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RESEARCH ARTICLE

Regenerations and applications

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Abstract
Chen-Gounelas-Liedtke recently introduced a powerful regeneration technique, a process opposite to specialization, to prove existence results for rational curves on projective $K3$ surfaces. We show that, for projective irreducible holomorphic symplectic manifolds, an analogous regeneration principle holds and provides a very flexible tool to prove existence of uniruled divisors, significantly improving known results.

1. Introduction

Rational curves on $K3$ surfaces have now been studied for decades, with motivations also coming from arithmetic geometry, (non-)hyperbolicity questions and general conjectures on 0-cycles. A natural generalization of $K3$ surfaces is given by irreducible holomorphic symplectic (IHS) manifolds, which are compact, simply connected Kähler manifolds with $H^{2,0}$ generated by a symplectic form. In any even dimension $2n$, $n \geq 2$, there are two known deformation classes (cf. [Bea83]), one is given by Hilbert schemes of points on $K3$ surfaces and their deformations (called varieties of $K3^{[n]}$ type), and the other is given by deformations of an analogous construction using abelian surfaces (called varieties of generalized Kummer type). Two more deformation classes discovered by O'Grady exist in dimension 6 and 10 (cf. [O'G99, O'G03]). For the basic theory of IHS manifolds we refer the reader to, for example, [Bea83, Huy99].

In recent years, rational curves on projective IHS manifolds have been actively investigated with different objectives and techniques; cf., for example, [AV15, BHT15, Voi16, LP19, MP18, CMP24, KLCM19a, OSY19, Ber21] and the references therein. Rational curves covering a divisor on an IHS manifold behave very well with respect to deformation theory (i.e., they deform in their Hodge locus inside the parameter space of deformations of the IHS manifold and keep covering a divisor). This has been one of the main properties used to prove existence results and, at the same time, one of the main limitations. Indeed, to produce a uniruled divisor in an ample linear system of a polarized IHS (X, H), one would try and exhibit such an example on a special point (X_0, H_0) in the same connected component of the corresponding moduli space. As proved in [OSY19], in some cases, it is impossible to do it with primitive rational curves. However, in [CMP24, MP18, MP21], this approach was successfully implemented to prove that outside at most a finite number of connected components (precisely those not satisfying the necessary conditions given in [OSY19]) of the moduli spaces of projective IHS manifolds of $K3^{[n]}$ or generalized Kummer type, for all the corresponding points (X, H), there exists a positive integer m such that the linear system $|mH|$ contains a uniruled divisor covered by rational curves of primitive homology class Poincaré-dual to that of $|H|$. For a completely different proof (based on Gromov-Witten theory) of the existence of uniruled divisors covered by primitive rational curves on deformations of $K3^{[n]}$, see [OSY19, Theorem 0.1]. Due to the cases left out by [CMP24, OSY19], respectively [MP18, MP21], one could reasonably wonder whether uniruled divisors on such manifolds do always exist.

More recently, Chen-Gounelas-Liedtke introduced in [CGL22] a new viewpoint to prove existence results for rational curves on projective $K3$ surfaces: regeneration, a process opposite to specialization. In this article, we show that for projective irreducible holomorphic symplectic manifolds, an analogous regeneration principle holds for uniruled divisors and provides a new and flexible tool to prove existence results. Combining this new viewpoint with results from [CMP24, MP18, MP21], we are able to improve significantly the available results, in some cases passing from no known existence result at all to density of uniruled divisors in the classical topology.

To state our results, we start with the following.

Definition 1.1. Let $\mathcal{X} \to B$ be a family of IHS manifolds over a connected base. Let $0 \in B$ and let X_0 be the corresponding fibre. Let $D_0 \subset X_0$ be an integral uniruled divisor. A *regeneration* of D_0 is a flat family $\mathcal{D}' \subset \mathcal{X}_{B'}$ of uniruled and generically integral divisors $\mathcal{D}' \to B'$, where $B' \to B$ is a finite cover,[1] such that D_0 is a component of the fiber $\mathcal{D}'_0$ of $\mathcal{D}'$ over (one preimage of) the central fibre 0.

A reducible divisor is called uniruled if all of its components are.

