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Time-optimal motions of robotic manipulators

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SUMMARY

An approach to planning time-optimal collision-free motions of robotic manipulators is presented. It is based on using a negative formulation of the Pontryagin Maximum Principle which handles efficiently various control and/or state constraints imposed on the manipulator motions, which arise naturally out of manipulator joint limits and obstacle avoidance. This approach becomes similar to that described by Weinreb and Bryson, as well as by Bryson and Ho if no state inequality constraints are imposed. In contrast to the penalty function method, the proposed algorithm does not require an initial admissible solution (i.e. an initial admissible trajectory) and finds manipulator trajectories with a smaller cost value than the penalty function approach. A computer example involving a planar redundant manipulator of three revolute kinematic pairs is included. The numerical results are compared with those obtained using an exterior penalty function method.

KEYWORDS: Time-optimal motions; Pontryagin's Maximum Principle; Redundant manipulator

1. INTRODUCTION

Optimal control of robotic manipulators is of great importance from both a theoretical and a practical point of view. It has particular significance in repetition technological processes and/or in the case when a robot performs complicated tasks in complex work spaces containing many obstacles. To avoid the collisions of manipulator links with the obstacles, state inequality constraints must be taken into account while determining optimal controls. Several approaches can be distinguished in this context. Application of potential, harmonic or magnetic field methods[1–4] to find manipulator motions realizing a given process seems to be especially attractive in view of real-time computations. However, these methods suffer from local optimality. It is also difficult to find a feasible joint trajectory (although it may exist) if the above methods were applied to a robotic manipulator performing complex processes in a work space containing complicated obstacles. This is a consequence of local minima of potential functions[1] and the stagnation points as well as the structural local minima of harmonic functions.[3] In order to eliminate the above shortcomings, methods involving global criteria of tasks performance seem to be the most appropriate.

The studies[5–7] use directly the Pontryagin Maximum Principle to find optimal motions in a workspace without obstacles. The penalty function method has been used to find time-optimal[8–10] and minimum-energy[11] motions in work spaces with obstacles. Minimum-energetic motions of a redundant manipulator whose end-effector is to follow a prescribed geometric path in a work space with obstacles were also determined in the study.[12] The discretization methods convert the continuous-time problem to a discrete-time one in order to exploit nonlinear programming algorithms. Singh and Leu[13] have found minimum-time control for non-redundant manipulators using a sequence of quadratic programming algorithms. Through the use of Lagrange multipliers parameterizing the controls by switching points and a concept of velocity obstacles, time-optimal motions have been found in reference [14].

This paper extends the results obtained by Galicki,[15] where optimal motions in a work space without obstacles have been considered. We propose a method of time-optimal collision-free motion planning for both non-redundant and redundant manipulators in work spaces with obstacles. Manipulator joint limits and obstacle avoidance conditions are incorporated through state inequality constraints. In turn, the requirement that a given final location of the end-effector be reached with zero final velocities results in a state equality constraint. Additionally, some constraints imposed directly on manipulator controls are taken into consideration. The performance index is introduced to determine the trajectory of manipulator limited by the aforementioned constraints.

In the case considered, it is very difficult to use Pontryagin's Maximum Principle in its classical form,[16] since it presents only a positive form of control. In fact, the strong variation algorithms based on the Maximum Principle[17,18] are not oriented towards optimal control problems involving state inequality constraints. The methods of sequential gradient restoration developed by Miele[19] increase the number of unknown functions to be found by converting this type of constraint to equality ones using slack variables. Sakawa and Shindo[20] have designed an algorithm which handles the state variable inequality constraints by using a prediction technique. Controls are computed by minimizing the penalized Hamiltonian. Strend and Balden[21] have shown ineffectiveness of this algorithm in the case of bang-bang control problem. The convergence proofs of many implementable algorithms are based on generating a sequence of controls (e.g. by maximizing the Hamiltonian) and searching for a limit control. However, the

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optimal control problems with state inequality constraints may not have limit controls (in a class of measurable controls) although the optimal trajectories for such problems may exist.[22,23] Therefore it is natural to make an attempt to solve the constrained time-optimal problem mentioned above by resorting to other techniques.

A new method, based on a negative formulation of Pontryagin's Maximum Principle[24–26] given in variational form, which makes it possible to handle the state inequality constraints efficiently, is proposed here to determine time-optimal controls. This approach, which consists in transforming the state constraints into control-dependent ones, in contrast to the penalty function method, does not require an initial admissible solution (whose determination may be very troublesome in practice). Additionally, the values of the performance index obtained are usually smaller than those determined by the penalty function method. The technique of the proof of the convergence for the algorithm presented here is different from that of references [17,20] and is based on the search for a limit trajectory whose existence is theoretically ensured. On the other hand, due to an affine dependence of the dynamic model of the robot on the control and the time criterion considered in the paper, a limit control exists, too.

An outline of the remainder of the paper is as follows. Section 2 describes how to employ the negative formulation of the Pontryagin Maximum Principle to determine the time-optional manipulator motions. Section 3 presents a computer example to find optimal collision-free motions of a planar manipulator with three revoltue kinematic pairs. A numerical comparison of the presented approach with the penalty function method is also given in the example.

