Proceedings of the Edinburgh Mathematical Society (1988) 31, 1–23 ©

UNSTABLE INVARIANT DISTRIBUTIONS FOR A CLASS OF STOCHASTIC DELAY EQUATIONS

by S. E. A. MOHAMMED

(Received 14th March 1986)

1. Introduction

In our article **[8]** we examined asymptotic mean square stability for linear retarded f.d.e.'s which are perturbed by white noise. It is shown in **[8]** and **[10]** that if the deterministic linear retarded f.d.e. is asymptotically stable, then so is the perturbed stochastic f.d.e.

In this note we wish to examine the asymptotic behaviour of the trajectory $\{x_t: t \geqq 0\}$ of the stochastic f.d.e.

$$\left.\begin{aligned} dx(t) &= H(x_t)\,dt + G(x_t)\,dW(t), && t \geqq 0 \\ x_t(s) &= x(t+s), && -r \leqq s \leqq 0 \end{aligned}\right\} \tag{I}$$

where the deterministic linear f.d.e.

$$\left.\begin{aligned} dy(t) &= H(y_t)\,dt, && t \geqq 0 \\ y_t(s) &= y(t+s), && -r \leqq s \leqq 0 \end{aligned}\right\} \tag{II}$$

is assumed to be *hyperbolic* with at least one asymptotically unstable solution. More specifically, following the notation of Hale **[5]** and Mohammed **[9, 8]**, let $J = [-r, 0]$, $C = C(J, \mathbb{R}^n)$ be the Banach space of all continuous paths $\eta: J \to \mathbb{R}^n$ given the supremum norm

$$\|\eta\|_C = \sup\{|\eta(s)| : s \in J\}.$$

Denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$. In (I) and (II), the *drift* coefficient $H: C \to \mathbb{R}^n$ is continuous linear and the *diffusion* coefficient $G: C \to L(\mathbb{R}^m, \mathbb{R}^n)$ is a (globally) Lipschitz map into the space $L(\mathbb{R}^m, \mathbb{R}^n)$ of linear maps $B: \mathbb{R}^m \to \mathbb{R}^n$ with the operator norm

$$\|B\| = \sup\{|B(v)| : v \in \mathbb{R}^m, |v| = 1\}.$$

The stochastic f.d.e. (I) is driven by m-dimensional Brownian motion $W: \mathbb{R}^{\geqq 0} \times \Omega \to \mathbb{R}^m$ on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqq 0}, P)$. It will be assumed throughout that the filtration $(\mathscr{F}_t)_{t \geqq 0}$ is right-continuous.

The set of all initial distributions $\mathscr{L}^2(\Omega, C; \mathscr{F}_0)$ is the space of all Bochner square-integrable $\mathscr{F}_0$-measurable maps $\theta:\Omega\to C$ with the complete semi-norm

$$\|\theta\|_{\mathscr{L}^2}=[E\|\theta(\cdot)\|^2]^{1/2}.$$

If $\theta\in\mathscr{L}^2(\Omega, C; \mathscr{F}_0)$, it is known that the stochastic f.d.e. (I) admits a unique sample continuous solution ${}^\theta x:[-r,\infty)\times\Omega\to\mathbb{R}^n$ which is $(\mathscr{F}_t)_{t\geqq 0}$-adapted and satisfies $x_0=\theta$. The trajectory $\{{}^\eta x_t:t\geqq 0, \eta\in C\}$ describe a continuous C-valued Feller process (Mohammed [9, Chapter III]).

The dynamics of the deterministic f.d.e. (II) is well understood through the work of J. K. Hale [5, pp. 165–190]. Indeed, according to [5], the space C has a topological splitting

$$C=\mathscr{U}\oplus\mathscr{S} \tag{1}$$

which is invariant under the semi-flow $T_t:C\to C$, $t\geqq 0$, of (II):

$$T_t(\eta)={}^\eta y_t, \qquad t\geqq 0, \quad \eta\in C \tag{2}$$

where the unique solution of (II) starting off at η is denoted by ${}^\eta y:[-r,\infty)\to\mathbb{R}^n$. The unstable subspace $\mathscr{U}$ is finite-dimensional and one has the following exponential estimates on $\mathscr{U}$ and the stable subspace $\mathscr{S}$:

$$\|T_t(\xi)\|\geqq L\|\xi\|e^{\delta t}, \qquad t\geqq 0, \quad \xi\in\mathscr{U} \tag{3}$$

$$\|T_t(\xi)\|\leqq K\|\xi\|e^{-\alpha t}, \qquad t\geqq 0, \quad \xi\in\mathscr{S} \tag{4}$$

where L, K, δ and α are positive constants independent of ξ and t (Hale [5, p. 181]).

The Markov trajectories $\{{}^\eta x_t:t\geqq 0, \eta\in C\}$ induce a contraction semi-group $P_t:C_b\to C_b$, defined on the Banach space C_b of all bounded uniformly continuous functions $\phi:C\to\mathbb{R}$ (Mohammed [9, pp. 66–69]). The space C_b is furnished with the supremum norm

$$\|\phi\|_{C_b}=\sup\{|\phi(\eta)|:\eta\in C\}.$$

Let $\mathscr{M}(C)$ be the Banach space of all finite (regular) Borel measures μ on C given the total variation norm

$$\|\mu\|=v(\mu)(C)=\sup\left\{\sum_{i=1}^k|\mu(B_i)|:B_i\in\text{Borel } C, \quad i=1,\dots,k, \text{ disjoint Borel partition of } C\right\}$$

$$=\sup\left\{\left|\int_C\phi\,d\mu\right|:\phi\in C_b, \|\phi\|_{C_b}\leqq 1\right\},$$

where $v(\mu)$ is the total variation measure of μ. The natural continuous bilinear pairing $\langle\cdot,\cdot\rangle: C_b\times\mathscr{M}(C)\to\mathbb{R}$

$$\langle\phi,\mu\rangle=\int_{\eta\in C}\phi(\eta)\,d\mu(\eta), \qquad \phi\in C_b, \quad \mu\in\mathscr{M}(C) \tag{5}$$

https://doi.org/10.1017/S0013091500006532 Published online by Cambridge University Press

defines the adjoint semi-group $P_t^*:\mathscr{M}(C)\to\mathscr{M}(C)$ through the relation

$$\langle\phi, P_t^*\mu\rangle=\langle P_t(\phi),\mu\rangle,\qquad \phi\in C_b,\quad \mu\in\mathscr{M}(C).$$

If $\{p(\eta,t,\cdot)=P\circ({}^\eta x_t)^{-1}:t\geqq 0,\eta\in C\}$ are the transition probabilities of the Markov process $\{{}^\eta x_t:t\geqq 0,\eta\in C\}$, then

$$(P_t^*\mu)(B)=\int_{\eta\in C} p(\eta,t,B)\,d\mu(\eta) \tag{6}$$

for every Borel set B in C and each $\mu\in\mathscr{M}(C)$. (Mohammed [9, pp. 68–69]).

Since the splitting (1) is completely determined by the drift H only, the unstable subspace $\mathscr{U}$ will *never* be invariant under the sample "stochastic flow" $\{{}^\eta x_t:t\geqq 0,\eta\in C\}$ unless G is identically zero. Even if the segment x_t momentarily visits $\mathscr{U}$, the Brownian forcing term will clearly "diffuse" it away from $\mathscr{U}$ during subsequent times. Moreover, when G is continuous linear and has a *positive memory*, the sample flow $\{{}^\eta x_t:t\geqq 0,\ \eta\in C\}$ has measurable versions $\Omega\times\mathbb{R}^+\times C\to C$ which are a.s. *non-linear* on C! (Mohammed [9, pp. 186–190]). Such flows will a.s. never respect the topological splitting in (1). It therefore seems reasonable that one should look for the "stable" and "unstable manifolds" of the stochastic f.d.e. (I) within the space of square integrable *distributions* $\mathscr{L}^2(\Omega,C)$ or $\mathscr{M}(C)$ rather than the underlying state space C, cf. deterministic results of Hale [5], Hale and Perello [6]. As a first step in this direction, we shall isolate a subset of distributions in $\mathscr{L}^2(\Omega,C)$ (or $\mathscr{M}(C)$) which is invariant under the stochastic flow $(\{P_t^*\}_{t\geqq 0})$ and on which trajectories of (I) diverge to infinity exponentially in the mean (square sense).

