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Further inequalities and properties of p-inner parallel bodies

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Abstract. A. R. Martínez Fernández obtained upper bounds for quermassintegrals of the p-inner parallel bodies: an extension of the classical inner parallel body to the L_p-Brunn-Minkowski theory. In this paper, we establish (sharp) upper and lower bounds for quermassintegrals of p-inner parallel bodies. Moreover, the sufficient and necessary conditions of the equality case for the main inequality are obtained, which characterize the so-called tangential bodies.

1 Introduction

An important observation in convex geometry is that the volume of a linear combination of convex bodies (compact convex set with nonempty interior) behaves as a polynomial. The coefficients of this polynomial are the so-called *mixed volumes*. That is to say, given convex bodies $K_1, \dots, K_m$, then

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} V(K_{i_1}, \dots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}, \tag{1.1}$$

where λ_i is non-negative, $\lambda_1 K_1 + \cdots + \lambda_m K_m$ means their linear combination, and $V(K_{i_1}, \dots, K_{i_n})$ are the *mixed volumes* of the convex bodies $K_{i_1}, \dots, K_{i_n}$. For the sake of brevity, we denote $(K_1[r_1], \dots, K_m[r_m]) \equiv (\underbrace{K_1, \dots, K_1}_{r_1}, \dots, \underbrace{K_m, \dots, K_m}_{r_m})$. If there are only two convex bodies K, E involved in the above sum, then formula (1.1) becomes the so-called relative Steiner formula:

$$V(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} W_i(K; E) \lambda^i,$$

where $\lambda \geq 0$, and the Minkowski addition $K + \lambda E$ of K and λE is the *outer parallel body* of K relative to E. The coefficients $W_i(K; E) = V(K[n-i], E[i])$ are called *relative quermassintegrals* of K (see [11, Section 5.1]). In particular, we have $W_0(K; E) = V(K)$ and $W_n(K; E) = V(E)$.

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Complementing the outer parallel body, we find the so-called *inner parallel body* K_λ of a convex body K relative to a convex body E for $-r(K;E) \le \lambda \le 0$:

$$K_\lambda = K \sim |\lambda|E = \{x \in \mathbb{R}^n : |\lambda|E + x \subseteq K\};$$

i.e., K_λ is the Minkowski difference of K and $|\lambda|E$. Here, $r(K;E) = \max\{r \ge 0 :$ there is $x \in \mathbb{R}^n$ with $x + rE \subseteq K\}$ is the *relative inradius* of K with respect to E. We write $r = r(K;E)$ for the sake of brevity. K_{-r} is the *kernel* of K with respect to E.

It is natural to consider whether there is a counterpart to the Steiner formula for the inner parallel bodies. However, the boundary structure of the inner parallel bodies is rather more difficult to control, and it also means that there is no direct way to compute their volume (quermassintegrals). Therefore, Hernández Cifre et al. [5] gave upper bounds for quermassintegrals of inner parallel bodies:

Theorem A *Let K, E be convex bodies, and let E be strictly convex and regular. For $-r \le \lambda \le 0$ and $i = 0, \dots, n-1$,*

$$W_i(K_\lambda;E) \le W_i(K;E) - |\lambda| \sum_{k=0}^{n-i-1} V(K_\lambda[k], K[n-i-k-1], K^*, E[i]). \tag{1.2}$$

If K is a tangential body of $K_{-r} + rE$ satisfying $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*)$, then equality holds in all the inequalities. Conversely, if equality holds in (1.2) for some $i \in \{0, \dots, n-1\}$, then K is a tangential body of $K_{-r} + rE$.

Here there are some important notions involved. A convex body K is called *strictly convex* if its boundary bd$K$ does not contain a segment and *regular* if the supporting hyperplane (see Section 2 for the detailed definition) of K at any boundary point is unique.

Let $\mathcal{U}_0(K)$ denote the set of the so-called 0-extreme normal vectors of K, i.e., those ones that cannot be written as a positive combination of two linearly independent normal vectors at one and the same boundary point of K.

Now we introduce the definition of *tangential body* which is closely related to the 0-extreme normal vectors: a convex body K containing a convex body E is called a *tangential body* of E, if each 0-extreme support plane (see Section 2 for the detailed definition) of K supports E. Given a convex body E, a special tangential body of E is the relative *form body* K^* of a convex body K, i.e., the intersection of the supporting half-spaces to E with outer normal vectors in $\mathcal{U}_0(K)$ (see Section 2 for the detailed definition).

Inspired by the work of Hernández Cifre et al., the aim of this paper is to give (sharp) upper and lower bounds for quermassintegrals of inner parallel bodies in the framework of the L_p-Brunn–Minkowski theory. This theory, extending the classical Brunn–Minkowski theory to the L_p setting, has its origin in the early 1960s when Firey [1] introduced the concept of L_p combinations of convex bodies. In [6] and [7], the L_p combinations of convex bodies were further investigated by Lutwak which leads to an embryonic L_p-Brunn–Minkowski theory. This new theory has attracted the interest of a large number of researchers in recent years (see e.g. the references in [11, Section 9.1]).