2 APPLICATION OF NEGATIVE FORMULATION OF THE PONTRYAGIN MAXIMUM PRINCIPLE

To express the robotic task in terms of an optimal control problem, the state vector

$$\chi(t) = \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \tag{1}$$

is introduced, where $q(t), \dot{q}(t) \in \mathbb{R}^n$ are the vectors of joint angles and velocities of the manipulator, respectively. Then the equation of the robot dynamic model can be written in state-space form as (see Appendix)

$$\dot{\chi}(t) = f(\chi(t), v(t)), \quad t \in [0, T] \tag{2}$$

where $v(t) \in \mathbb{R}^m$ denotes the vector of controls (torques/forces), with the initial condition

$$\chi(0) = \chi_0 = \begin{pmatrix} q_0 \\ 0 \end{pmatrix}$$

the vector q_0 being an initial collision-free manipulator configuration.

Constant limits on controls are admitted (a case of state-dependent limits is discussed in Section 4):

$$u_l \le v(t) \le u_u, \quad t \in [0, T] \tag{3}$$

where u_l and u_u are lower and upper limits on control $v(t)$, respectively. For now, T is an unknown final moment of performing the robotic task.

The motion of the robot is planned so as to bring the end-effector to a given final location in minimum time, i.e. our performance index is

$$J(v) = \int_0^T 1\,\mathrm{d}t \tag{4}$$

The robotic task of the manipulator may be generally expressed as follows:

$$\alpha(\chi(T)) = 0 \tag{5}$$

$$\forall t \in [0, T], \ \{\beta(\chi(t), t) \le 0\} \tag{6}$$

where $\alpha(\cdot)$ is a given scalar function expressing the fact of reaching the final location by the end-effector with zero manipulator velocity at this location, and $\{\beta(\chi(t), t) \le 0\}$ is a set of conditions with scalar functions $\beta(\cdot,\cdot)$, which involve the fulfilment of the constraints imposed by the robot mechanical limits and the collision-free conditions of the manipulator with (in general) moving obstacles.

Introducing a new variable τ related to the real time t by $t = \tau T$, where $\tau \in [0, 1]$, we obtain an optimal control problem with parameter T and the fixed final time $\tau = 1$. Consequently, the equation of robot dynamics (2) now becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}(\tau) = T f(x(\tau), u(\tau)) \tag{7}$$

where $x(\tau) = \chi(\tau T)$ and $u(\tau) = v(\tau T)$.

It should be noted that, through eqn. (7), the state vector x is functionally dependent on the control u and the final time T. In this way, the constraints (5) and (6) also depend on u and T. For further considerations, they will be reformulated in a functional form which is equivalent to the previous one. Therefore, the following notation is introduced:

$$a(x(1)) = \alpha(\chi(\tau T))_{\tau=1}, \quad b(x(\tau), \tau, T) = \beta(\chi(\tau T), \tau T))$$

Thus, the constraints (5) and (6) assume the following (functional) form: For the equality constraints:

$$g(u, T) = 0 \tag{8}$$

where $g(u, T) = a(x(1))$, and for the inequality constraints:

$$\{h(u, T) \le 0\} \tag{9}$$

where $h(u,T) = \max_{\tau \in [0,1]} \{b(x(t), \tau, T)\}$, with the performance index

$$J(u, T) = \int_0^1 T\,\mathrm{d}\tau \tag{10}$$

In order to use the negative formulation of Pontryagin's Maximum Principle, an initial performance time T^0 and an initial admissible control $u^0 = u^0(\cdot)$ in the sense of satisfying (3), but not necessarily state constraints (8) and (9), must be known. Moreover, it is assumed that T^0 and u^0 do not minimize the performance index (10).

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The use of the negative formulation of Pontryagin's Maximum Principle necessitates incrementation of the functionals given by the left-hand sides of relations (8) and (9), and the right-hand side of (10). Therefore we assume that the admissible control $u^0 = u^0(\tau)$ and the initial time T^0 of the task performance are perturbed by a small function (variation) $\delta u = \delta u(\tau) = (\delta u_1(\tau), \ldots, \delta u_n(\tau))^T$, and a small number of δT, respectively, where $\|\delta u\| = \max_{\tau \in [0,1]} \left\{ \max_{1 \le i \le n} |\delta u_i(\tau)| \right\} \le \rho$, $|\delta T| \le \epsilon$, ρ and ϵ being given small numbers making correctness of the presented method safe. According to the theory of small perturbations,[16] the value of the functional $J(u,T)$ for the perturbed control $u^0 + \delta u$ and time $T^0 + \delta T$ may simply be expressed by

$$J(u^0 + \delta u, T^0 + \delta T) = T^0 + \delta T \tag{11}$$

The value of the functional determined by the left-hand side of eqn. (8), for control $u^0 + \delta u$ and time $T^0 + \delta T$, after simple calculations, is given to first order by

$$g(u^0 + \delta u, T^0 + \delta T) = g(u^0, T^0) + \delta g(u^0, T^0; \delta u, \delta T) \tag{12}$$