2. Semi-groups and stopping times

Let $\mathscr{T}$ be the family of all P-essentially bounded stopping times (or Markov times) $\tau:\Omega\to\mathbb{R}^{\geqq 0}$ with respect to the filtration $(\mathscr{F}_t)_{t\geqq 0}$ For each $\tau\in\mathscr{T}$ define $\mathscr{F}_\tau$ to be the σ-algebra generated by all sets $A\in\mathscr{F}$ such that $A\cap(\tau<t)\in\mathscr{F}_t$ for every $t\geqq 0$. If $x:[-r,\infty)\times\Omega\to\mathbb{R}^n$ is the solution of (I), define the $\mathscr{F}_\tau$-measurable map $x_\tau:\Omega\to C$ by

$$x_\tau(\omega)(s)=x(\tau(\omega)+s,\omega),\qquad \omega\in\Omega,\quad s\in J.$$

Define also the stopped families $\{P_\tau:\tau\in\mathscr{T}\}$ and $\{P_\tau^*:\tau\in\mathscr{T}\}$ by setting

$$P_\tau(\phi)(\eta)=E\phi({}^\eta x_\tau),\qquad \phi\in C_b,\quad \eta\in C \tag{7}$$

and

$$\langle\phi, P_\tau^*\mu\rangle=\langle P_\tau(\phi),\mu\rangle,\qquad \phi\in C_b,\quad \mu\in\mathscr{M}(C).$$

To see that each P_τ maps C_b continuously into itself, we need the following two lemmas:

Lemma 1 (Doob's Inequality). *Let $X:\mathbb{R}^{\geq 0}\times\Omega\to\mathbb{R}^n$ be a sample continuous $\mathscr{L}^p$-integrable martingale on $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},P)$, where $p>1$. Let $\tau\in\mathscr{T}$. Then the functions*

$$\Omega\to\mathbb{R}^{\geq 0},\qquad\qquad \Omega\to\mathbb{R}^{\geq 0}$$

$$\omega\mapsto\sup_{0\leq u\leq\tau(\omega)}|X(u,\omega)|^p,\qquad\qquad \omega\mapsto|X(\tau(\omega),\omega)|^p$$

belong to $\mathscr{L}^1(\Omega,\mathbb{R};\mathscr{F}_\tau)$ and

$$E\sup_{0\leq u\leq\tau}|X(u)|^p\leq\left(\frac{p}{p-1}\right)^p E|X(\tau)|^p. \tag{8}$$

Proof. Suppose $\|\tau\|_\infty=\operatorname*{essup}_{\omega\in\Omega}|\tau(\omega)|<\infty$. For each $t\geq 0$, let $t\wedge\tau$ be the stopping time $\min(t,\tau)\in\mathscr{T}$. Define the stopped process $\hat{X}:\mathbb{R}^{\geq 0}\times\Omega\to\mathbb{R}^n$ by

$$\hat{X}(t,\cdot)=X(t\wedge\tau,\cdot),\qquad t\geq 0.$$

Since X is a sample continuous martingale, then so is $\hat{X}$ (Doob [3, p. 373], Lipster and Shiryayev [7, p. 69]).

Fix any $\omega\in\Omega$, then

$$\sup_{0\leq u\leq\|\tau\|_\infty}|\hat{X}(u,\omega)|^p=\max\left[\sup_{0\leq u\leq\tau(\omega)}|\hat{X}(u,\omega)|^p,|X(\tau(\omega),\omega)|^p\right]$$

$$=\sup_{0\leq u\leq\tau(\omega)}|X(u,\omega)|^p.$$

Since the function $|X(\|\tau\|_\infty,\cdot)|^p$ is integrable, then Doob's inequality for X tells us that

$$\omega\mapsto\sup_{0\leq u\leq\|\tau\|_\infty}|X(u,\omega)|^p$$

is also integrable. Now for each $\omega\in\Omega$,

$$|X(\tau(\omega),\omega)|^p\leq\sup_{0\leq u\leq\tau(\omega)}|X(u,\omega)|^p\leq\sup_{0\leq u\leq\|\tau\|_\infty}|X(u,\omega)|^p.$$

Thus $|X(\tau)|^p$ is integrable. We must prove that the function

$$\Omega\to\mathbb{R}$$

$$\omega\mapsto\sup_{0\leq u\leq\tau(\omega)}|X(u,\omega)|^p$$

is integrable. It suffices to indicate its $\mathscr{F}_\tau$-measurability. To see this define the process $Y:\mathbb{R}^{\geq 0}\times\Omega\to\mathbb{R}$ by

$$Y(t,\omega)=\sup_{0\leq u\leq t}|X(u,\omega)|^p,\qquad t\geq 0,\quad\omega\in\Omega.$$

https://doi.org/10.1017/S0013091500006532 Published online by Cambridge University Press

Since X is sample continuous and $(\mathscr{F}_t)_{t\geq 0}$-adapted, then so is Y. In particular, Y is progressively measurable (Stroock and Varadhan [**11**, p. 20]). Thus $Y(\tau)=\sup_{0\leq u\leq\tau}|X(u)|^p$ is $\mathscr{F}_\tau$-measurable and in fact belongs to $\mathscr{L}^1(\Omega,\mathbb{R};\mathscr{F}_\tau)$.

Finally, apply Doob's inequality to $\hat{X}$ so as to get

$$E\sup_{0\leq u\leq\tau}|X(u)|^p=E\sup_{0\leq u\leq\|\tau\|_\infty}|\hat{X}(u)|^p\leq\left(\frac{p}{p-1}\right)^p E|\hat{X}(\|\tau\|_\infty)|^p$$

$$=\left(\frac{p}{p-1}\right)^p E|X(\tau)|^p.$$

This completes the proof of the lemma. □

The next lemma describes the dependence of the "stopped" solution segment ${}^\theta x_\tau$ on the initial process θ.

Lemma 2. *For each $\theta\in\mathscr{L}^2(\Omega,C;\mathscr{F}_0)$, let ${}^\theta x:[-r,\infty)\times\Omega\to\mathbb{R}^n$ be the unique solution of the stochastic f.d.e.*

$$\left.\begin{aligned} dx(t)&=\hat{H}(x_t)\,dt+\hat{G}(x_t)\,dW(t), \qquad t>0\\ x_0&=\theta\end{aligned}\right\}\tag{III}$$

where $\hat{H}:C\to\mathbb{R}^n$, $\hat{G}:C\to L(\mathbb{R}^m,\mathbb{R}^n)$ are globally Lipschitz with a common Lipschitz constant $L>0$. Then for every $\tau\in\mathscr{T}$ and $\theta_1,\theta_2\in\mathscr{L}^2(\Omega,C;\mathscr{F}_0)$ we have

$$E\|{}^{\theta_1}x_\tau-{}^{\theta_2}x_\tau\|_C^2\leq 3E\|\theta_1-\theta_2\|_C^2\,e^{3(\|\tau\|_\infty+4K)L^2\|\tau\|_\infty}\tag{9}$$

where $K=4m^2$.

Proof. Let $\theta_1,\theta_2\in\mathscr{L}^2(\Omega,C;\mathscr{F}_0)$ and $\tau\in\mathscr{T}$. Fix $0\leq t\leq\|\tau\|_\infty$. Define the process $\Psi:[0,\|\tau\|_\infty]\times\Omega\to\mathbb{R}$ by

$$\Psi(u,\omega)=\begin{cases}1, & u<t\wedge\tau(\omega)\\ 0, & u>t\wedge\tau(\omega)\end{cases}$$

for $u\in[0,\|\tau\|_\infty]$ and $\omega\in\Omega$. Clearly Ψ is $(\mathscr{F}_t)_{t\geq 0}$-adapted.

Using the simple facts

$$\Psi(u,\cdot)\hat{H}({}^\theta x_u)=\Psi(u,\cdot)\hat{H}({}^\theta x_{u\wedge\tau})$$

$$\Psi(u,\cdot)\hat{G}({}^\theta x_u)=\Psi(u,\cdot)\hat{G}({}^\theta x_{u\wedge\tau})$$

and the previous lemma, one easily sees that

$$E\|{}^{\theta_1}x_{t\wedge\tau}-{}^{\theta_2}x_{t\wedge\tau}\|_C^2=E\sup_{s\in J}|{}^{\theta_1}x(t\wedge\tau+s)-{}^{\theta_2}x(t\wedge\tau+s)|^2$$

$$\leqq 3E\|\theta_1(\cdot)-\theta_2(\cdot)\|_C^2+3E\sup_{\substack{s\in J\\ t\wedge\tau+s\geqq 0}}\left|\int_0^{t\wedge\tau+s}\{\hat{H}({}^{\theta_1}x_u)-\hat{H}({}^{\theta_2}x_u)\}\,du\right|^2$$

$$+3E\sup_{\substack{s\in J\\ t\wedge\tau+s\geqq 0}}\left|\int_0^{t\wedge\tau+s}\{\hat{G}({}^{\theta_1}x_u)-\hat{G}({}^{\theta_2}x_u)\}\,dW(u)\right|^2$$

$$\leqq 3E\|\theta_1-\theta_2\|^2+3\left(E\int_0^{t\wedge\tau}|\hat{H}({}^{\theta_1}x_u)-\hat{H}({}^{\theta_2}x_u)|^2\,du\right)\|\tau\|_\infty$$

$$+3\cdot 4\cdot E\left|\int_0^{t\wedge\tau}\{\hat{G}({}^{\theta_1}x_u)-\hat{G}({}^{\theta_2}x_u)\}\,dW(u)\right|^2$$

$$=3E\|\theta_1-\theta_2\|^2+3\|\tau\|_\infty E\int_0^t\Psi(u,\cdot)|\hat{H}({}^{\theta_1}x_{u\wedge\tau})-\hat{H}({}^{\theta_2}x_{u\wedge\tau})|^2\,du$$

$$+12E\left|\int_0^t\Psi(u,\cdot)\{\hat{G}({}^{\theta_1}x_{u\wedge\tau})-\hat{G}({}^{\theta_2}x_{u\wedge\tau})\}\,dW(u)\right|^2$$

$$\leqq 3E\|\theta_1-\theta_2\|^2+3\|\tau\|_\infty L^2\int_0^t E\|{}^{\theta_1}x_{u\wedge\tau}-{}^{\theta_2}x_{u\wedge\tau}\|^2\,du$$

$$+12K\int_0^t E|\Psi(u,\cdot)\{\hat{G}({}^{\theta_1}x_{u\wedge\tau})-\hat{G}({}^{\theta_2}x_{u\wedge\tau})\}|^2\,du$$

$$\leqq 3E\|\theta_1-\theta_2\|^2+(3\|\tau\|_\infty L^2+12KL^2)\int_0^t E\|{}^{\theta_1}x_{u\wedge\tau}-{}^{\theta_2}x_{u\wedge\tau}\|^2\,du.$$

Therefore Gronwall's lemma implies that

$$E\|{}^{\theta_1}x_{t\wedge\tau}-{}^{\theta_2}x_{t\wedge\tau}\|^2\leqq 3E\|\theta_1-\theta_2\|^2\,e^{3(\|\tau\|_\infty+4K)L^2t}$$

for all $0\leqq t\leqq\|\tau\|_\infty$. Putting $t=\|\tau\|_\infty$ in the last inequality yields the required estimate (9). □

Remark. Estimates similar to (9) also hold if the coefficients $\hat{H}$ and $\hat{G}$ in the stochastic f.d.e. (III) are allowed to depend *explicitly* on $t\geqq 0$, $\omega\in\Omega$. We will not need these results here.

