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BOOK REVIEWS

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Emmanuel Breuillard and Hee Oh, *Thin groups and superstrong approximation* (Cambridge University Press, 2014), 376 pp., 978-1-107-03685-7 (hardback), £65.

This book constitutes articles by participants at the workshop 'Thin groups and superstrong approximation', held at MSRI in 2012. Many of the contributors are leading mathematicians, and the topic is one of great current interest with connections to many parts of mathematics: number theory, Lie theory, additive combinatorics and ergodic theory, to name a few.

Neither of the technical terms in the title are particularly standard yet, though they should become so in due course.

Let $\mathbf{G}$ be a connected semisimple algebraic group. For the purposes of reading this review, the reader may assume that $\mathbf{G} = \mathrm{SL}_n$. A *thin group* is a discrete subgroup Γ of $\mathbf{G}(\mathbb{R})$ that is Zariski-dense in $\mathbf{G}$ (that is, not contained in any proper subvariety) and yet has infinite covolume in $\mathbf{G}(\mathbb{R})$. One may contrast this with the notion of a *lattice*, which has the same properties except for having finite covolume in $\mathbf{G}(\mathbb{R})$. The articles of Fuchs, and of Long and Reid, give a variety of constructions and instances of such groups.

In order to understand what is meant by superstrong approximation, it is natural to first discuss strong approximation. A classical setting for this would involve looking at $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, which is a lattice in $\mathrm{SL}_n(\mathbb{R})$. Strong approximation in this setting refers to the phenomenon that the reduction map $\pi\colon \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ is surjective, for all $q \geqslant 1$. *Superstrong* approximation refers to the fact that, roughly speaking, this map is very efficiently surjective. Indeed, given a finite set S of generators for $\mathrm{SL}_n(\mathbb{Z})$, the reduction $\pi(S)$ generates $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ extremely rapidly and uniformly; in fact, the random walk on generating set $\pi(S)$ becomes highly equidistributed in time $O_n(\log q)$. (This property is known as *expansion*.)

Superstrong approximation for lattices is a fairly classical topic with contributions by Selberg, Burger, Sarnak, Clozel and many others. That the same phenomenon also exists for *thin groups* is more recent, and much progress has been possible in recent years due to advances in additive combinatorics (discussed in Breuillard's article, as well as in that by Pyber and Szabó) and the introduction of a powerful technique by Bourgain and Gamburd. There is now an extremely general theorem in this direction, due to Salehi and Varjú (a precise statement may be found in the articles of Bourgain, of Salehi and of Sarnak in the present volume).

Applications in number theory are one of the key justifications for studying thin groups and their approximation properties. A beautiful one is discussed in the articles by Bourgain and Kontorovich, reporting on joint work of the two of them. Given a positive integer $A > 1$, let D_A be the set of all $d \in \mathbb{N}$ for which at least one of the fractions b/d, $\mathrm{hcf}(b, d) = 1$, has all of the partial quotients in its continued fraction expansion bounded by A. A notorious conjecture of Zaremba asserts that in fact $D_A = \mathbb{N}$ if A is sufficiently large—perhaps even $A = 5$ will do. While this problem remains open, Bourgain and Kontorovich have proven that D_A has density 1 for $A \geqslant 50$. The connection with thin groups comes from the observation that D_A is precisely the set of lower right entries of the semigroup Γ_A generated by the matrices $\left(\begin{smallmatrix} 0 & 1 \\ 1 & a \end{smallmatrix}\right)$, $a \leqslant A$. (As A becomes larger, Γ_A remains thin but 'not too thin' in the sense that its *critical exponent* tends to 1.) To explain anything more about this application (including the link with superstrong approximation) would be too ambitious for this review, save to say that the analysis is a kind of variant of the Hardy–Littlewood circle method. The article of Kontorovich will be a very useful resource for those seeking to penetrate the very demanding papers.

Another beautiful application—also discussed in more than one article—is to various questions about the set of curvatures of Apollonian circle packings. In particular, one might be interested in making these curvatures prime or almost-prime, a problem that may be studied using the *affine sieve*, discussed in detail in the article of Salehi. Apollonian packings have been discussed in a number of other places, including by this reviewer [**1**].

The above is just a small selection of the topics on offer in the book. The breadth and centrality of the topics, as well as the high quality of writing and many open problems (benefitting greatly, one imagines, from the oversight of the editors Breuillard and Oh), means that the book should be of interest to a very wide range of mathematicians.

References

1. B. J. Green, Approximate groups and their applications: work of Bourgain, Gamburd, Helfgott and Sarnak, Preprint (arXiv.org/abs/0911.3354; 2009).

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