where

$$\delta g(u^0, T^0; \delta u, \delta T) = T^0 \int_0^1 \langle f_u^T(\tau)\psi_g(\tau), \delta u(\tau)\rangle \mathrm{d}\tau + \delta T \int_0^1 \langle \psi_g(\tau), f(\tau)\rangle \mathrm{d}\tau$$

is the Fréchet differential of the functional g (u, T), the adjoint mapping ψ_g being the solution of the Cauchy problem

$$\frac{\mathrm{d}\psi_g(\tau)}{\mathrm{d}\tau} + T^0 f_x^T(\tau)\psi_g(\tau) = 0, \quad \psi_g(1) = \left(\frac{\partial a}{\partial x}\right)_{x = x^0(1)}$$

with

$$f_x(\tau) = \left(\frac{\partial f}{\partial x}\right)_{\substack{x = x^0(\tau)\\ u = u^0(\tau)}}, \quad f_u(\tau) = \left(\frac{\partial f}{\partial u}\right)_{\substack{x = x^0(\tau)\\ u = u^0(\tau)}},$$

$$f(\tau) = f(x^0(\tau), \ u^0(\tau))$$

The values of the functional increments determined by the left-hand sides of inequalities (9), on account of the fact that the Fréchet derivatives do not exist in a general case, call for the Gâteaux derivatives. For the perturbed control $u^0 + \delta u = u^0 + \alpha w$, where α is a small positive number, $w = w(\tau)$ is a function from the same function space as u (the direction of differentiation), and the perturbed final time of task execution $T^0 + \delta T$, it may be concluded that

$$\{h(u^0 + \delta u, T^0 + \delta T) = h(u^0, T^0) + \delta h(u^0, T^0; \delta u, \delta T) \le 0\} \tag{13}$$

where

$$\delta h(u^0, T^0; \delta u, \delta T) = \max_{\tau' \in S}\left\{ T^0 \int_0^1 \langle f_u^T(\tau)\psi_h(\tau; \tau'), \delta u(\tau)\rangle \mathrm{d}\tau + \delta T\left[\left(\frac{\partial b}{\partial T}\right)_{x = x^0(\tau')} + \int_0^1 \langle \psi_h(\tau; \tau'), f(\tau)\rangle \mathrm{d}\tau\right]\right\}$$

is the Gâteaux differential of the functional $h(u, T)$,

$$\frac{\mathrm{d}\psi_h(\tau; \tau')}{\mathrm{d}\tau} + T^0 f_x^T(\tau)\psi_h(\tau; \tau') = -\left(\frac{\partial b}{\partial x}\right)_{x = x^0(\tau')} \delta(\tau - \tau'), \quad \psi_h(1; \tau') = 0$$

$\delta(\cdot)$ is the Dirac delta distribution, and $S = \{\tau' \in [0, 1]: b(x^0(\tau'), \tau', T) = h(u^0, T^0)\}$.

Note that the sets S include only the time moments which correspond to the values of the state trajectory activating the inequality constraints. This is in contrast to the discretization methods which usually take into account the inequality constraints in the optimization process for all the moments of discretization of the interval [0,1] or they must find an admissible triple $(x^0(\cdot), u^0(\cdot), T^0)$ satisfying relations (8) and (9) to decrease the total number of constraints.[13]

For properly selected variations δu and δT, the Fréchet and Gâteaux differentials of functionals in (8) and (9) can approximate the increments of these functionals with any desired accuracy. In this way, the negative formulation of Pontryagin's Maximum Principle just amounts to solving the problem: For given u^0 and T^0, minimize the functional

$$J(u^0 + \delta u^1, T^0 + \delta T^1) = T^0 + \delta T \tag{14}$$

with respect to δu^1 and δT^1, subject to the constraints:

$$\left\{\begin{array}{c} g(u^0, T^0) + \delta g(u^0, T^0, \delta u^1, \delta T^1) = 0 \\ \{h(u^0, T^0) + \delta h(u^0, T^0, \delta u^1, \delta T^1) \le 0\} \\ u_l \le u^0 + \delta u^1 \le u_u \\ \|\delta u^1\| \le \rho \\ |\delta T^1| \le \epsilon \\ 0 \le T^0 + \delta T^1 \end{array}\right. \tag{15}$$

The assumption of non-optimality of the admissible control $u^0(\cdot)$ and of the initial performance time T^0 implies the existence of sufficiently small variations δu^1 and δT^1, respectively, such that $J(u^0 + \delta u^1, T^0 + \delta T^1) < J(u^0, T^0)$, i.e. the existence of the control $u^0 + \delta u^1$ and of the time $T^0 + \delta T^1$ for the problem (14)–(15). The new control $u^1 = u^0 + \delta u^1$ and the new final time $T^1 = T^0 + \delta T^1$ result from solution of this infinite-dimensional linear programming problem. A finite-dimensional approximation of the control variable in (14)

