

ON NON-COMPACT p -ADIC DEFINABLE GROUPS

WILL JOHNSON  AND NINGYUAN YAO 

Abstract. In [16], Peterzil and Steinhorn proved that if a group G definable in an o -minimal structure is not definably compact, then G contains a definable torsion-free subgroup of dimension 1. We prove here a p -adic analogue of the Peterzil–Steinhorn theorem, in the special case of abelian groups. Let G be an abelian group definable in a p -adically closed field M . If G is not definably compact then there is a definable subgroup H of dimension 1 which is not definably compact. In a future paper we will generalize this to non-abelian G .

§1. Introduction. In [16], Peterzil and Steinhorn prove that if G is a definable group in an o -minimal structure M , and G is not definably compact, then G has a definable one-dimensional subgroup H that is not definably compact. To prove this, they take a continuous unbounded definable curve $I : [0, +\infty) \rightarrow G$ and take H to be the “tangent line at ∞ .” This can be made precise using the language of μ -types and μ -stabilizers developed later by Peterzil and Starchenko [15]. Say that two complete types $q, r \in S_G(M)$ are “infinitesimally close” if there are realizations $a \models q$ and $b \models r$ such that ab^{-1} is infinitesimally close to id_G (that is, ab^{-1} is contained in every M -definable neighborhood of id_G). This is an equivalence relation on $S_G(M)$, and equivalence classes are called “ μ -types.” The “ μ -stabilizer” $\text{stab}^\mu(q)$ of $q \in S_G(M)$ is the stabilizer of the μ -type of q .

With these definitions, the “tangent line of I at ∞ ” is simply the μ -stabilizer of the type on I at infinity, an unbounded one-dimensional definable type. (Here, we say that a type $q \in S_G(M)$ is “unbounded” if no formula in q defines a definably compact subset of G .) Peterzil and Steinhorn essentially show that the μ -stabilizer of an unbounded one-dimensional definable type is a torsion-free non-compact definable subgroup of dimension 1. More generally, in [15], Peterzil and Starchenko consider a general definable type $q \in S_G(M)$, showing that $\text{stab}^\mu(q)$ is a torsion-free definable group of a certain dimension.

It is natural to ask whether analogous results hold in the theory $p\text{CF}$ (p -adically closed fields). There are many formal similarities between $p\text{CF}$ and o -minimal theories, especially RCF (real closed fields). In both settings, definable groups can be regarded as real or p -adic Lie groups [17, 18], and are locally isomorphic to real or p -adic algebraic groups [8]. In both the real and p -adic contexts, definable sets have a dimension which has a topological description as well as an algebraic

Received March 23, 2021.

2020 *Mathematics Subject Classification.* 03C60, 03C98.

Key words and phrases. definable groups, NIP groups, p -adic numbers.

description (the algebro-geometric dimension of the Zariski closure). On the other hand, definable connectedness behaves very differently in the two settings.

In this paper, we restrict our attention to one-dimensional definable types, as in the original work of Peterzil and Steinhorn [16]. Unfortunately, we must also assume that G is “nearly abelian” for most of our theorems.

DEFINITION 1.1. Let G be a definable group in a model of p CF. G is *nearly abelian* if there is a definably compact definable normal subgroup $K \subseteq G$ with G/K abelian.

See Definition 2.1 for a precise definition of “definable compactness,” and Propositions 2.16 and 2.24 for some equivalent conditions.

Our main results are as follows:

THEOREM 1.2. *Let G be a definable group over a p -adically closed field M . If G is not definably compact and G is nearly abelian, then there is a one-dimensional definable subgroup $H \subseteq G$ that is not definably compact.*

We plan to generalize Theorem 1.2 to non-abelian groups in a future paper.

THEOREM 1.3. *Suppose that G is a definable group over an \aleph_1 -saturated p -adically closed field M . Then for any definable unbounded one-dimensional type $r \in S_G(M)$, the μ -stabilizer $\text{stab}^\mu(r)$ is a one-dimensional type-definable subgroup of G . If G is abelian (or nearly abelian), then $\text{stab}^\mu(r)$ is unbounded.*

Here, a set or type is “bounded” if it is contained in a definably compact set, and “unbounded” otherwise (Definition 2.9). The assumption on saturation is necessary. For example, suppose $M = \mathbb{Q}_p$, G is the multiplicative group, and $r \in S_G(\mathbb{Q}_p)$ is one of the definable types consistent with $\{x \mid v(x) < \mathbb{Z}\}$. Then $\text{stab}^\mu(r)$ is the intersection of all n -th powers $P_n = \{x \mid x \neq 0 \wedge \exists(x = y^n)\}$, which is the trivial group $\{1\}$.

We can also say something when M is not saturated, but we will need a few more definitions from [15]. Fix a group G definable in a p -adically closed field M . For any partial type $\Sigma(x)$ in G , and any \mathcal{L} -formula $\phi(x; y)$, let $\text{stab}_\phi(\Sigma)$ denote

$$\bigcap_{b \in M^k} \text{stab}\{g \in G(M) \mid \Sigma \vdash \phi(gx; b)\}.$$

(This can be understood as the stabilizer of the $\phi(z \cdot x; y)$ -type generated by $\Sigma(x)$.) It turns out that $\text{stab}(\Sigma) = \bigcap_{\phi \in \mathcal{L}} \text{stab}_\phi(\Sigma)$. Now suppose that r is a type in $S_G(M)$. Let μ be the partial type of “infinitesimals,” that is, the set of \mathcal{L}_M -formulas defining neighborhoods of id_G . Let $\mu \cdot r$ be the partial type such that $(\mu \cdot r)(N) = \mu(N) \cdot r(N)$ for sufficiently saturated $N \succ M$. It turns out that

$$\text{stab}^\mu(r) = \text{stab}(\mu \cdot r) = \bigcap_{\phi \in \mathcal{L}} \text{stab}_\phi(\mu \cdot r).$$

Moreover, when r is definable, the groups $\text{stab}_\phi(\mu \cdot r)$ are definable, and $\text{stab}(\mu \cdot r)$ is type-definable. (This is the reason why $\text{stab}^\mu(r)$ is type-definable in Theorem 1.3. In the o -minimal case, there is a descending chain condition on definable groups, which ensures that $\text{stab}^\mu(r)$ is *definable* in [15].)

THEOREM 1.4. *Suppose that G is a definable group over a p -adically closed field M . Let $r \in S_G(M)$ be a definable unbounded one-dimensional type. Then there is a*

formula $\phi \in \mathcal{L}$ such that $\text{stab}_\phi(\mu \cdot r)$ is a one-dimensional definable subgroup of G . When G is abelian (or nearly abelian), $\text{stab}_\phi(\mu \cdot r)$ is unbounded.

Our proofs of these theorems are based on the original proofs of Peterzil and Steinhorn [16], though several important changes are necessary. First of all, the μ -stabilizer $\text{stab}^\mu(r)$ is no longer definable, but merely type-definable, as mentioned above. For this reason, it is necessary to compute the stabilizers in an $|M|^+$ -saturated elementary extension $N \succ M$.

A more serious problem arises when trying to generalize [16, Lemma 3.8]. This lemma, which is used to show that $\text{stab}^\mu(p) \neq \{\text{id}_G\}$, roughly says the following: if I is a curve tending to infinity and B is an annulus around id_G , then $g \cdot B \cap I \neq \emptyset$ for all $g \in I$. This follows by a simple connectedness argument (I is connected, so it must cut across the annulus $g \cdot B$ on its way from g to infinity). This argument fails critically in the totally disconnected p -adic context. In Section 4 we develop an alternative argument to replace [16, Lemma 3.8]. Unfortunately, the argument only works properly in the abelian (or near-abelian) case.

1.1. Notation and conventions. We shall assume a basic knowledge of model theory, including basic notions such as definable types, saturation, heirs, and so on. Good references are [12, 21]. We refer to the excellent survey [1] as well as [8, 13] for the model theory of the p -adic field $(\mathbb{Q}_p, +, \times, 0, 1)$. In fact, [8, 13] are also good references for the model-theoretic background required for the current paper.

Let T be a theory in some language \mathcal{L} . We write \mathbb{M} for a monster model of T , in which every type over a small subset $A \subseteq \mathbb{M}$ is realized, where “small” means $|A| < \kappa$ for some big enough cardinal κ . The letters $M, N, M',$ and N' will denote small elementary submodels of \mathbb{M} . We will use x, y, z to mean arbitrary n -tuples of variables and $a, b, c \in \mathbb{M}$ to denote n -tuples in \mathbb{M}^n with $n \in \mathbb{N}$. Every formula is an $\mathcal{L}_{\mathbb{M}}$ -formula. For an \mathcal{L}_M -formula $\phi(x)$, $\phi(M)$ denotes the definable subset of $M^{|x|}$ defined by ϕ , and a set $X \subseteq M^n$ is definable if there is an \mathcal{L}_M -formula $\phi(x)$ such that $X = \phi(M)$. If $M \prec N \prec \mathbb{M}$, and $X \subseteq N^n$ is defined by a formula ψ with parameters from M , then $X(M)$ and $X(\mathbb{M})$ will denote $\psi(M)$ and $\psi(\mathbb{M})$, respectively; these are clearly definable subsets of M^n and \mathbb{M}^n , respectively.

Following [15, Definition 2.12], we say that a partial type Σ is A -definable or definable over A if for every formula $\phi(x; y)$, there is an \mathcal{L}_A -formula $\psi(y)$ such that

$$\Sigma(x) \vdash \phi(x; b) \iff M \models \psi(b),$$

for all $b \in M$. We will denote the formula $\psi(y)$ by $(d_{\Sigma}x)\phi(x, y)$, thinking of d_{Σ} as a quantifier. The map $\phi(x; y) \mapsto (d_{\Sigma}x)\phi(x, y)$ is called the *definition schema* of $\Sigma(x)$.