In particular, Martínez Fernández et al. [9] introduced the p-inner parallel body of convex bodies in the L_p-Brunn-Minkowski theory. For the sake of brevity, let $\mathcal{K}^n$ denote the set of convex bodies, and $\mathcal{K}_0^n$ denote the subset of $\mathcal{K}^n$ consisting in all convex bodies containing the origin. For $K \subseteq \mathbb{R}^n$, cl$K$ denotes its closure. When dealing with the p-inner parallel bodies, we need the subfamily of convex sets $\mathcal{K}_{00}^n(E) = \{K \in \mathcal{K}_0^n : 0 \in K_{-r}\}$ for $E \in \mathcal{K}_0^n$. We observe that if $K \in \mathcal{K}_{00}^n(E)$, then 0 lies in the interior of K. Let $E \in \mathcal{K}_0^n$ and $K \in \mathcal{K}_{00}^n(E)$. The *$p$-inner parallel body* K_λ^p of K relative to E is defined by

$$K_\lambda^p = \bigcap_{u \in S^{n-1}} \left\{x \in \mathbb{R}^n : x \cdot u \le \left[h(K,u)^p - |\lambda|^p h(E,u)^p\right]^{\frac{1}{p}}\right\} \text{ for } -r \le \lambda \le 0;$$

i.e., K_λ^p is the p-difference of K and $|\lambda|E$. Here, $x \cdot u$ denotes the standard inner product of x and u in $\mathbb{R}^n$, and $h(K,\cdot)$ denotes the support function of K (see Section 2 for the detailed definition). K_{-r}^p is the *p-kernel* of K with respect to E. When $p = 1$, we obtain the usual inner parallel body.

Our first main result is the following theorem.

Theorem 1.1 *Let $E \in \mathcal{K}_0^n$ be regular, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. Then $K = K_\lambda^p +_p |\lambda| K^*$ for any $-r \le \lambda \le 0$ if and only if K is a tangential body of $K_{-r}^p +_p rE$ satisfying that for any $-r \le \lambda \le 0$,*

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p +_p K^*). \tag{1.3}$$

Here, "$+_p$" denotes the L_p Minkowski addition (see Section 2 for the detailed definition). When $p = 1$, Hernández Cifre et al. [5] utilized the derivative of the support function of the inner parallel body to study the above theorem. However, we will use a different method to study it for all $p \ge 1$.

In the following main result, we give upper and lower bounds of $W_i(K_\lambda^p; E)$ and the sufficient and necessary conditions for the equality case of the upper bound are obtained.

Theorem 1.2 *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$.*

(1) *For all $-r < \lambda \le 0$ and $i = 0, \dots, n-1$, we have*

$$\begin{aligned} W_i(K_\lambda^p; E) \le{}& \left(1 - \left|\frac{\lambda}{r}\right|^p\right)^{\frac{-(n-i)}{q}} W_i(K;E) - |\lambda| \sum_{k=0}^{n-i-1} \left|\frac{\lambda}{r}\right|^{p-1} \\ &\times \left(1 - \left|\frac{\lambda}{r}\right|^p\right)^{\frac{-(n-i-k)}{q}} V(K_\lambda^p[k], K[n-i-k-1], K^*, E[i]), \end{aligned} \tag{1.4}$$

where q is the Hölder conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

(2) *Suppose E is regular and strictly convex. If K is homothetic to a tangential body of E satisfying (1.3) for all $-r < \lambda \le 0$, then equality holds in (1.4) for all $i = 0, \dots, n-1$ and all $-r < \lambda \le 0$. Conversely, if equality holds in (1.4) for some $i \in \{0, \dots, n-1\}$ and some $-r < \lambda < 0$, then K is homothetic to a tangential body of E.*

(3) *If* $cl\mathcal{U}_0(K^p_\lambda) = \mathcal{U}_0(K^p_\lambda +_p (K^p_\lambda)^*)$ *for all* $-r < \lambda \le 0$, *then, for any* $i = 0, \dots, n-1$,

(1.5)
$$W_i(K^p_\lambda; E) \ge W_i(K; E) - |\lambda| \sum_{k=0}^{n-i-1} V(K^p_\lambda[k], K[n-i-k-1], (K^p_\lambda)^*, E[i]).$$

When $p = 1$, then $q \to \infty$ and, in this case, the limit of the coefficients $(1 - |\frac{\lambda}{r}|^p)^{\frac{-(n-i)}{q}}$ and $|\frac{\lambda}{r}|^{p-1}(1 - |\frac{\lambda}{r}|^p)^{\frac{-(n-i-k)}{q}}$ in inequality (1.4) is 1. Inequality (1.4) reduces to inequality (1.2).

In [8], Martínez Fernández also gave an estimate for the upper bound of $W_i(K^p_\lambda; E)$, but without conditions for the equality case.

2 Background material

In this section, we will review some basic facts in convex geometry. Good general references for the theory of convex bodies include the books of Gardner [2] and Schneider [11].

If $K \subset \mathbb{R}^n$ is a closed convex set, then its support function, $h(K, \cdot) : \mathbb{R}^n \to (-\infty, +\infty)$, is defined by

$$h(K, u) = \max\{x \cdot u : x \in K\}, \quad u \in \mathbb{R}^n.$$

Obviously, for $K, L \in \mathcal{K}^n$,

(2.1)
$$K \subseteq L \quad \text{if and only if} \quad h(K, \cdot) \le h(L, \cdot).$$

Let $K \in \mathcal{K}^n$. For each $u \in \mathbb{R}^n \backslash \{0\}$, the hyperplane

$$H_K(u) = \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\}$$

is called the *supporting hyperplane* of K with outer normal u.

In the early 1960s, Firey [1] introduced the concept of L_p *Minkowski addition* (also known as the Minkowski–Firey addition) of convex bodies: if $K, L \in \mathcal{K}^n_0$ and $p \ge 1$, then the L_p Minkowski addition $K +_p E$ is defined by

$$h(K +_p L, \cdot)^p = h(K, \cdot)^p + h(L, \cdot)^p.$$

When $p = 1$, the usual Minkowski addition is obtained, namely,

$$K + L = \{x + y : x \in K, y \in L\}.$$

Moreover, Firey [1] showed that for any $1 \le s \le t$,

(2.2)
$$K +_t L \subseteq K +_s L.$$

In addition, from the definition of p-inner parallel body, it follows that for $1 \le p < \infty$ and all $-r \le \lambda \le 0$,

$$h(K^p_\lambda, u)^p \le h(K, u)^p - |\lambda|^p h(E, u)^p.$$

Notice that $h(K^p_\lambda, u)^p = h(K, u)^p - |\lambda|^p h(E, u)^p$ for $u \in \mathrm{cl}\mathcal{U}_0(K^p_\lambda)$ (see e.g. [11, p. 411]).