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and (15) seems to be very effective to solve this problem numerically. The process of approximation for an admissible control u^0 may be accomplished e.g. by using quasi-constant or quasi-linear functions, or splines. Then the interval [0,1] is divided into $N>0$ (e.g. equal) subintervals, i.e. partition $0=\tau_0<\tau_1<\tau_2<\ldots<\tau_N=1$ is formed, where $\tau_i=i/N$; $i=0, \ldots, N$. Our paper adopts looking for a sequence of controls which minimize the performance index (10) in a class of quasiconstant mappings

$$u(\tau)=\sum_{k=1}^{N} u_k\zeta_k(\tau), \quad \delta u(\tau)=\sum_{k=1}^{N} \delta u_k\zeta_k(\tau) \tag{16}$$

where

$$\zeta_k(\tau)=\begin{cases}1 & \text{if } \tau\in[\tau_{k-1}, \tau_k),\\ 0 & \text{otherwise.}\end{cases}$$

An important factor improving the speed of calculation of the minimizing sequence is a possibility of using parallel processors to calculate the Fréchet and Gâteaux functional differentials δg and δh.

The process of minimization is then repeated for the revised control u^1 and final time T^1. A sequence of pairs $\{(u^k, T^k)\}$ may thus be obtained as a result of solving the iterative approximation scheme (14)–(15). Each element of this sequence corresponds to a state trajectory x^k according to eqn. (7). From the practical point of view, it is not essential to know a limit control (which may not exist for some optimal control problems with state inequality constraints or may be non-unique for time-optimal control problems), whereas the convergence of the state trajectories x^k is of great importance.

We can now formulate our main result whose proof is given in Appendix.

Theorem 1.

The sequence of trajectories $\{x^k\}$, *corresponding to the sequence of pairs* $\{(u^k, T^k)\}$, *has a convergent subsequence.*

Thus the successive solutions of problems (14)–(15) converge to an optimal (limit) trajectory x^* and an optimal time T^*. Moreover, in the case considered, affine dependence of the dynamic model (2) on the control implies the existence of a time-optimal (limit) control u^*, i.e. the sequence $\{u^k\}$ has a convergent subsequence $\{u^{k_n}\}$ such that $\lim_{n\to\infty} u^{k_n}=u^*$. As can be shown, the control u^* maximizes (locally) the Hamiltonian.

3 COMPUTER EXAMPLE

A planar manipulator of three revolute kinematic pairs ($n=3$, $q=(q_1, q_2, q_3)$) shown in Figure 1 is considered. The data used in simulations are as follows:

- the components of the dynamic model of manipulator:
  - $D(q)=\text{diag}[5, 3, 2]$;
  - link lengths $l_1=3.0$; $l_2=2.0$, $l_3=1.0$;

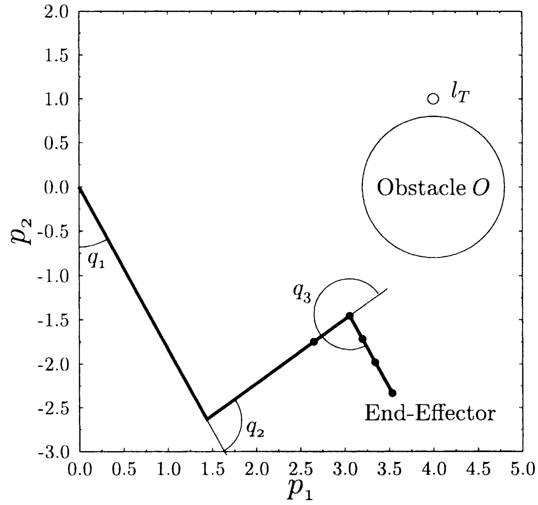

Fig. 1. Scheme of the manipulator and the obstracle O.

  - link masses $m_1=0.3$, $m_2=0.2$, $m_3=0.1$;
  - the mass centres of the manipulator links represented by the sections are located at their physical centres;
  - moments of inertia of links with respect to their mass centres $I_1=0.225$, $I_2=0.067$, $I_3=0.0083$;
- for simplicity, a static obstacle O has been assumed of the following boundary equation:

  $$o(p)=(0.8)^2-(p_1-4)^2-(p_2)^2=0, \quad p=(p_1, p_2)^T\in\mathbb{R}^2$$
- the discrete model of the manipulator assumed for numerical simulations is shown in Figure 1 (dots);
- the constraints (3), (5), and (6) correspond to

  $$q_0=(0.5,\ 1.7,\ -1.7)^T \qquad l_T=(4.0,\ 1.0)^2$$
  $$q_l=(0.18,\ -3.0,\ -3.0)^T, \qquad q_l=(3.0,\ 3.0,\ 3.0)^T$$
  $$u_l=(-20.0,\ -7.0,\ -2.3)^T, \quad u_u=(20.0,\ 7.0,\ 2.3)^T$$
  $$\rho=0.05,\ \epsilon=0.01$$

  where l_T is a given final location to be reached by the end-effector;
- the number of subintervals in the control discretization equals $N=75$

To draw a comparison with other approaches, simulations for the exterior penalty function method (EPFM) are also carried out. Figures 2 and 3 present, respectively, the initial controls and the corresponding initial manipulator motions for the method presented in the paper and for the EPFM. Let us note that the trajectory x^0 does not satisfy the constraints (8) and (9). The initial time equals $T^0=1.5$.