If $\Sigma(x)$ is a definable partial type over M , and $N \succ M$, then Σ^N will denote the canonical extension of Σ by definitions, i.e., the following partial type over N :

$$\Sigma^N = \{\phi(x; a) \in \mathcal{L}_N \mid N \models (d_{\Sigma}x)\phi(x, a)\}.$$

When p is a complete definable type over M , the canonical extension p^N is the same thing as the unique heir of p over N .

For a definable set $D \subseteq M^n$, and $\phi(x)$ an \mathcal{L}_M -formula, we say that $\phi(x)$ is a D -formula if $\mathbb{M} \models \phi(x) \implies x \in D(\mathbb{M})$. A partial type $q(x)$ (over a small subset) is a D -type if $q(x) \vdash x \in D(\mathbb{M})$. We write $S_D(M)$ for the space of complete D -types over M .

We consider \mathbb{Q}_p as a structure in the language of rings $\mathcal{L} = \mathcal{L}_r = \{+, \times, -, 0, 1\}$. The valuation ring \mathbb{Z}_p is definable in \mathbb{Q}_p . The valuation group $(\mathbb{Z}, +, <)$ and the valuation $v : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{\infty\}$ are interpretable. A p -adically closed field is a model of $p\text{CF} := \text{Th}(\mathbb{Q}_p)$. For any $M \models p\text{CF}$, $R(M)$ will denote the valuation ring, and Γ_M will denote the value group. By [11], $p\text{CF}$ admits quantifier elimination after adjoining predicates P_n for the n -th power of the multiplicative group for all $n \in \mathbb{N}^+$. The theory $p\text{CF}$ also has definable Skolem functions [3].

The p -adic field \mathbb{Q}_p is a locally compact topological field, with basis given by the sets

$$\mathcal{B}(a, n) = \{x \in \mathbb{Q}_p \mid x \neq a \wedge v(x - a) \geq n\},$$

for $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$. The valuation ring \mathbb{Z}_p is compact. The topology is definable (as in Section 2.1 below), so it extends to any p -adically closed field M , making M a topological field (usually not locally compact). Any definable set $X \subseteq M^n$ has a topological dimension, denoted by $\dim(X)$, which is the maximal $k \leq n$ such that the image of the projection $\pi : X \rightarrow M^k; (x_1, \dots, x_n) \mapsto (x_{r_1}, \dots, x_{r_k})$ has interior, for suitable $1 \leq r_1 < \dots < r_k \leq n$. As model-theoretic algebraic closure coincides with the field-theoretic algebraic closure, algebraic closure gives a pregeometry on M , and the algebraic dimension $\dim_{\text{alg}}(X)$ of X can be calculated in the usual way. The topological dimension coincides with the algebraic dimension.

1.2. Outline. In Section 2, we review the notion of definable compactness, and how it behaves in definable manifolds and definable groups in $p\text{CF}$. In Section 3 we review the theory of dp-rank, which is used in Section 4. In Section 4, we prove a technical statement about “gaps” in unbounded sets, which replaces the use of connectedness in Peterzil–Steinhorn [16, Lemma 3.8]. In Section 5, we review the theory of stabilizers and μ -stabilizers from [15]. Finally, we prove the main theorems in Section 6.

§2. Definable compactness. In this section, we review the notion of *definable compactness* for definable manifolds and definable groups in p -adically closed fields. The treatment of (p -adic) definable compactness in the literature is questionable, so we build up the theory from scratch, out of an abundance of caution.

In Section 2.1 we recall an abstract definition of definable compactness, which behaves well in any definable topological space. In the next two sections, we restrict our attention to p -adic definable manifolds. In Section 2.2 we show that our definition agrees with the definition in the literature in terms of curve completion. In Section 2.3 we give another characterization using specialization of definable types. Finally, in Section 2.4 we list some consequences for definable groups.

2.1. Abstract definable compactness. Let M be an arbitrary structure. A *definable topology* on a definable set $X \subseteq M^n$ is a topology with a (uniformly) definable basis of opens. A *definable topological space* is a definable set with a definable topology.

Recall that a topological space is compact if any filtered intersection of non-empty closed sets is non-empty.

DEFINITION 2.1. Let X be a definable topological space in a structure M . Say that X is *definably compact* if the following holds: for any definable family $\mathcal{F} = \{Y_t : t \in T\}$ of non-empty closed sets $Y_t \subseteq X$, if \mathcal{F} is downwards directed, then $\bigcap \mathcal{F} \neq \emptyset$.

More generally, say that a definable set $Y \subseteq X$ is definably compact if it is definably compact with respect to the induced subspace topology.

Definable compactness has many of the expected properties:

FACT 2.2.

1. If X is a compact definable topological space, then X is definably compact.
2. If X, Y are definably compact, then $X \times Y$ is definably compact.
3. If $f : X \rightarrow Y$ is definable and continuous, and X is definably compact, then the image $f(X) \subseteq Y$ is definably compact.
4. If X is a Hausdorff definable topological space and $Y \subseteq X$ is definably compact, then Y is closed.
5. If X is definably compact and $Y \subseteq X$ is closed and definable, then Y is definably compact.
6. If X is a definable topological space and $Y_1, Y_2 \subseteq X$ are definably compact, then $Y_1 \cup Y_2$ is definably compact.

Definition 2.1 and Fact 2.2 are due independently to Fornasiero [4] and the first author [9, Section 3.1].

REMARK 2.3. Suppose X is a definable topological space in a structure M , and $N \succ M$. Then $X(N)$ is naturally a definable topological space in the structure N , and $X(N)$ is definably compact if and only if X is definably compact. In other words, definable compactness is invariant in elementary extensions.

2.2. Definable compactness and definable manifolds in p CF. Let M be a p -adically closed field with valuation group Γ_M . Each power M^n is a definable topological space. We first characterize definable compactness for subsets of M^n .

LEMMA 2.4. *If $X \subseteq M^n$ is definably compact, then X is closed and bounded.*

PROOF. For $t \in M \setminus \{0\}$, let O_t be the n -dimensional ball $\mathcal{B}(0, v(t))^n$. Each O_t is clopen in M^n . Therefore $\{X \setminus O_t : t \in M \setminus \{0\}\}$ is a downwards-directed definable family of closed subsets of X , with empty intersection. By definable compactness, there is some t such that $X \setminus O_t = \emptyset$, or equivalently, $X \subseteq O_t$. Then X is bounded.

Closedness follows similarly, or by Fact 2.2(4). ←

LEMMA 2.5. *If $X \subseteq M^n$ is closed and bounded, then X is definably compact.*

PROOF. Equivalently, if $\{Y_t\}$ is a downwards-directed definable family of non-empty, closed, bounded sets, then $\bigcap_t Y_t \neq \emptyset$. This claim can be expressed as a countable conjunction of \mathcal{L} -sentences. (We need infinitely many sentences because there is no bound on the complexity of the definable family $\{Y_t\}$.) As a countable conjunction of \mathcal{L} -sentences, the claim holds in M if and only if it holds in \mathbb{Q}_p . Therefore, we may assume that $M = \mathbb{Q}_p$. In this case, the set X will be compact, and hence definably compact by Fact 2.2(1). ←

DEFINITION 2.6. Let X be a definable topological space. A Γ -exhaustion is a definable family $\{W_\gamma \mid \gamma \in \Gamma_M\}$ such that

- Each W_γ is an open, definably compact subset of X . In particular, W_γ is clopen.
- If $\gamma \leq \gamma'$, then $W_\gamma \subseteq W_{\gamma'}$.
- $X = \bigcup_{\gamma \in \Gamma_M} W_\gamma$.

LEMMA 2.7. *If $U \subseteq M^n$ is definable and open, then U has a Γ -exhaustion.*

PROOF. For any $\bar{x} = (x_1, \dots, x_n) \in M^n$ and $\gamma \in \Gamma_M$, let $\mathcal{B}(\bar{x}, \gamma)$ denote the ball of valuative radius γ around \bar{x} , i.e., $\prod_{i=1}^n \mathcal{B}(x_i, \gamma)$.

Let W_γ be the set of $x \in U$ such that $\mathcal{B}(x, \gamma) \subseteq U$ and $\bar{0} \in \mathcal{B}(x, -\gamma)$. We claim that the family W_γ is a Γ -exhaustion.

First of all, for all x' sufficiently close to x , we have $\mathcal{B}(x, \gamma) = \mathcal{B}(x', \gamma)$ and $\mathcal{B}(x, -\gamma) = \mathcal{B}(x', -\gamma)$, and so $x \in W_\gamma \iff x' \in W_\gamma$. Therefore W_γ is closed. Additionally,

$$x \in W_\gamma \implies \bar{0} \in \mathcal{B}(x, -\gamma) \iff x \in \mathcal{B}(\bar{0}, -\gamma).$$

Therefore W_γ is bounded. By Lemma 2.5, W_γ is definably compact.

If $\gamma' \geq \gamma$, then $\mathcal{B}(x, \gamma') \subseteq \mathcal{B}(x, \gamma)$ and $\mathcal{B}(x, -\gamma') \supseteq \mathcal{B}(x, -\gamma)$. Therefore

$$x \in W_\gamma \implies x \in W_{\gamma'},$$

and the family $\{W_\gamma\}$ is monotone.

