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Geography of simply connected nonspin symplectic 4-manifolds with positive signature. II

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Abstract. Building upon our earlier work with M. C. Hughes, we construct many new smooth structures on closed simply connected nonspin 4-manifolds with positive signature. We also provide numerical and asymptotic upper bounds on the function $\lambda(\sigma)$ that was defined in our earlier work.

1 Introduction

This is a companion paper to our earlier work [1] with M. C. Hughes and addresses the geography problem for closed simply connected nonspin symplectic 4-manifolds with positive signature. For some background and history, we refer the reader to the introduction in [1]. For the corresponding *spin* geography problem, we refer the reader to our papers [3, 4].

We start by setting up some basic notation. Given a closed smooth 4-manifold M, let $e(M)$ and $\sigma(M)$ denote the Euler characteristic and the signature of M, respectively. We define $\chi_h(M) = \frac{1}{4}(e(M) + \sigma(M))$ and $c_1^2(M) = 2e(M) + 3\sigma(M)$. When M is a complex surface, $\chi_h(M)$ is the holomorphic Euler characteristic of M, while $c_1^2(M)$ is the square of the first Chern class of M. Given an ordered pair of integers (a, b), the geography problem asks whether there exists a closed smooth 4-manifold M with the desired properties satisfying $\chi_h(M) = a$ and $c_1^2(M) = b$. We note that such M must satisfy $b = 8a + \sigma(M)$.

Given $x \in \mathbb{R}$, we define the ceiling function as

$$\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}. \tag{1.1}$$

Next we recall the following definition from [1, Definition 13].

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Definition 1.1 Given an integer $\sigma \geq 0$, let $\lambda(\sigma)$ be the smallest positive integer with the following properties.

(i) $\lambda(\sigma) \geq \lceil (\sigma+1)/2 \rceil$.

(ii) Every integral point (a, b) on the line $b = 8a + \sigma$ satisfying $a \geq \lambda(\sigma)$ is realized as $(\chi_h(M_i), c_1^2(M_i))$, where $\{M_i \mid i \in \mathbb{Z}\}$ is an infinite family of homeomorphic but pairwise nondiffeomorphic closed simply connected nonspin irreducible 4-manifolds such that M_i is symplectic for each $i \geq 0$ and M_i is nonsymplectic for each $i < 0$.

We also recall the following definition from [3, Definition 1].

Definition 1.2 We say that a 4-manifold M has ∞^2-*property* if there exist infinitely many pairwise nondiffeomorphic irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to M.

Let $\mathbb{CP}^2$ be the complex projective plane, and let $\overline{\mathbb{CP}}^2$ be the underlying smooth 4-manifold $\mathbb{CP}^2$ equipped with the opposite orientation. By Freedman's classification theorem (*cf.* [11]), if k is any odd integer satisfying $k \geq 2\lambda(\sigma) - 1$, then the nonspin 4-manifold $k\mathbb{CP}^2\#(k-\sigma)\overline{\mathbb{CP}}^2$, the connected sum of k copies of $\mathbb{CP}^2$, and $k - \sigma$ copies of $\overline{\mathbb{CP}}^2$, have ∞^2-property. The following conjecture from [1] remains open.

Conjecture 1.3 $\lambda(\sigma) = \lceil (\sigma+1)/2 \rceil$ for every integer $\sigma \geq 0$. Equivalently, given any integer $\sigma \geq 0$, $k\mathbb{CP}^2\#(k-\sigma)\overline{\mathbb{CP}}^2$ has ∞^2-property for every odd integer k satisfying

$$k \geq \begin{cases} \sigma & \textit{when } \sigma \textit{ is odd,} \\ \sigma + 1 & \textit{when } \sigma \textit{ is even.} \end{cases}$$

We note that Conjecture 1.3 postulates that there would be no constraint on $k\mathbb{CP}^2\#(k-\sigma)\overline{\mathbb{CP}}^2$ having ∞^2-property other than the positive integer k being odd, which is necessary for $k\mathbb{CP}^2\#(k-\sigma)\overline{\mathbb{CP}}^2$ to support a symplectic (and hence an almost complex) structure.