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For $K, K_1, \dots, K_{n-1} \in \mathcal{K}^n$, the mixed volume has the integral representation (see e.g. [11, Theorem 5.1.7])

$$V(K, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \dots, K_{n-1}; u), \tag{2.3}$$

where $S(K_1, \dots, K_{n-1}; \cdot)$ is the *mixed surface area measure* of $K_1, \dots, K_{n-1}$ on S^{n-1}. For an exhaustive study of mixed volumes and mixed surface area measures, we refer to [11, Section 5.1].

An outer normal vector of K is called r-extreme normal vector, $r = 0, 1, \dots, n-1$, if it cannot be written as a positive combination of $r+2$ linearly independent normal vectors at one and the same boundary point of K. We denote the set of r-extreme normal vectors of K by $\mathcal{U}_r(K)$. Notice that each r-extreme normal vector is also an s-extreme one for $r < s \le n-1$. When $r = 0$, we obtain the 0-extreme normal vector. A support plane is said to be 0-extreme if its outer normal vector is 0-extreme. A useful fact is that a convex body K is the intersection of the supporting half-spaces to K with outer normal vectors in $\mathcal{U}_0(K)$, namely,

$$K = \bigcap_{u \in \mathcal{U}_0(K)} \{x : x \cdot u \le h(K, u)\},$$

(see e.g. [10, (2.9)]). One can observe that the set $\mathcal{U}_0(K)$ can be replaced by $\mathrm{cl}\mathcal{U}_0(K)$ from the continuity of the support function.

For the r-extreme normal vectors, there is the following property (see [11, pp. 135–136]): if $K, E \in \mathcal{K}^n$, and E is regular and strictly convex, then $\mathrm{cl}\mathcal{U}_r(K) = \mathrm{supp}S(K[n-r-1], E[r]; \cdot)$. Here, $\mathrm{supp}\nu$ denotes the support of the measure ν.

The (relative) *form body* K^* of $K \in \mathcal{K}^n$ with respect to $E \in \mathcal{K}^n$ is defined as

$$K^* = \bigcap_{u \in \mathcal{U}_0(K)} \{x : x \cdot u \le h(E, u)\}.$$

In this definition, it is necessary to assume that K has interior points. If not so, the form body K^* could be even the whole $\mathbb{R}^n$. Notice that the form body K^* is always a tangential body of E. In addition, the set $\mathcal{U}_0(K)$ can also be replaced by $\mathrm{cl}\mathcal{U}_0(K)$ in the above definition, because of the continuity of the support function (see e.g. [11, p. 386]).

3 Proofs of the main results

In order to prove the main results, we will list some particular properties of the p-inner parallel bodies with respect to their 0-extreme normal vectors. Firstly, we need the following equivalent definition of 0-extreme normal vector, see e.g. [10, Lemma 2.3], for its proof.

Lemma 3.1 *If $K \in \mathcal{K}^n$, $u \in \mathcal{U}_0(K)$ if and only if for any $u_1, u_2 \in S^{n-1}$ and $\alpha, \beta > 0$ such that $u = \alpha u_1 + \beta u_2$, it holds $h(K, u) < \alpha h(K, u_1) + \beta h(K, u_2)$.*

The following two lemmas, proved in [8, Propositions 4.1.11 and 4.1.8], will be needed.

Lemma 3.2 *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. Then for any $-r < \lambda \le 0$*

$$\mathcal{U}_0(K_\lambda^p) \subseteq \mathcal{U}_0(K). \tag{3.1}$$

Lemma 3.3 *Let $K, L \in \mathcal{K}_0^n$ and $1 \le p < \infty$. Then*

$$\mathcal{U}_0(K) \cup \mathcal{U}_0(L) \subseteq \mathcal{U}_0(K +_p L).$$

Let $+_p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote the binary operation which was introduced in [9]:

$$a +_p b = \begin{cases} \operatorname{sgn}_2(a,b)(|a|^p + |b|^p)^{\frac{1}{p}} & \text{if } ab \ge 0, \\ \operatorname{sgn}_2(a,b)(\max\{|a|,|b|\}^p - \min\{|a|,|b|\}^p)^{\frac{1}{p}} & \text{if } ab \le 0, \end{cases}$$

where the function $\operatorname{sgn}_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\operatorname{sgn}_2(a,b) = \begin{cases} \operatorname{sgn}(a) = \operatorname{sgn}(b) & \text{if } ab > 0, \\ \operatorname{sgn}(a) & \text{if } ab \le 0 \text{ and } |a| \ge |b|, \\ \operatorname{sgn}(b) & \text{if } ab \le 0 \text{ and } |a| < |b|. \end{cases}$$

Here, sgn denotes the usual sign function. Obviously, this operation satisfies two facts:

1. $a +_p b = b +_p a$ for all $a, b \in \mathbb{R}$; i.e., $+_p$ is commutative.
2. $(a +_p b) +_p c = a +_p (b +_p c) = (a +_p c) +_p b$ for all $a, b, c \in \mathbb{R}$; i.e., $+_p$ is associative.

The next three lemmas are extensions from the classical Brunn-Minkowski theory to its L_p counterpart (see [5, Lemmas 3.1–3.3]). However, the proofs require quite different techniques.