Lastly, if $x \in U$, then for sufficiently large γ , we have $\mathcal{B}(x, \gamma) \subseteq U$, because U is open. Also, $\bar{0} \in \mathcal{B}(x, -\gamma)$ for sufficiently large γ . Thus $x \in W_\gamma$ for all sufficiently large γ . This shows $U = \bigcup_\gamma W_\gamma$. \dashv

An n -dimensional definable manifold over M is a Hausdorff definable topological space X with a covering by finitely many open subsets U_1, \dots, U_m , and a definable homeomorphism from U_i to an open set $V_i \subseteq M^n$ for each i .

PROPOSITION 2.8. *Let X be a definable manifold in M . Then X has a Γ -exhaustion.*

PROOF. Cover X with finitely many open sets U_i homeomorphic to open subsets of M^n . For each i , let $\{W_{i,\gamma}\}_{\gamma \in \Gamma_M}$ be a Γ -exhaustion of U_i . Let $V_\gamma = \bigcup_i W_{i,\gamma}$. Then the family $\{V_\gamma\}$ is a Γ -exhaustion of X . \dashv

DEFINITION 2.9. Let X be a definable manifold. An arbitrary subset $Y \subseteq X$ is *bounded* if $Y \subseteq D$ for some definably compact subset $D \subseteq X$.

Proposition 2.10(1) gives a more concrete definition of “bounded” in terms of Γ -exhaustions.

PROPOSITION 2.10. *Let X be a definable manifold and $Y \subseteq X$ be an arbitrary subset.*

1. *Let $\{W_\gamma\}$ be a Γ -exhaustion of X . Then Y is bounded if and only if there is $\gamma \in \Gamma$ such that $Y \subseteq W_\gamma$.*
2. *Suppose Y is definable. Then Y is definably compact if and only if Y is closed and bounded.*
3. *Suppose Y is definable. Then Y is bounded if and only if the closure \bar{Y} is definably compact.*

PROOF.

1. If $Y \subseteq W_\gamma$, then Y is contained in the definably compact set W_γ . Conversely, suppose Y is bounded, witnessed by a definably compact set $Z \subseteq X$ with $Y \subseteq Z$. The filtered intersection

$$\bigcap_\gamma (Z \setminus W_\gamma)$$

is empty, so there is some γ such that $W_\gamma \supseteq Z \supseteq Y$.

2. If Y is definably compact, then Y is closed (Fact 2.2(4)), and Y is bounded because $Y \subseteq \bar{Y}$. Conversely, suppose that Y is closed and bounded. Then Y is a definable closed subset of a definably compact set, so Y is definably compact by Fact 2.2(5).
3. If \bar{Y} is definably compact, then Y is bounded because $Y \subseteq \bar{Y}$. Conversely, suppose that Y is bounded. Then $Y \subseteq Z$ for some definably compact set $Z \subseteq X$. The closure \bar{Y} is a definable closed subset of Z , so \bar{Y} is definably compact by Fact 2.2(5). ⊢

REMARK 2.11. Definable compactness is a definable property: Let X_t be a definable manifold depending definably on some parameter $t \in T$. Then

$$\{t \in T : X_t \text{ is definably compact}\}$$

is definable. This can be proved from Proposition 2.10(1,2) by compactness, using Remark 2.3 to reduce to the case where M is highly saturated.

REMARK 2.12. When $M = \mathbb{Q}_p$, a definable manifold X is definably compact if and only if it is compact. One direction is Fact 2.2(1). Conversely, suppose X is definably compact. Cover X by definable open subsets U_1, \dots, U_n , each homeomorphic to an open subset of M^n . As in the proof of Proposition 2.8, let $\{W_{i,\gamma}\}_{\gamma \in \mathbb{Z}}$ be a Γ -exhaustion of U_i , and let $V_\gamma = \bigcup_{i=1}^n W_{i,\gamma}$, so that $\{V_\gamma\}_{\gamma \in \mathbb{Z}}$ is a Γ -exhaustion of X . By Proposition 2.10, there is some $\gamma \in \mathbb{Z}$ such that $X = V_\gamma$. Then $X = \bigcup_{i=1}^n W_{i,\gamma}$, where each $W_{i,\gamma}$ is definably compact. Lemmas 2.4 and 2.5 imply that definable compactness is equivalent to compactness for definable subsets of M^n . Therefore each $W_{i,\gamma}$ is compact. As X is covered by finitely many compact sets, X itself is compact.

We now try to relate our notion of definable compactness to the more familiar notions appearing in [13].

DEFINITION 2.13. Let X be a definable manifold. Let D be a definable subset of $M \setminus \{0\}$ with $0 \in \bar{D}$. Let $f : D \rightarrow X$ be a definable function. Then $a \in X$ is a *cluster point* of f if $(0, a)$ is in the closure of the graph of f . In other words, for every neighborhood U_1 of 0 and every neighborhood U_2 of a , there is $x \in U_1 \cap D$ such that $f(x) \in U_2$.

LEMMA 2.14. *Let X be a definable manifold. Let $f : R(M) \setminus \{0\} \rightarrow X$ be a definable function. Then f is continuous at all but finitely many points of $R(M)$.*

PROOF. An exercise using the fact that any definable function $M \rightarrow M^n$ is continuous off a finite set. ⊢

LEMMA 2.15. *Let X be a definable manifold. Let Y be a definable subset. The following are equivalent :*

1. Y is definably compact.
2. If D is a definable subset of $M \setminus \{0\}$ with $0 \in \bar{D}$, then every definable function $f : D \rightarrow Y$ has a cluster point.
3. Any definable continuous function $f : R(M) \setminus \{0\} \rightarrow Y$ has a cluster point in Y .
4. Any definable continuous function $f : \mathcal{B}(0, \gamma) \setminus \{0\} \rightarrow Y$ has a cluster point in Y .
5. Let $\{Z_\gamma\}_{\gamma \in \Gamma_M}$ be a definable family of non-empty closed subsets of Y , such that $\gamma \leq \gamma' \implies Z_\gamma \supseteq Z_{\gamma'}$. Then $\bigcap_{\gamma \in \Gamma_M} Z_\gamma \neq \emptyset$.

PROOF. (1) \Rightarrow (2): The set of cluster points is the intersection

$$\bigcap_{\gamma \in \Gamma_M} \overline{f(\mathcal{B}(0, \gamma) \cap D)}.$$

This is non-empty by definable compactness of Y .

(2) \Rightarrow (3) is trivial, and (3) \Rightarrow (4) follows by rescaling.

(4) \Rightarrow (5): By definable Skolem functions, there is some definable function $f : M \setminus \{0\} \rightarrow Y$ such that $f(x) \in Z_{v(x)}$ for all $x \in M \setminus \{0\}$. By Lemma 2.14, there is some $\delta \in \Gamma_M$ such that f is continuous on $\mathcal{B}(0, \delta) \setminus \{0\}$. By (4), f has a cluster point $a \in Y$. Then $a \in \bigcap_{\gamma} Z_{\gamma}$. Otherwise, take γ large enough that $a \notin Z_{\gamma}$. Because a is a cluster point and Z_{γ} is closed in Y , there is some $x \neq 0$ such that $v(x) \geq \gamma$ and $f(x) \notin Z_{\gamma}$. By choice of f , $f(x) \in Z_{v(x)} \subseteq Z_{\gamma}$, a contradiction.

(5) \Rightarrow (1): We first claim that Y is closed. Take $p \in \overline{Y}$. Because X is a definable manifold, we can identify a neighborhood of p in X with the closed ball $R(M)^n$ in M^n . For $\gamma \geq 0$, let B_{γ} be the closed ball of radius γ around p . For $\gamma \leq 0$ let $B_{\gamma} = B_0$. Then $B_{\gamma} \cap Y$ is a non-empty closed subset of Y for any γ , because $p \in \overline{Y}$. By (4), the intersection $\bigcap_{\gamma} (B_{\gamma} \cap Y)$ is non-empty, and so $p \in Y$. Therefore Y is closed.

Similarly, Y is bounded. Take a Γ -exhaustion $\{U_{\gamma}\}_{\gamma \in \Gamma_M}$ of the definable manifold X . If Y is unbounded, then $Y \setminus U_{\gamma}$ is a closed non-empty subset of Y for each γ . Applying (5) to the family of sets $Y \setminus U_{\gamma}$, we see that $Y \not\subseteq \bigcup_{\gamma} U_{\gamma} = X$, a contradiction. Therefore Y is closed and bounded. By Proposition 2.10(2), Y is definably compact. ⊖

Therefore, we could alternatively define definable compactness as follows:

PROPOSITION 2.16. *Let Y be a definable subset of a definable manifold X . Then Y is definably compact if and only if every definable continuous function $f : R(M) \setminus \{0\} \rightarrow Y$ has a cluster point.*

This is essentially the definition of “definable compactness” appearing in [13] (with the mistake fixed).

2.3. Definable compactness and definable one-dimensional types. Suppose that $N \succ M$. Let X be a definable manifold in M .

DEFINITION 2.17. For $a \in X(M)$ and $b \in X(N)$, say that a and b are *infinitesimally close* over M if b is contained in every M -definable neighborhood of a .

Suppose that X, Y are M -definable manifolds and $f : X \rightarrow Y$ is an M -definable continuous function. If $a \in X(M)$ is infinitesimally close to $b \in X(N)$, then $f(a)$ is infinitesimally close to $f(b)$.

DEFINITION 2.18.

- We let $\mathcal{O}_{X(M)}(N)$ denote the set of $b \in X(N)$ such that b is infinitesimally close to at least one $a \in X(M)$.
- There is a function $\text{st}_M^N : \mathcal{O}_{X(M)}(N) \rightarrow X(M)$ sending each b to the unique $a \in X(M)$ such that b and a are infinitesimally close. This is well defined because X is Hausdorff.