Hypothesis 1.2. Let X be a projective IHS manifold. There exists a constant $d \geq 0$ such that all primitive ample curve classes $[C] \in H_2(X, \mathbb{Z})$ satisfying $q(C) > d$ have a connected and rational representative $R \in [C]$ such that R rules a prime divisor of class proportional to $[C]^\vee$.

Here, $[C]^\vee$ denotes the divisor $[D] \in \mathrm{NS}(X) \otimes \mathbb{Q}$ such that $C \cdot E = q(D, E)$ for all divisors E, where q is the Beauville-Bogomolov-Fujiki form on X. A curve is said to be ample if its dual divisor is ample. Analogously, we define the curve dual to a divisor.

The above hypothesis, which may look slightly unnatural, is the higher dimension analogue of [CGL22, Theorem A.1] and, as we will see below, can be shown to hold for IHS manifolds of $K3^{[n]}$ and generalized Kummer type, thanks to previous work done in [CMP24, MP18, MP21]. Our main novel contribution is the following result which, despite the simplicity of its proof, seems to provide the right viewpoint to tackle these kinds of questions.

Regeneration principle 1.3. Let $\mathcal{X} \to B$ be a family of projective IHS manifolds with a central fibre $\mathcal{X}_0$ satisfying hypothesis 1.2. Let $D_0 \subset \mathcal{X}_0$ be an integral uniruled divisor on the central fibre. Then D_0 admits a regeneration.

The regeneration principle works perfectly on IHS manifold of $K3^{[n]}$ or generalized Kummer type.

Theorem 1.4. *Any integral uniruled divisor in a fiber of any family of projective IHS manifolds of $K3^{[n]}$ or generalized Kummer type admits a regeneration.*

Our first application is to show existence of ample uniruled divisors also for the connected components of the moduli spaces left out by [CMP24, MP18, MP21].

Theorem 1.5. *Let (X, H) be a polarized IHS manifold of $K3^{[n]}$ or generalized Kummer type. Then there exists $m \in \mathbb{N}$ and a uniruled divisor in $|mH|$.*

In particular, the applications to zero-cycles pointed out in [CMP24, Theorems 1.7 and 1.8] now hold for all polarized IHS manifolds of $K3^{[n]}$ or generalized Kummer type.

At the very general point in any component of the moduli space of polarized IHS manifolds of $K3^{[n]}$ or generalized Kummer type, we can significantly improve Theorem 1.5.

Theorem 1.6. *Let $\mathcal{M}$ be an irreducible component of the moduli space of polarized IHS manifolds of $K3^{[n]}$ or generalized Kummer type.*

[1] See the comment at the end of the proof of the Regeneration principle 1.3 for the need of this base change.

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1) *In the $K3^{[n]}$-type case, any polarized IHS manifold X outside a possibly countable union of subvarieties of $\mathcal{M}$ verifies the following: any pair of points $x_1, x_2 \in X$ and any $\epsilon > 0$, there exists a chain C of at most 2n rational curves, each of which deforms in a family covering a divisor, such that C intersects euclidean balls of radius ϵ centered in x_i, $i = 1, 2$.*
2) *In the generalized Kummer type case, any polarized IHS manifold X outside a possibly countable union of subvarieties of $\mathcal{M}$ contains infinitely many distinct ample uniruled divisors.*

Theorem 1.6, item 1) result can be seen as an effective non-hyperbolicity statement. The study of non-hyperbolicity of IHS manifolds dates back to Campana [Cam92], with more recent important contributions by Verbitsky [Ver15] and Kamenova-Lu-Verbitsky [KLV14]. We refer the interested reader to [KL22] for a thorough discussion and a complete list of references.

We can also show the following less strong but more precise result, which was previously known only in dimension 2 by [BT00, Theorem 4.10].

Theorem 1.7. *Let X be a projective IHS manifold of $K3^{[n]}$ or Kummer type such that $Bir(X)$ is infinite. Then X has infinitely many uniruled divisors.*

We hope that this new viewpoint via regenerations could also lead to progress towards the existence of higher codimension algebraically coisotropic subvarieties.