Lemma 3.4 *Let $K, L \in \mathcal{K}_0^n$, $\lambda > 0$, and $1 \le p < \infty$. Then*

$$\mathcal{U}_0(K +_p L) = \mathcal{U}_0(K +_p \lambda L).$$

Proof First, we assume that $0 < \lambda \le 1$. Let $u \in \mathcal{U}_0(K +_p L)$ and let $u_1, u_2 \in S^{n-1}$, $u_1 \ne u_2$ be such that $u = \alpha u_1 + \beta u_2$ with $\alpha, \beta > 0$. Then by Lemma 3.1, we have $h(K +_p L, u) < \alpha h(K +_p L, u_1) + \beta h(K +_p L, u_2)$. By Minkowski's inequality (see e.g. [3, p. 30]), we obtain

$$\begin{aligned} h(K +_p \lambda L, u) &= [h(K,u)^p + \lambda^p h(L,u)^p]^{\frac{1}{p}} \\ &= [\lambda^p h(K,u)^p + \lambda^p h(L,u)^p + (1-\lambda^p)h(K,u)^p]^{\frac{1}{p}} \\ &= [\lambda^p h(K +_p L, u)^p + (1-\lambda^p)h(K,u)^p]^{\frac{1}{p}} \\ &< \{\lambda^p[\alpha h(K +_p L, u_1) + \beta h(K +_p L, u_2)]^p \\ &\quad + (1-\lambda^p)[\alpha h(K,u_1) + \beta h(K,u_2)]^p\}^{\frac{1}{p}} \\ &\le [\alpha^p \lambda^p h(K +_p L, u_1)^p + \alpha^p(1-\lambda^p)h(K,u_1)^p]^{\frac{1}{p}} \\ &\quad + [\beta^p \lambda^p h(K +_p L, u_2)^p + \beta^p(1-\lambda^p)h(K,u_2)^p]^{\frac{1}{p}} \\ &= \alpha h(K +_p \lambda L, u_1) + \beta h(K +_p \lambda L, u_2). \end{aligned}$$

That implies $u \in \mathcal{U}_0(K +_p \lambda L)$, i.e., $\mathcal{U}_0(K +_p L) \subseteq \mathcal{U}_0(K +_p \lambda L)$.

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Using now that $h(K +_p L, u) = [h(K +_p \lambda L, u)^p + (1 - \lambda^p)h(L, u)^p]^{\frac{1}{p}}$, the same argument shows the reverse inclusion.

The case $\lambda \ge 1$ is analogous. ∎

In the following two lemmas, we discuss the equality conditions of (3.1). If $E \in \mathcal{K}_0^n$ is regular, we give a sufficient condition for the equality case of (3.1).

Lemma 3.5 *Let $E \in \mathcal{K}_0^n$ be regular, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. If K is a tangential body of $K_{-r}^p +_p rE$, then for any $-r < \lambda \le 0$,*

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p).$$

Proof First we prove that $\mathcal{U}_0(K) \subseteq \mathcal{U}_0(K_\lambda^p)$ for any $-r < \lambda \le 0$. Let $u \in \mathcal{U}_0(K)$. Since K is a tangential body of $K_{-r}^p +_p rE$, then $h(K, u) = h(K_{-r}^p +_p rE, u)$. it follows from the regularity of $E \in \mathcal{K}_0^n$ that $u \in \mathcal{U}_0(K) \subseteq \mathcal{U}_0(E)$. Let $u_1, u_2 \in S^{n-1}$, $u_1 \ne u_2$ be such that $u = \alpha u_1 + \beta u_2$ with $\alpha, \beta > 0$. Then, by Lemma 3.1, we have

$$h(E, u) < \alpha h(E, u_1) + \beta h(E, u_2).$$

As a direct consequence of [9, Proposition 4.2(ii)] one has $K_{-r}^p +_p (r +_p \lambda)E \subseteq K_\lambda^p$.

Combining these with the definition of p-inner parallel body and Minkowski's inequality, we obtain

$$\begin{aligned}
h(K_\lambda^p, u) &\le \Big[h(K, u)^p - |\lambda|^p h(E, u)^p\Big]^{\frac{1}{p}} \\
&= \Big[h(K_{-r}^p +_p rE, u)^p - |\lambda|^p h(E, u)^p\Big]^{\frac{1}{p}} \\
&= \Big[h(K_{-r}^p, u)^p + (r^p - |\lambda|^p)h(E, u)^p\Big]^{\frac{1}{p}} \\
&< \Big\{[\alpha h(K_{-r}^p, u_1) + \beta h(K_{-r}^p, u_2)]^p \\
&\quad + [\alpha h((r^p - |\lambda|^p)^{\frac{1}{p}} E, u_1) + \beta h((r^p - |\lambda|^p)^{\frac{1}{p}} E, u_2)]^p\Big\}^{\frac{1}{p}} \\
&\le \Big[\alpha^p h(K_{-r}^p, u_1)^p + \alpha^p h((r^p - |\lambda|^p)^{\frac{1}{p}} E, u_1)^p\Big]^{\frac{1}{p}} \\
&\quad + \Big[\beta^p h(K_{-r}^p, u_2)^p + \beta^p h((r^p - |\lambda|^p)^{\frac{1}{p}} E, u_2)^p\Big]^{\frac{1}{p}} \\
&= \alpha h(K_{-r}^p +_p (r +_p \lambda)E, u_1) + \beta h(K_{-r}^p +_p (r +_p \lambda)E, u_2) \\
&\le \alpha h(K_\lambda^p, u_1) + \beta h(K_\lambda^p, u_2).
\end{aligned}$$

Using the characterization of 0-extreme normal vectors which is given in Lemma 3.1, we get $u \in \mathcal{U}_0(K_\lambda^p)$. Thus, $\mathcal{U}_0(K) \subseteq \mathcal{U}_0(K_\lambda^p)$. From (3.1), we obtain $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p)$ for any $-r < \lambda \le 0$. ∎

If $K \in \mathcal{K}_{00}^n(E)$ is regular, we obtain the necessary and sufficient conditions for the equality case of (3.1).

