ON THE GENERAL SOLUTION OF THE HEIDEMAN–HOGAN FAMILY OF RECURRENCES

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Abstract   We consider a family of nonlinear rational recurrences of odd order which was introduced by Heideman and Hogan, and recently rediscovered in the theory of Laurent phenomenon algebras (a generalization of cluster algebras). All of these recurrences have the Laurent property, implying that for a particular choice of initial data (all initial values set to 1) they generate an integer sequence. For these particular sequences, Heideman and Hogan gave a direct proof of integrality by showing that the terms of the sequence also satisfy a linear recurrence relation with constant coefficients. Here we present an analogous result for the general solution of each of these recurrences.

Keywords: nonlinear recurrence; Laurent property; linearization

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1. Introduction

The theory of integer sequences generated by linear recurrences has a long history in number theory, and finds many applications in areas such as coding and cryptography [**6**], but the case of nonlinear recurrences is much less well studied. For some time there has been considerable interest in rational recurrence relations of the form

$$x_{n+N}\, x_n = F(x_{n+1}, \ldots, x_{n+N-1}), \qquad (1.1)$$

where F is a polynomial in $N - 1$ variables, which surprisingly generate integer sequences. Several quadratic recurrences of this kind were found by Somos, and this inspired others to find new examples, as described in the articles by Gale [**11**]. An important early observation was that if (1.1) has the Laurent property, meaning that it generates Laurent polynomials in the initial values with integer coefficients, i.e.

$$x_n \in \mathbb{Z}[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_{N-1}^{\pm 1}], \quad \forall n \in \mathbb{Z},$$

then an integer sequence is generated automatically by choosing the initial values to be $x_0 = x_1 = \cdots = x_{N-1} = 1$. Subsequently, as an offshoot of their development of cluster algebras, Fomin and Zelevinsky introduced the Caterpillar lemma [**7**], which is a useful

tool for proving the Laurent property for many recurrences of the form (1.1). In the special case where F is a sum of two monomials, Fordy and Marsh explained how such recurrences arise from cluster algebras associated with quivers that are periodic under cluster mutations [**9**], while for more general F a range of examples were found recently by Alman *et al.* [**1**], who considered mutation periodicity in the broader context of Laurent phenomenon (LP) algebras [**17**].

In this paper we are concerned with a particular family of nonlinear recurrences of odd order $N = 2k + 1$, given by

$$x_{n+2k+1}x_n = x_{n+2k}x_{n+1} + a(x_{n+k} + x_{n+k+1}), \tag{1.2}$$

where a is a non-zero parameter. This family was introduced in the case $a = 1$ by Heideman and Hogan [**12**], who proved that the sequence generated by (1.2) with the initial values $x_0 = x_1 = \cdots = x_{2k} = 1$ consists entirely of integers. (By rescaling $x_n \to ax_n$, the parameter a can always be removed, but it will be useful to retain it here for bookkeeping purposes.) One way to see the integrality of this particular sequence is to show that (1.2) has the Laurent property, which was noted in [**12**] and proved in [**13**]. More recently, the family (1.2) was rediscovered in a search for period 1 seeds in LP algebras: it is a particular case of the Little Pi polynomial, part 13 of Theorem 3.10 in [**1**], which provides an alternative proof of the Laurent property. However, Heideman and Hogan's original proof of integrality was based on the following result.

Theorem 1.1. *The terms of the sequence generated by the recurrence (1.2) with initial values $x_j = 1$ for $j = 0, 1, \ldots, 2k$ and $a = 1$ satisfy the linear relation*

$$x_{n+6k} - (2k^2 + 8k + 4)(x_{n+4k} - x_{n+2k}) - x_n = 0 \tag{1.3}$$

for all $n \in \mathbb{Z}$.

The integrality result in [**12**] is proved by starting from $x_j = 1$, $0 \le j \le 2k$, then determining the explicit form of the next $4k$ values x_j, $2k + 1 \le j \le 6k$ obtained by iterating (1.2) for $0 \le n \le 4k - 1$, where the value of x_{6k} is used to verify that (1.3) holds for $n = 0$, and finally showing by induction that if (1.3) is assumed to hold for all $n \ge 0$ then (1.2) also holds for all $n \ge 4k$. This particular sequence is also symmetric under reversal, in the sense that

$$x_{-n} = x_{n+2k}, \quad \forall n \in \mathbb{Z}. \tag{1.4}$$

In that case, the efficacy of this inductive approach can be seen directly from an operator identity (Lemma 1.3 below) connecting the linear operator in (1.3) with the nonlinear equation (1.2), which can be rewritten in the form $\xi_n = 0$, where

$$\xi_n := \begin{vmatrix} x_n & x_{n+2k} \\ x_{n+1} & x_{n+2k+1} \end{vmatrix} - a(x_{n+k} + x_{n+k+1}). \tag{1.5}$$

The main result of this paper is the analogue of Theorem 1.1 for the case of arbitrary initial data.