2. Regenerations

Proof of the Regeneration principle 1.3. We can suppose that $\mathcal{X}_0$ has Picard rank at least two and that D_0 is not proportional to the polarization; otherwise, by [CMP24, Corollary 3.5], we can deform a curve ruling D_0 over all of B, and obtain in this way a regeneration of D_0.

Let C_0 be the class of a minimal curve ruling D_0. Let $\mathcal{H} \in \mathrm{Pic}(\mathcal{X})$ be a relative polarization and H_0 its restriction to the central fibre $\mathcal{X}_0$. Let $H_0^\vee$ be the (ample) class of a curve dual to H_0. We can choose $m \in \mathbb{N}$ big enough so that $mH_0^\vee - C_0$ is ample, primitive and of square bigger than d_0. Therefore, by Hypothesis 1.2, we have a rational curve $R_0 \in [mH_0^\vee - C_0]$ which rules an ample divisor F_0 inside $\mathcal{X}_0$.

As the divisor F_0 is ample, we have $C_0 \cdot F_0 > 0$. Hence, we can fix a point in $C_0 \cap F_0$ and pick a curve R_0 in the ruling of F_0 passing through this point. Notice that C_0 cannot coincide with the ruling of F_0, as C_0 and R_0 are not proportional (because the divisors they rule are not). In this way, we obtain a connected rational curve of class $[C_0 + R_0]$. By abuse of notation, we denote this curve by $C_0 + R_0$. By [Ber21, Corollary 6.3], which generalizes [CMP24, Corollary 3.5] to the reducible case, the curve $C_0 + R_0$ deforms in its Hodge locus $\mathrm{Hdg}_{[\mathrm{C_0+R_0}]}$ of the class $[C_0 + R_0] = [mH_0^\vee]$ and keeps ruling a divisor on each point of $\mathrm{Hdg}_{[\mathrm{C_0+R_0}]}$. By construction, this Hodge locus coincides with B, as $C_0 + R_0$ is a multiple of $H_0^\vee$, and the result follows. Notice that here we might have to take a base change to a finite cover of B, in order to ensure the existence of a global family of divisors, whence the base change appearing in Definition 1.1. □

Notice that the proof of the Regeneration principle works as well in the surface case, where of course it has to be attributed to [CGL22].

The following can be seen as a concentration of some of the main contributions of [CMP24, MP18, MP21] – namely, the study of the monodromy orbits, constructions of examples and deformation theory.

Proposition 2.1. *Hypothesis 1.2 holds for any family of manifolds of $K3^{[n]}$ and Kummer type, and the constant d_0 is $(2n-2)^2(n-1)$ and $(2n+2)^2(n+1)$, respectively.*

Proof. Let (S, h_S) be a polarized K3 of genus p and (A, h_A) a polarized abelian surface of type $(1, p-1)$. We denote by r_n the class of an exceptional rational curve which is the general fiber of the Hilbert-Chow morphism $S^{[n]} \to S^{(n)}$ (resp. $K_n(A) \subset A^{[n+1]} \to A^{(n+1)}$) and by $h_S \in H_2(S^{[n]}, \mathbb{Z})$ (resp. $h_A \in H_2(K_n(A), \mathbb{Z})$) the image of the class $h_S \in H_2(S, \mathbb{Z})$ (resp. $h_A \in H_2(A, \mathbb{Z})$) under the inclusion $H_2(S, \mathbb{Z}) \hookrightarrow H_2(S^{[n]}, \mathbb{Z})$ (resp. $H_2(A, \mathbb{Z}) \hookrightarrow H_2(K_n(A), \mathbb{Z})$). Recall that $q(h_S) = 2p - 2 = q(h_A)$ and $q(r_n)$ equals $1/(2n-2)$ in the $K3^{[n]}$ case and $1/(2n+2)$ in the Kummer case.

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We take a primitive ample curve class $C \in H_2(X,\mathbb{Z})$ such that $q(C) > n-1$ (resp. $n+1$ for Kummer type). By [CMP24, Corollary 2.8] and [MP18, Theorem 4.2], the pair (X,C) is deformation equivalent to the pair $(S^{[n]}, h_S - 2gr_n)$ with $2g \leq n-1$ or $(S^{[n]}, h_S - (2g-1)r_n)$ with $2g \leq n$ (resp. $(K_n(A), h_A - 2gr_n)$ or $(K_n(A), h_a - (2g-1)r_n)$ with $2g \leq n-1$).