We can now prove

Theorem 1. *For each bounded stopping time $\tau \in \mathscr{T}$, $P_\tau: C_b \to C_b$ is a continuous linear map of C_b into itself. Similarly, $P_\tau^*: \mathscr{M}(C) \to \mathscr{M}(C)$ is continuous linear with respect to the total variation norm (or the vague topology). Furthermore, if $\tau \in \mathscr{T}$ and $t \geqq 0$, then*

$$P_\tau \circ P_t = P_{\tau+t} \tag{10}$$

and

$$P_t^* \circ P_\tau^* = P_{t+\tau}^*. \tag{11}$$

Proof. Let $\phi \in C_b$, we must show that $P_\tau(\phi): C \to \mathbb{R}$ is bounded and uniformly continuous on C. It is clear that the function

$$C \to \mathbb{R}$$

$$\eta \mapsto P_\tau(\phi)(\eta) = E\phi[{}^\eta x_\tau)$$

is bounded by $\|\phi\|_{C_b}$. To prove it is uniformly continuous, let $\varepsilon > 0$. By uniform continuity of ϕ, there is a $\delta' > 0$ such that $|\phi(\xi_1) - \phi(\xi_2)| < \varepsilon/2$ whenever $\xi_1, \xi_2 \in C$, $\|\xi_1 - \xi_2\| < \delta'$. Take $\delta > 0$ such that

$$\delta^2 < \frac{\varepsilon \delta'^2}{12\|\phi\|_{C_b}} e^{-3(\|\tau\|_\infty + 4K)L^2 \|\tau\|_\infty}.$$

Suppose $\eta_1, \eta_2 \in C$ are such that $\|\eta_1 - \eta_2\|_C < \delta$. Then

$$\begin{aligned}
&|P_\tau(\phi)(\eta_1) - P_\tau(\phi)(\eta_2)| \\
&\leqq \int_{\|{}^{\eta_2}x_\tau - {}^{\eta_2}x_\tau\|_C < \delta'} |\phi({}^{\eta_1}x_\tau) - \phi({}^{\eta_2}x_\tau)|\, dP + \int_{\|{}^{\eta_1}x_\tau - {}^{\eta_2}x_\tau\|_C \geqq \delta'} |\phi({}^{\eta_1}x_\tau) - \phi({}^{\eta_2}x_\tau)|\, dP \\
&< \frac{\varepsilon}{2} P(\|{}^{\eta_1}x_\tau - {}^{\eta_2}x_\tau\| < \delta') + 2\|\phi\|_{C_b} P(\|{}^{\eta_1}x_\tau - {}^{\eta_2}x_\tau\| \geqq \delta') \\
&< \frac{\varepsilon}{2} + \frac{2}{\delta'^2} \|\phi\|_{C_b} E\|{}^{\eta_1}x_\tau - {}^{\eta_2}x_\tau\|^2 \\
&< \frac{\varepsilon}{2} + \frac{2}{\delta'^2} \|\phi\|_{C_b} \cdot 3\|\eta_1 - \eta_2\|^2 e^{3(\|\tau\|_\infty + 4K)L^2 \|\tau\|_\infty} \\
&< \frac{\varepsilon}{2} + 6\|\phi\|_{C_b} \cdot \frac{\delta^2}{\delta'^2} e^{3(\|\tau\|_\infty + 4K)L^2 \|\tau\|_\infty} \\
&< \varepsilon.
\end{aligned}$$

Thus $P_\tau(\phi)$ is uniformly continuous and so belongs to C_b.

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Since $\|P_\tau(\phi)\|_{C_b} \leqq \|\phi\|_{C_b}$ for all $\phi \in C_b$, then P_τ is continuous linear and $\|P_\tau\| \leqq 1$. Because of the duality

$$\langle \cdot, \cdot \rangle : C_b \times \mathscr{M}(C) \to \mathbb{R}$$

in (5), the linear map $P_\tau^* : \mathscr{M}(C) \to \mathscr{M}(C)$ is continuous in the variation norm if and only if it is vaguely continuous. Also $\|P_\tau^*\| \leqq 1$, since $\|P_\tau\| \leqq 1$.

By duality the "semi-group" properties (10) and (11) are easily seen to be equivalent. But property (10) is a direct consequence of the time-homogeneity and the Feller property of the trajectory field $\{{}^\eta x_t : t \geqq 0, \eta \in C\}$. Indeed if $\phi \in C_b$, and $\eta \in C$, then

$$\begin{aligned} P_\tau\{P_t(\phi)\}(\eta) &= \int_{\omega \in \Omega} \int_{\omega' \in \Omega} \phi[{}^{{}^\eta x_{\tau(\omega)}(\omega)} x_t(\omega')] \, dP(\omega') \, dP(\omega) \\ &= \int_{\omega \in \Omega} \int_{\xi \in C} \phi(\xi) p[0, {}^\eta x_{\tau(\omega)}(\omega), t, d\xi] \, dP(\omega) \\ &= \int_{\omega \in \Omega} \int_{\xi \in C} \phi(\xi) p[\tau(\omega), {}^\eta x_{\tau(\omega)}(\omega), t + \tau(\omega), d\xi] \, dP(\omega) \\ &= \int_{\omega \in \Omega} \phi[{}^\eta x_{t + \tau(\omega)}(\omega)] \, dP(\omega) \\ &= P_{t+\tau}(\phi)(\eta). \end{aligned}$$

Hence the proof is complete. □

Remarks. (i) Define the family of Borel measures $\{p({}^\eta, \tau, \cdot) : \eta \in C, \tau \in \mathscr{T}\}$ by setting

$$p(\eta, \tau, B) = P\{({}^\eta x_\tau)^{-1}(B)\}, \qquad B \in \text{Borel}\, C.$$

Then Lemma 2 implies that for each $\tau \in \mathscr{T}$ the map

$$C \to \mathscr{M}(C)$$

$$\eta \mapsto p(\eta, \tau, \cdot)$$

is bounded and uniformly continuous (Elworthy [**4**, pp. 300–301]). Furthermore we may write

$$P_\tau(\phi)(\eta) = \int_{\xi \in C} \phi(\xi) p(\eta, \tau, d\xi), \qquad \phi \in C_b, \quad \eta \in C.$$

(ii) The last theorem is well known for Feller processes with values in a *locally compact* metric space (Chung [**2**, pp. 48–59]). In our infinite-dimensional case, the theorem yields the extra regularity property that the operator P_τ *leaves the set* C_b *of all bounded uniformly continuous functions invariant*. In fact, if the coefficients $\hat{H}$ and $\hat{G}$ in (III) are C^1 (or C^∞), the proof of Lemma 2 may be modified to show that the map

$$\mathscr{L}^2(\Omega, C; \mathscr{F}_0) \to \mathscr{L}^2(\Omega, C; \mathscr{F}_\tau)$$

$$\theta \mapsto {}^\theta x_\tau$$

is also C^1 (or C^∞) (cf. Mohammed [9, Theorem II3.2, pp. 41–45]). Hence P_τ leaves invariant the smooth functions in C_b whenever the coefficients of the stochastic f.d.e.'s (I) or (III) are smooth.

3. Unstable distributions

Let the coefficients H and G in the stochastic f.d.e. (I) be as in Section 1.

If $t_1 \geqq 0$, consider the set $\mathscr{U}_\omega^{t_1} \subset \mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1})$ of all initial distributions θ such that the trajectory $\{{}^\theta x_t^{t_1} : t \geqq t_1\}$ of the stochastic f.d.e.

$$\left.\begin{aligned} dx(t) &= H(x_t)\,dt + G(x_t)\,dW(t), \qquad t > t_1 \\ x_{t_1} &= \theta \end{aligned}\right\} \tag{I}$$

leaves the stable subspace $\mathscr{S}$ at some bounded random time after t_1 with positive probability. We shall prove that each trajectory of (I) starting off on $\mathscr{U}_\omega^{t_1}$ will diverge to infinity asymptotically in the mean (-square) with exponential speed.

Theorem 3.1. *Let* $\theta \in \mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1})$, $t_1 \geqq 0$. *Define* $\tau^\theta : \Omega \to \mathbb{R}^{\geqq 0}$ *by*

$$\tau^\theta = \inf\{t : t \geqq t_1, {}^\theta x_t^{t_1} \notin \mathscr{S}\}.$$

Suppose $L, \delta > 0$ *are as in* (3) *and let* $P_{\mathscr{U}} : C \to \mathscr{U}$ *denote the projection onto* $\mathscr{U}$ *given by the splitting* (1). *Then* τ^θ *is a stopping time. If, further* $\|\tau^\theta\|_\infty < \infty$ *then*

$$E[\|{}^\theta x_t^{t_1}\| \,|\, \mathscr{F}_{\tau^\theta}] \geqq L E^{\delta(t-\tau^\theta)} \|P_{\mathscr{U}}[{}^\theta x_{\tau^\theta}]\| \tag{12}$$

for all $t \geqq \|\tau^\theta\|_\infty$, $\mathscr{F}_{\tau^\theta}$*—a.s.*

Proof. Recall that the filtration $(\mathscr{F}_t)_{t \geqq 0}$ is right-continuous. Since $\{{}^\theta x_t^{t_1} : t \geqq t_1\}$ is sample continuous, then τ^θ is clearly a stopping time (Stroock and Varadhan [11, p. 22]).