Theorem 1.2. *The iterates of the recurrence* (1.2) *satisfy the linear relation*

$$x_{n+6k} - K(x_{n+4k} - x_{n+2k}) - x_n = 0 \tag{1.6}$$

for all $n \in \mathbb{Z}$, where

$$K = P^{(0)} + aP^{(1)} + a^2 P^{(2)}, \tag{1.7}$$

with

$$P^{(0)} = 1 + \frac{x_0}{x_{2k}} + \frac{x_{2k}}{x_0},$$

$$P^{(1)} = \left(1 + \frac{x_{2k}}{x_0}\right) \sum_{j=1}^{k} \frac{x_{j-1} + x_j}{x_{j+k-1}x_{j+k}} + \left(1 + \frac{x_0}{x_{2k}}\right) \sum_{j=1}^{k} \frac{x_{j+k-1} + x_{j+k}}{x_{j-1}x_j},$$

$$P^{(2)} = \frac{1}{x_k x_{2k}} + \sum_{j=0}^{k-1} \frac{1}{x_j}\left(\frac{1}{x_{j+k}} + \frac{1}{x_{j+k+1}}\right) + \sum_{\ell=1}^{k-1}\sum_{m=1}^{\ell} \frac{(x_\ell + x_{\ell+1})(x_{k+m-1} + x_{k+m})}{x_{k+\ell}x_{k+\ell+1}x_{m-1}x_m}.$$

In principle, it is possible to prove the above result directly by adapting the argument from [**12**] and using the following identity.

Lemma 1.3. *Let $\mathcal{S}$ denote the shift operator, such that $\mathcal{S}x_n = x_{n+1}$ for all n, and let*

$$\mathcal{L} = \mathcal{S}^{6k} - K(\mathcal{S}^{4k} - \mathcal{S}^{2k}) - 1,$$

where K is some fixed constant. Then

$$\mathcal{L}\xi_n = \mathcal{M}_n \cdot \mathcal{L}x_n, \tag{1.8}$$

where $\mathcal{M}_n$ is the linear operator

$$\mathcal{M}_n = x_{n+6k}\mathcal{S}^{2k+1} - x_{n+6k+1}\mathcal{S}^{2k} - x_{n+2k}\mathcal{S} + x_{n+2k+1} - a(\mathcal{S}^{k+1} + \mathcal{S}^k).$$

To apply the operator identity (1.8) one should take $2k + 1$ initial values $x_0, \ldots, x_{2k}$ for the nonlinear recurrence (1.2), which requires the $4k$ vanishing conditions $\xi_0 = \xi_1 = \cdots = \xi_{4k-1} = 0$ which fix $6k$ initial values $x_0, x_1, \ldots, x_{6k-1}$ for the linear equation $\mathcal{L}x_n = 0$ together with the value of K, determined as

$$K = \frac{x_{6k} - x_0}{x_{4k} - x_{2k}}, \tag{1.9}$$

and then further verify that $\xi_j = 0$ for a total of $6k$ adjacent values of j (including the range $0 \leq j \leq 4k - 1$ already assumed); this implies that the corresponding solution of the initial value problem for $\mathcal{L}\xi_n = 0$ is the zero solution $\xi_n = 0$ for all n. Heideman and Hogan made this argument effective with the use of computer algebra, which they used (for arbitrary k) to calculate explicit expressions for the values of $x_{2k+1}, \ldots, x_{6k}$ corresponding to $x_0 = x_1 = \cdots = x_{2k} = 1$, and hence to verify the value $K = 2k^2 + 8k + 4$ in (1.3) and other necessary identities; they also implicitly used the fact that this special sequence has the reversal symmetry (1.4) (although this fact was not stated in [**12**]), which means that once $\xi_j = 0$ holds for $0 \leq j \leq 4k - 1$ it automatically holds for $-2k + 1 \leq j \leq -1$ as

well, so it is enough to verify in addition that $\xi_{4k} = 0$ in order to show that $\xi_n = 0$ for all n by induction. However, this argument is much harder to apply to the case of generic initial data, because the corresponding sequence need not have the symmetry (1.4), so here we prefer to adopt a different approach. Nevertheless, we are able to exploit the fact that the recurrence (1.2) is itself reversible in the sense of [18], making the proof below much simpler than it might be otherwise.