If $p \leq g$, we would get a contradiction since

$$n-1 \leq q(C) = q(h_S) - 4g^2\frac{1}{2(n-1)} = 2(p-1) - 4g^2\frac{1}{2(n-1)} \leq 2(g-1) - 4g^2\frac{1}{2(n-1)} \leq n-2.$$

Notice that the class of C equals $\frac{D^\vee}{m}$ with $m \leq 2n-2$; hence, the condition $q(D) \geq (2n-2)^2(n \pm 1)$ ensures $q(C) \geq n \pm 1$ (where the sign is according to the deformation type).

Therefore, $p \geq g$, and by [CMP24, Section 4.1] and [MP21, Proof of Proposition 2.1], the curves we obtain in $S^{[n]}$ (resp. in $K_n(A)$) have a rational representative which covers a divisor by [CMP24, Proposition 4.1] and [MP21, Proposition 1.1]. Such rational curves then deform in its Hodge locus by [CMP24, Corollary 3.5], while still covering a divisor, and the proposition follows. ◻

Proof of Theorem 1.4. The result follows immediately from the combination of Proposition 2.1 and the Regeneration principle 1.3. ◻

3. Applications

In this section, we provide the proofs of the applications of the Regeneration principle to IHS manifolds of $K3^{[n]}$-type or generalized Kummer type.

Proof of Theorem 1.5. Again, the result follows from the combination of Proposition 2.1 and the Regeneration principle 1.3. Indeed, suppose that (X,H) is a polarized IHS manifold of $K3^{[n]}$-type, and let us consider a connected component $\mathcal{M}$ of the moduli space of polarized IHS manifolds containing (X,H). By [CMP24, Theorem 2.5], there exists a point in $\mathcal{M}$ which parametrizes the Hilbert scheme over a very general projective $K3(S,H_S)$. Let us choose any rational curve C in S, whose existence is guaranteed by Bogomolov-Mumford [MM06] (see also [BHPVdV04, Section VIII.23]), and let us consider the uniruled divisor $D_C = \{Z \in S^{[n]}$ such that $supp(Z) \cap C \neq \emptyset\}$. We then apply the Regeneration principle 1.3 to D_C and obtain a regeneration of it on all IHS manifolds corresponding to points of $\mathcal{M}$. As the very general element of $\mathcal{M}$ has Picard rank one, the class of this regeneration is proportional to this unique class; hence, our regeneration has class mH on X, for some m. For the generalized Kummer type, we proceed the same way, by using [MP18, Theorem 4.2] and [KLCM19b, Theorem 1.1] instead of the analogous results in the $K3^{[n]}$-type case. ◻

Notice that the uniruled divisors we produce are not necessarily irreducible.

More generally, we have the following result.

Proposition 3.1. *Let (X,H) be a projective IHS manifold of $K3^{[n]}$ or Kummer type, and let $D \in \mathrm{Pic}(X)$ be a divisor with $q(D) \geq 0$ and $(D,H) > 0$. Then there exists a uniruled divisor in $|mD|$ for some $m \in \mathbb{N}$.*

Proof. The proof is analogous to Theorem 1.5, with an extension to the case of square zero classes. If D has positive square, instead of the moduli space of polarized IHS manifolds, we consider the moduli space $\mathcal{M}$ of lattice polarized IHS manifolds such that $\mathrm{Pic}(X)$ contains a divisor of square $q(D)$, and pick the connected component containing (X,D). Let us choose a parallel transport operator γ on $\mathcal{M}$ such that $\gamma(X)$ has Picard rank 1. Therefore, $\gamma(D)$ is ample on $\gamma(X)$. By Theorem 1.5, a multiple of $\gamma(D)$ is uniruled by a rational curve $\gamma(C)$, which has class proportional to $\gamma(D)^\vee$. Therefore, by [CMP24, Proposition 3.1], $\gamma(C)$ deforms in its Hodge locus, which by construction contains (X,D), and we obtain a rational curve C covering a multiple of D. If $q(D) = 0$, we can suppose that D is nef by [Mar11, Proposition 5.6]; otherwise, we follow the same reasoning as above to reduce to the nef case. As X is projective, we have an ample divisor $H \in \mathrm{Pic}(X)$. Let L be the saturated lattice generated by D