Two types of optimal motions for both methods are considered. The first one is an unconstrained motion (only control limits (3) are taken into account) between the initial state χ_0 and final end-effector location l_T. The optimal controls are shown in Figure 4. As was expected, the controls for the method presented here are approximately of bang-bang type. The minimum time equals $T^*=0.884$ and is smaller than that determined by the EPFM ($T^*_{\text{EPFM}}=0.983$). Figure 5 shows the corresponding manipulator manipulatory motions for both algorithms.

The optimal motion for this case is shown in Figure 5.

The constrained time-optimal controls are shown in Figure 6. The optimal performance time $T^*=1.207$ is again

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smaller than that determined by the penalty function method ($T^*_{\mathrm{EPFM}} = 1.221$).

Let us note that, at each moment of the manipulator control, at least one of the actuators is saturated even when the collision-free constraints are active. This agrees with theoretical results obtained by Sontag and Sussman[27] for unconstrained time-optimal motions.

The optimal motion of manipulator is displayed in Figure 7.

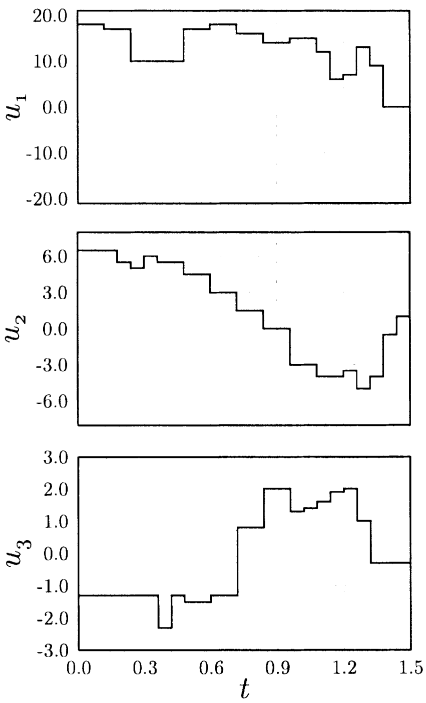

Fig. 2. Initial admissible control u^0.

4. CONCLUDING REMARKS

An application of the negative formulation of the Pontryagin Maximum Principle to find time-optimal controls of robotic manipulators is proposed in the paper. An important factor which affects the speed of determining the minimizing sequence of controls is a possibility of using parallel processors to calculate the Fréchet and Gâteaux functional differentials in the constraints (15). Moreover, the present approach adopts solution of the linear programming problem (14)–(16) through the so-called *Upper-Bounding Simplex Method*[28] exploiting the sparse structure of the coefficient matrix (let us note that after discretization the simple constraints $\max(-\rho, u_l - u_k^0) \leq \delta u_k \leq \min(\rho, u_u - u_k^0)$, $k = 1, \ldots, N$ hold an overwhelming majority). Such a technique enables us to consider non-trivial-sized dis-

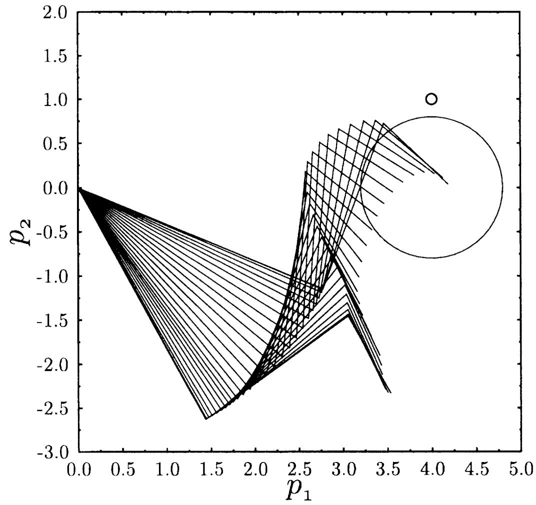

Fig. 3. Initial manipulator motion in the work space.