The result (1.2) can be restated as saying that the recurrence (1.2) is linearizable, with the coefficient K appearing in the linear relation (1.6) being a conserved quantity (this terminology is explained in more detail in the next section); the general solution for the case $k = 1$ was already covered in [14]. There are many other examples of nonlinear recurrences that are linearizable, which arise in diverse contexts ranging from cluster algebras associated with affine A-type Dynkin diagrams [8–10, 16], to frieze relations [2], Q-systems for characters in representation theory [4], and period 1 seeds in LP algebras [1, 15]. In all these examples, the key to obtaining the linear recurrences is provided by certain determinantal identities for discrete Wronskians. The fact that (1.2) can be rewritten as the vanishing of the expression (1.5) involving a 2×2 determinant permits a linear relation to be derived in a straightforward way, although it turns out that this approach is insufficient to obtain the precise form of (1.6).

In the next section we provide the proof of Theorem 1.2, and in §3 we present various corollaries, including some inhomogeneous linear relations (Corollary 3.2), and an expression for the solution in terms of Chebyshev polynomials (Corollary 3.3), before making some conclusions.

2. Proof of the main theorem

Before proceeding with the proof, we give some discussion of terminology, and describe properties of (1.2) that will be useful later on. First of all, note that iterating the nonlinear recurrence is equivalent to iterating a birational map in dimension $2k + 1$, namely

$$\varphi : (x_0, x_1, \ldots, x_{2k}) \mapsto \left(x_1, x_2, \ldots, \frac{x_1 x_{2k} + a(x_{k+1} + x_k)}{x_0} \right). \tag{2.1}$$

If we always use this map to iterate then it is useful to regard the terms in the sequence $(x_n)_{n \in \mathbb{Z}}$ as rational functions (in fact, Laurent polynomials, but we will not need this) in the initial coordinates $x_0, x_1, \ldots, x_{2k}$ and a, obtained by the pullback of φ (or its inverse) applied to these variables, so that

$$(\varphi^*)^n x_0 = x_n, \quad \forall n \in \mathbb{Z},$$

with $(\varphi^{-1})^* = (\varphi^*)^{-1}$. We say that a non-constant function $F(x_0, x_1, \ldots, x_{2k})$ is a conserved quantity, or first integral, for the map φ if it is invariant under pullback, i.e. $\varphi^* F = F \cdot \varphi = F$, and we say that it is a p-invariant if it is periodic with period p, i.e. $(\varphi^*)^p F = F$.

From Theorem 3.10 in [1], the map can be factored as $\varphi = \rho \cdot \mu$, where ρ is a cyclic permutation of the coordinates and μ is a mutation in an LP algebra, but more interesting for our purposes is the fact that it is a reversible map (it has the discrete analogue of time-reversal symmetry [18]), meaning that it is conjugate to its own inverse.

Lemma 2.1. *The map φ satisfies*

$$\varphi = \sigma \cdot \varphi^{-1} \cdot \sigma,$$

where the reversing symmetry σ is the involution

$$\sigma : (x_0, x_1, \ldots, x_{2k}) \mapsto (x_{2k}, x_{2k-1}, \ldots, x_0).$$

Reversibility means that the reversing symmetry σ can extended to the level of the whole sequence (x_n) by pullback, so that it acts according to

$$\sigma^* x_n = x_{2k-n}, \quad \forall n \in \mathbb{Z}. \tag{2.2}$$

In order to obtain linear relations for the terms of the sequence, it will be convenient to introduce the 3×3 discrete Wronskian matrix

$$\Psi_n := \begin{pmatrix} x_n & x_{n+2k} & x_{n+4k} \\ x_{n+1} & x_{n+2k+1} & x_{n+4k+1} \\ x_{n+2} & x_{n+2k+2} & x_{n+4k+2} \end{pmatrix}, \tag{2.3}$$

which has 2×2 minors of the form appearing in (1.5).

Proposition 2.2. *The determinant*

$$\delta_n := \det \Psi_n$$

is a k-invariant for the map φ.