The map st_M^N is the “standard part” map from $\mathcal{O}_{X(M)}(N)$ to $X(M)$.

DEFINITION 2.19. If p is a complete X -type over M , we say that p specializes to $a \in X(M)$ if $p(x) \vdash x \in U$ for every M -definable neighborhood $U \ni a$.

If $b \in X(N)$ is a realization of p , then p specializes to a if and only if $\text{st}_M^N(b) = a$.

FACT 2.20. If $a' \in N \setminus M$ is infinitesimally close to $a \in M$ over M , then there is a coset $C \subseteq N \setminus \{0\}$ of $\bigcap_{n \geq 1} P_n(N)$ such that $\text{tp}(a'/M)$ is determined by the partial type

$$\{v(x - a) > \gamma \mid \gamma \in \Gamma_M\} \cup \{x - a \in C\},$$

and $\text{tp}(a'/M)$ is definable over M .

This follows by a similar argument to Lemma 2.1 in [14].

LEMMA 2.21. Let \mathcal{C} be a definable (i.e., interpretable) family of balls $B \subseteq M$. Suppose the following conditions hold :

1. \mathcal{C} is non-empty.
2. \mathcal{C} is a chain : it is linearly ordered by \subseteq .
3. \mathcal{C} is upwards-closed : if $B \supseteq B' \in \mathcal{C}$ for balls B, B' , then $B \in \mathcal{C}$.
4. \mathcal{C} has no minimal element.

Then there is $d \in M$ such that \mathcal{C} is the set of balls containing d .

PROOF. We may assume $M = \mathbb{Q}_p$, in which case the lemma is an easy exercise using spherical completeness of \mathbb{Q}_p . ⊢

LEMMA 2.22. Let X be an M -definable set, and p be a one-dimensional definable type over M in X . Then there is an elementary extension $N \succ M$ and elements $a \in M$, $b \in X(M)$, such that a is infinitesimally close to 0, $b \in \text{dcl}(Ma)$, and $p = \text{tp}(b/M)$.

PROOF. Take $N \succ M$ containing a realization b of p . Because p is one-dimensional, there is some singleton $c \in N$ such that $\text{dcl}(Mb) = \text{dcl}(Mc)$. (In fact, we can take c to be a coordinate of the tuple b .) Replacing c with $1/c$ if necessary, we may assume that $v(c) \geq 0$. Then $\text{tp}(c/M)$ is definable and one-dimensional. Let \mathcal{C} be the family of M -definable balls which contain c . Then \mathcal{C} is definable, because $\text{tp}(c/M)$ is definable. Moreover, \mathcal{C} satisfies the four conditions of Lemma 2.21:

1. \mathcal{C} is non-empty, because it contains the ball $R(M)$ of radius 0.
2. \mathcal{C} is a chain, because any two balls which intersect are comparable, and \mathcal{C} cannot contain two disjoint balls.
3. \mathcal{C} is upwards-closed, trivially.
4. \mathcal{C} has no least element. Otherwise, if B were the smallest M -definable ball containing c , then we could write B as a disjoint union of smaller balls $B = B_1 \cup \dots \cup B_p$, and one of the B_i would belong to \mathcal{C} .

By Lemma 2.21, \mathcal{C} is the class of balls around some point d . So there is some $d \in M$ such that c is contained in every M -definable ball around d . Therefore, c is infinitesimally close to d over M . Take $a = c - d$. ⊢

LEMMA 2.23. Let X be a definable manifold over M . Let Y be a definably compact definable subset of X . Let p be a definable one-dimensional complete Y -type over M . Then p specializes to a point in Y .

PROOF. Let N be an \aleph_1 -saturated elementary extension of M , and let \mathbb{M} be a monster model extending N . Let p^N be the heir of p over N . We first show that p^N specializes to a point in $Y(N)$. Take $c \in Y(\mathbb{M})$ realizing p^N . By Lemma 2.22, we can write c as $g(a)$ for some N -definable function $g : \mathbb{M} \rightarrow Y(\mathbb{M})$ and some $a \in \mathbb{M}$ infinitesimally close to 0 over N . Because N is \aleph_1 -saturated, there is some $u \in N$ such that $a/u \in P_n(\mathbb{M})$ for all n . Replacing a with a/u , we may assume that $a \in P_n(\mathbb{M})$ for all n . For each n , let $S_n \subseteq Y(N)$ be the definable set of cluster points of $g \upharpoonright P_n(N)$. Each S_n is closed, and non-empty by Lemma 2.15(2). The intersection $\bigcap_n S_n$ is filtered, and therefore non-empty by \aleph_1 -saturation. Take $b \in \bigcap_n S_n$. Let $\Sigma(x)$ be the partial type saying that x is infinitesimally close to 0, $g(x)$ is infinitesimally close to b , and $x \in P_n$ for all n . Then $\Sigma(x)$ is finitely satisfiable, by choice of b . Take $a' \in \mathbb{M}$ realizing $\Sigma(x)$. By Fact 2.20, $\text{tp}(a'/N) = \text{tp}(a/N)$. Therefore a satisfies $\Sigma(x)$, and so $g(a)$ is infinitesimally close to b . It follows that $p^N(x)$ specializes to b .

Let Z be the set of $b \in Y(N)$ such that p^N specializes to b . The set Z is M -definable, because p^N is definable over M . The above argument shows $|Z| > 0$. On the other hand, $|Z| \leq 1$ because $Y(N)$ is Hausdorff. Therefore Z is a singleton $\{b\}$, and the element b lies in $Y(M)$. Then p specializes to b . \dashv

PROPOSITION 2.24. *Work in a model M . Let X be a definable manifold and Y be a definable subset. Then Y is definably compact if and only if every one-dimensional definable Y -type specializes to a point of Y .*

PROOF. One direction is Lemma 2.23. Conversely, suppose every one-dimensional definable type in Y specializes to a point. We claim that Y is definably compact. We use criterion (3) of Lemma 2.15. Let $f : R(M) \setminus \{0\} \rightarrow Y$ be a definable continuous function. Take a monster model $\mathbb{M} \succ M$ and a non-zero $a \in \mathbb{M}$ infinitesimally close to 0 over M . Let $b = f(a)$. By Fact 2.20, $\text{tp}(a/M)$ is definable. Therefore $\text{tp}(b/M)$ is one-dimensional and definable. Then $\text{tp}(b/M)$ specializes to a point $c \in Y(M)$. We claim that c is a cluster point of f . For any M -definable neighborhoods $U_1 \ni 0$ and $U_2 \ni c$, we have $(a, f(a)) = (a, b) \in U_1 \times U_2$. As $M \prec \mathbb{M}$, there must be some $(a', f(a')) \in U_1(M) \times U_2(M)$. This shows that c is a cluster point of f . \dashv

LEMMA 2.25. *Let X be an M -definable manifold and $\{O_t\}_{t \in \Gamma_M}$ be a Γ -exhaustion. Let p be a definable one-dimensional type in X over M , such that p does not concentrate on O_t for any $t \in \Gamma_M$. Suppose $\mathbb{M} \succ N \succ M$. Suppose that $b \in X(\mathbb{M})$ realizes p , and $b \notin O_t(\mathbb{M})$ for any $t \in \Gamma_N$. Then b realizes p^N , the heir of p over N .*

PROOF. By Lemma 2.22, we have $b = f(a)$ for some M -definable function $f : \mathbb{M} \rightarrow X$ and some $a \in \mathbb{M}$ infinitesimally close to 0 over M . By Lemma 2.14, f is continuous on $\mathcal{B}(0, \gamma_0)$ for some sufficiently large $\gamma_0 \in \Gamma_M$; note that $v(a) > \gamma_0$. We claim that a is infinitesimally close to 0 over N . Otherwise, there is some $\gamma \in \Gamma_N$ such that $v(a) < \gamma$. Let A be the definable set of $x \in \mathbb{M}$ such that $\gamma_0 < v(x) < \gamma$; note that $a \in A$. The set A is definably compact and N -definable. Also, f is N -definable and continuous on A . Therefore, the image $f(A)$ is N -definable, and definably compact. By Proposition 2.10, there is some $t \in \Gamma_N$ such that $f(A) \subseteq O_t$. Then $b = f(a) \in f(A) \subseteq O_t(\mathbb{M})$, contradicting the assumptions.

This shows that a is infinitesimally close to 0 over N . By Fact 2.20, $\text{tp}(a/N)$ is the heir of $\text{tp}(a/M)$, implying that $\text{tp}(b/N) = \text{tp}(f(a)/N)$ is the heir of $\text{tp}(f(a)/M) = p$. \dashv

2.4. Definable groups in pCF. By a *definable group* over M , we mean a definable set with a definable group operation. By [18], any group G definable in M admits a unique definable manifold structure making the group operations be continuous.

REMARK 2.26. In particular, there is a canonical notion of “definable compactness” for abstract definable groups and their definable subsets. As in Remarks 2.3 and 2.11, one can show that these notions are definable in families and invariant in elementary extensions.

DEFINITION 2.27. A *good neighborhood basis* is a definable neighborhood basis of the form $\{O_t : t \in \Gamma_M\}$ which is also a Γ -exhaustion, and such that $O_t = O_t^{-1}$ for each $t \in \Gamma_M$.

PROPOSITION 2.28. *Every definable group has a good neighborhood basis.*

PROOF. By Proposition 2.8, the group G admits a Γ -exhaustion $\{W_t : t \in \Gamma_M\}$. Replacing $\{W_t\}$ with $\{W_{t+\gamma}\}$, we may assume that W_0 is non-empty. Replacing $\{W_t\}$ with $\{a \cdot W_t\}$, we may assume that $\text{id}_G \in W_0$.