Lemma 3.6 *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ be regular and $1 \le p < \infty$. Then $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p)$ for any $-r < \lambda \le 0$ if and only if $K = K_{-r}^p +_p rE$.*

Proof If

$$K = K_{-r}^p +_p rE,$$

then, by [9, Proposition 4.2 (iv)], we get for any $-r < \lambda \le 0$

$$K_\lambda^p = K_{-r}^p +_p (r +_p \lambda)E.$$

This together with Lemma 3.4 implies

$$\mathcal{U}_0(K_\lambda^p) = \mathcal{U}_0(K_{-r}^p +_p (r +_p \lambda)E) = \mathcal{U}_0(K_{-r}^p +_p rE) = \mathcal{U}_0(K).$$

Conversely, suppose $\mathcal{U}_0(K_\lambda^p) = \mathcal{U}_0(K)$ for any $-r < \lambda \le 0$. Since K is regular, we get

$$\mathcal{U}_0(K_\lambda^p) = \mathcal{U}_0(K) = S^{n-1}.$$

If $u \in \mathcal{U}_0(K_\lambda^p) = \mathcal{U}_0(K) = S^{n-1}$, we know

$$h(K_\lambda^p, u)^p = h(K, u)^p - |\lambda|^p h(E, u)^p.$$

That is

$$\begin{aligned} h(K, u) &= [h(K_\lambda^p, u)^p + |\lambda|^p h(E, u)^p]^{\frac{1}{p}} \\ &= h(K_\lambda^p +_p |\lambda| E, u) \text{ for all } u \in S^{n-1}, \end{aligned}$$

i.e.,

$$K = K_\lambda^p +_p |\lambda| E \text{ for any } -r < \lambda \le 0.$$

Taking into account that $\lim_{\lambda \to -r} K_\lambda^p = K_{-r}^p$ (see [9, Proposition 4.3]) and the continuity of the support function with respect to the Hausdorff metric, we have

$$\begin{aligned} h(K, u)^p &= \lim_{\lambda \to -r} [h(K_\lambda^p, u)^p + |\lambda|^p h(E, u)^p] \\ &= h(K_{-r}^p, u)^p + r^p h(E, u)^p \text{ for all } u \in S^{n-1}. \end{aligned}$$

This implies

$$K = K_{-r}^p +_p rE.$$

■

Remark 3.7 Notice that if $K = K_\lambda^p +_p |\lambda| E$ for $-r < \lambda \le 0$, then it also holds that $K = K_{-r}^p +_p rE$.

Using an analogous argument as in the proof of Lemma 3.6, the following properties hold:

(1) If K is a tangential body of $K_\lambda^p +_p |\lambda| E$ for $-r < \lambda \le 0$, then K is a tangential body of $K_{-r}^p +_p rE$;

(2) If K is a tangential body of $K_\lambda^p +_p |\lambda| K^*$ for $-r < \lambda \le 0$, then K is a tangential body of $K_{-r}^p +_p rK^*$.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Suppose $K = K_\lambda^p +_p |\lambda| K^*$ for any $-r \le \lambda \le 0$, and in particular, that $K = K_{-r}^p +_p rK^*$. Thus, for any $u \in \mathcal{U}_0(K)$,

$$\begin{aligned} h(K,u) &= h(K_{-r}^p +_p rK^*, u) \\ &= [h(K_{-r}^p, u)^p + r^p h(K^*, u)^p]^{\frac{1}{p}} \\ &= [h(K_{-r}^p, u)^p + r^p h(E, u)^p]^{\frac{1}{p}} \\ &= h(K_{-r}^p +_p rE, u). \end{aligned}$$

Since $K_{-r}^p +_p rE \subseteq K$ (see [9, Proposition 4.2(ii)]), K is a tangential body of $K_{-r}^p +_p rE$. Moreover, from Lemma 3.4, we get $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p +_p |\lambda| K^*) = \mathcal{U}_0(K_\lambda^p +_p K^*)$ for all $-r \le \lambda \le 0$.

Conversely, suppose that K is a tangential body of $K_{-r}^p +_p rE$ satisfying $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p +_p K^*)$ for any $-r \le \lambda \le 0$. Since E is regular, by Lemma 3.5, we get $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p)$ for any $-r < \lambda \le 0$. Combining this with Lemma 3.4, we obtain for any $-r < \lambda \le 0$

$$\begin{aligned} K_\lambda^p +_p |\lambda| K^* &= \bigcap_{u \in \mathcal{U}_0(K_\lambda^p +_p |\lambda| K^*)} \{x : x \cdot u \le h(K_\lambda^p +_p |\lambda| K^*, u)\} \\ &= \bigcap_{u \in \mathcal{U}_0(K_\lambda^p +_p K^*)} \{x : x \cdot u \le h(K_\lambda^p +_p |\lambda| K^*, u)\} \\ &= \bigcap_{u \in \mathcal{U}_0(K)} \{x : x \cdot u \le [h(K_\lambda^p, u)^p + |\lambda|^p h(E,u)^p]^{\frac{1}{p}}\} \\ &= \bigcap_{u \in \mathcal{U}_0(K_\lambda^p)} \{x : x \cdot u \le [h(K,u)^p - |\lambda|^p h(E,u)^p + |\lambda|^p h(E,u)^p]^{\frac{1}{p}}\} \\ &= \bigcap_{u \in \mathcal{U}_0(K_\lambda^p)} \{x : x \cdot u \le h(K,u)\} \\ &= \bigcap_{u \in \mathcal{U}_0(K)} \{x : x \cdot u \le h(K,u)\} \\ &= K. \end{aligned}$$

Remark 3.7 ensures that $K_{-r}^p +_p rK^* = K$. Hence $K_\lambda^p +_p |\lambda| K^* = K$ for any $-r \le \lambda \le 0$.