Proof. Using Dodgson condensation [5] (also known as the Desnanot–Jacobi identity) to expand the 3×3 determinant in terms of its 2×2 connected minors yields

$$x_{n+2k+1}\delta_n = \begin{vmatrix} \xi_n + as_{n+k} & \xi_{n+2k} + as_{n+3k} \\ \xi_{n+1} + as_{n+k+1} & \xi_{n+2k+1} + as_{n+3k+1} \end{vmatrix}$$

$$= L_n + a^2 \begin{vmatrix} s_{n+k} & s_{n+3k} \\ s_{n+k+1} & s_{n+3k+1} \end{vmatrix},$$

where $s_n = x_n + x_{n+1}$, and the quantity L_n is a sum of homogeneous linear and quadratic terms in ξ_j for certain j. A direct calculation then shows that

$$x_{n+2k+1}x_{n+3k+1}(\delta_{n+k} - \delta_n) = x_{n+2k+1}L_{n+k} - x_{n+3k+1}L_n + \Delta_n,$$

where

$$\Delta_n = a^2(s_{n+2k+1}\xi_{n+2k} + s_{n+2k}\xi_{n+2k+1} - s_{n+3k+1}\xi_{n+k} - s_{n+3k}\xi_{n+k+1}),$$

which clearly vanishes, along with L_n and L_{n+k}, if $\xi_j = 0$ for all j. Therefore $\delta_{n+k} = (\varphi^*)^k \delta_n = \delta_n$ for all n, as required. $\square$

Remark 2.3. Working in the ambient field of rational functions, that is $\mathbb{C}(x_0, x_1, \ldots, x_{2k}, a)$, and using explicit expressions for the first few iterates (see below)

it can be verified directly that δ_0 and all its shifts $\delta_1, \ldots, \delta_{k-1}$ are non-zero rational functions (actually, Laurent polynomials), e.g.

$$\delta_0 = \delta_{-2k} = \begin{vmatrix} x_{-2k} & x_0 & x_{2k} \\ x_{-2k+1} & x_1 & x_{2k+1} \\ x_{-2k+2} & x_2 & x_{2k+2} \end{vmatrix}$$

can be calculated from the formulae in Lemma 2.7, and by periodicity none of the shifts $(\varphi^*)^n \delta_0$ can be identically zero (as a rational function).

Corollary 2.4. *The determinant of the 4×4 discrete Wronskian matrix*

$$\hat{\Psi}_n := \begin{pmatrix} x_n & x_{n+2k} & x_{n+4k} & x_{n+6k} \\ x_{n+1} & x_{n+2k+1} & x_{n+4k+1} & x_{n+6k+1} \\ x_{n+2} & x_{n+2k+2} & x_{n+4k+2} & x_{n+6k+2} \\ x_{n+3} & x_{n+2k+3} & x_{n+4k+3} & x_{n+6k+3} \end{pmatrix}$$

is zero.

Proof of Corollary. Using Dodgson condensation once more to expand the 4×4 determinant in terms of its 3×3 connected minors, which are shifts of the determinant of (2.3), gives

$$\det \hat{\psi}_n \begin{vmatrix} x_{n+2k+1} & x_{n+4k+1} \\ x_{n+2k+2} & x_{n+4k+2} \end{vmatrix} = \begin{vmatrix} \delta_n & \delta_{n+2k} \\ \delta_{n+1} & \delta_{n+2k+1} \end{vmatrix} = \begin{vmatrix} \delta_n & \delta_n \\ \delta_{n+1} & \delta_{n+1} \end{vmatrix} = 0$$

by Proposition 2.2. $\qquad\qquad\square$

The above results are almost, but not quite, sufficient to produce the linear relation in Theorem 1.2.

Theorem 2.5. *The iterates of the nonlinear recurrence (1.2) satisfy the linear recurrence*

$$x_{n+6k} - K^{(1)} x_{n+4k} + K^{(2)} x_{n+2k} - x_n = 0, \tag{2.4}$$

where $K^{(1)}, K^{(2)}$ are conserved quantities with

$$K^{(2)} = \sigma^* K^{(1)}, \tag{2.5}$$

as well as the linear recurrence

$$x_{n+3} - \gamma_n x_{n+2} + \beta_n x_{n+1} - \alpha_n x_n = 0, \tag{2.6}$$

where α_n is a k-invariant and β_n, γ_n are $2k$-invariants.

Proof. An element of the kernel of $\hat{\Psi}_n$ is given by a column vector $\mathbf{v}_n = (-K^{(3)}, K^{(2)}, -K^{(1)}, 1)^T$, where the last entry has been scaled to 1 (which is valid since δ_n is non-vanishing by Remark 2.3), and a priori the other entries $K^{(j)}$ depend on n.