In [1, 2, 5], numerical upper bounds for $\lambda(\sigma)$ were given when $0 \leq \sigma \leq 100$. In Section 3, we will present a new algorithm for constructing simply connected 4-manifolds starting from a surface fibration over a surface with a section, which need not be a fiber bundle nor a Lefschetz fibration. Using this algorithm, we will construct two new infinite families of closed, simply connected, nonspin, irreducible, symplectic 4-manifolds of positive signature, many of which have a smaller value of χ_h than the currently known upper bounds on $\lambda(\sigma)$. We cannot currently show that all of these 4-manifolds have ∞^2-property, but we suspect that they all do (see Remark 3.4 and Corollary 3.6). The new building blocks in our construction are certain complex surfaces of general type found in [6, 8, 16], and these will be reviewed in Section 2. In Section 4, we will also provide two explicit formulae for upper bounds on $\lambda(\sigma)$ that work for every nonnegative integer σ (see Corollaries 4.2 and 4.4). Asymptotically as $\sigma \to \infty$, we will prove that

$$\lambda(\sigma) \le \frac{8}{5}\sigma + O(\sigma^{1/2}). \tag{1.2}$$

Such an asymptotic upper bound has been missing in the literature, and we hope that our bound provides a useful benchmark for future works. Our ultimate goal is to decrease the coefficient of σ in (1.2) from 1.6 to a smaller number that is much closer to the coefficient 0.5 in Conjecture 1.3.

2 Building Blocks

In this section, we will collect all the 4-manifold building blocks that we will need for our constructions later. Our first family of building blocks are the so-called BCD surfaces constructed by Bauer, Catanese, and Dettweiler in [6, 8].

Lemma 2.1 *For each positive integer $n \ge 5$ that is coprime with 6, there exists a minimal complex surface $S(n)$ of general type with $c_1^2(S(n)) = 5(n-2)^2$, $e(S(n)) = 2n^2 - 10n + 15$, and $\sigma(S(n)) = (n^2 - 10)/3$. Each $S(n)$ admits a genus $n-1$ fibration over a genus $(n-1)/2$ curve. Moreover, $S(n)$ also contains four disjoint genus $(n-1)/2$ curves of self-intersection -1, one of which is a section of the fibration, and each of the other three is contained in a singular fiber and hence disjoint from regular fibers.*

Proof Recall from [8] that $S(n)$ arises as a $(\mathbb{Z}/n\mathbb{Z})^2$ Abelian Galois ramified cover (in the sense of [18]) over a del Pezzo surface $\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$ of degree 5. The branch divisor of this covering is a sum of ten rational curves, four of which are the exceptional divisors of the blow-ups. We note that the preimages of the exceptional divisors under this $(\mathbb{Z}/n\mathbb{Z})^2$ covering map are disjoint genus $(n-1)/2$ curves of self-intersection -1. The genus $n-1$ fibration structure on $S(n)$ and its singular fibers are discussed in [8, Proposition 4.2]. We recall that this fibration is obtained by lifting a pencil of lines going through a point of blow-up, and thus a section of the fibration is given by the inclusion of the preimage of the corresponding exceptional divisor. The characteristic numbers $c_1^2(S(n))$ and $e(S(n))$ were computed in [8, Proposition 4.3]. We can readily compute the signature of $S(n)$ using the well-known formula $c_1^2 = 2e + 3\sigma$. ■

Let Σ_b denote a closed connected 2-manifold with genus $b \ge 0$. Our second building block is a Σ_7 bundle over Σ_5 that was constructed in [16].

Lemma 2.2 *There exists a minimal complex surface Y of general type with $e(Y) = 96$ and $\sigma(Y) = 16$ such that Y is the total space of a surface bundle over a surface with base genus 5 and fiber genus 7. Moreover, this surface bundle admits a section whose image in Y has self-intersection -8.*

Proof In [16, Example 6.9], such Y was constructed as the double cover of $\Sigma_3 \times \Sigma_3$ branched over 4 disjoint graphs of involutions on Σ_3. Each graph in the branch locus gives rise to a section of the bundle whose image in Y has self-intersection equal to 2 times the self-intersection of the graph in $\Sigma_3 \times \Sigma_3$, which is -4. ■

Our next family of building blocks are the homotopy elliptic surfaces constructed by Fintushel and Stern in [9]. Let $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ denote a rational elliptic surface that is the complex projective plane blown up nine times. For a positive integer r, let $E(r)$ denote the fiber sum of r copies of $E(1)$. Then $E(r)$ is a simply connected elliptic surface without any multiple fiber. Let F be a smooth torus fiber of $E(r)$ and let K be a knot of genus $g(K)$ in S^3. Let $E(r)_K$ denote the result of performing a knot surgery on $E(r)$ along F:

$$E(r)_K = \big[E(r)\backslash \nu(F)\big] \cup \big[S^1 \times (S^3 \backslash \nu(K))\big], \tag{2.1}$$

where the ν's denote tubular neighborhoods. In (2.1), we glue the 3-torus boundaries in such a way that the meridians of F get identified with the longitudes of K.

