Math. Struct. in Comp. Science (2011), *vol.* 21, *pp.* 111–124.
doi:10.1017/S0960129510000460 First published online 2 December 2010

Noetherian orders

HERVÉ PERDRY[†] and PETER SCHUSTER[‡]

[†]*Université Paris-Sud UMR-S 669 and Inserm U 669, 16 avenue Paul-Vaillant-Couturier, Bâtiment Inserm 15/16, Villejuif F-94807*
[‡]*Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England*

Received 1 March 2010; revised 8 September 2010

Noether classes of posets arise in a natural way from the constructively meaningful variants of the notion of a Noetherian ring. Using an axiomatic characterisation of a Noether class, we prove that if a poset belongs to a Noether class, then so does the poset of the finite descending chains. When applied to the poset of finitely generated ideals of a ring, this helps towards a unified constructive proof of the Hilbert basis theorem for all Noether classes.

1. Introduction

This is the first of two papers whose objective is to deliver to constructive algebra *à la* Kronecker and Bishop (Edwards 2005; Lombardi and Quitté, to appear; Mines *et al.* 1988) a unified proof of several variants of the Hilbert basis theorem. The Hilbert basis theorem says that if a (commutative) ring A is Noetherian, then so is the polynomial ring $A[X]$.

> 'Standard classical proofs of the Hilbert basis theorem are constructive, if by *Noetherian* we mean that every ideal is finitely generated, but only trivial rings are Noetherian in this sense from a constructive point of view.' (Mines *et al.* 1988, page 193)

A similar problem occurs with the condition that every ascending chain of ideals is eventually constant.

Despite this, several constructively meaningful notions of a Noetherian ring have been put forward in the past 40 years, each of which allowed for a constructively provable variant of the Hilbert basis theorem (Jacobsson and Löfwall 1991; Perdry 2004; Richman 1974; Richman 2003; Seidenberg 1974; Tennenbaum 1973); see also Perdry (2008) and Schuster and Zappe (2006). The poset $\mathfrak{I}_A$ of the finitely generated ideals of the ring A need not be decidable for some of these variants of the Hilbert basis theorem (Perdry 2008; Richman 2003; Tennenbaum 1973), but in this, and the forthcoming second paper, we restrict our attention to the case where $\mathfrak{I}_A$ is decidable, which is to say that A is strongly discrete.

Many of the constructively provable variants of the Hilbert basis theorem rely purely on properties of the poset $\mathfrak{I}_A$, just as the ascending chain condition due to Noether does. In the present paper we therefore abstract from the ring context and consider the classes of posets that correspond to these properties. Each of these classes satisfies four characteristic conditions, which define what we call a Noether class of posets. The best known constructively meaningful property of $\mathfrak{I}_A$ is the chain condition introduced in

Richman (1974) and Seidenberg (1974): every descending[†] sequence $a_0 \geqslant a_1 \geqslant \dots$ halts, that is, there is n with $a_n = a_{n+1}$. The posets that possess this property form the prime example of a Noether class, the Richman–Seidenberg class (see Definition 3.1).

A less well-known example of a Noether class of posets is the one defined by the finite-depth property: every finitely branching tree labelled by the poset under consideration has finite depth; this defines the finite-depth class of posets (see Section 4). It contains the class of well-founded posets, which was used in Jacobsson and Löfwall (1991) to give a constructive proof of the Hilbert basis theorem[‡], and is contained in the class of posets that satisfy a variant of Richman's tree condition (Richman 2003). The latter was designed as a substitute for the chain condition introduced in Richman (1974) and Seidenberg (1974) in order to avoid countable choice. Moreover, the finite-depth class is equal to the Richman–Seidenberg class precisely when a fairly general form of Brouwer's fan theorem holds, which is the classical contrapositive of König's lemma.

Apart from investigating how the Noether classes in use are related to each other, we develop their theory up to a point where we are able to tackle the Hilbert basis theorem. In the vein of Coquand and Lombardi (2006), we undertake a constructive rereading of one of the classical proofs of the Hilbert basis theorem (for example, the first proof of Zariski and Samuel (1958, IV, Theorem 1)) in which the chain condition in question is proved to propagate from the poset of ideals to the poset of infinite chains of ideals. Our key observation is indeed that this also works with finite chains (our Theorem 3.1): if a poset E is in a Noether class $\mathscr{C}$, then the poset $E^\star$ of the eventually constant descending chains in E is also in $\mathscr{C}$. We also need the third condition we have imposed on the Noether classes (Definition 3.2 below): if a poset G is in a Noether class $\mathscr{C}$, then every poset F is in $\mathscr{C}$ that can be embedded into G along a (strictly) increasing mapping. In fact, once all this is applied to the posets $E = \mathfrak{I}_A$, $F = \mathfrak{I}_A^\star$ and $G = \mathfrak{I}_{A[X]}$, all we need to do to complete the proof of the Hilbert basis theorem is to show that the leading coefficients mapping from $\mathfrak{I}_{A[X]}$ to $\mathfrak{I}_A^\star$ are well-defined and strictly increasing.

Although we could well achieve the latter using some material provided in Mines *et al.* (1988), in the next paper we will do it using standard bases, since we find that approach more natural. More precisely, we will prove constructively that every (finitely generated) ideal of $A[X]$ has a standard basis. In other words, we will give a constructive termination proof for a variant of the otherwise well-known algorithm for computing the standard basis. This constructive existence proof of a standard basis will turn out to be the only missing link between the order-theoretic results of the present paper and the unified constructive proof of the Hilbert basis theorem for all Noether classes in the case of strongly discrete coherent rings.