The first three rows of the equation $\hat{\Psi}_n \mathbf{v}_n = \mathbf{0}$ give a linear system for the $K^{(j)}$, and by Cramer's rule the solution is

$$K^{(1)} = \frac{1}{\delta_n} \begin{vmatrix} x_n & x_{n+2k} & x_{n+6k} \\ x_{n+1} & x_{n+2k+1} & x_{n+6k+1} \\ x_{n+2} & x_{n+2k+2} & x_{n+6k+2} \end{vmatrix}, \qquad K^{(2)} = \frac{1}{\delta_n} \begin{vmatrix} x_n & x_{n+4k} & x_{n+6k} \\ x_{n+1} & x_{n+4k+1} & x_{n+6k+1} \\ x_{n+2} & x_{n+4k+2} & x_{n+6k+2} \end{vmatrix}$$

and $K^{(3)} = \delta_n^{-1} \delta_{n+2k} = 1$. The last three rows of $\hat{\Psi}_n \mathbf{v}_n = \mathbf{0}$ give the same linear system with all indices shifted by 1, implying that $K^{(1)}$ and $K^{(2)}$ are independent of n. Now applying σ^* to (2.4), replacing $n \to -n - 4k$ and adding the result back to the original relation produces

$$(\sigma^* K^{(2)} - K^{(1)}) x_{n+4k} + (K^{(2)} - \sigma^* K^{(1)}) x_{n+2k} = 0$$

for all n, hence (2.5) must hold. The same argument applied to the kernel of the transpose matrix $\hat{\Psi}_n^T$ yields the relation (2.6) where

$$\beta_n = \frac{1}{\delta_n} \begin{vmatrix} x_n & x_{n+2} & x_{n+3} \\ x_{n+2k} & x_{n+2k+2} & x_{n+2k+3} \\ x_{n+4k} & x_{n+4k+2} & x_{n+4k+3} \end{vmatrix}, \qquad \gamma_n = \frac{1}{\delta_n} \begin{vmatrix} x_n & x_{n+1} & x_{n+3} \\ x_{n+2k} & x_{n+2k+1} & x_{n+2k+3} \\ x_{n+4k} & x_{n+4k+1} & x_{n+4k+3} \end{vmatrix}$$

are $2k$-invariants and $\alpha_n = \delta_n^{-1} \delta_{n+1}$ is a k-invariant. $\qquad\square$

By considering the monodromy of the linear equation (2.6) with periodic coefficients, the coefficients in (2.4) for can be written in terms of $\alpha_j, \beta_j, \gamma_j$.

Proposition 2.6. *The conserved quantities $K^{(1)}$ and $K^{(2)}$ can be written as polynomials in $\alpha_j, \beta_j, \gamma_j$, given by*

$$K^{(1)} = \operatorname{tr} \mathbf{L}_{n+2k-1} \mathbf{L}_{n+2k-2} \cdots \mathbf{L}_n, \qquad K^{(2)} = \operatorname{tr} \mathbf{L}_n^{-1} \mathbf{L}_{n+1}^{-1} \cdots \mathbf{L}_{n+2k-1}^{-1}.$$

Proof. In terms of the matrix Ψ_n, the system (2.6) implies that

$$\Psi_{n+1} = \mathbf{L}_n \Psi_n, \qquad \mathbf{L}_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_n & -\beta_n & \gamma_n \end{pmatrix}, \qquad \mathbf{L}_n^{-1} = \frac{1}{\alpha_n} \begin{pmatrix} \beta_n & -\gamma_n & 1 \\ \alpha_n & 0 & 0 \\ 0 & \alpha_n & 0 \end{pmatrix},$$

so that

$$\Psi_{n+2k} = \mathbf{M}_n \Psi_n, \qquad \mathbf{M}_n = \mathbf{L}_{n+2k-1} \mathbf{L}_{n+2k-2} \cdots \mathbf{L}_n,$$

while on the other hand

$$\Psi_{n+2k} = \Psi_n \tilde{\mathbf{L}}, \qquad \tilde{\mathbf{L}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -K^{(2)} \\ 0 & 1 & K^{(1)} \end{pmatrix}, \qquad \tilde{\mathbf{L}}^{-1} = \begin{pmatrix} K^{(2)} & 1 & 0 \\ -K^{(1)} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then noting that $\operatorname{tr} \tilde{\mathbf{L}} = \operatorname{tr} \mathbf{M}_n$ and $\operatorname{tr} \tilde{\mathbf{L}}^{-1} = \operatorname{tr} \mathbf{M}_n^{-1}$, together with the observation that $\prod_{j=1}^k \alpha_j = 1$, we have the required result. $\qquad\square$

According to the result we are aiming for, Theorem 1.2, we expect to find $K^{(1)} = K^{(2)} = K$, a Laurent polynomial in $x_0, x_1, \ldots, x_{2k}$, which by (2.5) must be invariant under the reversal symmetry. However, none of the formulae for $K^{(1)}, K^{(2)}$ obtained so far makes this coincidence manifest, and none of them immediately yields a Laurent polynomial. Indeed, the preceding result is somewhat mysterious, since direct calculations for small values of k reveal that $\alpha_j, \beta_j, \gamma_j$ are not Laurent polynomials themselves. In order to prove the main result, we calculate explicit formulae for $2k$ iterates on either side of the initial data, which allows us to obtain K as a quadratic polynomial in a, by using (1.9) with suitably shifted indices.