We recall that $E(r)_K$ is homeomorphic to $E(r)$, so we have $\pi_1(E(r)_K) = 1$,

$$e(E(r)_K) = e(E(r)) = 12r \quad \text{and} \quad \sigma(E(r)_K) = \sigma(E(r)) = -8r.$$

We also recall that $E(r)$ and $E(r)_K$ are spin if and only if r is even. If K is a fibered knot, then $E(r)_K$ admits a symplectic structure, and a sphere section of $E(r)$ and a Seifert surface of K can be glued together to form a symplectic submanifold Σ_K of genus $g(K)$ and self-intersection $-r$ inside $E(r)_K$. Given a nonnegative integer m, let F_K^m be the genus $g(K) + m$ symplectic submanifold of $E(r)_K$ with self-intersection $2m - r$ that is obtained from the union of Σ_K and m copies of torus fiber by symplectically resolving their m intersection points. We note that $F_K^0 = \Sigma_K$.

Lemma 2.3 *Let $m \geq 0$ and $r > 0$ be integers, and let K be a fibered knot in S^3. Let $\nu(F_K^m)$ denote a tubular neighborhood of F_K^m in $E(r)_K$. Then the complement $E(r)_K \backslash \nu(F_K^m)$ is simply connected. If $r \geq 2$, then write $r = 2\rho + \varepsilon$ for integers $\varepsilon = 0, 1$ and $\rho \geq 1$. Then $E(r)_K \backslash \nu(F_K^m)$ contains 2ρ disjoint symplectic tori $T_j (j = 1, \ldots, 2\rho)$ of self-intersection 0 such that $\pi_1(E(r)_K \backslash (\nu(F_K^m) \cup (\cup_{j=1}^{2\rho} T_j))) = 1$.*

Proof Each surface F_K^m transversely intersects once a topological sphere in $E(r)_K$ coming from a cusp fiber of $E(r)$. Thus, any meridian of F_K^m is nullhomotopic in $E(r)_K \backslash \nu(F_K^m)$. Hence, we conclude that $\pi_1(E(r)_K \backslash \nu(F_K^m)) = \pi_1(E(r)_K) = 1$. Next, we recall from [13] that $E(2)$ contains 3 disjoint copies of the Gompf nucleus. If $r \geq 2$, then $E(r)$ can be viewed as the fiber sum of ρ copies of $E(2)$ and possibly a copy of $E(1)$. In each copy of $E(2)$, we have 2 copies of Gompf nuclei that are disjoint from the tori and sections used in the fiber sum, and thus $E(r)_K$ contains 2ρ Gompf nuclei that are all disjoint from $\nu(F_K^m)$. Let N_j denote one of these nuclei, and let T_j be a smooth torus fiber in N_j $(j = 1, \ldots, 2\rho)$. By changing the symplectic form on the $E(2)$ parts if necessary, we can arrange each T_j to be a symplectic submanifold of $E(r)_K$. Since T_j transversely intersects once a sphere section of N_j with self-intersection -2, every meridian of T_j is nullhomotopic. It follows that $\pi_1(E(r)_K \backslash (\nu(F_K^m) \cup (\cup_{j=1}^{2\rho} T_j))) = \pi_1(E(r)_K \backslash \nu(F_K^m)) = 1$. ∎

From the Seifert–Van Kampen theorem, we can also deduce that

$$\pi_1(E(r)_K \backslash (\nu(F_K^m) \cup (\cup_{j=1}^{\tau} T_j))) = 1$$

for any integer τ satisfying $0 \le \tau \le 2\rho$, *i.e.*, we can choose to take out less tori and still have the complement remain simply connected. Our final set of building blocks are certain families of symplectic 4-manifolds that were studied in [1].

Lemma 2.4 *For each positive integer u, there is a pair of closed simply connected nonspin irreducible symplectic 4-manifolds $Q(u)$ and $\tilde{Q}(u)$ satisfying*

$$\begin{aligned}\sigma(Q(u)) &= 26u - 2\lceil u/2\rceil - 2,\\ \chi_h(Q(u)) &= 27u + 32\lceil u/2\rceil - 2,\\ \sigma(\tilde{Q}(u)) &= 25u^2 + u - 2\lceil u/2\rceil - 2,\\ \chi_h(\tilde{Q}(u)) &= 25u^2 + \lceil u/2\rceil(30u+2) + 2u - 2,\end{aligned}$$

where $\lceil\ \rceil$ is given by (1.1). Let Q denote either $Q(u)$ or $\tilde{Q}(u)$. Then each Q contains a disjoint pair of symplectic tori T_1' and T_2' of self-intersection 0 satisfying $\pi_1(Q\backslash(T_1' \cup T_2')) = 1$.