[†] In order to give a uniform presentation, we work with descending chains rather than ascending ones, so we need to reverse the natural inclusion order on $\mathfrak{I}_A$.

[‡] A formal version of the Hilbert basis theorem over countable fields, which uses the construction 'every ideal is finitely generated' as the definition of a Noetherian ring, is equivalent to saying the ordinal number ω^ω is well-ordered (Simpson 1988; Simpson 1999).

2. Preliminaries

2.1. *Increasing mappings*

Every partially ordered set $(E, \leqslant)$ occurring in this paper will have a *decidable order* and thus be a *discrete set*: that is, $x \leqslant y$ and thus $x = y$ are decidable relations between the elements of E. We use $x < y$ to denote the conjunction of $x \leqslant y$ and $x \neq y$, where the latter stands for the negation of $x = y$.

Definition 2.1. Let E and F be posets. We say that a mapping $\varphi : E \to F$ is *increasing* if for all $a, b \in E$,

$$a \leqslant b \Longrightarrow \varphi(a) \leqslant \varphi(b),$$

and *strictly increasing* if for all $a, b \in E$,

$$a < b \Longrightarrow \varphi(a) < \varphi(b)).$$

It is clear that if the mappings $\varphi : E \to F$ and $\psi : F \to G$ between posets are (strictly) increasing, then $\psi \circ \varphi : E \to G$ is (strictly) increasing. If φ is injective in the sense that $a = b$ whenever $\varphi(a) = \varphi(b)$, then φ is strictly increasing if it is increasing. Conversely, if E is totally ordered and φ is strictly increasing, then φ is injective.

Remark 2.1. A mapping $\varphi : E \to F$ between posets is strictly increasing if and only if it is increasing and, in addition, for all $a, b \in E$,

$$a \leqslant b \wedge \varphi(a) = \varphi(b) \Longrightarrow a = b.$$

Analogous shorthands are defined, and analogous assertions hold, in the dual case. A mapping $\varphi : E \to F$ between posets is *decreasing* (respectively, *strictly decreasing*) if $\varphi : E \to F^\circ$ is increasing (respectively, strictly increasing) where F° stands for F with the reverse order.

2.2. *Direct and lexicographic products*

We next recall and expand some material from Mines *et al.* (1988, I.6).

Definition 2.2. Let E and F be posets. We write $E \times F$ for the direct product ordered by the *product order* for which

$$(x, y) \leqslant (x', y') \iff x \leqslant x' \wedge y \leqslant y',$$

and write $E \cdot F$ for the direct product ordered by the *lexicographic order* with

$$(x, y) \leqslant (x', y') \iff x < x' \vee (x = x' \wedge y \leqslant y').$$

For a poset E and $k \geqslant 1$, we write E^k for the k-fold product $E \times \ldots \times E$.

The identity $E \times F \to E \cdot F$ is strictly increasing. Since E, F have a decidable order, so too do $E \times F$, $E \cdot F$ and E^k. The following slightly more complex construction will prove useful.

Definition 2.3. Let $(E_i, \leqslant_i)_{i\in I}$ be a family of posets indexed by a poset $(I, \leqslant)$. We use $\sum_{i\in I} E_i$ to denote the disjoint union $\{(i,x) \,:\, i \in I,\ x \in E_i\}$ ordered by

$$(i,x) \leqslant (j,y) \iff i < j \vee (i = j \wedge x \leqslant_i y).$$

We further write $\pi : \sum_{i\in I} E_i \to I$ for the projection with $\pi((i,x)) = i$, and note that it is an increasing mapping.

In particular, $(i,x) \leqslant (j,y)$ implies $i \leqslant j$, and

$$(i,x) < (j,y) \iff i < j \vee (i = j \wedge x <_i y).$$

We sometimes identify E_i with the subset $\{i\} \times E_i$; accordingly, $u \in E_{\pi(u)}$ for every $u \in \sum_{i\in I} E_i$.

If $E_i = E$ for all $i \in I$, then $\sum_{i\in I} E_i$ is just the lexicographic product $I \cdot E$. Since the partial orders on I and on the E_i with $i \in I$ are decidable, so too is $\leqslant$ on $\sum_{i\in I} E_i$. The proofs of properties of $\sum_{i\in I} E_i$ will be very similar to the proofs of properties of $I \cdot E$.

Lemma 2.1. Let E be a poset and $(E_i)_{i\in I}$ be a family of posets indexed by a poset I. If $\eta : E \to I$ and all the $\varphi^i : E \to E_i$ with $i \in I$ are increasing mappings, then so too is

$$E \to \sum_{i\in I} E_i\,, \quad a \mapsto (\eta(a), \varphi^{\eta(a)}(a)).$$

2.3. *Finite chains of arbitrary length*

Now going beyond Mines *et al.* (1988, I.6), we study the eventually constant descending sequences. In order to have a concept with a finite nature, we formally suppress the constant tail of any such sequence.

Definition 2.4. By the set of the decreasing finite sequences in a poset E, we mean

$$E^\star = \bigcup_{n\in\mathbb{N}} \{(a_0, \ldots, a_n) \in E^{n+1} \,:\, a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n\}\,.$$

Every $(a_0, \ldots, a_n) \in E^\star$ can be extended, by setting $a_m = a_n$ for $m > n$, to a decreasing infinite sequence, with which we often identify it. With this convention, we define for any two $a, b \in E^\star$:

$$a \leqslant b \iff \forall m \in \mathbb{N}\,(a_m \leqslant b_m)$$
$$a = b \iff \forall m \in \mathbb{N}\,(a_m = b_m).$$

Note that:

— $E^\star$ does not contain the empty sequence;
— $\leqslant$ is a partial order with respect to the equality $=$; and
— $\leqslant$ on $E^\star$ is decidable since $\leqslant$ on E is.