and H, and let us consider the component $\mathcal{M}$ of the moduli space of L lattice polarized IHS manifolds containing (X, L). Inside of $\mathcal{M}$, by [MP23, Theorem 3.13], we can pick a point $\gamma(X)$ such that $\gamma(D)$ stays nef and there exists a prime exceptional divisor E on $\gamma(X)$ such that $q(\gamma(D), E) > 0$.[2] Let R be a curve ruling E. As $\gamma(D)$ is nef, there exists an $m \in \mathbb{N}$ such that $m\gamma(D)^\vee - R$ is an ample curve. Therefore, by Proposition 2.1, we produce a rational curve C of class $m\gamma(D)^\vee - R$ which rules an ample divisor and attach to it a rational tail R, so that the connected curve $C + R$ of class $m\gamma(D)^\vee$ rules a divisor and deforms in its Hodge locus by [Ber21, Corollary 6.3]. By construction, this Hodge locus contains $(X, D^\vee)$, and the result follows. □

To prove Theorem 1.6, we will use the following result of Chen and Lewis on $K3$ surfaces. Let $\mathcal{F}_g$ be the moduli space of polarized genus g K3 surfaces, and let $\mathcal{S}_g$ be the universal surface over $\mathcal{F}_g$. Let $\mathcal{C}_{g,n}$ be the scheme of relative dimension one whose fibre over a point $(S, L) \in \mathcal{F}_g$ consists of all irreducible rational curves contained in $|nL|$. Recall the following result.

Theorem 3.2 (Theorem 1.1, [CL13]). *The set $\cup_{n\in\mathbb{N}}\mathcal{C}_{g,n}$ is dense in the strong topology inside $\mathcal{S}_g$, for all $g \geq 2$.*

From this, one easily obtains the following.

Corollary 3.3. *Let S be a general projective $K3$ surface. Then for any pair of points ξ_1, ξ_2 on $S^{[n]}$ and any $\epsilon > 0$, there exists a chain C of at most $2n$ rational curves, each of which deforms in a family covering a divisor, such that C intersects balls of radius ϵ centered in ξ_i, $i = 1, 2$.*

Proof. Without loss of generality, we can suppose that the two points ξ_i, $i \in \{1, 2\}$ correspond to reduced subschemes and that $\mathrm{supp}(\xi_1) \cap \mathrm{supp}(\xi_2) = \emptyset$; otherwise, we can take arbitrarily close approximations by reduced subschemes with such property. Therefore, we write

$$\xi_i = p_1^i + \dots p_n^i,$$

with $p_1^i, \dots, p_n^i$ distinct points on S for $i = 1, 2$. By Theorem 3.2, we have two ample irreducible curves R_1^1, R_1^2 arbitrarily near p_1^1 and p_1^2, respectively. As these curves are ample, the rational curve $R_1 = R_1^1 \cup R_1^2$ is connected. Let us consider the rational curve $R_1 + p_2^1 + \dots p_n^1$ inside $S^{[n]}$: this can be used to approximate the subschemes $p_1^1 + p_2^1 + \dots p_n^1$ and $p_1^2 + p_2^1 + \dots p_n^1$. Iterating the argument, one obtains a rational curve (union of two irreducible ample curves) R_j for all $j \in \{1, \dots n\}$ which approximates the two points p_j^1 and p_j^2. Considering the curve $p_1^2 + \dots + p_{j-1}^2 + R_j + p_{j+1}^1 + \dots + p_n^1$, one can approximate the points $p_1^2 + \dots + p_{j-1}^2 + p_j^1 + p_{j+1}^1 + \dots + p_n^1$ and $p_1^2 + \dots + p_{j-1}^2 + p_j^2 + p_{j+1}^1 + \dots + p_n^1$. Therefore, by taking the union of these curves, we obtain a chain of $2n$ rational irreducible curves which approximate the two points ξ_1 and ξ_2. By construction, each of these rational curves C deforms in a family which covers the divisor $\{Z \in S^{[n]},\ \text{such that}\ \mathrm{supp}(Z) \cap C \neq \emptyset\}$, and the corollary follows. □