Proof We let $Q(u) = Q_n^m(W_{u_1,u_2}^{p,\nu})$ in [1, Example 12] with $m = 1$, $p = 5$, $u_1 = u$, $u_2 = 1$, $\nu = 1$, $t = \lceil u/2\rceil$, and $n = 16\lceil u/2\rceil + u + 1 \ge 18$. We let $\tilde{Q}(u) = Q_n^m(W_{u_1,u_2}^{p,\nu})$ in [1, Example 12] with $m = 1$, $p = 5$, $u_1 = u$, $u_2 = u$, $\nu = 1$, $t = \lceil u/2\rceil$, and $n = \lceil u/2\rceil(15u + 1) + u + 1 \ge 18$. The existence of T_1' and T_2' follows from [1, Theorem 9]. ■

3 New Symplectic 4-manifolds

We start the section with a general algorithm for producing simply connected 4-manifolds from a symplectic fibration. Let X be a closed symplectic 4-manifold that is the total space of a fibration $f : X \to \Sigma_b$ whose regular fiber is a 2-manifold Σ_a with genus $a \ge 0$. Assume that this fibration has a section $s : \Sigma_b \to X$ whose image $s(\Sigma_b)$ has self-intersection equal to d in X. Next let t and δ be nonnegative integers. By symplectically resolving the double points of the union of $s(\Sigma_b)$ and t copies of the fiber Σ_a, we obtain a symplectic submanifold Σ_{ta+b} in X with genus $ta + b$ and self-intersection $2t + d$. By symplectically blowing up δ points of Σ_{ta+b} in X, we obtain a genus $ta + b$ symplectic submanifold Σ'_{ta+b} in the blow-up $X\#\delta\overline{\mathbb{CP}}^2$ with self-intersection $2t + d - \delta$.

Let $E(r)_K$ and F_K^m be as in Section 2. Let $E(X)_{K,m,r}^{t,\delta}$ denote the symplectic normal sum (*cf.* [12, 17]) of $X\#\delta\overline{\mathbb{CP}}^2$ and $E(r)_K$ along symplectic submanifolds Σ'_{ta+b} and F_K^m:

$$E(X)_{K,m,r}^{t,\delta} = \big[(X\#\delta\overline{\mathbb{CP}}^2)\backslash\nu(\Sigma'_{ta+b})\big] \cup \big[E(r)_K\backslash\nu(F_K^m)\big].$$

For this symplectic normal sum to be well-defined, we require the genera of submanifolds to be equal and their self-intersections to have opposite signs, *i.e.*,

$$ta + b = g(K) + m \quad \text{and} \quad 2t + d - \delta = -(2m - r). \tag{3.1}$$

Theorem 3.1 *Assume that both conditions in (3.1) hold. Then $E(X)_{K,m,r}^{t,\delta}$ is a closed symplectic 4-manifold with*

$$e(E(X)_{K,m,r}^{t,\delta}) = e(X) + \delta + 12r + 4ta + 4b - 4,$$

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$$\sigma(E(X)^{t,\delta}_{K,m,r}) = \sigma(X) - \delta - 8r,$$
$$\chi_h(E(X)^{t,\delta}_{K,m,r}) = \chi_h(X) + r + ta + b - 1.$$

If $\delta > 0$, then $E(X)^{t,\delta}_{K,m,r}$ is nonspin. If $t > 0$, then $E(X)^{t,\delta}_{K,m,r}$ is simply connected. If $t > 0$ and $r \geq 2$, then $E(X)^{t,\delta}_{K,m,r}$ contains two disjoint symplectic tori, T_1 and T_2, of self-intersection 0 such that $\pi_1(E(X)^{t,\delta}_{K,m,r} \backslash (T_1 \cup T_2)) = 1$.

Proof We compute that $e(E(X)^{t,\delta}_{K,m,r}) = e(X\#\delta\overline{\mathbb{CP}}^2) + e(E(r)_K) - 2e(\Sigma'_{ta+b})$ and $\sigma(E(X)^{t,\delta}_{K,m,r}) = \sigma(X\#\delta\overline{\mathbb{CP}}^2) + \sigma(E(r)_K)$. When $\delta > 0$, we have a punctured 2-sphere in the $[(X\#\delta\overline{\mathbb{CP}}^2)\backslash\nu(\Sigma'_{ta+b})]$ half coming from an exceptional divisor of a blow-up. We can glue this disk to a punctured torus fiber in the $[E(r)_K\backslash\nu(F^m_K)]$ half and obtain a torus with self-intersection -1, which implies that the intersection form of $E(X)^{t,\delta}_{K,m,r}$ is not even.