Definition 2.5. Let $a = (a_0, \ldots, a_n) \in E^\star$. We write $\lambda(a)$ for the *limit* $\lambda(a) = a_n$ of a and set

$$\eta(a) = \min\{m \in \mathbb{N} \,:\, a_m = \lambda(a)\}$$

and $\varphi^k(a) = (a_0, \ldots, a_k) \in E^{k+1}$ for every $k \in \mathbb{N}$. For each $e \in E$, let

$$E^\star_{(e)} = \{a \in E^\star \; : \; \lambda(a) = e\}$$

be the subset of $E^\star$ consisting of all the sequences with the same limit e.

Note that for each $m \in \mathbb{N}$, we have $a_m \geqslant \lambda(a)$ with

$$a_m = \lambda(a) \iff m \geqslant \eta(a)\,.$$

Lemma 2.2. Let E be a poset. Then $\lambda \; : \; E^\star \to E$ is increasing, and so is $\eta : E^\star_{(e)} \to \mathbb{N}$ for every $e \in E$. Moreover, $\varphi^i : E^\star \to E^{i+1}$ is increasing for every $i \in \mathbb{N}$.

Proof. We only need to verify that if $a \leqslant b$ in $E^\star$ such that $\lambda(a) = \lambda(b)$, then $\eta(a) \leqslant \eta(b)$. To this end, let $i \in \mathbb{N}$. If $i \geqslant \eta(b)$, that is, $b_i = \lambda(b)$, then

$$\lambda(a) \leqslant a_i \leqslant b_i = \lambda(b),$$

and thus we also have $a_i = \lambda(a)$, which is to say that $i \geqslant \eta(a)$. □

3. Noether classes

The following notion was introduced, for ideals, in Richman (1974) and Seidenberg (1974); it is a way of putting the absence of strictly decreasing infinite sequences positively.

Definition 3.1. A poset E is in the *Richman–Seidenberg class of posets* whenever if $a_0 \geqslant a_1 \geqslant \cdots$ in E, there is $n \in \mathbb{N}$ such that $a_n = a_{n+1}$. We use $\mathscr{RS}$ to denote the Richman–Seidenberg class of posets.

If $E \in \mathscr{RS}$, then for any descending chain $a_0 \geqslant a_1 \geqslant \cdots$ in E one can pin down the least n with $a_n = a_{n+1}$ simply because E is a discrete set. In this way one can avoid having to choose any such n.

Definition 3.2. We say that a class $\mathscr{C}$ of posets is a *Noether class* if it satisfies the following four conditions:

(1) $\mathscr{C} \subseteq \mathscr{RS}$.
(2) $\mathbb{N} \in \mathscr{C}$.
(3) If there is a strictly increasing mapping from E to F, then $E \in \mathscr{C}$ whenever $F \in \mathscr{C}$.
(4) Let I be a poset in $\mathscr{C}$. If $(E_i)_{i \in I}$ is a family of posets in $\mathscr{C}$, then $\sum_{i \in I} E_i$ is in $\mathscr{C}$.

In the rest of this paper we will simply refer to these four conditions as *Conditions 1, 2, 3 and 4.*

Proposition 3.1. The class $\mathscr{RS}$ is a Noether class.

Proof. It is clear that $\mathscr{RS}$ satisfies Conditions 1 and 2. To see that it fulfils Condition 3, let $\varphi \; : \; E \to F$ be an increasing mapping with $F \in \mathscr{RS}$. If $a_0 \geqslant a_1 \geqslant \cdots$ in E, then $\varphi(a_0) \geqslant \varphi(a_1) \geqslant \cdots$ in F. So there is an n such that $\varphi(a_n) \geqslant \varphi(a_{n+1})$, and for this n we also have $a_n = a_{n+1}$ (Remark 2.1).

We still need to show that $\mathscr{RS}$ satisfies Condition 4. To do this, we let $(E_i)_{i\in I}$ be a family of posets indexed by a poset I, and set $E = \sum_{i\in I} E_i$. Suppose $I \in \mathscr{RS}$, and that $E_i \in \mathscr{RS}$ for all $i \in I$. To prove that E is in $\mathscr{RS}$, we use the projection $\pi : E \to I$. Let $u_0 \geqslant u_1 \geqslant \cdots$ be a sequence in E. We need to find an integer $N \geqslant 1$ with $u_{N-1} = u_N$. Observe that $\pi(u_0) \geqslant \pi(u_1) \geqslant \cdots$ in I.

We first show that for each $n \in \mathbb{N}$ we can find $m > n$ such that either $\pi(u_m) < \pi(u_n)$ or $u_{m-1} = u_m$. To this end, we define the sequence $(v_{n,m})_{m\geqslant n}$ in $E_{\pi(u_n)}$ by setting $v_{n,n} = u_n$ and

$$v_{n,m} = \begin{cases} u_m & \text{if } \pi(u_m) = \pi(u_n) \\ v_{n,m-1} & \text{otherwise} \end{cases}$$

for $m > n$. Note that $v_{n,n} \geqslant v_{n,n+1} \geqslant \cdots$ in $E_{\pi(u_n)}$, and this poset is in $\mathscr{RS}$. So there is $m > n$ such that $v_{n,m-1} = v_{n,m}$. Either $\pi(u_m) < \pi(u_n)$ or $\pi(u_m) = \pi(u_n)$. In the latter case, $v_{n,m} = u_m$, but also $\pi(u_{m-1}) = \pi(u_n)$, so $v_{n,m-1} = u_{m-1}$, and thus $u_{m-1} = u_m$.

