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SEMILINEAR CALDERÓN PROBLEM ON STEIN MANIFOLDS WITH KÄHLER METRIC

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Abstract

We extend existing methods which treat the semilinear Calderón problem on a bounded domain to a class of complex manifolds with Kähler metric. Given two semilinear Schrödinger operators with the same Dirchlet-to-Neumann data, we show that the integral identities that appear naturally in the determination of the analytic potentials are enough to deduce uniqueness on the boundary up to infinite order. By exploiting the assumed complex structure, this information allows us to apply the method of stationary phase and recover the potentials in the interior as well.

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1. Introduction

We study the semilinear elliptic equation

$$\begin{cases} \Delta_g u + V(p,u) = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases} \tag{NCP$_{f,V}$}$$

on a complex n-dimensional compact connected Kähler manifold (M, g) with smooth boundary ∂M. Here $V : M \times \mathbb{C} \to \mathbb{C}$ is a $C^\infty(M)$ function for every complex variable such that

$$V(p,u) = \sum_{k\geq 1} \frac{V_k(p)}{k!} u^k, \quad \text{where } V_k(p) \overset{\text{def}}{=} \partial_u^k V(p,0), \tag{1.1}$$

converges in the $C^s(M)$ topology for noninteger $s > 2$. We assume zero is not an eigenvalue for the operator $\Delta_g + V_1$. Moreover,

$$\Delta_g u \overset{\text{def}}{=} -\frac{1}{\sqrt{\det g}} \partial_i (g^{ij} \sqrt{\det g} \partial_j u)$$

defined locally is the positive Laplace–Beltrami operator. We assume in addition that M is holomorphic separable and has local charts given by holomorphic functions, in the sense that:

- $*$ for any $p, q \in M$, with $p \neq q$, there exists $f \in C^\infty(M) \cap \mathcal{O}(\operatorname{Int} M)$ such that $f(p) \neq f(q)$;
- $*$ for any $p \in M$, there exist $f_1, \ldots, f_n \in C^\infty(M) \cap \mathcal{O}(\operatorname{Int} M)$ which form a complex coordinate system centred at p.

In particular, these assumptions cover the case of Stein manifolds.

It was shown in [5] that there exist $\delta, r, C > 0$ depending on (M, g) such that if we consider the sets

$$U_\delta \stackrel{\text{def}}{=} \{h \in C^s(\partial M) \mid \|h\|_{C^s(\partial M)} \leq \delta\} \quad \text{and} \quad V_r \stackrel{\text{def}}{=} \{w \in C^s(M) \mid \|w\|_{C^s(M)} \leq r\},$$

then for any $f \in U_\delta$ there exists a unique $u_f \in V_r$ which solves ($\mathrm{NCP}_{f,V}$), with the estimate

$$\|u\|_{C^s(M)} \leq C\|f\|_{C^s}(M). \tag{1.2}$$

In particular, u_f is analytic with respect to a small complex perturbation of f. If $f_\epsilon = \epsilon_1 f_1 + \cdots + \epsilon_k f_k$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \mathbb{C}^k$ is a complex parameter and $f_1, \ldots, f_k$ are in $C^s(M)$, then the solution u_{f_ϵ} admits a power series representation with respect to the parameter ϵ in the C^k topology. Thus for sufficiently small boundary data, we can define the Dirchlet-to-Neumann map

$$\mathcal{N} : U_\delta \to C^{s-1}(\partial M), \quad f \longmapsto \partial_\nu u_f|_{\partial M}.$$

We will prove the following theorems.

THEOREM 1.1. *Let (M, g) be specified as above. Let $\mathcal{N}_V$ and $\mathcal{N}_W$ be the Dirchlet-to-Neumann maps corresponding to V and W, where V and W are smooth and satisfy condition (1.1). If $\mathcal{N}_V = \mathcal{N}_W$ and $V_1 = W_1 = 0$, then $V = W$.*

The case where only partial data are available is much harder. Let Γ be an arbitrarily small open subset of ∂M and let Γ^c denote the complement of Γ in ∂M. We consider the Dirchlet-to-Neumann map with partial data

$$\mathcal{N}^\Gamma : \{h \in U_\delta \mid \operatorname{Supp} h \subseteq \Gamma\} \to C^{s-1}(\partial M), \quad f \longrightarrow \partial_\nu u_f|_\Gamma.$$

Using the tools developed in [10], we will extend Theorem 1.1 to solve the semilinear Calderón problem on Riemann surfaces where only partial data are available.

THEOREM 1.2. *Let (Σ, g) be a Riemann surface. Let $\mathcal{N}_V^\Gamma$ and $\mathcal{N}_W^\Gamma$ be the Dirchlet-to-Neumann map with partial data corresponding to V and W, where V and W are smooth and satisfy condition (1.1). If $\mathcal{N}_V^\Gamma = \mathcal{N}_W^\Gamma$, then $V = W$.*

Since every Riemann surface is also Stein, Theorem 1.2 in particular covers the situation of Theorem 1.1.