Lemma 2.7. *The first $2k$ terms obtained by iterating (1.2) forwards from the initial values $x_0, x_1, \ldots, x_{2k}$ are given by*

$$x_{2k+j} = x_0^{-1} x_j x_{2k} + a F_{2k+j}^{(1)},$$
$$x_{3k+j} = x_0^{-1} x_{k+j} x_{2k} + a F_{3k+j}^{(1)} + a^2 F_{3k+j}^{(2)}, \quad 1 \leq j \leq k,$$

where the coefficients of the linear and quadratic terms in a are specified by

$$F_{2k+j}^{(1)} = x_j \sum_{\ell=1}^{j} (x_{\ell-1} x_\ell)^{-1} (x_{k+\ell-1} + x_{k+\ell}),$$

$$F_{3k+j}^{(1)} = x_0^{-1} x_{k+j} x_{2k} \sum_{\ell=1}^{j} (x_{k+\ell-1} x_{k+\ell})^{-1} (x_{\ell-1} + x_\ell) + x_k^{-1} x_{k+j} F_{3k}^{(1)},$$

$$F_{3k+j}^{(2)} = x_{k+j} \sum_{\ell=1}^{j} (x_{k+\ell-1} x_{k+\ell})^{-1} (F_{2k+\ell-1}^{(1)} + F_{2k+\ell}^{(1)}),$$

for the same range of the index j, with $F_{2k}^{(1)} = 0$. The first $2k$ terms obtained by iterating (1.2) backwards from the same initial values are

$$x_{-j} = x_{2k}^{-1} x_{2k-j} x_0 + a F_{-j}^{(1)},$$
$$x_{-k-j} = x_{2k}^{-1} x_{k-j} x_0 + a F_{-k-j}^{(1)} + a^2 F_{-k-j}^{(2)}, \quad 1 \leq j \leq k,$$

where, for the same range of j values,

$$F_{-j}^{(1)} = \sigma^* F_{2k+j}^{(1)}, \qquad F_{-k-j}^{(1)} = \sigma^* F_{3k+j}^{(1)}, \qquad F_{-k-j}^{(2)} = \sigma^* F_{3k+j}^{(2)}.$$

Proof. The first $2k+1$ iterations of (1.2), either forwards or backwards, only require multiplication and addition of previous terms, as well as division by one of $x_0, x_1, \ldots, x_{2k}$, so for the division there is no need to consider any cancellations between numerator and denominator (which are required for the Laurent property to hold at subsequent steps). The first k terms produced by iterating are linear in a, while the next k terms are quadratic. By expanding x_{2k+j} in a and substituting into (1.2) with $n = j - 1$, the

leading order term is obtained recursively, while the coefficient of the linear term satisfies the recursion

$$x_j^{-1}F_{2k+j}^{(1)} - x_{j-1}^{-1}F_{2k+j-1}^{(1)} = (x_{j-1}x_j)^{-1}(x_{k+j-1} + x_{k+j}), \quad 1 \le j \le k,$$

which can be summed telescopically (with $F_{2k}^{(1)} = 0$) to obtain the above formula for $F_{2k+j}^{(1)}$. Similarly, expanding in a for the next k iterations, the leading order term is found directly, while for the term linear in a the recursion is

$$x_{k+j}^{-1}F_{3k+j}^{(1)} - x_{k+j-1}^{-1}F_{3k+j-1}^{(1)} = (x_{k+j-1}x_{k+j}x_0)^{-1}x_{2k}(x_{j-1} + x_j), \quad 1 \le j \le k.$$

which immediately yields the above expression for $F_{3k+j}^{(1)}$; and for the quadratic term the formula for the coefficient is found by solving the recursion

$$x_{k+j}^{-1}F_{3k+j}^{(2)} - x_{k+j-1}^{-1}F_{3k+j-1}^{(2)} = (x_{k+j-1}x_{k+j})^{-1}\bigl(F_{2k+j-1}^{(1)} + F_{2k+j}^{(1)}\bigr), \quad 1 \le j \le k.$$

Similarly, iterating $2k$ steps backwards produces the images of the forward iterates under the action of the reversing map, so the formulae for x_{-j} and x_{-k-j} follow by direct application of σ^*. $\qquad\square$

Proposition 2.8. *If K is defined by*

$$K = \frac{x_{4k} - x_{-2k}}{x_{2k} - x_0}, \tag{2.7}$$

where x_{4k} and x_{-2k} are given as Laurent polynomials in $x_0, x_1, \ldots, x_{2k}$ and a according to Lemma 2.7, then it is given explicitly by (1.7).