Starting from $n(0) = 0$, we can now recursively define a sequence $n(0) < n(1) < \cdots$ in $\mathbb{N}$ such that each $n(k+1)$ is the least $m > n(k)$ such that either $\pi(u_m) < \pi(u_{n(k)})$ or $u_{m-1} = u_m$. Since $\pi(u_{n(0)}) \geqslant \pi(u_{n(1)}) \geqslant \cdots$ in I, and I is in $\mathscr{RS}$, there is $K \in \mathbb{N}$ with $\pi(u_{n(K)}) = \pi(u_{n(K+1)})$ for which $u_{n(K+1)-1} = u_{n(K+1)}$. In other words, $N = n(K+1)$ is as required above. □

3.1. *Propagation results*

Lemma 3.1. Let $\mathscr{C}$ be a Noether class, and let E, F be posets. If E and F are in $\mathscr{C}$, then $E \cdot F$ and $E \times F$ are in $\mathscr{C}$. In particular, E^k is in $\mathscr{C}$ for all $k \geqslant 1$.

Proof. The implication from $E \in \mathscr{C}$ and $F \in \mathscr{C}$ to $E \cdot F \in \mathscr{C}$ is a special case of Condition 4. Applying Condition 3 to the strictly increasing mapping $\mathrm{id} : E \times F \to E \cdot F$, we see that $E \cdot F \in \mathscr{C}$ implies $E \times F \in \mathscr{C}$. Induction on k then yields the final statement. □

Theorem 3.1. Let $\mathscr{C}$ be a Noether class and E be a poset. If E is in $\mathscr{C}$, then so is $E^\star$.

Proof. Let E be a poset. By Lemmas 2.1 and 2.2, the two mappings

$$\varphi \;:\; E^\star_{(e)} \to \sum_{n\in\mathbb{N}} E^{n+1}, \quad a \mapsto (\eta(a), \varphi^{\eta(a)}(a))$$

$$\psi \;:\; E^\star \to \sum_{e\in E} E^\star_{(e)}, \quad a \mapsto (\lambda(a), a)$$

are increasing. Since both mappings are also injective, they are *strictly* increasing.

Now let E be in $\mathscr{C}$. By Lemma 3.1, we have $E^{n+1} \in \mathscr{C}$ for all n. We use Conditions 2–4 of a Noether class. Now, $E^\star_{(e)}$ is in $\mathscr{C}$ for every $e \in E$ (use φ), so $E^\star$ is in $\mathscr{C}$ (use ψ). □

3.2. *Further examples*

3.2.1. *Well foundedness.* This is the constructively meaningful equivalent of a well-known concept (see Mines *et al.* (1988, I.6)), which proved fruitful for the Hilbert basis theorem discussed in Jacobsson and Löfwall (1991).

Definition 3.3. A subset H of a poset E is *hereditary* if

$$\{y \in E \ : \ y < x\} \subseteq H \Longrightarrow x \in H$$

for every $x \in E$. A poset E is *well-founded* if every hereditary subset H equals E. A well-founded and totally ordered poset is *well-ordered*. We use $\mathscr{WF}$ to denote the class of well-founded posets.

Proposition 3.2. The class $\mathscr{WF}$ is a Noether class.

Proof. It is easy to see that $\mathscr{WF}$ satisfies Condition 1 (this is Mines *et al.* (1988, Chapter I, Exercise 4)). The proofs that $\mathscr{WF}$ satisfies Conditions 2–4 are given in Mines *et al.* (1988, I.6.1–3). □

3.2.2. *Strong Noetherianity.* Perdry (2004) introduced and investigated the following notion of strong Noetherianity.

Definition 3.4. A poset E is *strongly Noetherian* if there exists a strictly increasing mapping $\varphi : E \to F$ for a well-ordered poset F. We use $\mathscr{SN}$ to denote the class of strongly Noetherian posets.

The next remark is a consequence of Condition 3 for $\mathscr{WF}$.

Remark 3.1. The class $\mathscr{SN}$ is a subclass of $\mathscr{WF}$.

Proposition 3.3. The class $\mathscr{SN}$ is a Noether class.

Proof. For every poset E, $E \in \mathscr{SN}$ implies $E \in \mathscr{WF}$ (Remark 3.1), which implies $E \in \mathscr{RS}$ (Proposition 3.2), so $\mathscr{SN}$ fulfils Condition 1.

Since $\mathbb{N}$ is well-ordered and $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is strictly increasing, Condition 2 is fulfilled. To see that $\mathscr{SN}$ satisfies Condition 3, let $\varphi : E \to F$ be a strictly increasing mapping with $F \in \mathscr{SN}$. The latter means that there is a strictly increasing mapping $\psi : F \to G$ for a well-ordered poset G, for which $\psi \circ \varphi : E \to G$ is also strictly increasing, and thus E is in $\mathscr{SN}$.