Our strategy will be as follows. In Section 2 we formulate our main theorems in terms of a class of integral identities analogous to the linearised Calderón problem [7, 8, 11, 23]. Starting from Section 3, our method will begin to differ from [11]. We prove directly a boundary determination result assuming nothing but the integral

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identity in Section 2. This will be done using special solutions to the Laplace equation constructed in [22] via the WKB method. The standard techniques in proving such results are usually based on the theory of pseudodifferential operators. Then in Section 4, assuming sufficient regularity on the potential, we can simplify the proofs in [10] and recover the interior potential based on the boundary results obtained in Section 3.

A historical account of the semilinear Calderón problem is in order. The linear Calderón problem on domains in $\mathbb{R}^n$ has been studied intensively (see [15] for a recent survey). The setting has been extended to conformally transversally anisotropic (CTA) manifolds by the authors of [6, 9, 16], but the authors of [1, 2] show that there are Riemannian manifolds for which these methods fail to apply. Complex methods were first employed in solving the Calderón problem in two dimensions [21]. For Riemann surfaces, based on [4], the partial-data Calderón problem was solved completely by the authors of [10].

For the semilinear Schrödinger equation $\Delta_g u + V(p,u)$, the problem of recovering the potential V was studied in two dimensions in [12, 13] and in higher-dimensional settings in [14]. Our method of linearisation is based on [20], generalised in [18, 19] and extended to CTA manifolds in [5]. In the very recent work [17], the problem has been extended to more general gradient nonlinearities. On the other hand, the linearised Calderón problem has been completely solved in the case of real bounded domains as well as on CTA manifolds [7, 8]. In the complex case, the authors of [7, 10] completely solved the partial-data linearised Calderón problem on Riemann surfaces and the full-data problem on Stein manifolds.

Throughout this paper, $d\omega_g$ denotes the Riemannian volume element of (M, g) and $d\sigma_g$ the associated boundary element, $C^s(M)$ denotes the space of Hölder continuous functions of order s with the usual topology and, for a complex manifold M without boundary, $\mathcal{O}(M)$ denotes the space of holomorphic functions on M.

2. Integral identities

In this section we reformulate Theorem 1.1 in terms of a collection of integral identities. The procedure will be similar to that in [18] for the case of a real bounded domain, but we prove the result in the form which will be convenient for our purpose.

Proposition 2.1. *Let (M, g) be a compact, oriented manifold and $\Gamma \subseteq \partial M$ be an arbitrarily small open subset. Let V, W be smooth functions satisfying condition (1.1) such that the corresponding Dirchlet-to-Neumann partial-data maps $\mathcal{N}_V^\Gamma = \mathcal{N}_W^\Gamma$ agree. Assume $V_1 = W_1$ and that, for every $f \in C^\infty(M)$,*

$$\int_M f u_1 \cdots u_k u_{k+1}\, d\omega_g = 0, \tag{2.1}$$

for all C^s solutions $u_1, \ldots, u_k$ to the linear Schrödinger equation with potential $V_1 = W_1$ and harmonic functions u_{k+1}, all with boundary data supported in Γ, implies $f = 0$. Then $V = W$.

It is convenient to formulate the following lemmas.

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Lemma 2.2. *Assume the setting of Proposition 2.1. Let* $f_\epsilon = \epsilon_1 f_1 + \cdots + \epsilon_k f_k$ *where* $\epsilon_1, \ldots, \epsilon_k$ *are small complex parameters and* $f_1, \ldots, f_k$ *are in* $C^s(\Gamma)$. *Let* v_{f_ϵ} *and* w_{f_ϵ} *be solutions to the boundary value problems* $(\mathrm{NCP}_{f_\epsilon,V})$ *and* $(\mathrm{NCP}_{f_\epsilon,W})$, *respectively. If* $V_j = W_j$ *for* $1 \le j \le k-1$, *then for all such* j,

$$\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_j}|_{\epsilon_{\ell_1}=\cdots=\epsilon_{\ell_j}=0}} v_{f_\epsilon} = \partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_j}|_{\epsilon_{\ell_1}=\cdots=\epsilon_{\ell_j}=0}} w_{f_\epsilon},$$

where $\epsilon_{\ell_1}, \ldots, \epsilon_{\ell_j}$ *belong to* $\{\epsilon_1, \ldots, \epsilon_k\}$.