Since we know from Lemma 2.3 that

$$\pi_1(E(r)_K\backslash\nu(F^m_K)) = 1, \tag{3.2}$$

the Seifert–Van Kampen theorem implies that

$$\pi_1(E(X)^{t,\delta}_{K,m,r}) \cong \frac{\pi_1((X\#\delta\overline{\mathbb{CP}}^2)\backslash\nu(\Sigma'_{ta+b}))}{\langle\pi_1(\partial\nu(\Sigma'_{ta+b}))\rangle}, \tag{3.3}$$

where $\partial\nu(\Sigma'_{ta+b})$ is the boundary of $\nu(\Sigma'_{ta+b})$ and $\langle\pi_1(\partial\nu(\Sigma'_{ta+b}))\rangle$ is the normal subgroup of $\pi_1((X\#\delta\overline{\mathbb{CP}}^2)\backslash\nu(\Sigma'_{ta+b}))$ generated by the image of $\pi_1(\partial\nu(\Sigma'_{ta+b}))$ under the inclusion induced homomorphism.

Note that $\partial\nu(\Sigma'_{ta+b})$ is a circle bundle over Σ'_{ta+b} with Euler number $2t + d - \delta$. It is well known (*cf.* [10, Proposition 10.4]) that

$$\pi_1(\partial\nu(\Sigma'_{ta+b})) = \Big\langle \alpha_i, \beta_i, \mu \,\Big|\, \prod_{i=1}^{ta+b}[\alpha_i, \beta_i] = \mu^{2t+d-\delta},\ \alpha_i\mu\alpha_i^{-1} = \mu,\ \beta_i\mu\beta_i^{-1} = \mu \Big\rangle,$$

where the index i ranges over $1, \dots, ta + b$. Here, μ is represented by a fiber circle that is a meridian of Σ'_{ta+b}, and α_i, β_i are the parallel push-offs of the standard generators of $\pi_1(\Sigma'_{ta+b})$.

In (3.3), we have $\mu = 1$ in the quotient group, since $\mu \in \langle\pi_1(\partial\nu(\Sigma'_{ta+b}))\rangle$. Thus, we can write

$$\pi_1(E(X)^{t,\delta}_{K,m,r}) \cong \frac{\pi_1(X\#\delta\overline{\mathbb{CP}}^2)}{\langle\pi_1(\Sigma'_{ta+b})\rangle} \cong \frac{\pi_1(X)}{\langle\pi_1(\Sigma_{ta+b})\rangle}. \tag{3.4}$$

From the long exact sequence of the fibration, we have an exact sequence

$$\pi_1(\Sigma_a) \longrightarrow \pi_1(X) \longrightarrow \pi_1(\Sigma_b) \longrightarrow 1,$$

where the first and second arrows are induced by the inclusion of a regular fiber and the fibration map, respectively. When $t > 0$, the image of $\pi_1(\Sigma_{ta+b})$ in $\pi_1(X)$ contains all the generators of the images of $\pi_1(\Sigma_a)$ and $\pi_1(s(\Sigma_b))$ under the inclusion induced homomorphisms. Thus, we can conclude that the quotient group (3.4) is trivial.

When $r \geq 2$, Lemma 2.3 tells us that the $[E(r)_K \backslash \nu(F_K^m)]$ half contains (at least) two disjoint symplectic tori T_1 and T_2 of self-intersection 0 such that

$$\pi_1([E(r)_K \backslash \nu(F_K^m)] \backslash (T_1 \cup T_2)) = 1. \tag{3.5}$$

To show that $\pi_1(E(X)_{K,m,r}^{t,\delta} \backslash (T_1 \cup T_2)) = 1$, we can apply the above argument to show that $\pi_1(E(X)_{K,m,r}^{t,\delta}) = 1$ with the only change being the replacement of (3.2) with (3.5). ■

Next we apply Theorem 3.1 to the BCD surface $S(n)$ from Lemma 2.1, now viewed as a symplectic 4-manifold.