We still need to verify Condition 4, which we do by mimicking the proof of the second item of Perdry (2004, Corollary 2.3). Let $(E_i)_{i\in I}$ be a family of posets in $\mathscr{SN}$, indexed by a poset $I \in \mathscr{SN}$. For every $i \in I$ we have a strictly increasing mapping $\varphi_i : E_i \to F_i$, where F_i is a well-founded poset. Now

$$\sum_{i\in I} E_i \to \sum_{i\in I} F_i, \quad (i, x) \mapsto (i, \varphi_i(x))$$

is strictly increasing. Since I is in $\mathscr{SN}$, it is in $\mathscr{WF}$, and thus $\sum_{i\in I} F_i$ is well-founded (Proposition 3.2), so $\sum_{i\in I} E_i \in \mathscr{SN}$. □

4. Labelled trees

Definition 4.1. A *finitely branching tree* is a poset T satisfying the following three conditions:

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— It has a least element ε (the *root* of the tree).
— For every $a \in T$ the set $D_a = \{x \in T \ : \ a < x\}$ (of the *descendants* of a) has a finite number of minimal elements (the *children* of a).
— For every $a \in T$, the set $\{x \in T \ : \ x < a\}$ (of the *ancestors* of a) is a finite chain (whose greatest element is the *parent* of a).

As every tree will be finitely branching, from now on we will just say *tree* without any further qualification.

Definition 4.2. Let T be a tree. If $D_a = \emptyset$, then a is a *leaf* of T. The elements of T are the *nodes* of the tree. A *branch* of T is a (possibly finite) sequence $a_0 = \varepsilon, a_1, a_2, \dots$ in T such that a_{i+1} is a child of a_i for all i.

If $u = a_0, a_1, a_2, \dots, a_n$ is a finite branch of T, then we say the node a_n *terminates* u or the branch u *ends with* a_n, and $|u| = n$ is the *length* of u. We use the conventions that the length of the empty sequence () is < 0, and that an infinite branch of T has length $\geqslant n$ for all $n \in \mathbb{N}$.

If every branch of T is finite, then T is a *well-founded tree*. If there is $N \in \mathbb{N}$ such that every branch of T is finite and has length $\leqslant N$, then T is a *finite tree*.

Note that the length of a finite branch is the number of steps rather than the number of elements.

Definition 4.3. Let T be a tree. A mapping φ from T to a set E is called a *labelling* of (the nodes of) T by (the elements of) E.

Let T be labelled by a poset E with labelling $\varphi : T \to E$. We say that T is a *(strictly) decreasing* tree if φ is a (strictly) decreasing mapping.

Let $N \in \mathbb{N}$. A (finite or infinite) branch $u = a_0, a_1, a_2, \dots$ of T *halts before* N if either $|u| < N$ or $|u| \geqslant N$ and there is $n < N$ with $\varphi(a_n) = \varphi(a_{n+1})$. We say that T has *depth* $\leqslant N$ if every branch of T halts before N. Finally, T has *finite depth* if it has depth $\leqslant N$ for some $N \in \mathbb{N}$.

This notion of depth is essentially the one given in Mines *et al.* (1988, I.5). If a branch halts before N, then it halts before M for every $M \geqslant N$; a finite branch $u = a_0, a_1, a_2, \dots, a_N$ halts before $|u| = N$ precisely when $\varphi(a_n) = \varphi(a_{n+1})$ for some $n < N$. Last but not least, only () halts before 0.

4.1. *The finite-depth property*

Definition 4.4. A poset E has the *finite-depth property* if every decreasing tree T labelled by E has finite depth. We use $\mathscr{FD}$ to denote the class of posets with the finite-depth property.

Proposition 4.1. The class $\mathscr{WF}$ is a subclass of $\mathscr{FD}$.

Proof. We need to prove that if E is in $\mathscr{WF}$, then it is in $\mathscr{FD}$. Let $H \subseteq E$ consist of the $z \in H$ for which every decreasing tree with root labelled by z has finite depth. To show that H is hereditary, we assume that $\forall x < y\ (x \in H)$, and deduce $y \in H$ as follows. Let T

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be a decreasing tree with root labelled by y. To prove that T has finite depth, let $a_1, \dots, a_k$ with $k \geqslant 0$ be the children of the root of T, labelled by $x_1, \dots, x_k$. If $y = x_i$ for some i, then T has depth $\leqslant 1$. If, otherwise, $y > x_i$ for all i, then $x_1, \dots, x_k \in H$ by hypothesis, so all the subtrees of T with roots $a_1, \dots, a_k$ have finite depth, and thus T also has finite depth. □

If a branch in a strictly decreasing tree halts before n, then it has length $< n$.

Remark 4.1. If a poset E is in $\mathscr{F}\mathscr{D}$, then every strictly decreasing tree T labelled by E is finite.

We next check that $\mathscr{F}\mathscr{D}$ is a Noether class – the difficult point is Condition 4.

Lemma 4.1. Let I be a poset and E_i be a family of posets indexed by I. If $I \in \mathscr{C}$ and $E_i \in \mathscr{C}$ for all $i \in I$, then $\sum_{i \in I} E_i \in \mathscr{C}$.

Proof. Let T be a tree and φ be a decreasing mapping $\varphi : T \to \sum_{i \in I} E_i$. Let $\psi = \pi \circ \varphi : T \to I$, that is, $\psi(a) = i$ precisely when $\varphi(a) \in E_i$. Note that ψ is a decreasing mapping.

For each $a \in T$, the set

$$T_a = \{x \in T \ : \ a \leqslant x \text{ and } \psi(x) = \psi(a)\}$$

is a tree with root a. Moreover, the restriction of φ to T_a assumes its values in $E_{\psi(a)}$ and is decreasing. Since $E_{\psi(a)}$ is in $\mathscr{C}$, the tree T_a has finite depth N_a. Let L_a be the finite set consisting of all the leaves of T_a that terminate a branch of length $\leqslant N_a$. Considering the elements of L_a as elements of T, we use C_a to denote the finite set of their children.