Proof. Since, for sufficiently small parameters $\epsilon_1, \ldots, \epsilon_k$, a unique solution v_{f_ϵ} to $(\mathrm{NCP}_{f_\epsilon,V})$ exists in the class V_r whenever $f_\epsilon \in U_\delta$, we can specify the conditions above. Taking the first-order linearisation at zero of the equation

$$\begin{cases} \Delta_g v_{f_\epsilon} + V(p, v_{f_\epsilon}) = 0 & \text{in } M, \\ \qquad\qquad v_{f_\epsilon} = f_\epsilon & \text{on } \partial M \end{cases} \tag{2.2}$$

with respect to the parameters $\epsilon_1, \ldots, \epsilon_k$ and using condition (1.1),

$$\begin{cases} \Delta_g \partial_{\epsilon_j|_{\epsilon_j=0}} v_{f_\epsilon} = -V_1 \partial_{\epsilon_j|_{\epsilon_j=0}} v_{f_\epsilon} & \text{in } M, \\ \quad \partial_{\epsilon_j|_{\epsilon_j=0}} v_{f_\epsilon} = f_j & \text{on } \partial M \end{cases}$$

for $1 \le j \le k$. Thus, $\partial_{\epsilon_j|_{\epsilon_j=0}} v_{f_\epsilon}$ solves the linear Schrödinger equation with potential V_1 and Dirichlet data f_j. The same calculation works for the solution w_{f_ϵ} of $(\mathrm{NCP}_{f_\epsilon,W})$. In particular, since $V_1 = W_1$ by assumption, this proves the lemma for $j = 1$ via elliptic regularity. Assume now that the claim holds for $j \le k-2$. Then we write

$$\Delta_g(v_{f_\epsilon} - w_{f_\epsilon}) = \sum_{j \le k-1} \frac{W_j}{j!}(w_{f_\epsilon}^j - v_{f_\epsilon}^j) + \frac{W_k}{k!} w_{f_\epsilon}^k - \frac{V_k}{k!} v_{f_\epsilon}^k + \sum_{j>k} \frac{W_j}{j!} w_{f_\epsilon}^j - \frac{V_j}{j!} v_{f_\epsilon}^j.$$

Since $v_{f_\epsilon|_{\epsilon_1=\cdots=\epsilon_k=0}} = 0$ by estimate (1.2), taking a $(k-1)$th-order linearisation and by considering the terms in $\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}|_{\epsilon_{\ell_1}=\cdots=\epsilon_{\ell_{k-1}}=0}} v_{f_\epsilon}^j$ for $j \ge k-1$ which do not contain positive powers of v_{f_ϵ}, we find

$$\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}|_{\epsilon_{\ell_1}=\cdots=\epsilon_{\ell_{k-1}}=0}} v_{f_\epsilon}^{k-1} = (k-1)!(\partial_{\epsilon_{\ell_1}|_{\epsilon_{\ell_1}=0}} v_{f_\epsilon}) \cdots (\partial_{\epsilon_{\ell_{k-1}}|_{\epsilon_{\ell_{k-1}}=0}} v_{f_\epsilon}),$$
$$\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}|_{\epsilon_{\ell_1}=\cdots=\epsilon_{\ell_{k-1}}=0}} v_{f_\epsilon}^j = 0, \quad \text{for } j > k+1.$$

On the other hand, for $j \le k-2$ the expression $\partial_{\epsilon_{\ell_1}} \cdots \partial_{\epsilon_{\ell_{k-1}}|_{\epsilon_{\ell_1}=\cdots=\epsilon_{\ell_{k-1}}=0}} v_{f_\epsilon}^j$ contains only lower-order derivatives of v_{f_ϵ}. The same calculation works for w_{f_ϵ}. Taking these derivatives of $\Delta_g(v_{f_\epsilon} - w_{f_\epsilon})$ and applying elliptic regularity and the induction hypothesis concludes the proof of the lemma. □

Lemma 2.3. *Assume the setting of Proposition 2.1 and Lemma 2.2. Then for all* $k \ge 2$,

$$\sum_{j \le k-1} \int_M \frac{W_j}{j!} u_{k+1} \partial_{\epsilon_1} \cdots \partial_{\epsilon_k|_{\epsilon_1=\cdots=\epsilon_k=0}} (v_{f_\epsilon} - w_{f_\epsilon})\, d\omega_g + \int_M (V_k - W_k) u_1 u_2 \cdots u_{k+1}\, d\omega_g = 0,$$

where $u_1, \ldots, u_k$ *are* C^s *solutions to the linear Schrödinger operator with potential* $V_1 = W_1$, u_{k+1} *is* C^s *harmonic, and the Dirichlet data of these solutions are supported in* Γ.