Corollary 3.2 *For any positive integer $n \geq 5$ such that $n \equiv \pm 1 \pmod 6$ and any fibered knot $K \subset S^3$ of genus $3(n-1)/2$, there is a simply connected irreducible symplectic 4-manifold $M(n)_K$ that is homeomorphic to*

$$\left(\frac{7}{6}n^2 - 2n + \frac{11}{6}\right)\mathbb{CP}^2 \# \left(\frac{5}{6}n^2 - 2n + \frac{79}{6}\right)\overline{\mathbb{CP}}^2. \tag{3.6}$$

Proof An integer n is coprime with 6 if and only if $n \equiv \pm 1 \pmod 6$. We let $M(n)_K = E(X)_{K,m,r}^{t,\delta}$ with $X = S(n)$, $a = n-1$, $b = (n-1)/2$, $d = -1$, $t = 1$, $\delta = 0$, $g(K) = 3(n-1)/2$, $m = 0$, and $r = 1$. We can easily check that both conditions in (3.1) are satisfied. We note that $M(n)_K$ is nonspin, since it contains three curves of square -1 in the $[(X \# \delta \overline{\mathbb{CP}}^2) \backslash \nu(\Sigma'_{ta+b})]$ half by Lemma 2.1.

Since $e(M(n)_K) = 2n^2 - 4n + 17$ and $\sigma(M(n)_K) = (n^2 - 34)/3$, Freedman's classification theorem (*cf.* [11]) implies that $M(n)_K$ must be homeomorphic to (3.6). Since $S(n)$ is minimal, the symplectic normal sum $M(n)_K$ is also minimal by Usher's theorem in [19]. We recall from [14, 15] that any simply connected minimal symplectic 4-manifold is irreducible. ■

We note that $M(n)_K$ has positive signature except when $n = 5$. For many values of n, Corollary 3.2 gives a new symplectic (and thus exotic) smooth structure on (3.6). For example, when $n = 7, 11, 13, 17$, we get an exotic smooth structure on each of $45\mathbb{CP}^2\#40\overline{\mathbb{CP}}^2$, $121\mathbb{CP}^2\#92\overline{\mathbb{CP}}^2$, $173\mathbb{CP}^2\#128\overline{\mathbb{CP}}^2$, and $305\mathbb{CP}^2\#220\overline{\mathbb{CP}}^2$. These 4-manifolds have signature equal to 5, 29, 45, and 85, and χ_h equal to 23, 61, 87, and 153, respectively. For comparison, we showed in [1, Table 2] that $\lambda(5) \leq 47$, $\lambda(29) \leq 87$, $\lambda(45) \leq 85$ and $\lambda(85) \leq 166$. Thus these exotic smooth structures are new solutions to the symplectic geography problem when $n = 7, 11, 17$ as far as we know.

Similarly, we can apply Theorem 3.1 to the surface bundle Y from Lemma 2.2 and obtain the following corollary.

Corollary 3.3 *For any fibered knot $K \subset S^3$ of genus 8, there is a simply connected irreducible symplectic 4-manifold Z_K that is homeomorphic to $79\mathbb{CP}^2\#72\overline{\mathbb{CP}}^2$.*

Proof We let $Z_K = E(X)_{K,m,r}^{t,\delta}$ with $X = Y$, $a = 7$, $b = 5$, $d = -8$, $t = 1$, $\delta = 1$, $g(K) = 8$, $m = 4$, and $r = 1$. We have $e(Z_K) = 153$ and $\sigma(Z_K) = 7$. The rest of the proof is similar to that of Corollary 3.2 and is left to the reader. ■

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In [1], we showed that $\lambda(7) \leq 49$. Since $\chi_h(Z_K) = 40$, the symplectic smooth structure in Corollary 3.3 is new.

Remark 3.4 It is well known (*cf.* [7]) that for a fixed genus $g > 1$, there are infinitely many genus g fibered (and nonfibered) knots that are distinguished by their Alexander polynomials. By varying the knot K while fixing the genus $g(K)$, we expect the resulting collection of $M(n)_K$'s and Z_K's to provide infinitely many distinct smooth structures on (3.6) and $79\mathbb{CP}^2\#72\overline{\mathbb{CP}}^2$. At present, it is not clear to us how to compute the Seiberg–Witten invariants of these 4-manifolds completely so as to distinguish their smooth structures.

We end this section by constructing another family of simply connected irreducible symplectic 4-manifolds.

Corollary 3.5 *For any positive integer $n \geq 5$ such that $n \equiv \pm 1 \pmod 6$ and any fibered knot $K \subset S^3$ of genus $\frac{3}{2}(n-1)-1$, there is a simply connected irreducible symplectic 4-manifold $X(n)_K$ that is homeomorphic to*

$$\left(\frac{7}{6}n^2 - 2n + \frac{23}{6}\right)\mathbb{CP}^2\#\left(\frac{5}{6}n^2 - 2n + \frac{145}{6}\right)\overline{\mathbb{CP}}^2. \tag{3.7}$$

Moreover, each $X(n)_K$ contains two disjoint symplectic tori, T_1 and T_2, of self-intersection 0 such that $\pi_1(X(n)_K \backslash (T_1 \cup T_2)) = 1$.

