the evolution of eddies in a rapidly rotating system. *J. Fluid Mech.* **557**, 135–144.

GERKEMA, T. & SHRIRA, V. I. 2005 Near-inertial waves in the ocean: beyond the 'traditional approximation'. *J. Fluid Mech.* **529**, 195–219.

GERKEMA, T., ZIMMERMAN, J. T. F., MAAS, L. R. M. & VAN HAREN, H. 2008 Geophysical and astrophysical fluid dynamics beyond the traditional approximation. *Rev. Geophys.* **46**, RG2004.

GREENSPAN, H. P. 1968 *Theory of Rotating Fluids*. Cambridge University Press.

GRIMSHAW, R. H. J. 1975 A note on the β-plane approximation. *Tellus* **27**, 351–357.

VAN HAREN, H. & MILLOT, C. 2005 Gyroscopic waves in the Mediterranean Sea. *Geophys. Res. Lett.* **32**, L24614.

KAMENKOVICH, V. M. 1977 *Fundamentals of Ocean Dynamics*. Elsevier.

KASAHARA, A. 2003 The roles of the horizontal component of the Earth's angular velocity in nonhydrostatic linear models. *J. Atmos. Sci.* **60**, 1085–1095.

KLEIN, P. & LLEWELLYN-SMITH, S. 2001 Horizontal dispersion of near-inertial oscillations in a turbulent mesoscale eddy field. *J. Mar. Res.* **59**, 697–723.

KLEIN, P., LLEWELLYN-SMITH, S. & LAPEYRE, G. 2004 Organization of near-inertial energy by an eddy field. *Q. J. R. Meteorol. Soc.* **130**, 1153–1166.

KUNZE, E. 1985 Near-inertial wave propagation in geostrophic shear. *J. Phys. Oceanogr.* **15**, 544–565.

LEBLOND, P. H. & MYSAK, L. A. 1978 *Waves in the Ocean*. Elsevier.

MIROPOL'SKY, Y. Z. 2001 *Dynamics of Internal Gravity Waves in the Ocean*. Kluwer.

REZNIK, G. M. 2013*a* Linear dynamics of a stably-neutrally stratified ocean. *J. Mar. Res.* **71**, 253–288.

REZNIK, G. M. 2013*b* Geostrophic adjustment with gyroscopic waves: stably-neutrally stratified fluid without traditional approximation. *J. Fluid Mech.* (submitted).

REZNIK, G. M., ZEITLIN, V. & BEN JELLOUL, M. 2001 Nonlinear theory of geostrophic adjustment. Part 1. Rotating shallow-water model. *J. Fluid Mech.* **445**, 93–120.

STAPLEHURST, P. J., DAVIDSON, P. A. & DALZIEL, S. B. 2008 Structure formation in homogeneous freely decaying turbulence. *J. Fluid Mech.* **598**, 81–105.

TIMMERMANS, M. L., MELLING, H. & RAINVILLE, L. 2007 Dynamics in the Deep Canada Basin, Arctic Ocean, inferred by thermistor chain time series. *J. Phys. Oceanogr.* **37**, 1066–1076.

WINGATE, B. A., EMBID, P., HOLMES-CERFON, M. & TAYLOR, M. A. 2011 Low Rossby limiting dynamics for stably stratified flow with finite Froude number. *J. Fluid Mech.* **676**, 546–571.

YOUNG, W. R. & BEN JELLOUL, M. 1997 Propagation of near-inertial oscillations through a geostrophic flow. *J. Mar. Res.* **55**, 735–766.

ZEITLIN, V., REZNIK, G. M. & BEN JELLOUL, M. 2003 Nonlinear theory of geostrophic adjustment. Part 2. Two-layer and continuously stratified primitive equations. *J. Fluid Mech.* **491**, 207–228.