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PROOF. Let u_{k+1} be a harmonic function with Dirichlet data f_{k+1} supported in Γ. Then

$$\int_{\partial M} f_{k+1}\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\cdots=\epsilon_k=0}(\mathcal{N}_V^\Gamma - \mathcal{N}_W^\Gamma)f_\epsilon \, d\sigma_g$$
$$= \int_{\partial M\setminus\Gamma} f_{k+1}\partial_\nu\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\cdots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\, d\sigma_g = 0$$

since f_{k+1} is supported away from the set integrated. The last integral is also equal to

$$-\int_M u_{k+1}\Delta_g\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\cdots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\, d\omega_g$$
$$+ \int_M \langle du_{k+1}, d\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\cdots=\epsilon_k}(v_{f_\epsilon} - w_{f_\epsilon})\rangle_g \, d\omega_g. \tag{2.3}$$

The second integral in (2.3) is

$$\int_M \partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\cdots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\Delta_g u_{k+1}\, d\omega_g + \int_{\partial M} \partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\cdots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\partial_\nu u_{k+1}\, d\sigma_g$$

which vanishes. Applying the calculation of Lemma 2.2 to the first integral in (2.3) gives

$$\int_M u_{k+1}(W_k - V_k)u_1\cdots u_k\, d\omega_g + \sum_{j\le k-1}\int_M \frac{W_j}{j!}u_{k+1}\partial_{\epsilon_1}\cdots\partial_{\epsilon_k}|_{\epsilon_1=\cdots=\epsilon_k=0}(v_{f_\epsilon} - w_{f_\epsilon})\, d\omega_g = 0$$

as desired. □

PROOF OF PROPOSITION 2.1. By virtue of Lemma 2.2 we will prove the claim via induction on k. For $k = 2$, the assumption $V_1 = W_1$ allows us to invoke Lemma 2.3 to conclude that

$$\int_M (V_2 - W_2)u_1u_2u_3\, d\omega_g = 0,$$

so our assumption ensures that $V_2 = W_2$. Now assume that $V_j = W_j$ holds for $j \le k - 1$. Combining the statements of Lemmas 2.2 and 2.3 again yields

$$\int_M (V_k - W_k)u_1u_2\cdots u_{k+1}\, d\omega_g = 0.$$

Applying our assumption once more shows that $V_k = W_k$ and concludes the proof of the claim. □

Therefore, in order to solve the semilinear Calderón problem it suffices to prove the assumption in Proposition 2.1. In the next section we first recover some useful information on the boundary, which will be our key step towards interior identification.

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3. Boundary determination

To recover uniqueness in the interior, the first step is often to do so on the boundary. In this section we will show that the integral identity assumed in Proposition 2.1 is valid up to infinite order on the boundary. If (Σ, g) is a Riemann surface, then, using conformal coordinates, the zeroth-order result will follow from the proof in the appendix of [10], which will turn out to be enough for our purpose. For the general case it suffices to consider a complex manifold (M, g) with real dimension $\dim M > 2$. For our problem, this is equivalent to assuming that $V_1 = W_1 = 0$. This means that the following proposition is sufficient.

PROPOSITION 3.1. *Let (M, g) be a compact, connected Riemannian manifold with smooth boundary. Suppose that $f \in C^\infty(M)$ satisfies*

$$\int_M f uv \, d\omega_g = 0 \tag{3.1}$$

for all C^s harmonic functions u, v with Dirichlet data supported in an arbitrarily small open subset $\Gamma \subseteq \partial M$. Then $\partial_\nu^k f|_\Gamma = 0$ for all $k \in \mathbb{N}$.

We will follow the idea developed in [3] by exploiting harmonic functions with prescribed boundary data which concentrate at an arbitrary boundary point $p \in \Gamma$ to very fine orders. For this purpose, let $(\chi, \rho) \in C_c^\infty(\mathbb{R}^{n-1}) \times C_c^\infty(\mathbb{R})$ be such that

$$\operatorname{Supp}\chi \times \operatorname{Supp}\rho \subseteq B \quad \text{and} \quad \|\chi\|_{L^2(\mathbb{R}^{n-1})} = \|\rho\|_{L^2(\mathbb{R})} = 1,$$

where $B \overset{\text{def}}{=} (|x'| < 1) \times (0 \leq x_n < 1)$ is a relatively open half ball in $\mathbb{R}^n_+$, such that $\chi = \rho = 1$ near the origin and χ, ρ vanish near ∂B. For $h > 0$, we also define

$$\chi_h(x') \overset{\text{def}}{=} \chi(x'/\sqrt{h}), \quad \rho_h(x_n) \overset{\text{def}}{=} \rho(x_n/h) \quad \text{and} \quad B_h \overset{\text{def}}{=} (|x'| < \sqrt{h}) \times (0 \leq x_n < \sqrt{h}),$$

where $x' = (x_1, \ldots, x_{n-1})$ denotes the local boundary coordinates. Without loss of generality, for $h > 0$ small, we can assume that B_h is contained in a boundary normal coordinate chart of (M, g) centred at $p \in \Gamma$. Thus locally, the metric satisfies

$$g^{\alpha n} = 0 \quad \text{for} \quad \alpha \leq n-1, \quad \text{and} \quad g^{nn} = 1.$$