Proof We let $X(n)_K = E(X)^{t,\delta}_{K,m,r}$ with $X = S(n)$, $a = n-1$, $b = (n-1)/2$, $d = -1$, $t = 1$, $\delta = 1$, $g(K) = \frac{3}{2}(n-1)-1$, $m = 1$, and $r = 2$. We have $e(X(n)_K) = 2n^2 - 4n + 30$ and $\sigma(X(n)_K) = \frac{1}{3}(n^2 - 61)$. The rest of the proof is similar to the proof of Corollary 3.2 and is left to the reader. ∎

We note that the signature of $X(n)_K$ is positive when $n \geq 11$. By performing knot surgeries along T_1 (and/or T_2) on $X(n)_K$, we can obtain infinitely many distinct smooth structures on (3.7).

Corollary 3.6 *For any positive integer $n \geq 5$ such that $n \equiv \pm 1 \pmod 6$, the 4-manifold (3.7) in Corollary 3.5 has ∞^2-property (cf. Definition 1.2).*

Proof This follows immediately from [1, Theorem 16]. ∎

For example, when $n = 17$, we obtain infinitely many exotic smooth structures on $307\mathbb{CP}^2\#231\overline{\mathbb{CP}}^2$, which have signature equal to 76 and χ_h equal to 154. For comparison, we only showed in [1] that $\lambda(76) \leq 167$, so these exotic smooth structures are new (*cf.* Remark 4.5).

4 Upper Bounds on $\lambda(\sigma)$

The goal of this section is to exhibit concrete formulae for upper bounds on $\lambda(\sigma)$ that are valid for any nonnegative integer σ. First, we recall the following theorem, which was proved in [1, Corollary 17].

Theorem 4.1 *Let X be a closed, simply connected, nonspin, minimal, symplectic 4-manifold with $b_2^+(X) > 1$ and $\sigma(X) \geq 0$. Assume that X contains disjoint symplectic tori T_1 and T_2 of self-intersection 0 such that $\pi_1(X \backslash (T_1 \cup T_2)) = 1$. Suppose σ is a fixed integer satisfying $0 \leq \sigma \leq \sigma(X)$. If $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$ and if we define*

$$\ell(\sigma) = \left\lceil \frac{\sigma(X) - \sigma}{8} - 1 \right\rceil,$$

then

$$\lambda(\sigma) \leq \chi_h(X) + \ell(\sigma) + 1.$$

Now we apply Theorem 4.1 to the 4-manifolds $Q(u)$ in Lemma 2.4 and obtain the following corollary.

Corollary 4.2 *If $\lambda(\sigma)$ is as in Definition 1.1, then we have*

$$\lambda(\sigma) < \frac{43}{25}\sigma + \frac{6813}{100} = 1.72\sigma + 68.13. \tag{4.1}$$

Proof Given a nonnegative integer σ, let u be the smallest positive integer such that $\sigma \leq \sigma(Q(u)) = 26u - 2\lceil u/2 \rceil - 2$. It follows that when $u > 1$,

$$\sigma(Q(u)) - \sigma < \sigma(Q(u)) - \sigma(Q(u-1)) = 26 - 2\big(\lceil u/2 \rceil - \lceil (u-1)/2 \rceil\big) \leq 26, \tag{4.2}$$

since $\lceil u/2 \rceil - \lceil (u-1)/2 \rceil$ is either 0 or 1 depending on whether u is even or odd. Note that $\sigma(Q(1)) = 22$ so that we still have $\sigma(Q(u)) - \sigma < 26$ even when $u = 1$. Thus, we have $\ell(\sigma) < (\sigma(Q(u)) - \sigma)/8 < 13/4$. Since

$$\sigma > \sigma(Q(u)) - 26 = 26u - 2\lceil u/2 \rceil - 28$$

by (4.2) and $2\lceil u/2 \rceil$ is u or $u+1$ depending on whether u is even or odd, we conclude that $\sigma > 26u - (u+1) - 28 = 25u - 29$. Thus, we have $u < (\sigma + 29)/25$ and

$$\begin{aligned}\lambda(\sigma) &\leq \chi_h(Q(u)) + \ell(\sigma) + 1 < 27u + 32\left\lceil \frac{u}{2} \right\rceil - 2 + \frac{13}{4} + 1\\ &\leq 43u + \frac{73}{4} < \frac{43}{25}\sigma + \frac{6813}{100},\end{aligned}$$

since $32\lceil u/2 \rceil$ is $16u$ or $16u + 16$ depending on whether u is even or odd. ∎

Remark 4.3 For a specific value of σ, (4.1) may not provide the optimal bound procured from $Q(u)$. For example, when $\sigma = 76$, we can apply Theorem 4.1 to $Q(4)$ directly and obtain $\lambda(76) \leq 173$, which is better than the bound $\lambda(76) \leq 198$ coming from (4.1).