Without loss of generality, suppose that $t_1 = 0$.

We first prove the following version of the stochastic variation of parameters formula (Mohammed [8], Mohammed, Scheutzow and Weizsäcker [10]): *Let* $H : C \to \mathbb{R}^n$ *be continuous linear and* $g : \mathbb{R}^{\geqq 0} \times \Omega \to L(\mathbb{R}^m, \mathbb{R}^n)$ *a measurable* $(\mathscr{F}_t)_{t \geqq 0}$*-adapted process such that* $\int_0^a E\|g(t, \cdot)\|^2\,dt < \infty$ for each $a > 0$. *Suppose* $z : [-r, \infty) \times \Omega \to \mathbb{R}^n$ *is a solution of the stochastic f.d.e.*

$$dz(t) = H(z_t)\,dt + g(t)\,dW(t), \qquad t > 0. \tag{IV}$$

Let $\tau_0 \in \mathcal{T}$ be any essentially bounded stopping time. Then

$$z_t = T_{t-\tau_0}(z_{\tau_0}) + \int_{\tau_0}^{t} \hat{T}_{t-u}\Delta g(u)\, dW(u) \tag{13}$$

a.s. on $(t \geqq \tau_0)$.

In (13), $\Delta = \chi_{\{0\}} \mathrm{id}_{\mathbb{R}^n}$, $\chi_{\{0\}}: J \to \mathbb{R}$ is the indicator function of $\{0\}$, $\Delta g(u): J \to L(\mathbb{R}^m, \mathbb{R}^n)$ denotes the map

$$\Delta g(u)(s) = \begin{cases} g(u) & s = 0, \quad u \geqq 0 \\ 0 & -r \leqq s < 0 \end{cases}$$

and $(\hat{T}_t)_{t \geqq 0}$ is the natural semi-flow induced by $(T_t)_{t \geqq 0}$ on the Banach space $B(J, L(\mathbb{R}^m, \mathbb{R}^n))$ of all bounded Borel-measurable maps $J \to L(\mathbb{R}^m, \mathbb{R}^n)$ with supremum norm. The stochastic integral in (13) is a "stopping-time-version" of the Pettis–Itô integral introduced in [**8**]. In fact, when $\tau_0 \equiv t_0 \in \mathbb{R}^{\geqq 0}$ is deterministic, it is proved in [**8**, Theorem 3.1] that a.s.

$$z_t = T_{t-t_0}(z_{t_0}) + \int_{t_0}^{t} \hat{T}_{t-u}\Delta g(u)\, dW(u), \qquad t \geqq t_0. \tag{13$'$}$$

Since (IV) admits sample continuous trajectories $\{z_t : t \geqq 0\}$ on C, it is evident from the equality in (13)$'$ that the Pettis–Itô integral on the righthand side has the C-valued version $z_t - T_{t-t_0}(z_{t_0})$ which is jointly sample continuous in $t_0, t, t_0 \leqq t$. Therefore one can substitute a simple stopping time $\hat{\tau}_0$ for t_0 in (13)$'$. Hence by approximation with simple stopping times it follows that (13) holds a.s. on $(t \geqq \tau_0)$ for any (bounded) stopping time τ_0 (cf. Elworthy [**4**, Corollary 7A, pp. 44–48]). If $\{x_t : t \geqq 0\}$ is a trajectory of the stochastic f.d.e.

$$dx(t) = H(x_t)\, dt + G(x_t)\, dW(t), \qquad t > 0 \tag{I}$$

then a.s. on $(t \geqq \tau_0)$

$$x_t = T_{t-\tau_0}(x_{\tau_0}) + \int_{\tau_0}^{t} \hat{T}_{t-u}\Delta g(x_u)\, dW(u), \tag{14}$$

Apply the continuous linear projection $P_{\mathcal{U}}$ to both sides of (14) and use the properties of the Pettis–Itô integral to get

$$P_{\mathcal{U}}(x_t) = T_{t-\tau_0}[P_{\mathcal{U}}(x_{\tau_0})] + \int_{\tau_0}^{t} \hat{T}_{t-u}\{P_{\mathcal{U}}(\Delta)\}G(x_u)\, dW(u) \tag{15}$$

a.s. on $(t>\tau_0)$ (Mohammed [8]). Now in (14) and (15) take C-valued *Bochner* conditional expectations with respect to $\mathscr{F}_{\tau_0}$. Thus

and
$$\left.\begin{aligned} E((x_t)|\mathscr{F}_{\tau_0}) &= T_{t-\tau_0}(x_{\tau_0}) \\ E(P_{\mathscr{U}}(x_t)|\mathscr{F}_{\tau_0}) &= T_{t-\tau_0}[P_{\mathscr{U}}(x_{\tau_0})] \end{aligned}\right\} \tag{16}$$

a.s. for all $t \geqq \|\tau_0\|_\infty$.

Next we proceed to prove the estimate (12). So let $\tau^\theta \equiv \inf\{t: t \geqq 0,\ {}^\theta x_t \notin \mathscr{S}\}$ be essentially bounded. In (16) put $\tau_0 = \tau^\theta$ and estimate the (Bochner) conditional expectations to obtain a.s.

$$E(\|{}^\theta x_t\| | \mathscr{F}_{\tau^\theta}) \geqq E(\|P_{\mathscr{U}}[{}^\theta x_t]\| | \mathscr{F}_{\tau^\theta}) \tag{17}$$

$$\geqq \|E(P_{\mathscr{U}}[{}^\theta x_{\tau^\theta}] | \mathscr{F}_{\tau^\theta})\| \quad \text{(Elworthy [4, p. 5])}$$

$$\geqq \|T_{t-\tau^\theta}[P_{\mathscr{U}}({}^\theta x_{\tau^\theta})]\|$$

$$= L e^{\delta(t-\tau^\theta)} \|P_{\mathscr{U}}[{}^\theta x_{\tau^\theta}]\| \tag{18}$$

for all $t \geqq \|\tau^\theta\|_\infty$. Taking expectations in the above inequality yields the conclusions of the theorem. □

Corollary 1. *If* $\theta \in \mathscr{L}^2(\Omega, \mathscr{U}; \mathscr{F}_0)$, *then*

$$E\|{}^\theta x_t\| \geqq L e^{\delta t} E\|\theta\| \tag{19}$$

for all $t \geqq 0$.

Proof. In the theorem, take $\tau^\theta \equiv 0$. □

For each $t \geqq 0$, define the subsets $\mathscr{U}^t_\omega$ and $\mathscr{S}^t_\omega$ of $\mathscr{L}^2(\Omega, C; \mathscr{F}_t)$ by

$$\mathscr{U}^t_\omega = \{\theta: \theta \in \mathscr{L}^2(\Omega, C; \mathscr{F}_t), \quad \exists\, \tau \in \mathscr{T} \ni P(P_{\mathscr{U}} \circ {}^\theta x^t_{t+\tau} \neq 0) > 0\}$$

and

$$\mathscr{S}^t_\omega = \mathscr{L}^2(\Omega, C; \mathscr{F}_t) \backslash \mathscr{U}^t_\omega.$$

Note that ${}^\theta x^t$ is the solution of the stochastic f.d.e. (I) satisfying ${}^\theta x^t_t = \theta$. Denote by $\Phi^{t_1}_{t_2}: \mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1}) \to \mathscr{L}^2(\Omega, C; \mathscr{F}_{t_2})$, $t_1 \leqq t_2$, the "stochastic flow":

$$\Phi^{t_1}_{t_2}(\theta) = {}^\theta x^{t_1}_{t_2}, \quad \theta \in \mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1}).$$

https://doi.org/10.1017/S0013091500006532 Published online by Cambridge University Press

Then one knows that

$$\Phi_{t_3}^{t_2} \circ \Phi_{t_2}^{t_1} = \Phi_{t_3}^{t_1}, \qquad t_1 \leqq t_2 \leqq t_3$$

(Mohammed [9, Theorem II.2.2, pp. 39–40]).

Our next corollary shows that the family of distributions $\{\mathcal{U}_\omega^t : t \geqq 0\}$ plays the role of a *stochastic unstable manifold* for the stochastic f.d.e. (I) viz. it is invariant under the flow $\Phi_{t_2}^{t_1}, t_1 \leqq t_2$, and for every $\theta \in \mathcal{U}_\omega^{t_1}$ the trajectory $\Phi_t^{t_1}(\theta)$, $t \geqq t_1$, diverges to infinity asymptotically in the mean square. Within the space of measures $\mathcal{M}(C)$ the family $\{\mathcal{U}_\omega^t : t \geqq 0\}$ corresponds to a set of probability measures

$$\mathcal{M}_{\mathcal{U}} = \{\mu : \mu \in m(C), \text{ there exist } t_1 \geqq 0 \text{ and } \theta \in \mathcal{U}_\omega^{t_1} \text{ such that } \mu = P \circ \theta^{-1}\}$$

which is invariant under the semi-group $\{P_t^*\}_{t \geqq 0}$.