In particular, we may assume that $g_{ij}(0) = \delta_{ij}$. We will now proceed to the proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. Following [22], for every $k \in \mathbb{N}$ there exist polynomials $Q_{j,h}$ in x_n such that

$$v_h = \sum_{0 \leq j \leq 2k+6} h^{j/2} Q_{j,h} e^{(ix'\cdot\tau - x_n)/h},$$

$$Q_{0,h} = \chi_h, \quad Q_{j,h} = \sum_{0 \leq i \leq j} h^{-i} q_{ij}\Big(\frac{x'}{\sqrt{h}}\Big) x_n^i \quad \text{for } j \geq 1,$$

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where $\tau \in \mathbb{R}^{n-1}$ is a vector locally tangential to ∂M, q_{ij} are compactly supported in $(|x'| \le 1)$ and $|\Delta_g v_h| = o(h^{k+1})$ uniformly on B_h as $h \to 0$.

Consider harmonic functions $u_h = \rho_h v_h + R_h$ where $\|R_h\|_{L^2} \le C\|\Delta_g(\rho_h v_h)\|_{L^2}$. Substituting these functions into the integral identity (2.1) yields

$$\int_M f|u_h|^2\,d\omega_g = \int_M f|\rho_h v_h|^2\,d\omega_g + \int_M f\rho_h v_h \overline{R}_h\,d\omega_g + \int_M f\rho_h \overline{v}_h R_h\,d\omega_g + \int_M f|R_h|^2\,d\omega_g.$$

Without loss of generality, we may assume that

$$\partial_\nu f|_\Gamma = \cdots = \partial_\nu^{k-1} f|_\Gamma = 0.$$

Therefore, by Taylor's theorem, a careful calculation on the order of h yields

$$\begin{aligned}\int_M f|\rho_h v_h|^2\,d\omega_g &= (\partial_\nu^k f)(0)\int_{B_h} x_n^k|\rho_h v_h|^2\sqrt{\det g}\,dx + o(h^{(n+2k+1)/2})\\ &= \frac{h^{(n+2k)/2}(\partial_\nu^k f)(0)}{(-2)^k}\int_{|x'|\le 1}|\chi(x')|^2\int_0^1|\rho(x_n)|^2 e^{-2x_n/h}\sqrt{\det g}\,dx_n\,dx' + o(h^{(n+2k+1)/2})\\ &= \frac{-h^{(n+2k+1)/2}(\partial_\nu^k f)(0)}{(-2)^k}\int_{|x'|\le 1}|\chi(x')|^2\sqrt{\det g_h(x',0)}\,dx' + o(h^{(n+2k+1)/2})\end{aligned}$$

as $h \to 0$. This follows by expanding v_h in the integral, applying convexity and using

$$h^{j-2i}\int_{|x'|\le\sqrt{h}}\left|q_{ij}\Big(\frac{x'}{\sqrt{h}}\Big)\right|^2\int_0^{\sqrt{h}} x_n^{k+2i}|\rho_h|^2 e^{-2x_d/h}\sqrt{\det g}\,dx_n dx' = o(h^{(n+2k+1)/2}),$$

for $j \ge 1$, $i \le j$. Indeed, it suffices to apply integration by parts with respect to the normal direction and change of variables, as well as choosing $\rho = 0$ in a neighbourhood of $x_n = 1$, to obtain this asymptotic estimate. Next we look at the L^2 estimate of $\Delta(\rho_h v_h)$. By the Leibniz rule,

$$\|\Delta_g(\rho_h v_h)\|_{L^2} \le \|\rho_h \Delta_g v_h\|_{L^2} + \|[\rho_h, \Delta_g]v_h\|_{L^2},$$

where $[\rho_h, \Delta_g]v_h = v_h\Delta_g\rho_h + \langle d\rho_h, dv_h\rangle_g$ is the commutator. By construction it is obvious that $\|\rho_h\Delta_g v_h\|_{L^2} = o(h^{(n+4k+4)/2})$. On the other hand, directly,

$$\int_M |v_h\Delta_g\rho_h|^2\,d\omega_g \le \sum_{j\le 2k+6}\sum_{i\le j} Ch^{j-2i}\int_{|x'|\le\sqrt{h}}\left|q_{ij}\Big(\frac{x'}{\sqrt{h}}\Big)\right|^2\int_0^{\sqrt{h}} x^{2i}e^{-2x_n}|\Delta_g\rho_h|^2\,dx_n\,dx',$$

$$\begin{aligned}\int_M |\langle d\rho_h, dv_h\rangle|^2\,d\omega_g &\le \sum_{j\le 2k+6}\sum_{i\le j} Ch^{j-2i}\int_{|x'|\le\sqrt{h}}\left|q_{ij}\Big(\frac{x'}{\sqrt{h}}\Big)\right|^2\int_0^{\sqrt{h}}|\partial_n x_n^i\partial_n\rho_h|^2 e^{-2x_n/h}\,dx_n\,dx'\\ &\quad + \sum_{j\le 2k+6}\sum_{i\le j} Ch^{j-2i-2}\int_{|x'|\le\sqrt{h}}\left|q_{ij}\Big(\frac{x'}{\sqrt{h}}\Big)\right|^2\int_0^{\sqrt{h}}|x_n^i\partial_n\rho_h|^2 e^{-2x_n/h}\,dx_n\,dx'.\end{aligned}$$