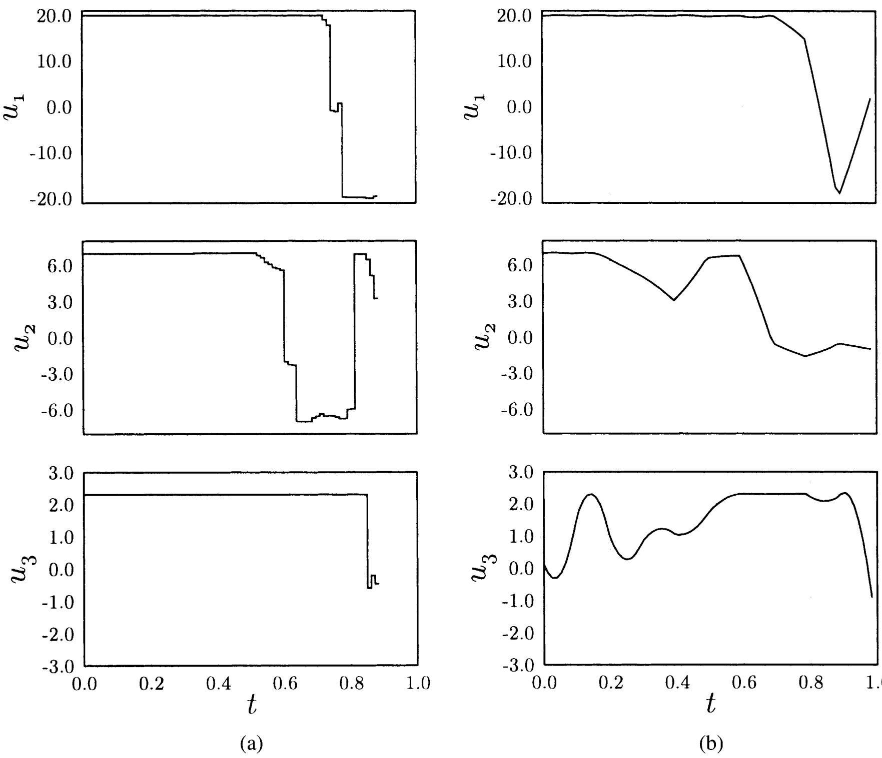

Fig. 4. Controls for unconstrained motion. (a) Proposed method and (b) EPFM.

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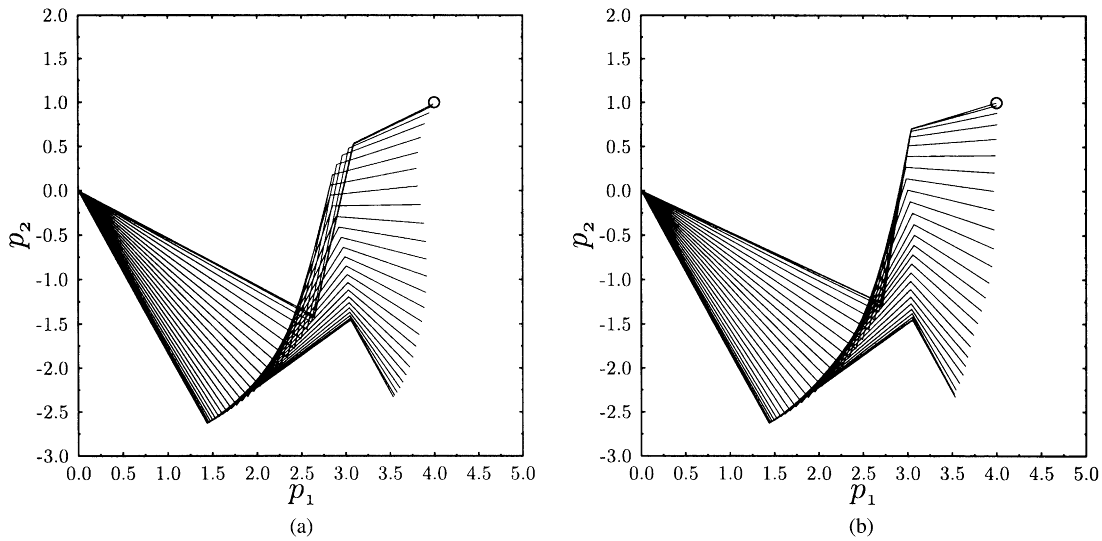

Fig. 5. Unconstrained manipulator motion. (a) Proposed method and (b) EPFM.

cretizations on one hand, and on the other hand to reduce significantly the time of computations. In this way, the minimum hardware is needed to run the corresponding code (in the case considered here, a low-cost PC-486 has been used).

A case of state-dependent control constraints can be tackled by introducing a new control vector v such that $\dot{u} = v$ for $u_l(x) \leq u \leq u_u(x)$, where $u_l(x)$ and $u_u(x)$ are state dependent lower and upper limits imposed on vector u, respectively. Thus, a modified control problem is obtained with a new trajectory (x, u) and the control vector v.

It is important to note that the method presented here does not call for knowledge of an initial solution satisfying the constraints (8) and (9). It is only required that the admissible control should satisfy relations (3). In such a case, at initial iterations the trajectory x is forced by the algorithm to satisfy (8) and (9), and then the proper minimization of the functional (10) follows. Numerical simulations show that