Let S be a finite subset of T. We use S' to denote the set $S' = \bigcup_{a \in S} C_a$, and N_S to denote the supremum $N_S = \sup_{a \in S} N_a$ with $N_\varnothing = 0$. Let $S_0 = \{\varepsilon\}$ and $S_{n+1} = S'_n$, and set $N_n = N_{S_n}$. By construction, a branch of length greater than N_0 that does not halt before N_0 cuts S_1; a branch of length greater than $N_0 + N_1$ that does not halt before $N_0 + N_1$ cuts S_2; and so on.

(We can give a more precise argument for the induction step as follows. Let $n \in \mathbb{N}$ and $u = a_0, a_1, \dots$ a branch in T of length $\geqslant N_0 + \cdots + N_{n+1}$, and $a_p \in S_n$ with $1 \leqslant p \leqslant N_0 + \cdots + N_n$, and if $n = 0$, then $p = 0$. If u does not halt before $N_0 + \cdots + N_{n+1}$, then $a_q \notin T_{a_p}$ for some q with

$$p + 1 \leqslant q \leqslant p + N_{n+1} \leqslant N_0 + \cdots + N_{n+1}\,.$$

If q is the least number with this property, then $a_{q-1} \in L_{a_p}$ and $a_q \in C_{a_p} \subseteq S_{n+1}$.)

The union $S_\omega = \bigcup_n S_n$ is a tree. The restriction of ψ to S_ω is a strictly decreasing mapping, and I is in $\mathscr{F}\mathscr{D}$, so S_ω is finite. Hence, there is $N \in \mathbb{N}$ such that $S_N = \varnothing$, so T has finite depth $\sum N_n$. □

Proposition 4.2. The class $\mathscr{F}\mathscr{D}$ is a Noether class.

Proof. We first prove Condition 1. Given $x_0 \geqslant x_1 \geqslant \cdots$ in a poset E, consider $T = \mathbb{N}$ as the decreasing tree labelled by E with $n \mapsto x_n$. The one and only infinite branch of T is $0, 1, 2, \dots$ Now, if $E \in \mathscr{F}\mathscr{D}$, this branch halts, that is, $x_i = x_{i+1}$ for some i.

Condition 2 is readily seen from the fact that if T is a decreasing tree with labelling $\varphi : T \to \mathbb{N}$, then T has depth $\leqslant N = \varphi(\varepsilon) + 1$.

To verify Condition 3, let $\varphi : T \to E$ be an increasing mapping, and $\psi : E \to F$ be a strictly increasing mapping. Suppose that F is in $\mathscr{F}\mathscr{D}$, so T as labelled by $\psi \circ \varphi : T \to F$ has depth $\leqslant N$. Since ψ is strictly increasing, T also has depth $\leqslant N$, as labelled by φ (Remark 2.1).

Finally, Condition 4 is Lemma 4.1. □

By Proposition 4.2, $\mathscr{F}\mathscr{D}$ is a subclass of $\mathscr{R}\mathscr{S}$; we will now classify the reverse inclusion. Every finite tree is well–founded; the converse is (a general form of) Brouwer's fan theorem.

Definition 4.5. The *generalized fan theorem* (GFT) says that for every tree T, if T is well-founded, then T is finite.

Proposition 4.3. The GFT is equivalent to the assertion that $\mathscr{R}\mathscr{S}$ is a subclass of $\mathscr{F}\mathscr{D}$.

Proof. Assume first that GFT is valid. Let E be a poset in $\mathscr{R}\mathscr{S}$. To show that E is in $\mathscr{F}\mathscr{D}$, let T be a decreasing tree with labelling $\varphi : T \to E$. By $\mathscr{R}\mathscr{S}(E)$, every branch of T halts. The tree

$$T_\varphi = \{a \in R \ : \ \forall x < a, (\varphi(x) > \varphi(a))\}$$

is obtained from T by dropping every subtree of T whose root b is a child of some $a \in T$ with $\varphi(a) = \varphi(b)$. Now T_φ is well-founded. By the GFT, T_φ is finite, so T has finite depth.

Conversely, suppose $\mathscr{R}\mathscr{S}$ is a subclass of $\mathscr{F}\mathscr{D}$. To prove the GFT, let T be a tree, and assume that T is well-founded. With T° for T with the reverse order, we have $T^\circ \in \mathscr{R}\mathscr{S}$, as we shall show below. From this we arrive at $T^\circ \in \mathscr{F}\mathscr{D}$ by hypothesis. By applying the latter to T labelled by the identity mapping $\mathrm{id} \ : \ T \to T^\circ$, we have that T with this labelling has finite depth. Since T with this labelling is *strictly* decreasing, T is finite (Remark 4.1).

To prove that T° is in $\mathscr{R}\mathscr{S}$, let $a_0 \leqslant a_1 \leqslant \dots$ be a non–decreasing sequence in T, that is, a decreasing sequence in T°. Set

$$N_i = |\{x \in T \ : \ x < a_i\}|$$

for every $i \geqslant 0$, and for which $N_0 \leqslant N_1 \leqslant \dots$ and

$$N_i < N_{i+1} \iff a_i < a_{i+1}\,.$$

Arrange the $x \in T$ with $x < a_0$ as the finite branch $u_0 = b_0, b_1, \dots, b_{N_0-1}$. In particular, $b_0 = \varepsilon$ unless $N_0 = 0$, in which case $u_0 = ()$. Extend u_0 to the branch $u = b_0, b_1, b_2, \dots$ as follows. Assume that the $b_0, b_1, \dots, b_{N_i-1}$ are already constructed; this is the case for $i = 0$. If $a_j < a_{j+1}$ for all $j < i$ but $a_i = a_{i+1}$, then let u terminate with $b_{N_i} = a_i$, for which $|u| = N_i$. If, however, $a_j < a_{j+1}$ for all $j \leqslant i$, then extend u by the $b_{N_i}, \dots, b_{N_{i+1}-1}$ that exhaust the $x \in T$ with $a_i \leqslant x < a_{i+1}$. Now if T is well–founded, then u is finite, so $|u| = N_i$ for some i for which $a_i = a_{i+1}$. □