The following lemma can be found in [8, Proposition 4.2.3].

Lemma 3.8 *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. Then for any $-r \le \lambda \le 0$*

$$K_\lambda^p +_p |\lambda| K^* \subseteq K. \tag{3.2}$$

In order to get lower bounds for the quermassintegrals of the p-inner parallel bodies of a convex body, we need the following lemma.

Lemma 3.9 *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}_{00}^n(E)$ and $1 \le p < \infty$. If for any $-r < \lambda \le 0$*

$$cl\mathcal{U}_0(K_\lambda^p) = \mathcal{U}_0(K_\lambda^p +_p (K_\lambda^p)^*),$$

then the following property holds:

$$K \subseteq K_\lambda^p +_p |\lambda| (K_\lambda^p)^*. \tag{3.3}$$

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If E is regular, equality holds for any $-r < \lambda \le 0$ if and only if K is a tangential body of $K^p_{-r} +_p rE$.

Proof Since $\mathrm{cl}\mathcal{U}_0(K^p_\lambda) = \mathcal{U}_0(K^p_\lambda +_p (K^p_\lambda)^*)$ for any $-r < \lambda \le 0$, then, by Lemma 3.4 and Lemma 3.2, we get

$$\begin{aligned}
K^p_\lambda +_p |\lambda|(K^p_\lambda)^* &= \bigcap_{u\in\mathcal{U}_0(K^p_\lambda +_p|\lambda|(K^p_\lambda)^*)} \{x : x\cdot u \le h(K^p_\lambda +_p |\lambda|(K^p_\lambda)^*, u)\} \\
&= \bigcap_{u\in\mathcal{U}_0(K^p_\lambda +_p(K^p_\lambda)^*)} \{x : x\cdot u \le h(K^p_\lambda +_p |\lambda|(K^p_\lambda)^*, u)\} \\
&= \bigcap_{u\in\mathrm{cl}\mathcal{U}_0(K^p_\lambda)} \{x : x\cdot u \le [h(K^p_\lambda, u)^p + |\lambda|^p h((K^p_\lambda)^*, u)^p]^{\frac{1}{p}}\} \\
&= \bigcap_{u\in\mathrm{cl}\mathcal{U}_0(K^p_\lambda)} \{x : x\cdot u \le [h(K,u)^p - |\lambda|^p h(E,u)^p + |\lambda|^p h(E,u)^p]^{\frac{1}{p}}\} \\
&= \bigcap_{u\in\mathrm{cl}\mathcal{U}_0(K^p_\lambda)} \{x : x\cdot u \le h(K,u)\} \\
&\supseteq \bigcap_{u\in\mathrm{cl}\mathcal{U}_0(K)} \{x : x\cdot u \le h(K,u)\} \\
&= K.
\end{aligned}$$

Now we discuss the equality case of (3.3). Since E is regular, if K is a tangential body of $K^p_{-r} +_p rE$, Lemma 3.5 ensures that $\mathcal{U}_0(K^p_\lambda) = \mathcal{U}_0(K)$ for any $-r < \lambda \le 0$, and hence equality holds in (3.3) for any $-r < \lambda \le 0$.

Conversely, if $K = K^p_\lambda +_p |\lambda|(K^p_\lambda)^*$ for any $-r < \lambda \le 0$, then, by Lemma 3.4, we obtain $\mathcal{U}_0(K) = \mathcal{U}_0(K^p_\lambda +_p |\lambda|(K^p_\lambda)^*) = \mathcal{U}_0(K^p_\lambda +_p (K^p_\lambda)^*) = \mathrm{cl}\mathcal{U}_0(K^p_\lambda)$, and hence $K^* = (K^p_\lambda)^*$. Thus, $K = K^p_\lambda +_p |\lambda|K^*$ for $-r < \lambda \le 0$ and hence, by Remark 3.7, also $K = K^p_{-r} +_p rK^*$. Then Theorem 1.1 shows that K is a tangential body of $K^p_{-r} +_p rE$. ■

Finally, we give (sharp) upper and lower bounds for quermassintegrals of p-inner parallel bodies following the same idea as the one in [5]: we prove Theorem 1.2.

Proof of Theorem 1.2 (1) If $\lambda = 0$ the result is trivial. Therefore, from now, on we will work in the range $-r < \lambda < 0$. From (2.3), combining with (2.1), (3.2), and Hölder's inequality (see e.g. [3, p. 21]), we get for any $-r \le \lambda < 0$ that

$$\begin{aligned}
W_i(K;E) &= V(K[n-i], E[i]) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(K[n-i-1], E[i]; u) \\
&\ge \frac{1}{n}\int_{S^{n-1}} h(K^p_\lambda +_p |\lambda|K^*, u)dS(K[n-i-1], E[i]; u) \\
&= \frac{1}{n}\int_{S^{n-1}} [h(K^p_\lambda, u)^p + |\lambda|^p h(K^*, u)^p]^{\frac{1}{p}} dS(K[n-i-1], E[i]; u) \\
&\ge \frac{1}{n}\int_{S^{n-1}} \left[\left(1 - \left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{q}} h(K^p_\lambda, u) + \left|\frac{\lambda}{r}\right|^{\frac{p}{q}} |\lambda| h(K^*, u)\right] dS(K[n-i-1], E[i]; u) \\
&= \left(1 - \left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{q}} V(K^p_\lambda, K[n-i-1], E[i]) + \left|\frac{\lambda}{r}\right|^{p-1} |\lambda| V(K[n-i-1], K^*, E[i])
\end{aligned}$$