Proof. The formula (1.7) is readily checked at each order in a. At leading order this is trivial, while at order a and a^2 the identities

$$(x_{2k} - x_0)P^{(j)} = F_{4k}^{(j)} - F_{-2k}^{(j)}$$

are seen to hold for $j = 1, 2$. $\qquad\square$

Theorem 2.9. *The quantity K in (1.7) is a first integral for the map φ.*

Proof. To verify that K is a conserved quantity, observe that

$$\varphi^*K = \frac{x_{4k+1} - x_{-2k+1}}{x_{2k+1} - x_1}$$

from (2.7), and this is equal to K if and only if $\mathcal{L}x_{-2k+1} = 0$, where $\mathcal{L}$ is the operator in Lemma 1.3. Now $\mathcal{L}x_{-2k}$ vanishes by (2.7), so from the identity

$$x_{2k}\mathcal{L}x_{-2k+1} - x_{2k+1}\mathcal{L}x_{-2k} - \xi_{2k} + K\xi_0 = a(x_{3k} + x_{3k+1})$$
$$- Ka(x_k + x_{k+1}) + x_{2k+1}x_{-2k} - x_{2k}x_{-2k+1}$$

we see that it is sufficient to check that the right-hand above is zero. Substitution of the explicit expressions from Lemma 2.7 yields a cubic polynomial in a; the order zero

term clearly vanishes, while at order one, two and three it is straightforward to verify the identities

$$(x_k + x_{k+1})(x_0^{-1} x_{2k} - P^{(0)}) + x_0^{-1} x_1 x_{2k} F_{-2k}^{(1)} + x_{2k}^{-1} x_0^2 F_{2k+1}^{(1)} - x_{2k} F_{-2k+1}^{(1)} = 0,$$

$$F_{3k}^{(1)} + F_{3k+1}^{(1)} + x_0^{-1} x_1 x_{2k} F_{-2k}^{(2)} + F_{2k+1}^{(1)} F_{-2k}^{(1)} - x_{2k} F_{-2k+1}^{(2)} - (x_k + x_{k+1}) P^{(1)} = 0,$$

$$F_{3k+1}^{(2)} + (x_k + x_{k+1})(x_0^{-1} F_{-2k}^{(2)} - P^{(2)}) = 0.$$

This shows that $\varphi^* K = K$, and completes the proof of Theorem 1.2. □

3. General solution and other corollaries

The result of Theorem 1.2 has many implications for the recurrence (1.2). Theorem 1.1 is just a special case: one finds the value $K = 2k^2 + 8k + 4$ by substituting $a = x_0 = x_1 = \cdots = x_{2k} = 1$ into (1.7). Also, note that we have not made use of the Laurent property in the proof, yet by the formulae in Lemma 2.7 we see that $x_{-2k}, x_{-2k+1}, \ldots, x_{4k}$ provide $6k$ initial data for the linear recurrence $\mathcal{L} x_n = 0$, and these are all Laurent polynomials in $x_0, x_1, \ldots, x_{2k}$ with coefficients in $\mathbb{Z}[a]$, as is K given by (1.7), so we arrive at the following.

Corollary 3.1. *The nonlinear recurrence (1.2) has the Laurent property, i.e. $x_n \in \mathbb{Z}\left[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_{2k}^{\pm 1}, a\right] \ \forall n \in \mathbb{Z}$.*

In addition to the homogeneous linear recurrences in Theorem 2.5, various inhomogeneous linear recurrences now follow.

Corollary 3.2. *The iterates of (1.2) satisfy the linear recurrences*

$$x_{n+4k} - (K-1) x_{n+2k} + x_n = \nu_n, \tag{3.1}$$

where ν_n is a $2k$-invariant,

$$\sum_{j=0}^{2k-1} x_{n+4k+j} - (K-1) x_{n+2k+j} + x_{n+j} = K', \tag{3.2}$$

where

$$K' = \nu_0 + \nu_1 + \cdots + \nu_{2k-1} \tag{3.3}$$

is a conserved quantity, and

$$x_{n+2} + \eta_n x_{n+1} + \zeta_n x_n = \epsilon_n, \tag{3.4}$$

where ϵ_n, ζ_n and η_n are all $2k$-invariants.