Because G is a definable manifold, there is some definable neighborhood basis $\{N_t : t \in \Gamma_M, t < 0\}$ such that each N_t is clopen, and N_t depends monotonically on t . Define

$$B_t = \begin{cases} W_t & t \geq 0, \\ W_0 \cap N_t & t < 0. \end{cases}$$

Then $\{B_t : t \in \Gamma_M\}$ is a definable neighborhood basis and a Γ -exhaustion. Lastly, define $O_t = B_t \cap B_t^{-1}$. Then $\{O_t : t \in \Gamma_M\}$ has all the desired properties. \dashv

PROPOSITION 2.29. *Let $\{O_t : t \in \Gamma_M\}$ be a good neighborhood basis of a definable group G .*

1. For any $t \in \Gamma_M$, there is $t' \in \Gamma_M$ such that $O_{t'} \cdot O_{t'} \subseteq O_t$.
2. For any $t \in \Gamma_M$, there is $t'' \in \Gamma_M$ such that $O_t \cdot O_t \subseteq O_{t''}$.

PROOF. (1) is by continuity. For (2), note that the set $O_t \cdot O_t$ is an image of the definably compact space $O_t \times O_t$ under the definable continuous map $(x, y) \mapsto x \cdot y$. Therefore $O_t \cdot O_t$ is definably compact. Then t'' exists by Proposition 2.10. \dashv

LEMMA 2.30. *Let $\{O_t : t \in \Gamma_M\}$ be a good neighborhood basis of a definable group G . For every $t, \varepsilon \in \Gamma_M$, there is $\delta \in \Gamma_M$ such that if $a \in O_\delta$ and $b \in O_t$, then $b^{-1}ab \in O_\varepsilon$.*

PROOF. Define $S_\delta = \{(a, b) \in O_\delta \times O_t : b^{-1}ab \notin O_\varepsilon\}$. Suppose for the sake of contradiction that $S_\delta \neq \emptyset$ for all δ . The family S_δ is definable, and depends monotonically on δ . Each set S_δ is closed, because O_ε, O_δ , and O_t are clopen. By definable compactness of $O_\delta \times O_t$, the intersection $\bigcap_\delta S_\delta$ is non-empty. Therefore there are $a, b \in G$ such that

1. $a \in O_\delta$ for all δ .
2. $b \in O_t$.
3. $b^{-1}ab \notin O_\varepsilon$.

The first point implies $a = \text{id}_G$, which then implies $b^{-1}ab = \text{id}_G \in O_\varepsilon$, a contradiction. \dashv

§3. Review of dp-rank. In Section 4 we will make extensive use of dp-rank, so we review its basic properties here.

DEFINITION 3.1. Let κ be a cardinal and $\Sigma(x)$ be a partial type. An *ict-pattern* of depth κ in $\Sigma(x)$ consists of

- a family of formulas $\{\phi_\alpha(x, y_\alpha)\}_{\alpha < \kappa}$,
- an array of parameters $\{b_{\alpha,i}\}_{\alpha < \kappa, i < \omega}$ with $|b_{\alpha,i}| = |y_\alpha|$,

such that for any function $\eta : \kappa \rightarrow \omega$, the following type is consistent:

$$\Sigma(x) \cup \{\phi_\alpha(x, b_{\alpha,i}) : \alpha < \kappa, i = \eta(\alpha)\} \cup \{\neg\phi_\alpha(x, b_{\alpha,i}) : \alpha < \kappa, i \neq \eta(\alpha)\}.$$

DEFINITION 3.2. The *dp-rank* of a partial type $\Sigma(x)$ is the supremum of cardinals κ such that, in some elementary extension $N \supseteq M$, there is an ict-pattern of depth κ in $\Sigma(x)$. When there is no supremum, the dp-rank is defined to be ∞ , a formal symbol greater than all cardinals.

We write the dp-rank of $\Sigma(x)$ as $\text{dp-rk}(\Sigma)$. When $\Sigma(x)$ is a complete type $\text{tp}(b/A)$, we write the dp-rank as $\text{dp-rk}(b/A)$.

The following facts can be found in [10], or alternatively [22, Chapter 4].

FACT 3.3. *The following are equivalent in a structure M :*

1. M is NIP.
2. $\text{dp-rk}(x = x) < \infty$.
3. Every partial type has dp-rank $< \infty$.

FACT 3.4. *If $\Sigma(x)$ is a partial type over A , and if the ambient model M is $|A|^+$ -saturated, then $\text{dp-rk}(\Sigma)$ is the supremum of $\text{dp-rk}(b/A)$ as b ranges over realizations of $\Sigma(x)$.*

FACT 3.5. *If $b \in \text{acl}(A)$, then $\text{dp-rk}(b/A) = 0$. If $b \notin \text{acl}(A)$, then $\text{dp-rk}(b/A) > 0$.*

FACT 3.6. *For any b, c, A , we have*

$$\text{dp-rk}(b/A) \leq \text{dp-rk}(bc/A) \leq \text{dp-rk}(b/cA) + \text{dp-rk}(c/A).$$

It is also helpful to view dp-rank as a property of definable sets:

DEFINITION 3.7. If D is a definable set, then the dp-rank of D , written $\text{dp-rk}(D)$, is $\text{dp-rk}(\phi(x))$ for any formula $\phi(x)$ defining D .

The following facts are easy exercises using Facts 3.3–3.6.

FACT 3.8. $\text{dp-rk}(D) > 0$ if and only if D is infinite.

FACT 3.9. *If D_1, D_2 are definable sets, then $\text{dp-rk}(D_1 \times D_2) = \text{dp-rk}(D_1) + \text{dp-rk}(D_2)$.*

FACT 3.10. *If $f : D_1 \rightarrow D_2$ is a definable injection, then $\text{dp-rk}(D_1) \leq \text{dp-rk}(D_2)$. If $f : D_1 \rightarrow D_2$ is a definable surjection, then $\text{dp-rk}(D_1) \geq \text{dp-rk}(D_2)$.*

We will need the following about dp-rank in p -adically closed fields:

FACT 3.11 ([2, Theorem 6.6]). *If M is a p -adically closed field, then $\text{dp-rk}(M) = 1$.*

COROLLARY 3.12. *If M is a p -adically closed field, then every n -type in M has dp -rank at most n .*

In fact, dp -rank in pCF agrees with the natural notion of dimension (topological dimension or acl -dimension), by [22, Exercise 4.38]. We will not need this fact, however.

§4. Large gaps. In order to apply the strategy of Peterzil and Steinhorn, we need a technical statement about “gaps” in unbounded curves:

CONJECTURE 4.1. *Let G be a definable group over a p -adically closed field M , with a good neighborhood basis $\{O_t \mid t \in \Gamma_M\}$. Let I be a one-dimensional unbounded definable subset of G . Then for every $t_0 \in \Gamma_M$, there is $t \in \Gamma_M$ such that*

$$\{g \in I \mid g(O_t \setminus O_{t_0}) \cap I = \emptyset\}$$

is bounded.

The o -minimal analogue of Conjecture 4.1 holds by an easy connectedness argument [16, Lemma 3.8]. But in a p -adically closed field, everything is totally disconnected and we need a completely different approach. In the end, we will prove Conjecture 4.1 only in a special case (Proposition 4.14), namely when G is nearly abelian (Definition 1.1).

REMARK 4.2. It is useful to consider what a counterexample to Conjecture 4.1 would look like. For each $t \gg t_0$, there would be unboundedly many $g \in I$ such that

$$g(O_t \setminus O_{t_0}) \cap I = \emptyset,$$

or equivalently $gO_t \cap I = gO_{t_0} \cap I$. Around g , the set I looks like an “island” $gO_{t_0} \cap I$ surrounded by a very large empty space $g(O_t \setminus O_{t_0})$. Since I is unbounded, there must be infinitely many of these “islands.” Because this holds for *any* t , the gaps between the islands must become greater and greater as we move towards “ ∞ .”

The behavior described above is reminiscent of the behavior of the set $2^{\mathbb{Z}}$ in the group $(\mathbb{R}, +, <)$. The structure $(\mathbb{R}, +, <, 2^{\mathbb{Z}})$ is NIP [5, Theorem 6.5] but it does not have finite dp -rank, and this is a direct consequence of the “large gaps” in $2^{\mathbb{Z}}$. In a non-standard elementary extension, by choosing $a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n$ carefully, one can ensure that the map

$$\prod_{i=1}^n (2^{\mathbb{Z}} \cap [a_i, b_i]) \rightarrow \mathbb{R}$$

$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i$$

is injective and each set $2^{\mathbb{Z}} \cap [a_i, b_i]$ is infinite, showing that the model has dp -rank at least n (for arbitrary finite n).

Our approach for attacking Conjecture 4.1 is based on this line of argument: take a set I with large gaps and obtain infinite dp -rank. Unfortunately, the argument only works in the nearly abelian case (Proposition 4.14), though we can salvage a much weaker statement in the non-abelian case (Proposition 4.15).

4.1. Notation. Let G be a group. If H is a subgroup of G , we let G/H denote the set of left cosets of H . If $A \subseteq G$, we will write A/H to indicate the image of A in G/H . If $A, B \subseteq G$, we let A^B indicate $\{b^{-1}ab : a \in A, b \in B\}$. Notation like “ $X \setminus Y$ ” will always mean set subtraction, rather than quotienting by a group action on the left.