Corollary 2. *Let $t_1 \geqq 0$. Then*

(i) *$\mathcal{U}_\omega^{t_1}$ is open in $\mathcal{L}^2(\Omega, C; \mathcal{F}_{t_1})$;*

(ii) *$\Phi_t^{t_1}\{\mathcal{U}_\omega^{t_1}\} \subseteq \mathcal{U}_\omega^t$ for all $t \geqq t_1$;*

(iii) *$P_t^*\{\mathcal{M}_{\mathcal{U}}\} \subseteq \mathcal{M}_{\mathcal{U}}$ for all $t \geqq 0$;*

(iv) *if $\theta = \mathcal{U}_\omega^{t_1}$, there exist $L_\theta > 0$, $t_2 \geqq t_1$ such that*

$$E\|\Phi_t^{t_1}(\theta)\| \geqq L_\theta e^{\delta(t - t_1)} \text{ for all } t \geqq t_2. \tag{20}$$

Proof. We first prove (ii).
Fix $t_1 \geqq 0$ and take $\theta \in \mathcal{U}_\omega^{t_1}$. Pick $\tau_1 \in \mathcal{T}$ such that

$$P(P_{\mathcal{U}} \circ {}^\theta x_{t_1 + \tau_1}^{t_1} \neq 0) > 0.$$

We contend that if $\tau \in \mathcal{T}$ is such that $\tau \geqq t_1 + \tau_1$ a.s. then $P(P_{\mathcal{U}} \circ {}^\theta x_\tau^{t_1} \neq 0) > 0$. Suppose, if possible, that there exist $\tau \in \mathcal{T}$ so that $\tau \geqq t_1 + \tau_1$ a.s. and $P_{\mathcal{U}} \circ {}^\theta x_\tau^{t_1} = 0$ a.s.

Recall that, on $\mathcal{U}$, each $\hat{T}_t | \mathcal{U}$ is a linear homeomorphism. For simplicity let $\hat{T}_{-t}$ denote $(\hat{T}_t | \mathcal{U})^{-1}$. Then putting $\tau_0 = t_1 + \tau_1$ in (15) gives

$$P_{\mathcal{U}} \circ {}^\theta x_t^{t_1} = T_{t - t_1 - \tau_1}(P_{\mathcal{U}} \circ {}^\theta x_{t_1 + \tau_1}^{t_1}) + T_t \int_{t_1 + \tau_1}^{t} \hat{T}_{-u}(P_{\mathcal{U}} \circ \Delta) G({}^\theta x_u^{t_1})\, dW(u)$$

a.s. on $(t \geqq t_1 + \tau_1)$. We shall replace t in the above equation by the Markov time τ. To do so suppose $\tau' = \sum_i t^i \chi_{\Omega_i}$ is a simple Markov time such that $\tau' \geqq t_1 + \tau_1$ a.s. and each $\Omega_i \in \mathcal{F}_{t_1 + \tau_1}$. Therefore

$$\begin{aligned} P_{\mathcal{U}} \circ {}^\theta x_{\tau'}^{t_1} &= \sum_i P_{\mathcal{U}} \circ {}^\theta x_{t^i}^{t_1} \chi_{\Omega_i} \\ &= \sum_i T_{t^i - t_1 - \tau_1}(P_{\mathcal{U}} \circ {}^\theta x_{t_1 + \tau_1}^{t_1}) \chi_{\Omega_i} + \sum_i \chi_{\Omega_i} T_{t^i} \int_{t_1 + \tau_1}^{t^i} \hat{T}_{-u}(P_{\mathcal{U}} \circ \Delta) G({}^\theta x_u^{t_1})\, dW(u) \\ &= T_{\tau' - t_1 - \tau_1}(P_{\mathcal{U}} \circ {}^\theta x_{t_1 + \tau_1}^{t_1}) + T_{\tau'} \int_{t_1 + \tau_1}^{\tau'} \hat{T}_{-u}(P_{\mathcal{U}} \circ \Delta) G({}^\theta x_u^{t_1})\, dW(u) \end{aligned} \tag{21}$$

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a.s. on $(\tau' \geqq t_1 + \tau_1)$. Approximate $\tau \in \mathscr{T}$ a.s. by a decreasing sequence $\{\tau_k\}_{k=1}^{\infty}$ of simple Markov times viz. $\tau_k \geqq \tau$ a.s. for all $k \geqq 1$ and $\tau_k(\omega) \downarrow \tau(\omega)$ as $k \to \infty$ for a.a. $\omega \in \Omega$ (Lipster and Shiryayev [7, p. 60]). Hence we may replace τ' in (21) by τ_k and then pass to the limit (in probability) as $k \to \infty$ so as to get

$$P_{\mathscr{U}} \circ {}^{\theta}x_{\tau}^{t_1} = T_{\tau - t_1 - \tau_1}(P_{\mathscr{U}} \circ {}^{\theta}x_{t_1+\tau_1}^{t_1}) + T_{\tau} \int_{t_1+\tau_1}^{\tau} \hat{T}_{-u}(P_{\mathscr{U}} \circ \Delta) G({}^{\theta}x_u^{t_1})\, dW(u) \tag{22}$$

a.s. on $(\tau \geqq t_1 + \tau_1)$ (Elworthy [4, pp. 45–48]).

Now $P_{\mathscr{U}} \circ {}^{\theta}x_{\tau}^{t_1} = 0$ a.s. and $T_{\tau(\omega)}$ is a linear (bijection) for a.a. ω in $(\tau \geqq t_1 + \tau_1)$; thus (22) gives

$$T_{-t_1-\tau_1}(P_{\mathscr{U}} \circ {}^{\theta}x_{t_1+\tau_1}^{t_1}) + \int_{t_1+\tau_1}^{\tau} \hat{T}_{-u}(P_{\mathscr{U}} \circ \Delta) G({}^{\theta}x_u^{t_1})\, dW(u) = 0 \tag{23}$$

a.s. on $(\tau \geqq t_1 + \tau_1)$. Taking $\mathscr{U}$-valued conditional expectations with respect to $\mathscr{F}_{t_1+\tau_1}$ then yields

$$T_{-t_1-\tau_1}(P_{\mathscr{U}} \circ {}^{\theta}x_{t_1+\tau_1}^{t_1}) = E[T_{-t_1-\tau_1}(P_{\mathscr{U}} \circ {}^{\theta}x_{t_1+\tau_1}^{t_1}) \mid \mathscr{F}_{t_1+\tau_1}] = 0$$

a.s. Since $T_{-t_1-\tau_1}$ is 1:1 a.s., then

$$P_{\mathscr{U}} \circ {}^{\theta}x_{t_1+\tau_1}^{t_1} = 0$$

a.s. This contradicts our initial choice of τ_1, and so our contention is proved. In particular, if $t \geqq t_1$, then

$$P(P_{\mathscr{U}} \circ {}^{\theta}x_{t+\tau_1}^{t_1} \neq 0) > 0.$$

Therefore,

$$P(P_{\mathscr{U}} \circ \Phi_{t+\tau_1}^{t}\{\Phi_t^{t_1}(\theta)\} \neq 0) = P(P_{\mathscr{U}} \circ {}^{\theta}x_{t+\tau_1}^{t_1} \neq 0) > 0$$

i.e., $\Phi_t^{t_1}(\theta) \in \mathscr{U}_{\omega}^{t}$. This proves the invariance property (ii).

Next we prove that $\mathscr{U}_{\omega}^{t_1}$ is open in $\mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1})$. Fix any $\tau \in \mathscr{T}$. Then by Lemma 2 of Section 2 the map

$$\mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1}) \to \mathscr{L}^2(\Omega, \mathscr{U}; \mathscr{F}_{t_1+\tau})$$

$$\theta \mapsto P_{u} \circ {}^{\theta}x_{t_1+\tau}^{t_1}$$

is continuous. Since $\mathscr{U} \backslash \{0\}$ is open in $\mathscr{U}$, it is easy to see that the function

$$\mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1}) \to \mathbb{R}$$

$$\theta \mapsto P(P_{\mathscr{U}} \circ {}^{\theta}x_{t_1+\tau}^{t_1} \neq 0)$$

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is lower semi-continuous (Stroock and Varadhan [11, pp. 7–10]). Thus the set

$$\mathscr{U}_{\omega,\tau}^{t_1}=:\{\theta:\theta\in\mathscr{L}^2(\Omega,C;\mathscr{F}_{t_1}),\ P(P_{\mathscr{U}}\circ{}^{\theta}x_{t_1+\tau}^{t_1}\neq 0)>0\}$$

is open in $\mathscr{L}^2(\Omega,C;\mathscr{F}_{t_1})$. Therefore

$$\mathscr{U}_{\omega}^{t_1}=\bigcup_{\tau\in\mathscr{T}}\mathscr{U}_{\omega,\tau}^{t_1}$$

is also open in $\mathscr{L}^2(\Omega,C;\mathscr{F}_{t_1})$. This proves assertion (i).

Now we prove the estimate (20). Suppose $\theta\in\mathscr{U}_{\omega}^{t_1}$. Then there is a $\tau_1\in\mathscr{T}$ such that the event

$$A_1=\{\omega:\omega\in\Omega,\ P_{\mathscr{U}}\circ{}^{\theta}x_{t_1+\tau_1(\omega)}^{t_1}(\cdot,\omega)\neq 0\}$$

has positive probability in $\mathscr{F}_{t_1+\tau_1}$. Take $t_2=t_1+\|\tau_1\|_\infty$. Then by the theorem we get

$$E\|\Phi_t^{t_1}(\theta)\|\geqq L\,e^{\delta(t-t_1)}\int_{A_1}e^{-\delta\tau_1}\|P_{\mathscr{U}}\circ{}^{\theta}x_{t_1+\tau_1}^{t_1}\|\,dP$$

for all $t\geqq t_2$. Thus (20) is valid where

$$L_\theta=:L\int_{A_1}e^{-\delta\tau_1}\|P_{\mathscr{U}}\circ{}^{\theta}x_{t_1+\tau_1}^{t_1}\|\,dP>0.$$