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Since the derivatives of ρ vanish at the end points of $[0, \sqrt{h}]$, in all cases the interior integrals in x_n decay to order $o(h^\infty)$ as $h \to 0$ and therefore so do $\|v_h \Delta_g \rho_h\|_{L^2}$ and $\|\langle d\rho_h, dv_h\rangle\|_{L^2}$. From the Cauchy–Schwartz inequality,

$$\left|\int_M f\rho_h v_h \overline{R}_h \, d\omega_g + \int_M f\rho_h \bar{v}_h R_h \, d\omega_g\right| \le 2\|f\|_{L^\infty}\|v_h\|_{L^2}\|\Delta_g(\rho_h v_h)\|_{L^2}$$
$$\le 2\|f\|_{L^\infty}\|v_h\|_{L^2}\|\rho_h \Delta_g v_h\|_{L^2} + 2\|f\|_{L^\infty}\|v_h\|_{L^2}\|[\rho, \Delta_g]v_h\|_{L^2} = o(h^{(n+2k+1)/2}),$$
$$\left|\int_M f|R_h|^2 \, d\omega_g\right| \le \|f\|_{L^\infty}\|R_h\|_{L^2}^2 \le \|f\|_{L^\infty}\|\Delta_g(\rho_h v_h)\|_{L^2}^2 = o(h^{(n+2k+1)/2}).$$

Putting everything together, we arrive at

$$\int_M f|u_h|^2 \, d\omega_g = \frac{-h^{(n+2k+1)/2}}{(-2)^k}(\partial_\nu^k f)(0) \int_{|x'|\le 1} |\chi(x')|^2 \sqrt{\det g_h(x', 0)} \, dx' + o(h^{(n+2k+1)/2}),$$

that is,

$$0 = (\partial_\nu^k f)(0) \int_{|x'|\le 1} |\chi(x')|^2 \sqrt{\det g_h(x', 0)} \, dx' + o(1).$$

Taking the limit as $h \to 0$. we conclude that $(\partial_\nu^k f)(0) = 0$. Since this holds for every $p \in \Gamma$, the claim follows. □

4. Interior determination

We recall key results from [10, 11] on the existence of special holomorphic functions with prescribed critical points and real boundary conditions.

PROPOSITION 4.1. *Let M be a compact complex manifold with smooth boundary. Assume that M has local charts given by global holomorphic functions. For $k \ge 2$, we can find a dense subset S of M such that for any $p \in S$, there exists Φ in $C^k(M) \cap \mathcal{O}(\mathrm{Int}\, M)$ having a critical point at p such that both the real and imaginary parts of Φ are Morse functions in M.*

In the case of a Riemann surface Σ, the same is true and Φ is real on Γ^c, where Γ^c is the complement of an arbitrarily small open subset $\Gamma \subseteq \partial\Sigma$. Moreover, for every set of discrete points $\{p, p_1, \ldots, p_N\}$ we can find a holomorphic function a on Σ such that $a(p) = 0$ and $a(p_1) = \cdots = a(p_N) = 0$ up to large orders, with the boundary condition that $a|_{\Gamma^c}$ is purely imaginary.

4.1. The case $n > 1$. Full data. Using Proposition 2.1 and the boundary uniqueness result of Proposition 3.1, the proof of Theorem 1.1 is now a straightforward application of the result in [11].

PROOF OF THEOREM 1.1. By Proposition 2.1, it suffices to show that if $f \in C^\infty(M)$ satisfies condition (2.1) for all C^s harmonic functions u, v, then $f = 0$. Since we only consider the full-data case, by taking identity functions it suffices to replace (2.1) by

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(3.1). By Proposition 3.1, in this case f vanishes on ∂M up to infinite orders, therefore we follow the idea in [11] and apply the stationary-phase argument.