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$$\geq \left(1-\left|\frac{\lambda}{r}\right|^p\right)^{\frac{2}{q}} V(K_\lambda^p[2],K[n-i-2],E[i])$$
$$+\left(1-\left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{q}}\left|\frac{\lambda}{r}\right|^{p-1}|\lambda|V(K_\lambda^p,K[n-i-2],K^*,E[i])$$
$$+\left|\frac{\lambda}{r}\right|^{p-1}|\lambda|V(K[n-i-1],K^*,E[i]) \geq \cdots \geq \left(1-\left|\frac{\lambda}{r}\right|^p\right)^{\frac{n-i}{q}} W_i(K_\lambda^p;E)$$
$$+|\lambda|\sum_{k=0}^{n-i-1}\left(1-\left|\frac{\lambda}{r}\right|^p\right)^{\frac{k}{q}}\left|\frac{\lambda}{r}\right|^{p-1} V(K_\lambda^p[k],K[n-i-k-1],K^*,E[i]). \tag{3.4}$$

This gives inequality (1.4) when $-r < \lambda \leq 0$.

(2) Now we deal with the equality case of (1.4). Again we assume that $\lambda < 0$, otherwise the result is trivial. If K is homothetic to a tangential body of E, then [4, Lemma 19] yields $\left(1-\left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{p}} K = K_\lambda^p$ for all $-r < \lambda < 0$ and hence $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p)$. It is known that $h(K_\lambda^p,u)^p = h(K,u)^p - |\lambda|^p h(E,u)^p$ for any $u \in \mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda^p)$. That is $h(K_\lambda^p +_p |\lambda|E,u) = h(K,u)$ for any $u \in \mathcal{U}_0(K)$, which means that K is a tangential body of $K_\lambda^p +_p |\lambda|E$ for all $-r < \lambda < 0$. Remark 3.7 ensures that K is a tangential body of $K_{-r}^p +_p rE$. Moreover, since K satisfies (1.3) for all $-r < \lambda < 0$, by Theorem 1.1, we obtain $K = K_\lambda^p +_p |\lambda|K^*$ for all $-r < \lambda < 0$. Therefore, we have equality in the first inequality of (3.4).

On the other hand, since $\left(1-\left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{p}} K = K_\lambda^p$ for all $-r < \lambda < 0$, then $1-\left|\frac{\lambda}{r}\right|^p = \frac{h(K_\lambda^p,u)^p}{h(K,u)^p} = \frac{h(K_\lambda^p,u)^p}{h(K_\lambda^p,u)^p+|\lambda|^p h(K^*,u)^p}$ for every $u \in S^{n-1}$. That is $\frac{1-\left|\frac{\lambda}{r}\right|^p}{\left|\frac{\lambda}{r}\right|^p} = \frac{h(K_\lambda^p,u)^p}{|\lambda|^p h(K^*,u)^p}$ for all $-r < \lambda < 0$. Therefore, by the equality conditions of Hölder's inequality, equality holds in the second inequality of (3.4). In conclusion, equality holds in all the inequalities of (3.4) for all $-r < \lambda < 0$ and hence equality holds in (1.4) for all $i = 0,\ldots,n-1$.

Conversely, since (1.4) is established from (3.4), if equality holds in (1.4) for some $i \in \{0,\ldots,n-1\}$ and some $-r < \lambda < 0$, we must have, in particular

$$\int_{S^{n-1}} h(K,u)dS(K[n-i-1],E[i];u)$$
$$= \int_{S^{n-1}} h(K_\lambda^p +_p |\lambda|K^*,u)dS(K[n-i-1],E[i];u), \tag{3.5}$$

and

$$\int_{S^{n-1}} [h(K_\lambda^p,u)^p + |\lambda|^p h(K^*,u)^p]^{\frac{1}{p}} dS(K[n-i-1],E[i];u)$$
$$= \int_{S^{n-1}} \left[\left(1-\left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{q}} h(K_\lambda^p,u) + \left|\frac{\lambda}{r}\right|^{\frac{p}{q}} |\lambda| h(K^*,u)\right] dS(K[n-i-1],E[i];u).$$

Firstly, by (3.5) and (3.2), we deduce $h(K,u) = h(K_\lambda^p +_p |\lambda|K^*,u)$ for all $u \in \operatorname{supp} S(K[n-i-1],E[i];u)$. Since $E \in \mathcal{K}_0^n$ is regular and strictly convex, then

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$\mathrm{supp}S(K[n-i-1],E[i];\cdot) = \mathrm{cl}\mathcal{U}_i(K) \supseteq \mathrm{cl}\mathcal{U}_0(K)$. Hence, for all $u \in \mathcal{U}_0(K)$, we have $h(K,u) = h(K_\lambda^p +_p |\lambda|K^*, u)$.

Secondly, from Hölder's inequality, we get

$$\left[h(K_\lambda^p,u)^p + |\lambda|^p h(K^*,u)^p\right]^{\frac{1}{p}} \geq \left(1 - \left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{q}} h(K_\lambda^p,u) + \left|\frac{\lambda}{r}\right|^{\frac{p}{q}} |\lambda| h(K^*,u).$$