DEFINITION 4.3. Let X, Y be subsets of a group G . Define

$$X \diamond Y = \{g \in X : gY \cap X = \emptyset\}.$$

Note that $X \diamond Y$ depends negatively on Y . We will write “ $A \diamond B \setminus C$ ” to mean “ $A \diamond (B \setminus C)$.”

REMARK 4.4. Suppose X, Y are subgroups of G , $S \subseteq G$, and $a, b \in S \diamond X \setminus Y$. Then

$$aX = bX \implies aY = bY.$$

Otherwise, $b = a\delta$ for some $\delta \in X \setminus Y$, and so $b \in a(X \setminus Y) \cap S$, contradicting the fact that $a(X \setminus Y) \cap S = \emptyset$.

4.2. The bad gap configuration. Recall that an *externally definable* set X in a structure M is a set of the form $Y \cap M^n$ for some elementary extension $N \succ M$ and definable set $Y \subseteq N^n$. The *Shelah expansion* M^{Sh} is the expansion of M by all externally definable sets. When M is NIP, the Shelah expansion M^{Sh} has elimination of quantifiers [22, Proposition 3.23]. Using this, it is easy to see that M^{Sh} has the same dp-rank as M .

REMARK 4.5. Let \mathcal{F} be a collection of definable subsets of M^n . If the sets in \mathcal{F} are uniformly definable, and \mathcal{F} is linearly ordered by inclusion, then the sets $\bigcup \mathcal{F}$ and $\bigcap \mathcal{F}$ are externally definable [6, Kaplan’s Lemma 3.4].

Later, we will use Remark 4.5 in conjunction with Proposition 2.29 to construct externally definable subgroups of definable groups.

DEFINITION 4.6. Let G be a definable group in a structure M . A *bad gap configuration* in G consists of the following:

- a finite subgroup $F \subseteq G$,
- externally definable subgroups,

$$\dots \subseteq Y_2 \subseteq Y_1 \subseteq Y_0 \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq G,$$

- an externally definable subset $I \subseteq G$,

such that the following conditions hold:

- $Y_i^F \subseteq Y_i$, for all i .
- $Y_i^{X_i} \subseteq Y_{i-1}$, for $i > 0$.
- $(X_i \cap (I \diamond X_{i-1} \setminus FY_{i-1}))/X_{i-1}$ is infinite, for $i > 0$.

We say that a bad gap configuration is (A -)definable if all of F , the X_i , Y_i , and I are (A -)definable.

LEMMA 4.7. *If G has finite dp-rank, then there is no bad gap configuration in G .*

PROOF. Let $(F, \{Y_i\}, \{X_i\}, I)$ be a bad gap configuration. Replacing M with the Shelah expansion M^{Sh} , we may assume that the bad gap configuration is definable. Passing to an elementary extension and naming parameters, we may assume that M is \aleph_1 -saturated and the bad gap configuration is \emptyset -definable.

Note that $FY_i = Y_iF$ is a subgroup of G , and that the index of Y_i if FY_i is finite, no more than $|F|$. Let D_i be the definable set $X_i \cap (I \diamond X_{i-1} \setminus FY_{i-1})$. By assumption, D_i/X_{i-1} is infinite.

CLAIM 1. Suppose $a_i, a'_i \in D_i$ for $i = 1, \dots, n$, and suppose

$$Y_n a_n a_{n-1} \cdots a_1 = Y_n a'_n a'_{n-1} \cdots a'_1. \tag{1}$$

Then $a_n FY_{n-1} = a'_n FY_{n-1}$. If moreover $a_n Y_{n-1} = a'_n Y_{n-1}$, then

$$Y_{n-1} a_{n-1} \cdots a_1 = Y_{n-1} a'_{n-1} \cdots a'_1. \tag{2}$$

PROOF. Note that $a_i, a'_i \in X_i$. Equation (1) implies that

$$a'_n a'_{n-1} \cdots a'_1 = \varepsilon a_n a_{n-1} \cdots a_1 = a_n \varepsilon^{a_n} a_{n-1} \cdots a_1, \tag{3}$$

for some $\varepsilon \in Y_n$. Then $\varepsilon^{a_n} \in Y_n^{X_n} \subseteq Y_{n-1} \subseteq X_{n-1}$. For $i < n$, we have $a_i, a'_i \in X_i \subseteq X_{n-1}$. Therefore (3) implies that $a'_n X_{n-1} = a_n X_{n-1}$. Both a'_n and a_n are in $I \diamond X_{n-1} \setminus FY_{n-1}$, so by Remark 4.4 we have $a'_n FY_{n-1} = a_n FY_{n-1}$ as desired. Now suppose that $a_n Y_{n-1} = a'_n Y_{n-1}$. Then $a'_n = a_n \delta$ for some $\delta \in Y_{n-1}$. Then Equation (3) implies

$$\begin{aligned} a_n \varepsilon^{a_n} a_{n-1} \cdots a_1 &= a_n \delta a'_{n-1} a'_{n-2} \cdots a'_1, \\ \varepsilon^{a_n} a_{n-1} \cdots a_1 &= \delta a'_{n-1} a'_{n-2} \cdots a'_1. \end{aligned}$$

Both ε^{a_n} and δ are in Y_{n-1} , so Equation (2) holds. ⊖

For any $n \in \mathbb{N}$, we claim that $\text{dp-rk}(G) \geq n$. By assumption, the interpretable set D_i/X_{i-1} is infinite. The interpretable set D_i/Y_{i-1} is even bigger, because $Y_{i-1} \subseteq X_{i-1}$. By the properties of dp-rank in Section 3, $\prod_{i=1}^n D_i/Y_{i-1}$ has dp-rank at least n . Take a tuple $\bar{b} \in \prod_{i=1}^n D_i/Y_{i-1}$ such that $\text{dp-rk}(\bar{b}/\emptyset) \geq n$. Each b_i is a coset $a_i Y_{i-1}$ for some $a_i \in D_i$. Let $c = a_n a_{n-1} \cdots a_1 \in G$.

CLAIM 2. For each i , we have $b_i \in \text{acl}(c, b_{i+1}, \dots, b_n)$.

PROOF. Let S be the set of $(a'_1, \dots, a'_n) \in \prod_j D_j$ such that

- $a'_n a'_{n-1} \cdots a'_1 = c$.
- $a'_j Y_{j-1} = b_j = a_j Y_{j-1}$, for $j > i$.

Then $(a_1, \dots, a_n) \in S$ and S is definable over c, b_{i+1}, \dots, b_n . If $(a'_1, \dots, a'_n) \in S$, then

$$Y_n a'_n a'_{n-1} \cdots a'_1 = Y_n c = Y_n a_n a_{n-1} \cdots a_1.$$

By Claim 1 applied $(n - i + 1)$ times, we see that $a_i FY_{i-1} = a'_i FY_{i-1}$. We have shown

$$\{a'_i FY_{i-1} : (a'_1, \dots, a'_n) \in S\} = \{a_i FY_{i-1}\}.$$

It follows that $a_i FY_{i-1}$ is definable over c, b_{i+1}, \dots, b_n . The fibers of the map $G/Y_{i-1} \rightarrow G/(FY_{i-1})$ are finite, and so $a_i Y_{i-1} = b_i$ is algebraic over c, b_{i+1}, \dots, b_n . ⊖

By Claim 2 and induction, $\bar{b} \in \text{acl}(c)$. Therefore

$$n \leq \text{dp-rk}(\bar{b}/\emptyset) \leq \text{dp-rk}(c/\emptyset) \leq \text{dp-rk}(G).$$

As n was arbitrary, G has infinite dp-rank, a contradiction. ⊥

4.3. The saturated case. Until Section 4.4, we will work in a monster model $\mathbb{M} \models p\text{CF}$. Fix a definable group G , not definably compact, and fix a good neighborhood basis $\{O_t : t \in \Gamma_{\mathbb{M}}\}$ in the sense of Definition 2.27.

LEMMA 4.8. *There is no bad gap configuration in G .*

PROOF. For definable sets in $p\text{CF}$, dp-rank agrees with dimension. In particular, dp-rank is finite. Therefore Lemma 4.7 applies to G . ⊥

Recall from Definition 2.9 and Proposition 2.10(1) that a subset $S \subseteq G(\mathbb{M})$ is bounded if $S \subseteq O_t$ for some $t \in \Gamma_{\mathbb{M}}$.

LEMMA 4.9. *If $S \subseteq G(\mathbb{M})$ is bounded, then $S \subseteq X$ for some bounded externally definable subgroup $X \subseteq G(\mathbb{M})$.*

PROOF. Take $t_0 \in \Gamma_{\mathbb{M}}$ such that $S \subseteq O_{t_0}$. By Proposition 2.29, we can build an ascending sequence

$$t_0 < t_1 < t_2 < \dots$$

in $\Gamma_{\mathbb{M}}$ such that $O_{t_i} \cdot O_{t_i} \subseteq O_{t_{i+1}}$ for each i . By saturation, we can also find some $t_\omega > t_i$ for all finite i . Set $X = \bigcup_{i < \omega} O_{t_i}$. The set X is externally definable (Remark 4.5). The set X is bounded, because $X \subseteq O_{t_\omega}$. We have $X = X^{-1}$ because $O_{t_i} = O_{t_i}^{-1}$ for each i . Lastly, X is closed under the group operation by choice of the t_i 's. ⊥

LEMMA 4.10. *Let I be an unbounded subset of G . Let $X \subseteq G(\mathbb{M})$ be a bounded subgroup. Then there is an externally definable bounded subgroup $X' \supseteq X$ such that $(X' \cap I)/X$ is infinite.*

PROOF. We claim that I/X is infinite. Otherwise, I is contained in a finite union of cosets: $I \subseteq \bigcup_{i=1}^n a_i X$. Take $t \in \Gamma_{\mathbb{M}}$ such that $X \subseteq O_t$. Then I is a subset of the definably compact set $\bigcup_{i=1}^n a_i O_t$, so I is bounded, a contradiction.