Proof of Corollary. The operator $\mathcal{L}$ can be factorized as

$$\mathcal{L} = (\mathcal{S}^{2k} - 1)(\mathcal{S}^{4k} - (K-1)\mathcal{S}^{2k} + 1) = (\mathcal{S} - 1)\bigg(\sum_{j=0}^{2k-1} \mathcal{S}^j\bigg)(\mathcal{S}^{4k} - (K-1)\mathcal{S}^{2k} + 1),$$

which means that (1.6) can be 'integrated' in two different ways to yield (3.1) and (3.2). By shifting and summing $2k$ copies of (3.1), the conserved quantity K' is given as a symmetric function of the $2k$ independent shifts of ν_n according to (3.3).

The equation (1.6) also implies that, for all n, the vector $(1, -K, K, -1)^T$ belongs to the kernel of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_n & x_{n+2k} & x_{n+4k} & x_{n+6k} \\ x_{n+1} & x_{n+2k+1} & x_{n+4k+1} & x_{n+6k+1} \\ x_{n+2} & x_{n+2k+2} & x_{n+4k+2} & x_{n+6k+2} \end{pmatrix}$$

so by writing a vector in the kernel of its transpose as the row vector $(\epsilon_n, -\zeta_n, \eta_n, -1)$, the relation (3.4) follows, and the same argument as in the proof of Theorem 2.5 shows that $\epsilon_n, \zeta_n, \eta_n$ are invariant under shifting $n \to n + 2k$. $\qquad\square$

The fact that the iterates of (1.2) satisfy a linear relation with constant coefficients means that they can be written explicitly in terms of the roots of the associated characteristic polynomial. Due to the particular form of (1.6), the general solution can also be written using either trigonometric functions or Chebyshev polynomials (with the latter form of the solution for $k = 1$ being included in the results of [**14**]). In order to do this, we introduce quantities θ and t such that

$$t = \frac{K-1}{2} = \cos\Theta, \quad \Theta = 2k\theta,$$

and recall that the Chebyshev polynomials of the first and second kinds are defined by

$$T_n(t) = \cos(n\Theta), \qquad U_{n-1}(t) = \frac{\sin(n\Theta)}{\sin\Theta}$$

respectively, so that $T_0 = U_0 = 1$, $T_1 = T_{-1} = t$, $U_1 = 2t$, $U_{-1} = 0$.

Corollary 3.3. *The general solution of (1.2) can be written in the form*

$$x_n = a_n + b_n \cos(n\theta + \phi_n),$$

where a_n, b_n, ϕ_n are all periodic in n with periodic $2k$, or as

$$x_n = q_j + r_j T_m(t) + s_j U_m(t), \quad m = \left\lfloor \frac{n}{2k} \right\rfloor,$$

where $j = n \mod 2k$ and for $j = 0, 1, \ldots, 2k-1$ the coefficients are

$$\begin{pmatrix} q_j \\ r_j \\ s_j \end{pmatrix} = \frac{1}{2t(1-t)} \begin{pmatrix} t & -2t^2 & t \\ -1 & 2t & 1-2t \\ 1-t & 0 & t-1 \end{pmatrix} \begin{pmatrix} x_{2k+j} \\ x_j \\ x_{-2k+j} \end{pmatrix}.$$

4. Conclusions

We have proved that all sequences generated by the nonlinear recurrence (1.2) satisfy a linear relation, which was left as an open problem in [**12**]. A key feature in the proof was to use the reversibility property in Lemma 2.1. In fact, all the cluster maps obtained from period 1 quivers in [**9**], and many of the recurrences considered in [**1**], are also reversible with the same sort of reversing symmetry, which means that the above approach can be applied in those cases too, and may prove useful for finding explicit formulae for conserved quantities (when they exist).

One question that remains open is whether there is any natural Poisson structure which is preserved by the map (2.1), since Poisson structures (or presymplectic structures) arise naturally in the context of cluster algebras, but whether there is something similar for LP algebras in general is an open question. In fact we expect that there is a Poisson bracket (albeit a rather degenerate one, of rank two) for a combination of two reasons: first, the map should have many conserved quantities in addition to K and K' given by (3.3), since any symmetric function of the shifts of a $2k$-invariant is conserved; and second, φ has the logarithmic volume form

$$\omega = \frac{1}{x_0 x_1 \cdots x_{2k}} \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{2k}$$

which is anti-invariant, in the sense that $\varphi^* \omega = -\omega$; so if there are at least $2k - 2$ independent conserved quantities, then the corresponding co-volume can be contracted with their differentials to construct a Poisson bracket, by one of the results in [**3**].

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