This together with (3.6) implies $[h(K_\lambda^p,u)^p + |\lambda|^p h(K^*,u)^p]^{\frac{1}{p}} = \left(1 - \left|\frac{\lambda}{r}\right|^p\right)^{\frac{1}{q}} h(K_\lambda^p,u) + \left|\frac{\lambda}{r}\right|^{\frac{p}{q}}|\lambda| h(K^*,u)$ for all $u \in \mathrm{supp}S(K[n-i-1],E[i];u)$. According to the equality conditions of Hölder's inequality, then $\frac{1-\left|\frac{\lambda}{r}\right|^p}{\left|\frac{\lambda}{r}\right|^p} = \frac{h(K_\lambda^p,u)^p}{|\lambda|^p h(K^*,u)^p}$ for all $u \in \mathrm{supp}S(K[n-i-1],E[i];\cdot) = \mathrm{cl}\mathcal{U}_i(K) \supseteq \mathrm{cl}\mathcal{U}_0(K)$ and hence $1 - \left|\frac{\lambda}{r}\right|^p = \frac{h(K_\lambda^p,u)^p}{h(K_\lambda^p,u)^p + |\lambda|^p h(K^*,u)^p} = \frac{h(K_\lambda^p,u)^p}{h(K,u)^p}$ for all $u \in \mathcal{U}_0(K)$. Thus,

$$\begin{aligned}
K_\lambda^p &= \bigcap_{u \in S^{n-1}} \{x \cdot u \leq h(K_\lambda^p,u)\} \\
&\subseteq \bigcap_{u \in \mathcal{U}_0(K)} \{x \cdot u \leq (1 - |\frac{\lambda}{r}|^p)^{\frac{1}{p}} h(K,u)\} \\
&= (1 - |\frac{\lambda}{r}|^p)^{\frac{1}{p}} K.
\end{aligned}$$

Since [4, Lemma 19] shows that $(1 - \left|\frac{\lambda}{r}\right|^p)^{\frac{1}{p}} K \subseteq K_\lambda^p$, we have $(1 - \left|\frac{\lambda}{r}\right|^p)^{\frac{1}{p}} K = K_\lambda^p$. By the equality conditions of [4, Lemma 19], we obtain that K is homothetic to a tangential body of E.

(3) We finally prove (1.5). Let $\mathrm{cl}\mathcal{U}_0(K_\lambda^p) = \mathcal{U}_0(K_\lambda^p +_p (K_\lambda^p)^*)$ for all $-r < \lambda \leq 0$. Using (2.3), (2.1), (2.2) and Lemma 3.9, we get

$$\begin{aligned}
W_i(K;E) &= V(K[n-i],E[i]) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(K[n-i-1],E[i];u) \\
&\leq \frac{1}{n}\int_{S^{n-1}} h(K_\lambda^p +_p |\lambda|(K_\lambda^p)^*,u)dS(K[n-i-1],E[i];u) \\
&\leq \frac{1}{n}\int_{S^{n-1}} [h(K_\lambda^p,u) + |\lambda| h((K_\lambda^p)^*,u)]dS(K[n-i-1],E[i];u) \\
&= V(K_\lambda^p, K[n-i-1],E[i]) + |\lambda| V(K[n-i-1],(K_\lambda^p)^*,E[i]) \\
&\leq V(K_\lambda^p[2],K[n-i-2],E[i]) + |\lambda| V(K_\lambda^p,K[n-i-2],(K_\lambda^p)^*,E[i]) \\
&\quad + |\lambda| V(K[n-i-1],(K_\lambda^p)^*,E[i]) \leq \cdots \\
&\leq W_i(K_\lambda^p;E) + |\lambda| \sum_{k=0}^{n-i-1} V(K_\lambda^p[k],K[n-i-k-1],(K_\lambda^p)^*,E[i]),
\end{aligned}$$

which implies

$$W_i(K;E) - |\lambda| \sum_{k=0}^{n-i-1} V(K_\lambda^p[k],K[n-i-k-1],(K_\lambda^p)^*,E[i]) \leq W_i(K_\lambda^p;E).$$

References

[1] W. J. Firey, *p-Means of convex bodies*. Math. Scand. **10**(1962), 17–24. https://doi.org/10.7146/math.scand.a-10510

[2] R. J. Gardner, *Geometric tomography*. 2nd ed., Encyclopedia of Mathematics and its Applications, 58, Cambridge University Press, Cambridge, MA, 2006. http://doi.org/10.1017/CBO9781107341029

[3] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*. 2nd ed., Cambridge University Press, Cambridge, MA, 1952.

[4] M. A. Hernández Cifre, A. R. Martínez Fernández, and E. Saorín Gómez, *Differentiability properties of the family of p-parallel bodies*. Appl. Anal. Discrete Math. **10**(2016), no. 1, 186–207. https://doi.org/10.2298/aadm160325005c

[5] M. A. Hernández Cifre and E. Saorín, *On inner parallel bodies and quermassintegrals*. Israel J. Math. **177**(2010), 29–47. https://doi.org/10.1007/s11856-010-0037-6

[6] E. Lutwak, *The Brunn–Minkowski–Firey theory. I. Mixed volumes and the Minkowski problem*. J. Differ. Geom. **38**(1993), no. 1, 131–150. https://doi.org/10.4310/jdg/1214454097

[7] E. Lutwak, *The Brunn–Minkowski–Firey theory II: Affine and geominimal surface areas.* Adv. Math. **118**(1996), no. 2, 244–294. https://doi.org/10.1006/aima.1996.0022

[8] A. R. Martínez Fernández, *Going further in the L_p-Brunn–Minkowski theory: a p-difference of convex bodies*. Ph. D. thesis, University of Murcia, 2016. https://doi.org/10.13140/RG.2.2.22938.44482

[9] A. R. Martínez Fernández, E. Saorín Gómez, and J. Yepes Nicolás, *p-difference: a counterpart of Minkowski difference in the framework of the L_p-Brunn–Minkowski theory*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **110**(2016), no. 2, 613–631. https://doi.org/10.1007/s13398-015-0253-3

[10] R. Schneider, *On the Aleksandrov–Fenchel inequality*. Ann. N. Y. Acad. Sci. **440**(1985), no. 1, 132–141. https://doi.org/10.1111/j.1749-6632.1985.tb14547.x

[11] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*. 2nd ed., Encyclopedia of Mathematics and its Applications, 151, Cambridge University Press, Cambridge, MA, 2014.

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