We complete the proof of the corollary by indicating the invariance of $\mathscr{M}_{\mathscr{U}}$ under $\{P_t^*\}_{t\geqq 0}$. Let $\mu\in\mathscr{M}_{\mathscr{U}}$ be such that $\mu=P\circ\theta^{-1}$ where $\theta\in\mathscr{U}_{\omega}^{t_1}$ for some $t_1\geqq 0$. Let $t\geqq 0$ and $\phi\in C_b$. Then by time-homogeneity and the Markov property we get

$$\begin{aligned}\langle\phi,P_t^*\mu\rangle&=\langle P_t(\phi),P\circ\theta^{-1}\rangle\\&=\int_{\omega'\in\Omega}\int_{\omega\in\Omega}\phi[{}^{\theta(\omega')}x_{t_1+t}^{t_1}(\omega)]\,dP(\omega)\,dP(\omega')\\&=\int_{\omega\in\Omega}\phi[{}^{\theta}x_{t_1+t}^{t_1}(\omega)]\,dP(\omega)\\&=\langle\phi,P\circ({}^{\theta}x_{t_1+t}^{t_1})^{-1}\rangle\end{aligned}$$

(Mohammed [9, pp. 51–69]). Therefore

$$P_t^*\mu=P\circ({}^{\theta}x_{t_1+t}^{t_1})^{-1}.$$

But ${}^{\theta}x_{t_1+t}^{t_1}\in\mathscr{U}_{\omega}^{t_1+t}$, so $P_t^*\mu\in\mathscr{M}_{\mathscr{U}}$. □

Our next corollary gives a version of a "*stochastic stable manifold theorem*" for small linear or delayed diffusions. Indeed all solutions starting off on $\mathscr{S}_{\omega}^0$ (i.e., outside $\mathscr{U}_{\omega}^0$) are exponentially asymptotically stable in the mean square. The result is a generalization of those in [8, 10].

Corollary 3. *Let*

$$l = \mathrm{Li}\,p(G) =: \inf\left\{\frac{\|G(\eta_1) - G(\eta_2)\|}{\|\eta_1 - \eta_2\|} : \eta_1, \eta_2 \in C, \eta_1 \neq \eta_2\right\}$$

and suppose G satisfies one of the following conditions:

(i) *G is continuous linear;*

(ii) *G is of the form*

$$G(\eta) = g(\eta(-d_0))$$

or

$$G(\eta) = \int_{-r}^{0} g(\eta(s))\,d\lambda(s), \qquad \eta \in C,$$

where $g: \mathbb{R}^n \to L(\mathbb{R}^m, \mathbb{R}^n)$, *is a given map,* $0 \leqq d_0 \leqq r$ *and* λ *is a finite Borel measure on* J.

Suppose $t_1 \geqq 0$. *Then there is an* $\varepsilon > 0$ *such that if* $l < \varepsilon$ *and* $\theta \in \mathscr{S}^{t_1}_{\omega}$, *there exist* $M, \gamma > 0$ *so that*

$$E|{}^{\theta}x^{t_1}(t)|^2 \leqq M\,e^{-\gamma t} E\|\theta\|^2 \tag{24}$$

for all $t \geqq t_1$.

Proof. We prove the estimate (24) when $G: C \to L(\mathbb{R}^m, \mathbb{R}^n)$ is continuous linear and $\|G\|$ is sufficiently small. The proof resembles that of Theorem (5.1) in [**8**].

Let G correspond to a Borel $L(\mathbb{R}^n, L(\mathbb{R}^m, \mathbb{R}^n))$-valued measure ν with finite total variation $v(\nu)$, viz.

$$G(\eta) = \int_{-r}^{0} d\nu(s)(\eta(s))$$

for every $\eta \in C$. Fix $t_1 \geqq 0$. It is clear from the definition of $\mathscr{S}^{t_1}_{\omega}$ that

$$\mathscr{S}^{t_1}_{\omega} = \{\theta : \theta \in \mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1}), {}^{\theta}x^{t_1}_{t_1+\tau} \in \mathscr{S} \text{ a.s. for all } \tau \in \mathscr{T}\}.$$

In particular ${}^{\theta}x^{t_1}_t \in \mathscr{S}$ a.s. for all $t \geqq t_1$. Without loss of generality assume that $t_1 = 0$. Apply the continuous linear projection $P_{\mathscr{S}}: C \to \mathscr{S}$ to the stochastic variation of parameters formula (14) ($\tau_0 \equiv 0$) so as to get

$${}^{\theta}x_t = T_t(\theta) + \int_0^t \hat{T}_{t-u}\{P_{\mathscr{S}} \circ \Delta\} G({}^{\theta}x_u)\,dW(u) \tag{25}$$

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a.s. for all $t \geqq 0$. Note that $\theta(\omega) \in \mathscr{S}$ for a.a. $\omega \in \Omega$, so (4) implies

$$E\left|{}^\theta x(t)\right|^2 \leqq 2E\left|T_t(\theta)(0)\right|^2 + 2E\left|\int_0^t \hat{T}_{t-u}(P_{\mathscr{S}} \circ \Delta)(0) \int_{-r}^0 dv(s)({}^\theta x(u+s))\, dW(u)\right|^2$$

$$\leqq 2K^2 e^{-2\alpha t} E\|\theta\|^2 + 2m^2K^2 \int_0^t e^{-2\alpha(t-u)} E\left|\int_{-r}^0 dv(s)[{}^\theta x(u+s)]\right|^2 du$$

$$\leqq 2K^2 e^{-2\alpha t} E\|\theta\|^2 + 2m^2K^2 e^{-2\alpha t} \int_0^t e^{2\alpha u} v(v)(J) E \int_{-r}^0 \left|{}^\theta x(u+s)\right|^2 dv(v)(s)\, du$$

(Hölder's inequality)

$$= 2K^2 e^{-2\alpha t} E\|\theta\|^2 + 2m^2K^2 e^{-2\alpha t} v(v)(J) \int_{-r}^0 \int_s^{s+t} e^{2\alpha u} e^{-2\alpha s} E\left|{}^\theta x(u)\right|^2 du\, dv(v)(s)$$

(Fubini's theorem).

Hence

$$e^{2\alpha t} E\left|{}^\theta x(t)\right|^2 \leqq 2K^2 E\|\theta\|^2 + \frac{m^2K^2}{\alpha} v(v)(J) E\|\theta\|^2 \int_{-r}^0 (e^{-2\alpha s} - 1)\, dv(v)(s)$$

$$+ 2m^2K^2 v(v)(J) \int_{-r}^0 e^{-2\alpha s}\, dv(v)(s) \int_0^t e^{2\alpha u} E\left|{}^\theta x(u)\right|^2 du$$

for all $t \geqq 0$. Gronwall's lemma then yields the required estimate (24) where

$$M = 2K^2 + \frac{m^2K^2}{\alpha} v(v)(J) \int_{-r}^0 (e^{-2\alpha s} - 1)\, dv(v)(s)$$

$$\gamma = 2\alpha - 2m^2K^2 v(v)(J) \int_{-r}^0 e^{-2\alpha s}\, dv(v)(s).$$

The proof is quite similar when G takes one of the forms in (ii), cf. Mohammed [**8**]. □

Remarks. (i) It is not clear whether the *segment* ${}^\theta x_t^{t_1}$, $t \geqq t_1$, satisfies an estimate similar to (24) when $\theta \in \mathscr{S}_\omega^{t_1}$.

(ii) It is not hard to see that the above corollary holds when G has several discrete and/or distributed random delays (cf. [**8**, Theorems (1.1), (4.1), (4.2)]).

(iii) If $t \geqq t_1$, then $\Phi_t^{t_1}\{\mathscr{S}_\omega^{t_1}\} \subseteq \mathscr{S}_\omega^t$.

(iv) Define the set of probability measures $\mathscr{M}_{\mathscr{S}} \subset \mathscr{M}(C)$ by

$$\mathscr{M}_{\mathscr{S}} = \{\mu : \mu \in \mathscr{M}(C), \text{ there exist } t_1 \geqq 0 \text{ and } \theta \in \mathscr{S}_\omega^{t_1} \text{ such that } \mu = P \circ \theta^{-1}\}.$$

Then $P_t^*\{\mathscr{M}_{\mathscr{S}} \subseteq \mathscr{M}_{\mathscr{S}}$ for every $t \geqq 0$.

We close this section by giving sufficient conditions on the drift and diffusion coefficients $H:C\to\mathbb{R}^n$ and $G:C\to L(\mathbb{R}^m,\mathbb{R}^n)$ for the family $\mathcal{S}^t_\omega$ to be trivial—so that one almost always gets unstable solutions of the stochastic f.d.e. (I). To this end we need the following lemma:

Lemma 1. *Let $A\in L(\mathbb{R}^m,\mathbb{R}^n)$ be onto. Then $(P_{\mathcal{U}}\circ\Delta)A=0$ if and only if $\mathcal{U}=\{0\}$.*

1 **Proof.** We need only prove that $(P_{\mathcal{U}}\circ\Delta)A=0$ implies $\mathcal{U}=\{0\}$. Assume that $A:\mathbb{R}^m\to\mathbb{R}^n$ is onto and let $((P_{\mathcal{U}}\circ\Delta)A)=0$. We adopt the notation of Hale [5, pp. 168–182]). Recall that the unstable subspace $\mathcal{U}$ has finite dimension d. Define $C^*=C([0,r],\mathbb{R}^{n*})$ where $\mathbb{R}^{n*}$ is the space of all n-dimensional row vectors. Set the bilinear map $(\cdot,\cdot):C^*\times C\to\mathbb{R}$ to be

$$(\psi,\phi)=\psi(0)\phi(0)-\int_{-r}^{0}\int^{u}\psi(s-u)\,d\mu(u)\phi(s)\,ds$$

when $\mu:\text{Borel}\,J\to L(\mathbb{R}^n)$ is the measure representing $H:C\to\mathbb{R}^n$. Let $A^H:D(A^H)\subset C\to C$ be the infinitesimal generator of the semi-flow $T_t:C\to C$, $t\geqq 0$ of (II)

$$dy(t)=H(y_t)\,dt. \tag{II}$$

Denote by $A^{H*}:D^*\subset C^*\to C^*$ the adjoint of A^H under the above bilinear form: $(A^{H*}\psi,\phi)=(\psi,A^H\phi)$, $\psi\in D^*$, $\phi\in D$ (Hale [5, p. 174]).