Now take $a_1, a_2, a_3, \dots \in I$ such that the cosets $a_i X$ are pairwise distinct. By saturation, there is some $t \in \Gamma$ such that $\{a_1, a_2, \dots\} \subseteq O_t$. Then $\{a_1, a_2, \dots\}$ and X are bounded. By Lemma 4.9, there is an externally definable bounded subgroup X' containing $\{a_1, a_2, \dots\} \cup X$. Then $(X' \cap I)/X$ is infinite, witnessed by the $a_i X$. ⊥

Recall from Definition 1.1 that G is *nearly abelian* if there is a definably compact definable normal subgroup $K \subseteq G$ with G/K abelian. Equivalently, G is nearly abelian if there is a definably compact subgroup K containing the derived group $[G, G]$.

LEMMA 4.11. *Suppose that G is nearly abelian. Let I be an unbounded definable subset of $G(\mathbb{M})$. For any bounded set A , there is $t \in \Gamma_{\mathbb{M}}$ such that $I \diamond O_t \setminus A$ is bounded.*

PROOF. Suppose not.

CLAIM. *For any bounded sets $C \supseteq B \supseteq A$, the set $I \diamond C \setminus B$ is unbounded.*

PROOF. Take $t \in \Gamma_{\mathbb{M}}$ such that $C \subseteq O_t$. Then $I \diamond C \setminus B$ contains the unbounded set $I \diamond O_t \setminus A$, because $O_t \setminus A \supseteq C \setminus B$. \dashv

Let K be the normal subgroup witnessing near-abelianity. By Lemma 4.9, there is a bounded externally definable subgroup $X_0 \supseteq A \cup K$. By Lemma 4.10 we can recursively build an increasing chain of bounded externally definable subgroups

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

such that

- $(X_1 \cap I)/X_0$ is infinite.
- For $n > 1$, $(X_n \cap (I \diamond X_{n-1} \setminus X_0))/X_{n-1}$ is infinite. This is possible because $I \diamond X_{n-1} \setminus X_0$ is unbounded by the claim.

Let $Y_i = X_0$ for all i , and let $F = \{\text{id}_G\}$. Note X_0 is normal, because it contains K which contains $[G, G]$. We have constructed a bad gap configuration in G , contradicting Lemma 4.8. \dashv

LEMMA 4.12. *If $S \subseteq G(\mathbb{M})$ is a neighborhood of id_G , then $S \supseteq X$ for some externally definable open subgroup $X \subseteq G(\mathbb{M})$. If, in addition, $B \subseteq G(\mathbb{M})$ is a bounded set, then we can choose the group X to ensure $X^B \subseteq X$.*

PROOF. Take $t_0 \in \Gamma_M$ such that $S \supseteq O_{t_0}$. By Proposition 2.29 and Lemma 2.30, there is a descending sequence

$$t_0 > t_1 > t_2 > \dots$$

in $\Gamma_{\mathbb{M}}$ such that $O_{t_{i+1}} \cdot O_{t_{i+1}} \subseteq O_{t_i}$ and also $O_{t_{i+1}}^B \subseteq O_{t_i}$. Take $X = \bigcap_{i=1}^{\infty} O_{t_i}$. Then X is an externally definable subgroup with $X^B \subseteq X$. We can take some t_{ω} less than all the t_i 's, and then $O_{t_{\omega}} \subseteq X$. Therefore X has interior, and is an open subgroup. \dashv

LEMMA 4.13. *Let I be an unbounded definable subset of $G(\mathbb{M})$. Let F be a finite subgroup of $G(\mathbb{M})$. Then there exist $t, t' \in \Gamma_{\mathbb{M}}$ such that $I \diamond O_t \setminus (F \cdot O_{t'})$ is bounded.*

PROOF. Suppose not.

CLAIM. *For any neighborhood $A \ni \text{id}_G$ and any bounded set $B \subseteq G$, the set $I \diamond B \setminus FA$ is unbounded.*

PROOF. Take t, t' such that

$$\begin{aligned} O_{t'} &\subseteq A \text{ and } B \subseteq O_t, \\ O_t \setminus (F \cdot O_{t'}) &\supseteq B \setminus (F \cdot A), \\ I \diamond O_t \setminus (F \cdot O_{t'}) &\subseteq I \diamond B \setminus (F \cdot A). \end{aligned} \quad \dashv$$

Take any bounded open externally definable subgroup $X_0 \subseteq G$. By Lemma 4.12 there is an externally definable open subgroup $Y_0 \subseteq X_0$ such that $Y_0^F \subseteq Y_0$. Recursively build chains

$$\begin{aligned} X_0 &\subseteq X_1 \subseteq \dots, \\ Y_0 &\supseteq Y_1 \supseteq \dots, \end{aligned}$$

where

- X_i is a bounded externally definable subgroup, chosen large enough to ensure that $(X_i \cap (I \diamond X_{i-1} \setminus FY_{i-1}))/X_{i-1}$ is infinite (Lemma 4.10).
- Y_i is an open externally definable subgroup with $Y_i^F = Y_i$, chosen small enough that $Y_i^{X_i} \subseteq Y_{i-1}$ (Lemma 4.12).

This gives a bad gap configuration in G , contradicting Lemma 4.8. ⊖

4.4. The general case.

PROPOSITION 4.14. *Let M be any model of pCF . Let G be a definable non-compact group and $\{O_t : t \in \Gamma_M\}$ be a good neighborhood basis. Suppose that G is nearly abelian. Let I be an unbounded definable set. Then for any $t \in \Gamma_M$, there is $t' \in \Gamma_M$ such that $I \diamond O_{t'} \setminus O_t$ is bounded.*

PROOF. We may replace M with a monster model, and then apply Lemma 4.11. ⊖

PROPOSITION 4.15. *Let M be any model of pCF . Let G be a definable non-compact group. Let I be an unbounded definable set. Let F be a finite subgroup of G . Then for any sufficiently small s and sufficiently large t , the set $I \diamond O_t \setminus (FO_s)$ is bounded.*

PROOF. We may replace M with a monster model, and then apply Lemma 4.13. ⊖

§5. Stabilizers and μ -stabilizers. In this section we review some notation and facts from [15].

5.1. Stabilizers. Let G be a group definable in a structure M .

NOTATION 5.1. (1) *If $\phi(x)$ and $\psi(x)$ are G -formulas then $\phi \cdot \psi$ denotes the G -formula*

$$(\phi \cdot \psi)(x) := \exists u \exists v (\phi(u) \wedge \psi(v) \wedge x = u \cdot v).$$

Thus $(\phi \cdot \psi)(M) = \phi(M) \cdot \psi(M)$.

(2) *More generally, if $q(x)$ and $r(x)$ are partial G -types then $q \cdot r$ denotes the G -type*

$$(q \cdot r)(x) := \{\phi \cdot \psi(x) \mid q(x) \vdash \phi(x), r(x) \vdash \psi(x)\}.$$

Thus $(q \cdot r)(N) = q(N) \cdot r(N)$ for an $|M|^+$ -saturated elementary extension $N \succ M$.

(3) *If $g \in G(M)$ and $\phi(x)$ is a G -formula, then $g \cdot \phi$ denotes the G -formula*

$$(g \cdot \phi)(x) := \exists u (\phi(u) \wedge x = g \cdot u).$$

Thus $(g \cdot \phi)(M) = g \cdot \phi(M)$.

(4) *If $g \in G(M)$ and $p(x)$ is a partial G -type then $g \cdot p$ denotes the G -type*

$$(g \cdot p)(x) := \{\phi \cdot \psi(x) \mid p(x) \vdash \phi(x)\}.$$

Thus $(g \cdot p)(N) = g \cdot p(N)$ for an $|M|^+$ -saturated $N \succ M$.

Note that for partial G -types q_1, q_2, q_3 over M , we have

$$(q_1 \cdot q_2) \cdot q_3 = q_1 \cdot (q_2 \cdot q_3),$$

as $((q_1 \cdot q_2) \cdot q_3)(N) = q_1(N) \cdot q_2(N) \cdot q_3(N) = (q_1 \cdot (q_2 \cdot q_3))(N)$ for $|M|^+$ -saturated $N \succ M$.

DEFINITION 5.2. Given a partial type $\Sigma(x)$ over M , define $\text{stab}(\Sigma)$ to be the stabilizer, i.e.,

$$\text{stab}(\Sigma) := \{g \in G(M) \mid g\Sigma \equiv \Sigma\},$$

where $\Sigma \equiv \Sigma'$ if $\Sigma(x) \vdash \Sigma'(x)$ and $\Sigma'(x) \vdash \Sigma(x)$. Equivalently, $\text{stab}(\Sigma)$ is $\{g \in G(M) \mid g\Sigma(N) = \Sigma(N)\}$ for $|M|^+$ -saturated $N \succeq M$.

DEFINITION 5.3. Given a partial type $\Sigma(x)$ over M and an \mathcal{L} -formula $\phi(x, y)$, we define

$$\text{stab}_\phi(\Sigma) = \bigcap_{b \in M^k} X_{\phi,b},$$

where each $X_{\phi,b}$ is the stabilizer of $\{g \in G(M) \mid \Sigma \vdash (g\phi)(x, b)\}$.

REMARK 5.4. Given $\phi(x; y)$, let $\phi'(x; y, z)$ be the formula $\phi(z \cdot x; y)$. Then G acts on ϕ' -types by left translation, and $\text{stab}_\phi(\Sigma)$ is the stabilizer of the ϕ' -type generated by Σ .