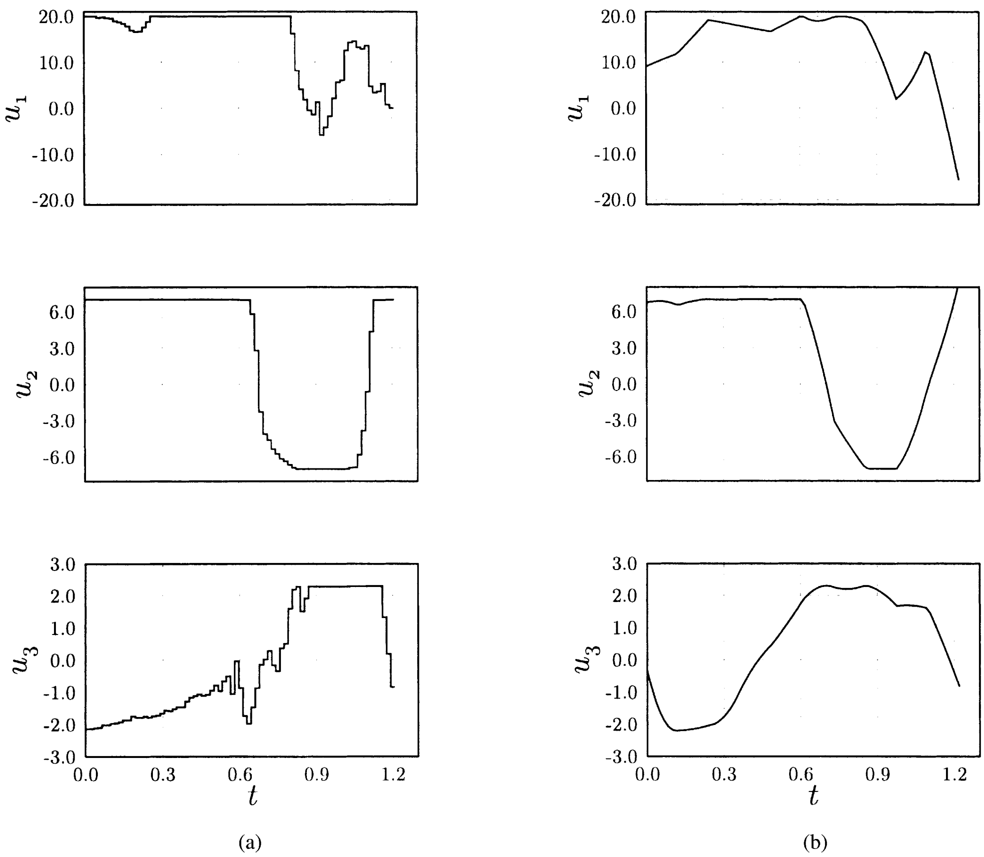

Fig. 6. Controls for constrained motion. (a) Proposed method and (b) EPFM.

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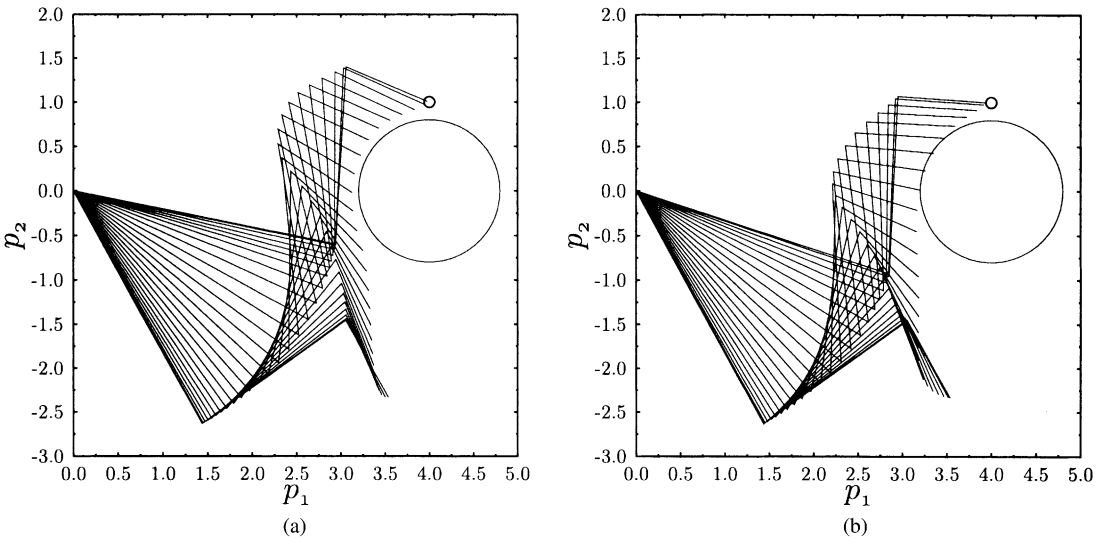

Fig. 7. Constrained manipulator motion. (a) Proposed method and (b) EPFM.

the presented method finds smaller values of the performance index that the penalty-function approach for both unconstrained and collision-free motions of the robot. The problem formulation and the given approach to its solution may be directly applicable to multiple manipulators interacting in a three-dimensional workspace with obstacles. This method is also useful in finding admissible trajectories (in the sense of satisfying state constraints (8) and (9)), which is often encountered in practice.

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APPENDIX

We outline only the main ideas of the proof of Theorem 1. The existence of a finite optimal time less than a fixed number $A>0$ follows from standard existence theorems.[29] For the state vector (1) we have to show that $\|\dot q\|$ is bounded (the boundedness of $\|q\|$ is a consequence of inequalities (6), i.e. $q-q_u\le 0$, $q_l-q\le 0$ where q_l and q_u are, respectively, the given lower and upper limits on the configuration q. The robot dynamics is described by:

$$f(\chi,\upsilon)=\begin{pmatrix}\dot q\\ -I^{-1}(q)(\dot q^T C(q)\dot q+D(q)\dot q+G(q))\end{pmatrix}+\begin{pmatrix}0\\ I^{-1}(q)\end{pmatrix}\upsilon \tag{A1}$$