The unstable subspace $\mathcal{U}$ is the sum of the generalized eigenspaces of A corresponding to the eigenvalues with non-negative real parts. The space $\mathcal{U}^*$ is constructed in a similar way (Hale [5, pp. 174–179]). Furthermore, pick a basis

$$\Phi=(\phi_1,\ldots,\phi_d)\text{ of }\mathcal{U}\text{ and a basis }\phi=\begin{bmatrix}\psi_1\\ \vdots\\ \psi_d\end{bmatrix}\text{ for }\mathcal{U}^*.$$

Approximate ΔA pointwise by a uniformly bounded sequence of continuous functions and use the identity

$$P_{\mathcal{U}}(\phi)=\Phi(\Psi,\phi),\qquad \phi\in C$$

(Hale [5, p. 178]) to obtain

$$(P_{\mathcal{U}}\circ\Delta)A=\Phi(\Psi,\Delta A)A=\Phi\Psi(0)A$$

(Mohammed [9, p. 207]). Therefore by assumption

$$\Phi\Psi(0)A=0$$

where products in the above formulae are "matrix-like". Hence, for every $v \in \mathbb{R}^m$, we have

$$\sum_{i=1}^{d} \phi_i \psi_i(0) A v = 0.$$

Since $\{\phi_i\}_{i=1}^d$ are linearly independent, then

$$\psi_i(0) A v = 0, \qquad 1 \leqq i \leqq d, \quad v \in \mathbb{R}^m.$$

But $A(\mathbb{R}^m) = \mathbb{R}^n$, so $\psi_i(0)\hat{v} = 0$ for every $\hat{v} \in \mathbb{R}^n$. Therefore $\chi_i(0) = 0$ for every $1 \leqq i \leqq d$. Thus $\psi(0) = 0$ for all $\psi \in \mathscr{U}^*$. Now $\mathscr{U}^* \subset D^*$ and $A^{H*}\mathscr{U}^* \subset \mathscr{U}^*$, by construction (Hale [5, pp. 173–179]). Hence for every $\psi \in \mathscr{U}^*$, we get $A^{H*}\psi \in \mathscr{U}^*$ and $(A^{H*}\psi)(0) = \psi'(0) = 0$. Similarly $[(A^{H*})^2\psi(0) = \psi''(0) = 0, \ldots, \psi^{(p)}(0) = 0$ for every integer $p \geqq 1$. Since $\mathscr{U}^*$ is a finite sum of subspaces of the form $\ker(A^{H*} - \lambda I)^k$ where λ is an eigenvalue of A^{H*}, we may assume without loss of generality that $\psi \in \mathscr{U}^*$ has the form

$$\psi(s) = \sum_{j=1}^{k} \beta_j \frac{(-s)^{k-j}}{(k-j)!} e^{-\lambda s}, \qquad 0 \leqq s \leqq r$$

for some $\beta_j \in \mathbb{R}^{n*}$, $1 \leqq j \leqq k$, (Hale [5, Lemma 3.3, p. 177]). Now differentiate the above relation with respect to s p-times for $p \geqq k$ and put $s = 0$ to obtain

$$\beta_k = \beta_{k-1} = \cdots = \beta_2 = \beta_1 = 0.$$

Therefore $\psi(s) = 0$ for all $0 \leqq s \leqq r$, $\mathscr{U}^* = \{0\}$ and $\mathscr{U} = \{0\}$. This proves the lemma.

The diffusion coefficient $G: C \to L(\mathbb{R}^m, \mathbb{R}^n)$ is said to be *non-degenerate at* $\eta \in C$ if $G(\eta): \mathbb{R}^m \to \mathbb{R}^n$ is onto. Our final result in this section says that if the deterministic linear f.d.e. (II) is unstable and the diffusion coefficient G is non-degenerate, then the family $\mathscr{S}^t_\omega$ is trivial:

Theorem 3.2. *Assume that the deterministic drift*

$$dy(t) = H(y_t)\,dt, \qquad t > 0 \tag{II}$$

is unstable, i.e., $\mathscr{U} \neq \{0\}$, *or equivalently the characteristic equation*

$$\det\{\lambda I - H(e^{\lambda(\cdot)})\} = 0$$

has at least one root with positive real part.

(i) *Suppose there is a* $\delta \in J = [-r, 0]$ *such that* G *is non-degenerate at every* $\eta \in C$ *with* $\eta(-\delta) \neq 0$. *Then, for every* $t_1 \geqq 0$,

$$\mathscr{S}^{t_1}_\omega = \{\theta : \theta \in \mathscr{L}^2(\Omega, C; \mathscr{F}_{t_1}),\ {}^\theta x^{t_1}_t = 0 \text{ a.s. for all } t \geqq t_1 - \delta + r\}.$$

(ii) *Suppose* G *is non-degenerate at every* $\eta \neq 0$. *Then for* $t_1 \geqq 0$, $\mathscr{S}^{t_1}_\omega$ *is either empty or* $\{0\}$.

(iii) *Suppose* G *is non-degenerate at every* $\eta \in C$. *Then* $\mathscr{S}^{t_1}_\omega = \varnothing$ *for all* $t_1 \geqq 0$.

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Proof. Assume throughout the proof that $\mathscr{U} \neq \{0\}$.

(i) Let $G(\eta)$ be onto for every η with $\eta(-\delta)\neq 0$ where $-r \leqq \delta \leqq 0$ is fixed. Fix $t_1 \geqq 0$ and let $\theta \in \mathscr{S}_{\omega}^{t_1}$. Then

$$T_{t-t_1}[P_{\mathscr{U}} \circ \theta] + T_t \int_{t_1}^{t} \hat{T}_{-u}(P_{\mathscr{U}} \circ \Delta) G({}^{\theta}x_u^{t_1})\, dw(u) = 0$$

a.s. for all $t \geqq t_1$ (cf. (23)). This implies that

$$\int_{t_1}^{t} \hat{T}_{-u}(P_{\mathscr{U}} \circ \Delta) G({}^{\theta}x_u^{t_1})\, dw(u) = 0$$

a.s. for every $t \geqq t_1$. Therefore $\hat{T}_{-u}(P_{\mathscr{U}} \circ \Delta) G({}^{\theta}x_u^{t_1})\, dw(u) = 0$ a.s. for all $u \geqq t_1$. As $\hat{T}_{-u}$ is injective, then $(P_{\mathscr{U}} \circ \Delta) G({}^{\theta}x_u^{t_1} = 0$ a.s. for every $u \geqq t_1$. Suppose—if possible—there is a $t_0 > t_1$ such that

$$P({}^{\theta}x_{t_0}^{t_1}(-\delta) \neq 0) > 0.$$

Therefore by the non-degeneracy of G and Lemma 1, we must have $\mathscr{U} = \{0\}$. This contradicts the instability of (II). Hence ${}^{\theta}x_t^{t_1}(-\delta) = 0$ a.s. for all $t \geqq t_1$. So ${}^{\theta}x_t^{t_1} = 0$ a.s. for every $t \geqq t_1 - \delta + r$. This completes the proof of assertion (i).

(ii) To prove assertion (ii), suppose $G(\eta)$ is onto whenever $\eta \neq 0$. Let $\theta \in \mathscr{S}_{\omega}^{t_1}$. Then as before $(P_{\mathscr{U}} \circ \Delta) G({}^{\theta}x_t^{t_1}) = 0$ a.s. for every $t \geqq t_1$. If there were a $t_0 \geqq t_1$ such that $P({}^{\theta}x_{t_0}^{t_1} \neq 0) > 0$, then we must have $\mathscr{U} = \{0\}$; which is a contradiction. Therefore ${}^{\theta}x_t^{t_1} = 0$ a.s. for all $t \geqq t_1$. In particular $\theta = 0$; so $\mathscr{S}_{\omega}^{t_1} = \{0\}$.

(iii) The proof of the final assertion (iii) is similar.

The proof of the theorem is complete. □

Remark. In the case $m = n$, it is easy to formulate ellipticity conditions on G which are sufficient to give the non-degeneracy requirements of Theorem 3.2. In particular, let $g: \mathbb{R}^n \to L(\mathbb{R}^n)$ be a Lipschitz map satisfying

$$\langle g(x)(v), v \rangle \geqq \varepsilon(x)|v|^2, \qquad x, v \in \mathbb{R}^n$$

where $\varepsilon: \mathbb{R}^n \to \mathbb{R}^+$ is continuous with $\varepsilon(x) > 0$ for every $x \neq 0$. The Euclidean inner product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$ and the corresponding norm by $|\cdot|$, as usual. The diffusion coefficient

$$G(\eta) = g(\eta(-d_0)), \qquad 0 \leqq d_0 \leqq r,$$

is easily seen to satisfy assertion (i) of Theorem 3.2. Similarly

$$G(\eta) = \int_{-r}^{0} g(\eta(s))\, ds$$

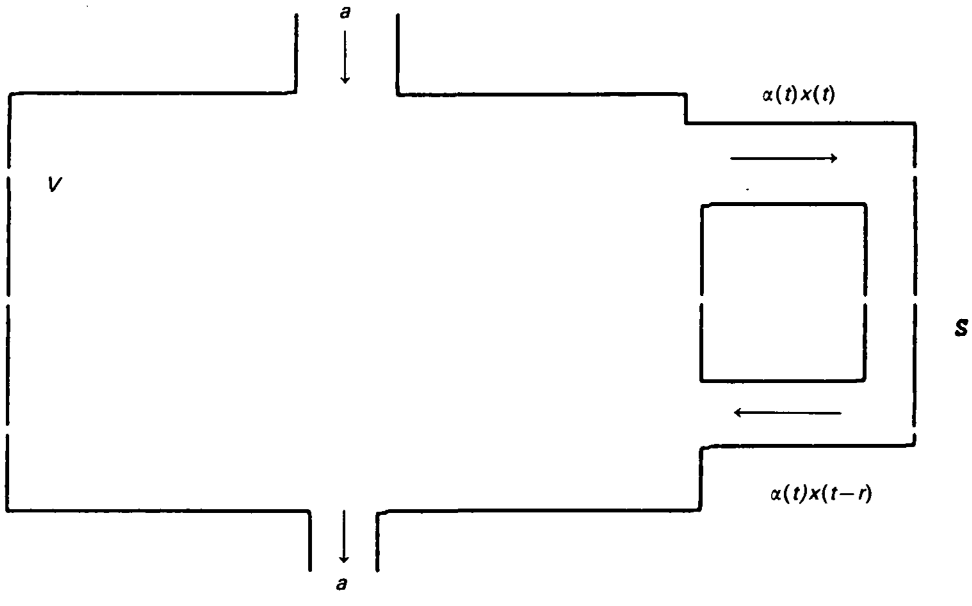
satisfies assertion (ii) of the above theorem.