Assume that (3.1) holds. By the results of Proposition 4.1, there exists a dense subset $S \subseteq M$ such that for every $p \in S$, there exists a holomorphic function $\Phi \in C^k(M) \cap \mathcal{O}(\operatorname{Int} M)$ for some large k such that p is a critical point of Φ and $\operatorname{Im}\Phi$ is a Morse function. Choose

$$u_h = e^{\Phi/h} a \quad \text{and} \quad v_h = e^{-\bar{\Phi}/h}\bar{a} \quad \text{for all } h > 0, \tag{4.1}$$

where $a \in \mathcal{O}(\operatorname{Int} M)$ satisfies $a(p) = 1$ and $a(p_1) = \cdots = a(p_N) = 0$ and $\{p, p_2, \ldots, p_N\}$ is the set of critical points of $\operatorname{Im}\Phi$ in $\operatorname{Int} M$. Such an amplitude can be constructed by the assumption that M is holomorphic separable. In particular, u_h and v_h are respectively holomorphic and anti-holomorphic and so harmonic for all $h > 0$. It follows that

$$\int_M f e^{2i \operatorname{Im}\Phi/h} |a|^2 \, d\omega_g = 0 \quad \text{for all } h > 0.$$

Take a partition of unity (χ_j) of M such that p is contained in $\operatorname{Supp}\chi$ but not $\operatorname{Supp}\chi_j$ for any $j \neq 0$ and $\chi_1(p) = 1$. Since f vanishes up to infinite order on ∂M by Proposition 3.1, the method of stationary phase yields

$$\left| (2\pi)^n f(p) \sqrt{\det g(p)} \exp\left(\frac{i\pi}{4} \operatorname{sgn} \nabla_g^2 \operatorname{Im}\Phi(p)\right) (\det\, \nabla_g^2 \operatorname{Im}\Phi(p))^{-1/2} \right| = o(1)$$

as $h \to 0$. Taking this limit, we see that $f = 0$ on S. Because S is dense in M and f is continuous, $f = 0$ on M as well. □

4.2. The case $n = 1$. Partial data. We now move on to the consideration of Riemann surfaces. Now the special structure has only one complex dimension and adapting the techniques in [10] allows us to prove Theorem 1.2 with partial data.

Proof of Theorem 1.2. The Dirchlet-to-Neumann map of $(\mathrm{NCP}_{f,V})$ determines the Dirchlet-to-Neumann map of the linear Calderón problem for V_1. Indeed, for $\epsilon > 0$ small enough and $\tilde{f} \in C^s(\Gamma)$,

$$\partial_{\epsilon|_{\epsilon=0}} u_{\epsilon\tilde{f}} + V_1 \partial_{\epsilon|_{\epsilon=0}} u_{\epsilon\tilde{f}} = 0 \;\text{ in } M \quad \text{and} \quad \partial_{\epsilon|_{\epsilon=0}} u_{\epsilon\tilde{f}} = \tilde{f} \;\text{ on } \Gamma.$$

Thus, if the Dirchlet-to-Neumann maps of $(\mathrm{NCP}_{f,V})$ and $(\mathrm{NCP}_{f,W})$ are the same, then $V_1 = W_1 = U$ by the result in [10]. This reduces the claim to the assumption in Proposition 2.1. Now, appealing to Proposition 4.1 once more, we can find a dense subset $S \subseteq \Sigma$ such that for every $p \in S$, we can choose a holomorphic function $\Phi \in C^k(\Sigma) \cap \mathcal{O}(\operatorname{Int}\Sigma)$ for some large k such that p is a critical point of Φ and φ, ψ are Morse functions up to the boundary. Moreover, $\Phi|_{\Gamma^c}$ is purely real and we can choose $a \in \mathcal{O}(\operatorname{Int}\Sigma)$ such that $a|_{\Gamma^c}$ is purely imaginary, $a(p) = 1$ and $a(p') = 0$ up to arbitrarily large orders for any other critical points p' of $\operatorname{Im}\Phi$. Therefore, choosing

$$\tilde{u}_h = e^{\Phi/h} a + \overline{e^{\Phi/h} a} \quad \text{and} \quad \tilde{v}_h = e^{-\Phi/h} a + \overline{e^{-\Phi/h} a}, \quad \text{for all } h > 0,$$

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ensures that $\tilde{u}_h$ and $\tilde{v}_h$ are harmonic and

$$\tilde{u}_h = e^{\varphi/h}(\operatorname{Im} a - \operatorname{Im} a) = 0 \quad \text{and} \quad \tilde{v}_h = e^{-\varphi/h}(\operatorname{Im} a - \operatorname{Im} a) = 0 \quad \text{on } \Gamma^c,$$

so that $\operatorname{Supp} \tilde{u}_h$ and $\operatorname{Supp} \tilde{v}_h \subseteq \Gamma$ as well. This is not quite enough because we need to extend these solutions to become solutions to the linear Schrödinger equation with potential U. For that we will assume the technical results proved in [10, Chapter 5]. We can extend the construction by means of a Carleman estimate to obtain H^2 solutions to the Schrödinger equation with potential U, of the form

$$\begin{aligned} u_h &= e^{\Phi/h}(a + ha_0 + r_1) + \overline{e^{\Phi/h}(a + ha_0 + r_1)} + e^{\varphi/h} r_2, \\ v_h &= e^{-\Phi/h}(a + hb_0 + s_1) + \overline{e^{-\Phi/h}(a + hb_0 + s_1)} + e^{-\varphi/h} s_2, \end{aligned}$$