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4.2. *The descending tree property*

Finally, we adapt Richman's ascending tree property (Richman 2003) to our setting. Unlike Richman, we restrict our attention to trees that are finitely branching, but focus on trees that are spreads (that is, without leaves) after a while. For simplicity, and to follow Richman's terminology more closely, from now on we will write *descending tree* whenever we mean a decreasing tree.

Definition 4.6. A descending tree *halts* if either T has a leaf or there is a finite branch u in T that halts before $|u|$. A poset E has the *descending tree property* if every descending tree labelled by E halts. We use $\mathscr{DT}$ to denote the class of posets with the descending tree property.

Note that every finite tree halts since it has a leaf – in general it has more than one.

Proposition 4.4. The class $\mathscr{FD}$ is a subclass of $\mathscr{DT}$.

Proof. Let T be a descending tree labelled by a poset E. Since E is in $\mathscr{FD}$, there is $N \in \mathbb{N}$ such that every branch in T halts before N. We look at the finitely many branches u in T with $|u| \leqslant N$. Either $|u| < N$ for every branch u of this sort, in which case T is finite and thus halts, or there is a branch u in T with $|u| = N$, which halts before $|u|$ by our choice of N. This proves that E is in $\mathscr{DT}$. □

We will now relate $\mathscr{DT}$ to Richman's original concept (Richman 2003).

Definition 4.7. A *spread* is a tree in which every node has a successor. A *spread family* in a poset E is a family $(e_t)_{t\in T}$ of elements of E indexed by a spread T. Any such family is *descending* if $e_s \geqslant e_t$ whenever $s \leqslant t$ for all $s,t \in T$; it *halts* if there are $s,t \in T$ with $s < t$ such that $e_s = e_t$.

A spread has no leaves at all. If a descending spread family $(e_t)_{t\in T}$ halts, then there are $s,t \in T$ with $s < t$ such that $e_s = e_t$ and, in addition, s is the parent of t.

Lemma 4.2. A poset E is in $\mathscr{DT}$ if and only if every descending spread family in E halts.

Proof. Let E be a poset. Assume first that E is in $\mathscr{DT}$ and let $(e_t)_{t\in T}$ be a descending family in E indexed by a spread T. View T as labelled by E with labelling $t \mapsto e_t$. Since E is in $\mathscr{DT}$, this descending tree halts. As a spread, T has no leaves, so $(e_t)_{t\in T}$ halts.

Assume next that every descending spread family in E halts, and let T be a descending tree labelled by $\varphi : T \to E$. We extend T to a spread T' by attaching a linear spread to each leaf of T: for instance, let

$$T' = T \cup \bigcup_{t\in L} \{(t,n) : n \in \mathbb{N}\}$$

where L is the set of leaves of T and

$$t < (t,0) < (t,1) < \dots$$

for every $t \in L$. Also, we extend φ to a labelling $\varphi' : T' \to E$ by setting $\varphi'(b) = \varphi(a)$ whenever b is a successor of a leaf a of T. Now $\left(\varphi'(t)\right)_{t\in T'}$ is a descending spread family

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in E, so, by hypothesis, there are $s, t \in T'$ with $s < t$ such that $\varphi'(s) = \varphi'(t)$. If $t \notin T$, then T has a leaf. If $t \in T$, the finite branch u in T leading from ε through s to t halts before $|u|$. □

A descending chain is just a descending family indexed by the spread $\mathbb{N}$.

Corollary 4.1. The class $\mathscr{DT}$ is a subclass of $\mathscr{RS}$.

By dependent choice, every spread has an infinite branch.

Remark 4.2. By dependent choice, $\mathscr{RS}$ is a subclass of $\mathscr{DT}$.

It is noteworthy that, in view of Propositions 4.4 and 4.3, the inclusion $\mathscr{RS} \subseteq \mathscr{DT}$ also follows from the GFT. Moreover, the statement

$$\text{'every spread has an infinite branch'}, \tag{$*$}$$

which is sufficient for $\mathscr{RS} \subseteq \mathscr{DT}$, is a consequence of König's Lemma, since every spread is an infinite tree (that is, has finite branches of arbitrary length). As recalled in Berger *et al.* (2009) for detachable binary trees, $(*)$ can be proved without any choice whenever one can distinguish a child of each node (for example, 'the first child'), in which case an infinite branch of a spread can be defined recursively by taking the distinguished child as the next node at any step.

Lemma 4.3. Let E and F be posets such that there is an increasing mapping $\varphi : E \to F$. If F is in $\mathscr{DT}$, then so is E.

Proof. We understand $\mathscr{DT}$ to be as characterised by Lemma 4.2. Let $(e_t)_{t\in T}$ be a descending family of elements of E indexed by a spread T. Since φ is increasing, the family $(\varphi(e_t))_{t\in T}$ of elements of F indexed by the same spread T is also descending. Since F is in $\mathscr{DT}$, there are $s, t \in T$ with $s < t$ such that $\varphi(e_s) = \varphi(e_t)$. Since $e_s \geqslant e_t$ and φ is increasing, by Remark 2.1, we have $e_s = e_t$. □

Lemma 4.4. Let $(E_i)_{i\in I}$ be a family of posets indexed by a poset I. In the presence of dependent choice, if $I \in \mathscr{RS}$ and $E_i \in \mathscr{DT}$ for all $i \in I$, then $\sum_{i\in I} E_i \in \mathscr{DT}$.