4. Examples

(i) **Dye mixing** (Pollution)

A model for the circulation of dye from a tank was proposed by Bailey and Williams [1]. In [1] the effect of diffusion on the circulation has been neglected. We would like to investigate the effect of (small) diffusion on the dye-mixing model.

For simplicity, we consider the set-up:



and introduce the following notation:

V = volume of the tank (cc)
$x(t)$ = concentration of dye in the tank at time t; (gm/cc)
r = time taken by the dye to traverse the side tube S
$\alpha(t)$ = random diffusion of dyed solution in S (cc per sec)
$0 < a$ = constant rate of flow of water in V (cc per sec)
= constant rate of flow of dyed water leaving the tank.

In the time interval $(t, t+\Delta t)$ the amount of dye entering the tank is equal to the amount leaving it so that the change in dye concentration $\Delta x(t)$ at time t is given by

$$V\Delta x(t) = -\alpha(t)x(t)\Delta t + \alpha(t)x(t-r)\Delta t - ax(t)\Delta t.$$

Therefore,

$$\Delta x(t) = -\frac{a}{V}x(t)\Delta t + \frac{1}{V}[x(t-r) - x(t)]\alpha(t)\Delta t. \tag{26}$$

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Now $\{\alpha(t):t\geqq 0\}$ is a random process which we may assume to be white noise viz.

$$\alpha(t)=\alpha_0+\beta(t)$$

where $\alpha_0>0$ is a constant and $\beta(t)\Delta t=\Delta W(t)$, for some one-dimensional Brownian motion W. Letting $\Delta t\to 0$, (26) then becomes the stochastic linear d.d.e.

$$dx(t)=\frac{1}{V}[\alpha_0 x(t-r)-(a+\alpha_0)x(t)]\,dt+\frac{1}{V}[x(t-r)-x(t)\,dW(t). \tag{V}$$

We wish to find suitable condition(s) on the parameters α_0, V, a and r so that the zero solution of the above equation is asymptotically stable in the mean square.

First consider the deterministic d.d.e.

$$dy(t)=\frac{1}{V}[\alpha_0 y(t-r)-(a+\alpha_0)y(t)]\,dt. \tag{VI}$$

Since $a,\alpha_0,V>0$, it follows from [1] that the zero solution of (VI) is globally asymptotically stable. Indeed the roots of the characteristic equation

$$V\lambda\,e^{\lambda r}+(a+\alpha_0)\,e^{\lambda r}-\alpha_0=0 \tag{27}$$

have all real parts negative. By the notation of Section 1, Section 3, the unstable subspace $\mathscr{U}$ of (VI) is trivial and $\mathscr{S}=C$. Thus $\mathscr{U}^t_\omega=\varnothing$ and $\mathscr{S}^t_\omega=\mathscr{L}^2(\Omega,C;\mathscr{F}_t)$ for all $t\geqq 0$. Furthermore applying Corollary 3 of Theorem 3.1, there is a $V_0>0$ so that if $V\geqq V_0$, then for all initial distributions $\theta\in\mathscr{L}^2(\Omega,C;\mathscr{F}_0)$ of dye concentration, the dye in the tank will eventually be washed out at the exponential speed

$$E|^\theta x(t)|^2\leqq M\,e^{-\gamma t}E\|\theta\|^2$$

for all $t>0$ and some constants M, $\gamma>0$.

It is interesting to note here that the above conclusion is *independent* of the “size” of the perturbation $\beta(t)$ i.e., if V is sufficiently large, the effect of diffusion may always be ignored (cf. [1]).

It is an interesting question to determine the asymptotic behaviour of $x(t)$ when the volume V is *possibly small.*

The stochastic d.d.e. (V) is derived under the assumption that dye mixes *instantaneously* within the tank V. Suppose, however, that mixing in V is equivalent to dye “dissolving” in the tank at white noise rate with constant variance $b>0$. Then the model becomes

$$dx(t)=\frac{1}{V}[\alpha_0 x(t-r)-(a+\alpha_0)x(t)]+b\,dW(t). \tag{VII}$$

For the sake of simplicity, we have ignored in (VII) the effect of "small" diffusion into the side tube S. According to Theorem VI (4.2) of [**8**, p. 208], or [**10**, Theorem 3], the dye concentration $x(t)$ is asymptotically close in mean square to a stationary Gaussian distribution.

Equations like (V) and (VII) may also serve as some possible pollution models on river banks. We shall not go into this here.

(ii) **Population growth**

Consider a large population $x(t)$ at time t evolving with a (constant) birth rate $\mu>0$, and a fixed development (incubation) period $r>0$. Suppose that there is a Gaussian migration $\alpha(t)$ from the population viz.

$$\alpha(t)=\alpha_0+\beta(t)$$

where $\alpha_0<0$ and $\beta(t)$ is white noise. Then the change in population Δx over a small time interval $(t, t+\Delta t)$ is given by

$$\begin{aligned}\Delta x &= \mu x(t-r)\Delta t+[\alpha_0+\beta(t)]x(t)\Delta t\\ &=[\mu x(t-r)+\alpha_0 x(t)]\Delta t+x(t)\Delta W.\end{aligned}$$

Letting $\Delta t\to 0$ yields the stochastic d.d.e.

$$dx(t)=[\mu x(t-r)+\alpha_0 x(t)]\,dt+x(t)\,dW(t). \tag{VIII}$$

If $\mu<|\alpha_0|$, the deterministic d.d.e.

$$dy(t)=[\mu y(t-r)+\alpha_0 y(t)]\,dt \tag{IX}$$

is exponentially asymptotically stable, whatever the development period r. Thus if β (or W) has sufficiently small variance, one expects the population to become extinct with exponential speed in the mean square (Corollary 3, Theorem 3.1). On the other hand if $\mu>|\alpha_0|$ the d.d.e. is unstable (Bailey and Williams [**1**]) and so the unstable subspace $\mathscr{U}$ has *positive* dimension. From Corollary 1 (Theorem 3.1) it follows that the population becomes exponentially unstable whenever it starts off with an initial distribution in $\mathscr{L}^2(\Omega,\mathscr{U})$ (or even on $\mathscr{U}$).

An interesting statistical problem is to estimate the time τ required by x_t to hit the unstable subspace $\mathscr{U}$.

From a control-theory viewpoint, one would like to "stabilize" the population x_t by "guiding" it away from the unstable subspace $\mathscr{U}$.

Acknowledgements. The author is most grateful to the anonymous referee for helpful suggestions and criticism of an earlier version of the article and to Linda Macak for the typing.

REFERENCES

1. H. R. Bailey and M. Z. Williams, Some results on the differential-difference equation $\dot{x}(t)=\sum_{i=1}^{N} A_i x(t-T_i)$, *J. Math. Anal. Appl.* **15** (1966), 569–587.

2. K. L. Chung, *Lectures from Markov processes to Brownian motion* (Springer-Verlag, New York–Heidelberg–Berlin, 1982).

3. J. L. Doob, *Stochastic processes* (Wiley, New York, 1967).

4. K. D. Elworthy, *Stochastic differential equations on manifolds* (LMS Lecture Note Series **70**, Cambridge University Press, Cambridge, 1982).

5. J. K. Hale, *Theory of functional differential equations* (Springer-Verlag, New York–Heidelberg–Berlin, 1977).

6. J. K. Hale and C. Perello, The neighborhood of a singular point of functional-differential equations, *Cont. Differential Equations* **3** (1964), 351–375.

7. R. S. Lipster and A. N. Shiryayev, *Statistics of random processes I, General theory* (Springer-Verlag, New York–Heidelberg–Berlin, 1977).

8. S. E. A. Mohammed, Stability of linear delay equations under a small noise, *Proc. Edinburgh Math. Soc.* **29** (1986), 233–254.

9. S. E. A. Mohammed, *Stochastic functional differential equations* (Research Notes in Mathematics **99**, Pitman Advanced Publishing Program, Boston–London–Melbourne, 1984).

10. S. E. A. Mohammed, M. Scheutzow and H. v. Weizsäcker, Hyperbolic state space decomposition for a linear stochastic delay equation, *SIAM J. Control Optim.* **24** (1986), 543–551.

11. D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes* (Springer-Verlag, Berlin–Heidelberg–New York, 1979).

Mathematics Department
Southern Illinois University at Carbondale
Carbondale
Illinois 62901
U.S.A.