Proof of Theorem 1.6. 1) Let X be a very general IHS manifold in $\mathcal{M}$. Let $x_1, x_2 \in X$ be two points on it. Thanks to [MP23, Corollary 1.2], we can pick a point in $\mathcal{M}$ which parametrizes the punctual Hilbert scheme of a very general projective $K3(S, H)$ arbitrarily close to X and two points $\xi_1, \xi_2 \in S^{[n]}$ approximating x_1 and x_2, respectively. We take the chain R of $2n$ rational curves approximating ξ_1 and ξ_2 given by Corollary 3.3. We can now apply the Regeneration principle 1.3 to regenerate the union of the divisors ruled by the deformations of the irreducible components of R to obtain a chain of rational curves on X satisfying the statement.

2) Let A be an abelian surface, ι the $-(1)$-involution and $S \longrightarrow A/\iota$ the associated projective Kummer K3 surface. On S, we have infinitely many (ample) rational curves by [CGL22]. These rational curves yield infinitely many (singular) hyperelliptic curves on the abelian surface A. By taking the fibers of the degree two map onto $\mathbb{P}^1$ to these infinitely many hyperelliptic curves, we associate infinitely many

[2]By the above cited theorem, the locus where a given extra class is algebraic is dense in $\mathcal{M}$, and the locus where this class E has a fixed intersection with $\gamma(D)$ is a proper Zariski closed subset of $\mathcal{M}$; therefore, the locus where the intersection is positive is non-empty.

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rational curves in $Kum_2(A)$. Notice that, by translation, deformations of each of these rational curves cover a uniruled divisor. By adding $(n-2)$ arbitrary points, we obtain the same conclusion on $Kum_n(A)$, for all $n \geq 2$. These divisors have positive square. Regenerating these infinitely many uniruled divisors, we obtain the conclusion on the very general deformation of $Kum_n(A)$. □

Remark 3.4. Actually, using [CGL22, Theorem A] and the Regeneration principle, a simpler version of the proof of Theorem 1.6 yields the existence of infinitely many uniruled divisors for the very general point of *any* family $\mathcal{X} \to B$ of projective IHS manifolds such that one of the fibres is the Hilbert scheme over a projective $K3$.

Proof of Theorem 1.7. To prove the theorem, we will show the existence of an ample uniruled divisor with infinite $\mathrm{Bir}(X)$-orbit. By [Ogu06, Theorem 1.1], as $\mathrm{Bir}(X)$ is infinite, there exists an element $g \in \mathrm{Bir}(X)$ of infinite order. Let D be an ample uniruled divisor, whose existence is granted by Theorem 1.5. We claim that the orbit of D via g is infinite, as otherwise, a multiple of g would give an isometry of the lattice $D^\perp \subset \mathrm{NS}(X)$. The latter is negative definite as D is ample and has therefore finite isometry group. Hence, g would act with finite order on both D and $D^\perp$, which is absurd, and the claim follows. □

We recall now the following well-known result for the reader's convenience. This tells us that Theorem 1.7 yields its conclusion only for a codimension at least one locus in the moduli space of projective IHS manifolds.

Lemma 3.5. *Let X be a projective IHS manifold with $\rho(X) = 1$. Then* $\mathrm{Aut}(X) = \mathrm{Bir}(X)$*, and it is a finite group.*

Proof. First of all, recall that a birational map between two IHS manifolds sending an ample class into an ample class can be extended to an isomorphism. As such, when $\rho(X) = 1$, we have $\mathrm{Aut}(X) = \mathrm{Bir}(X)$. By [Fuj78, Theorem 4.8] the group of automorphisms of a compact Kähler manifold that fix a Kähler class has only finitely many connected components. On the other hand the group of automorphisms of an IHS manifold X is discrete, since $h^0(X, T_X) = h^0(X, \Omega^1_X) = 0$. Hence $\mathrm{Aut}(X)$ must be finite. □

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Competing interest. The authors have no competing interest to declare.

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