Proof. Suppose $I \in \mathscr{RS}$ and $E_i \in \mathscr{DT}$ for all $i \in I$, and set $E = \sum_{i\in I} E_i$. To prove that E is in $\mathscr{DT}$ as characterised by Lemma 4.2, let $(e_t)_{t\in T}$ be a descending family in E indexed by a spread T. Let $e_t = (i_t, x_t)$ where $i_t \in I$ and $x_t \in E_{i_t}$ for every $t \in T$, and note that $(i_t)_{t\in T}$ is a descending family in I indexed by T. For each $t \in T$, consider the subtree $T_t = \{s \in T : s \geqslant t\}$ of T with root t. We write $p(s)$ for the parent of any $s \in T_t$ with $s > t$ and define a descending family $\left(y_{t,s}\right)_{s\in T_t}$ in E_{i_t} indexed by T_t by setting $y_{t,s} = x_s$ when $i_s = i_t$ (in particular, $y_{t,t} = x_t$) and $y_{t,s} = y_{t,p(s)}$ whenever $i_s < i_t$ (in which case $s > t$). Note that T_t is a spread for every $t \in T$.

Next, we construct a chain $t(0) < t(1) < \ldots$ in T with

$$y_{t(k),p(t(k+1))} = y_{t(k),t(k+1)} \tag{1}$$

for every $k \geqslant 0$ as follows. Set $t(0) = \varepsilon$. If $t(k)$ has already been constructed, consider the descending family $\left(y_{t(k),s}\right)_{s\in T_{t(k)}}$ in $E_{i_{t(k)}}$ indexed by $T_{t(k)}$. Since $E_{i_{t(k)}}$ is in $\mathscr{DT}$, there is

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$s > t(k)$ with $y_{t(k),p(s)} = y_{t(k),s}$. Set $t(k+1) = s$ for any such s. (This construction requires a certain amount of dependent choice.)

Since I is in $\mathscr{RS}$, there is $k \geqslant 0$ with $i_{t(k)} = i_{t(k+1)}$. Take the least k of this sort and set $s = p(t(k+1))$. Note that $i_{t(k)} = i_s = i_{t(k+1)}$ because $t(k) \leqslant s < t(k+1)$. Now $y_{t(k),s} = x_s$ and $y_{t(k),t(k+1)} = x_{t(k+1)}$ by their definition, so, in view of (1), we have $x_s = x_{t(k+1)}$ and thus $e_s = e_{t(k+1)}$ as required (recall that $i_s = i_{t(k+1)}$). □

Note that $I \in \mathscr{RS}$ was sufficient for this proof, which, however, required some dependent choice. This leads us to ask whether this can be done without any choice by using $I \in \mathscr{DT}$ instead, which we anyway have at our disposal when Condition 4 of '$\mathscr{DT}$ is a Noether class' is at stake.

Proposition 4.5. The class $\mathscr{DT}$ is a Noether class.

Proof. Corollary 4.1 says that $\mathscr{DT}$ satisfies Condition 1. Since $\mathbb{N}$ is in $\mathscr{FD}$ by Proposition 4.2, and thus in $\mathscr{DT}$ by Lemma 4.4, Condition 2 is fulfilled. Conditions 3 and 4 are just Lemmas 4.3 and 4.4, respectively. □

5. Discussion

We have proved the following inclusions between Noether properties:

$$\mathscr{SN} \subseteq \mathscr{WF} \subseteq \mathscr{FD} \subseteq \mathscr{DT} \subseteq \mathscr{RS}.$$

While the converse of $\mathscr{DT} \subseteq \mathscr{RS}$ can be proved using dependent choice (Remark 4.2), the converse of $\mathscr{FD} \subseteq \mathscr{RS}$ is equivalent to the generalised fan theorem (Proposition 4.3). One may ask whether there are any (Brouwerian) counterexamples, or (sheaf/topos) countermodels, for the converses of $\mathscr{SN} \subseteq \mathscr{WF}$ and $\mathscr{WF} \subseteq \mathscr{FD}$. And is the statement 'every spread has an infinite branch', which we used to prove $\mathscr{RS} \subseteq \mathscr{DT}$, also necessary for this implication, or to which form of dependent choice is this implication equivalent?

In the follow-up paper, we will show how, with reasonable conditions imposed on the ring, a minimal prime decomposition *à la* Lasker–Noether can be achieved with $\mathscr{FD}$ in place of $\mathscr{SN}$, which was used for this purpose prior to Perdry (2004). One may ask whether $\mathscr{FD}$, or perhaps $\mathscr{DT}$, is enough to prove the termination of further algorithms in commutative algebra, or elsewhere? And last, but not least, does Theorem 3.1 have any applications in proof theory or in descriptive set theory?

Acknowledgements

Some of the questions listed in the Discussion were posed by Peter Aczel, or inspired by his comments. Henri Lombardi and Alban Quadrat enabled the two authors of this paper to get together in Besançon and Oberwolfach to finish the present work. The final version was produced when the second author was hosted in Genoa by Pino Rosolini. Last, but not least, the authors are grateful to the anonymous referees.

https://doi.org/10.1017/S0960129510000460 Published online by Cambridge University Press

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