where $I(q)$ is an $n\times n$ inertia matrix, $C(q)$ is an $n\times n\times n$ tensor whose coordinates are Christoffel's coefficients, $D(q)$ stands for an $n\times n$ viscous friction matrix, and $G(q)$ denotes an n-dimensional vector representing gravity forces. We assume that $q(\cdot)$ and $\dot q(\cdot)$ belong to the class of mappings which are absolutely continuous with respect to time t. Additionally, the control $\upsilon(\cdot)$ is an integrable Lebesgue function. It follows that

$$\ddot q=I^{-1}(q)(\upsilon-\dot q^T C\dot q-D(q)\dot q-G(q)) \tag{A2}$$

Hence

$$\|\ddot q\|\le\alpha+\beta\|\dot q\|+\gamma\|\dot q\|^2 \tag{A3}$$

where $\|\cdot\|$ *denotes the Euclidean norm*,

$$\alpha=\max_{q,\upsilon}\|I^{-1}(q)(\upsilon-G(q))\|,$$

$$\beta=\max_{q}\|I^{-1}(q)D(q)\|,$$

$$\gamma=\max_{q}\|I^{-1}(q)C(q)\|,$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\|\dot q\|\le\|\ddot q\|$, we see that

$$\frac{\mathrm{d}(\|\dot q\|)}{\alpha+\beta\|\dot q\|+\gamma\|\dot q\|^2}\le \mathrm{d}t \tag{A4}$$

Not decreasing the character of considerations, we assume that $\Delta=\beta^2-4\alpha\gamma<0$ (the case $\Delta\ge 0$ is considered analogously). Integrating both sides of (A4) over the time interval $[0,A]$, we obtain the following bound on $\|\dot q\|$:

$$\|\dot q(t)\|\le M \tag{A5}$$

where

$$M=\max_{t\in[0,A]}\frac{1}{2\gamma}\left[\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}t=\arctan\frac{\beta}{\sqrt{-\Delta}}\right)-\beta\right]$$

and $\|\mathrm{d}q/d\tau\|\le AM$. Hence the state vector x is bounded, i.e.

$$\begin{pmatrix}q_l\\ -AM\mathbf{1}_n\end{pmatrix}\le x\le\begin{pmatrix}q_u\\ AM\mathbf{1}_n\end{pmatrix} \tag{A6}$$

where $\mathbf{l}_n$ is the n-dimensional vector with all the coordinates equal to 1.

By solving the successive minimization problems (14)–(15), the sequence of triples $\{(u^k,x^k,T^k)\}$ which satisfy the conditions

$$a(x^k(1))=0,\quad \max_{\tau\in[0,1]}\{b(x^k(\tau),\tau,T^k)\}\le 0,\quad 0\le T^k\le A \tag{A7}$$

is generated.

Let us note that

$$\begin{pmatrix}q_l\\ -AM\mathbf{1}_n\end{pmatrix}\le x^k\le\begin{pmatrix}q_u\\ AM\mathbf{1}_n\end{pmatrix} \tag{A8}$$

and

$$\left\|\frac{\mathrm{d}x^k}{\mathrm{d}\tau}\right\|=\|T^k f(x^k,u^k)\|\le\sigma \tag{A9}$$

where

$$\sigma=A\max_{x,u}\|f(x,u)\|$$

The inequality (A8) means that the set $\{x^k: k=0,1,2,\dots\}$ is uniformly bounded, whereas the inequality (A9) means that this set is also equi-continuous. Thus, the Arzela-Ascoli theorem[30] holds for $\{x^k\}$ and the Bolzano theorem holds for $\{T^k\}$. Convergent subsequences $\{x^{k_n}\}$ and $\{T^{k_n}\}$, for which $\lim_{n\to\infty}x^{k_n}=x^*$ and $\lim_{n\to\infty}T^{k_n}=T^*$, may be selected from the sequences $\{x^k$ and $\{T^k\}$, respectively. It follows from Fillipov's theorem[22] that x^* is an absolutely continuous function. Moreover, the continuity of $a(x(1))$ and $\max_{\tau\in[0,1]}\{b(x(\tau),\tau,T^*)\}$ with respect to x and T implies the relations $a(x^*(1))=0$ and $\max_{\tau\in[0,1]}\{b(x^*(\tau),\tau,T^*)\}\le 0$.

If an admissible pair (u^0,T^0) does not satisfy the state constraints (8) and (9), then we may replace the equality (8) by an equivalent functional inequality of the form

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$g'(u,T) = [a(x(1))]^2 \leqslant 0$. Thus, there exist sufficiently small variations δu^1 and δT^1 for which $T^0 + \delta T^1 = T^0 + \min \{\delta T\}$ and $\delta g'(u^0, T^0; \delta u^1, \delta T^1) \leq 0$, $\{\delta h(u^0, T^0; \delta u^1, \delta T^1) \leq 0\}$, where $\delta g'(u^0, T^0; \delta u^1, \delta T^1)$ is the Fréchet differential of $g'(u, T)$. The fulfilment of the constraints (8) and (9) is then forced by successive solution of the minimization problems (14)–(15) which generate decreasing sequences $\{g'(u^k, t^k)\}$ and $\{h(u^k, T^k)\}$.

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