Similarly, we apply Theorem 4.1 to the 4-manifolds $\tilde{Q}(u)$ in Lemma 2.4 and obtain the following corollary.

Corollary 4.4 *If $\lambda(\sigma)$ is as in Definition 1.1, then we have*

$$\lambda(\sigma) < \frac{8}{5}\sigma + \frac{417}{20}\sqrt{\sigma+4} + \frac{1353}{20} = 1.6\sigma + 20.85\sqrt{\sigma+4} + 67.65. \tag{4.3}$$

Proof Given a nonnegative integer σ, let u be the smallest positive integer such that $\sigma \leq \sigma(\tilde{Q}(u)) = 25u^2 + u - 2\lceil u/2 \rceil - 2$. Note that

$$\sigma(\tilde{Q}(u)) - \sigma(\tilde{Q}(u-1)) = 50u - 24 - 2\big(\lceil u/2 \rceil - \lceil (u-1)/2 \rceil\big) \leq 50u - 24.$$

It follows that

$$\sigma(\tilde{Q}(u)) - \sigma < 50u - 24. \tag{4.4}$$

Note that $\sigma(\tilde{Q}(1)) = 22$ so that (4.4) still holds when $u = 1$. Thus, we have $\ell(\sigma) < (\sigma(\tilde{Q}(u)) - \sigma)/8 < \frac{25}{4}u - 3$. Since $\lceil u/2 \rceil \leq (u+1)/2$, we get

$$\begin{aligned}\lambda(\sigma) &\leq \chi_h(\tilde{Q}(u)) + \ell(\sigma) + 1 \\ &< 25u^2 + \lceil u/2 \rceil(30u + 2) + 2u - 2 + \frac{25}{4}u - 2 \\ &\leq 25u^2 + (u+1)(15u+1) + \frac{33}{4}u - 4 = 40u^2 + \frac{97}{4}u - 3. \end{aligned} \tag{4.5}$$

From (4.4), we also obtain

$$\begin{aligned}\sigma &> \sigma(\tilde{Q}(u)) - 50u + 24 = 25u^2 - 2\lceil u/2 \rceil - 49u + 22 \\ &\geq 25u^2 - (u+1) - 49u + 22 = 25u^2 - 50u + 21.\end{aligned}$$

Thus, we must have $u < 1 + \frac{1}{5}\sqrt{\sigma+4}$, and plugging this into (4.5), we obtain (4.3). ∎

We observe that (4.1) is a better (*i.e.*, lower) upper bound than (4.3) when $\sigma \leq 30185$ and (4.3) is better than (4.1) when $\sigma \geq 30186$.

Remark 4.5 If we apply Theorem 4.1 to our 4-manifolds $X(n)_K$ in Corollary 3.5 with $n \geq 11$ and argue as in the proof of Corollary 4.4, then we can deduce an upper bound

$$\lambda(\sigma) < \frac{7}{4}\sigma + 4\sqrt{3\sigma + 61} + 45,$$

which is always worse than (4.1). However, we note that it is still possible to get a new and better upper bound for $\lambda(\sigma)$ from $X(n)_K$ for individual σ. For example, by applying Theorem 4.1 to $X(17)_K$, we obtain

$$\lambda(75) \leq 155 \text{ and } \lambda(76) \leq 154, \tag{4.6}$$

which are better than the bounds $\lambda(75) \leq 197$ and $\lambda(76) \leq 198$ coming from (4.1). The upper bounds in (4.6) are also better than the bound $\lambda(\sigma) \leq 173$ for $\sigma = 75, 76$ that is obtained by applying Theorem 4.1 to $Q(4)$ (*cf.* Remark 4.3), and the bound $\lambda(\sigma) \leq 167$ for $\sigma = 75, 76$ in [1, Table 2], which was obtained by applying Theorem 4.1 to $\tilde{Q}(2)$.

We finish our paper by observing that (4.1) does not give the least known upper bound on $\lambda(\sigma)$ for very low values of σ. For example, (4.1) gives $\lambda(0) \leq 68$, whereas we already know from [5] that $\lambda(0) \leq 12$. We still hope that (4.1) and (4.3) provide baselines of comparison for future research.

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