where a_0, b_0 are holomorphic and independent of h and the remainders satisfy

$$\begin{aligned} &e^{-\Phi/h}(\Delta_g + V_1)e^{\Phi/h}(a + ha_0 + r_1) = O_{L^2}(h|\log h|), \quad \|r_1\|_{L^2} = O(h), \\ &(e^{\Phi/h}(a + r_1 + ha_0) + \overline{e^{\Phi/h}(a + r_1 + ha_0)})_{|\Gamma^c} = 0, \quad \|r_2\|_{L^2} \le Ch^{3/2}|\log h|, \end{aligned}$$

and likewise for b_0, s_1 and s_2. The Dirichlet boundary data of u_h and v_h can be constructed to have support on Γ.

As stated, the regularities of u_h and v_h are insufficient for an application of (2.1). For this we consider a sequence of smooth approximations $(f_{h,j})_j$ defined on the boundary such that $\operatorname{Supp} f_{h,j} \subseteq \Gamma$ and $\lim_{j\to\infty} f_{h,j} = u_h|_{\partial M}$ in $H^1(\partial\Sigma)$. Let $(\phi_{h,j})_j$ be the corresponding smooth solutions to $(\Delta_g + U)u = 0$, $u|_{\partial\Sigma} = f_{h,j}$. Elliptic boundary regularity estimates ensure that $\lim_{j\to\infty} \phi_{h,j} = u_h$ in H^1. By making a similar calculation for v_h, we conclude that there exist smooth approximations $(\psi_{h,j})_j$ and remainders R_{u_h}, R_{v_h} depending on j such that

$$\phi_{h,j} = u_h + R_{u_h}, \quad \psi_{h,j} = v_h + R_{v_h} \quad \text{and} \quad \lim_{j\to\infty} R_{u_h} = \lim_{j\to\infty} R_{v_h} = 0 \quad \text{in } H^1.$$

Assume first that

$$\int_\Sigma fuv \, d\omega_g = 0$$

for all C^s solutions to the Schrödinger equation with Dirichlet data supported on Γ as in (2.1). Then

$$\begin{aligned} 0 &= \int_\Sigma f\phi_{h,j}\varphi_{h,j} \, d\omega_g \\ &= \int_\Sigma fu_h v_h \, d\omega_g + \int_\Sigma fu_h R_{v_h} \, d\omega_g + \int_\Sigma fv_h R_{u_h} \, d\omega_g + \int_\Sigma R_{u_h} R_{v_h} \, d\omega_g. \end{aligned}$$

The last three integrals converge to zero as $j \to \infty$. By taking this limit we arrive at

$$\int_\Sigma fu_h v_h \, d\omega_g = 0.$$

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Next we get an expansion

$$0 = I_1 + I_2 + o(h),$$

where

$$I_1 \stackrel{\text{def}}{=} \int_\Sigma f(a^2 + \overline{a}^2)\, d\omega_g + 2\mathrm{Re} \int_\Sigma e^{2i\psi/h} f|a|^2\, d\omega_g,$$
$$I_2 \stackrel{\text{def}}{=} 2h\mathrm{Re} \int_\Sigma af\Big(e^{2i\psi/h}\Big(\overline{\frac{s_1}{h} + b_0}\Big) + e^{-2i\psi/h}\Big(\overline{a_0 + \frac{r_1}{h}}\Big) + b_0 + a_0 + \frac{s_1 + r_1}{h}\Big) d\omega_g.$$

Applying the theorem of stationary phase as in [10],

$$0 = 2Cf(p)|a(p)|^2 + o(1), \quad C \neq 0,$$

as $h \to 0$. Hence $f = 0$.

To prove the general case, we assume that

$$\int_M fuvw\, d\omega_g = 0$$

for all C^s solutions u, v to the Schrödinger equation and harmonic functions w with Dirichlet data supported on Γ. If w is smooth, then by what was proved above, we deduce $fw = 0$, thus

$$\int_\Sigma fuv\, d\omega_g = 0$$

for all smooth harmonic functions u and v with Dirichlet data supported in Γ. By virtue of the smooth approximation argument above, $f = 0$. Now suppose inductively that

$$\int_\Sigma fu_1 \cdots u_{k-1} w\, d\omega_g = 0$$

for all C^s solutions $u_1, \ldots, u_{k-1}$ to the Schrödinger equation and harmonic functions w implies $f = 0$, and assume

$$\int_\Sigma fu_1 \cdots u_{k-1} u_k w\; d\omega_g = 0$$

for an additional C^∞ solution to the Schrödinger equation as above. Choosing u_k smooth again implies $fu_{k-1} = 0$, and another smooth approximation argument yields $f = 0$. This concludes the proof of the claim. □

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