REMARK 5.5. Note that our stab_ϕ is slightly different from the Stab_ϕ considered in [15], which is more like the set $X_{\phi,b}$ appearing in Definition 5.3 above.

The following two facts are easy exercises.

FACT 5.6. $\text{stab}_\phi(\Sigma)$ is a definable subgroup of G if Σ is definable.

FACT 5.7. For every partial type Σ over M ,

$$\text{stab}(\Sigma) = \bigcap_{\phi \in \mathcal{L}} \text{stab}_\phi(\Sigma).$$

In particular, if Σ is definable then $\text{stab}(\Sigma)$ is an intersection of definable subgroups.

Recall the notation Σ^N for the canonical extension of a definable type Σ to an elementary extension $N \succ M$, and the notation $(d_\Sigma x)\phi(x; y)$ for the ϕ -definition of Σ .

LEMMA 5.8. If Σ is definable and $N \succ M$, then $\text{stab}_\phi(\Sigma^N) = \text{stab}_\phi(\Sigma)(N)$, and so

$$\text{stab}(\Sigma^N) = \bigcap_{\phi \in \mathcal{L}} \text{stab}_\phi(\Sigma)(N).$$

PROOF. Indeed, $\text{stab}_\phi(\Sigma)(M)$ is defined by the formula

$$\forall y \forall g : ((d_\Sigma z)\phi(g \cdot z; y)) \leftrightarrow ((d_\Sigma z)\phi(x \cdot g \cdot z; y)),$$

and $\text{stab}_\phi(\Sigma^N)$ is defined by the same formula, because Σ^N and Σ have the same definition schema. ⊣

5.2. μ -types and μ -stabilizers. In this section we assume that G is a Hausdorff topological group definable in M with a uniformly definable basis $\{O_t \mid t \in T\}$ of open neighborhoods of the identity. For each $N \succ M$, the group $G(N)$ is again a topological group and the definable family $\{O_t(N) \mid t \in T(N)\}$ again forms a basis for the open neighborhoods of id_G .

DEFINITION 5.9. The *infinitesimal type* of G , denoted $\mu(x)$, is the partial type consisting of all formulas $x \in U$ with U an M -definable neighborhood of id_G .

Thus, if $N \succeq M$, then $\mu(N)$ is the set of elements of $G(N)$ which are infinitesimally close to id_G :

$$\begin{aligned} \mu(N) &= \bigcap \{U(N) \mid U \text{ is an } M\text{-definable neighborhood of } \text{id}_G\} \\ &= \bigcap_{t \in T(M)} O_t(N). \end{aligned}$$

FACT 5.10 ([15, Corollary 2.5 and Claim 2.15]).

1. If $N \succ M$, then $\mu(N)$ is a subgroup of $G(N)$ normalized by $G(M)$.
2. For any definable $q \in S_G(M)$, the partial type $\mu \cdot q$ is definable.

Partial types of the form $\mu \cdot q$ for $q \in S_G(M)$ are called μ -types. The μ -stabilizer of $q \in S_G(M)$ is the stabilizer of the associated μ -type:

$$\text{stab}^\mu(q) := \text{stab}(\mu \cdot q).$$

Note that if μ is the infinitesimal type of $G = G(M)$, and $N \succeq M$, then the canonical extension μ^N is the infinitesimal type of $G(N)$.

FACT 5.11 ([15, Remark 2.16]). *If p is a definable type over M and $N \succ M$, then the product of the canonical extensions is equal to the canonical extension of the product:*

$$\mu^N \cdot p^N = (\mu \cdot p)^N.$$

REMARK 5.12. Let N be an $|M|^+$ -saturated extension of M , and μ^N and p^N be the canonical extensions of μ and p . Then

$$\text{stab}(\mu^N \cdot p^N) = \text{stab}((\mu \cdot p)^N) = \bigcap_{\phi \in \mathcal{L}} \text{stab}_\phi(\mu \cdot p)(N),$$

by Lemma 5.8 and Fact 5.11.

By Fact 5.10, $\mu(N) \cdot G(M)$ is a subgroup of $G(N)$ as $\mu(N) \subseteq G(N)$ is normalized by $G(M)$. This subgroup is the $\mathcal{O}_{G(M)}(N)$ of Definition 2.18. Because $\mu(N) \cap G(M) = \{\text{id}_G\}$, the group $\mu(N) \cdot G(M)$ is a semidirect product of $\mu(N)$ and $G(M)$, and there is a natural homomorphism

$$\mathcal{O}_{G(M)}(N) = \mu(N) \cdot G(M) \rightarrow G(M).$$

This map is exactly the “standard part” map st_M^N of Definition 2.18. For $Y \subseteq G(N)$, we will write $\text{st}_M^N(Y)$ as a shorthand for $\text{st}_M^N(Y \cap \mathcal{O}_{G(M)}(N))$, following [15].

LEMMA 5.13. *Let $p \in S_G(M)$ be a definable type and let $\beta \in G(\mathbb{M})$ realize p^N . Then*

1. $\text{stab}(\mu^N \cdot p^N) = \text{st}_N^{\mathbb{M}}(p^N(\mathbb{M})\beta^{-1})$;
2. $\text{stab}(\mu^N \cdot p^N) = \bigcap_{\psi \in p^N} \text{st}_N^{\mathbb{M}}(\psi(\mathbb{M})\beta^{-1})$.

PROOF. Clause (1) is by Claim 2.22 in [15].

For (2), we must show

$$\text{st}_N^{\mathbb{M}}(p^N(\mathbb{M})\beta^{-1}) = \bigcap_{\psi \in p^N} \text{st}_N^{\mathbb{M}}(\psi(\mathbb{M})\beta^{-1}).$$

The \subseteq direction is clear. For \supseteq , suppose that $g \in \bigcap_{\psi \in p^N} \text{st}_N^{\mathbb{M}}(\psi(\mathbb{M})\beta^{-1})$. Then for any $\psi \in p^N$, there is $h_\psi \in \psi(\mathbb{M})$ such that $h_\psi \cdot \beta^{-1} \cdot g^{-1}$ satisfies μ^N . By compactness there is $h \in p^N(\mathbb{M})$ such that $h \cdot \beta^{-1} \cdot g^{-1}$ satisfies μ^N . Then $g \in \text{st}_N^{\mathbb{M}}(p^N(\mathbb{M})\beta^{-1})$. ◻

§6. Proof of main theorems. From now on M is a p -adically closed field, $\mathbb{M} \succ M$ is the monster model, $G \subseteq M^n$ denotes a group definable in M , and μ denotes the infinitesimal type of G over M . All formulas and types will be G -formulas and G -types. We assume G is not definably compact. Fix a good neighborhood basis $\{O_t : t \in \Gamma_M\}$ of G .

Fix a one-dimensional definable type $p \in S_G(M)$ which does not specialize to any point of $G(M)$. Such a type p exists by Proposition 2.24. Fix a small $|M|^+$ -saturated model N with $M \prec N \prec \mathbb{M}$. As usual, p^N and μ^N denote the canonical extensions to N . Fix an element $\beta \in G(\mathbb{M})$ realizing p^N .

REMARK 6.1. The types p and p^N are “unbounded” in the following sense:

1. If $t \in \Gamma_M$, then $O_t \not\subseteq p$.
2. If $t \in \Gamma_N$, then $O_t \not\subseteq p^N$.
3. If X is a bounded M -definable subset of $G(M)$, then $X \not\subseteq p$.
4. If X is a bounded N -definable subset of $G(N)$, then $X \not\subseteq p^N$.

Point (1) follows by Proposition 2.24: if $O_t \in p$ then p specializes to a point in $O_t(M)$, because O_t is definably compact. Point (2) then follows because p^N is the heir of p . Points (3) and (4) reduce to (1) and (2), respectively.

LEMMA 6.2. $\text{stab}(\mu^N \cdot p^N) = \bigcap_{\phi \in p} \text{st}_N^{\mathbb{M}}(\phi(N)\beta^{-1})$.

PROOF. By Lemma 5.13, it suffices to show

$$\bigcap_{\phi \in p} \text{st}_N^{\mathbb{M}}(\phi(\mathbb{M})\beta^{-1}) \subseteq \bigcap_{\phi \in p^N} \text{st}_N^{\mathbb{M}}(\phi(\mathbb{M})\beta^{-1}).$$

Suppose g belongs to the left-hand side. In particular, $g \in G(N)$. By a compactness argument similar to Lemma 5.13, we see that $g = \varepsilon b \beta^{-1}$ for some $\varepsilon \in \mu^N(\mathbb{M})$ and $b \in p(\mathbb{M})$. It suffices to show $b \in p^N(\mathbb{M})$. By Lemma 2.25, it suffices to show $b \notin O_t(\mathbb{M})$ for any $t \in \Gamma_N$. Suppose $b \in O_t(\mathbb{M})$. Since $g \in N$, there is some $t' \in \Gamma_N$ such that $g^{-1} \cdot O_0(\mathbb{M}) \cdot O_t(\mathbb{M}) \subseteq O_{t'}(\mathbb{M})$. Then

$$\beta = g^{-1} \varepsilon b \in g^{-1} O_0(\mathbb{M}) O_t(\mathbb{M}) \subseteq O_{t'}(\mathbb{M}),$$

contradicting the fact that $\text{tp}(\beta/N)$ is unbounded. ◻

Note that a similar argument to the proof of Lemma 2.31 of [23] shows the following:

FACT 6.3. Suppose that $b \in \mathbb{M}^k$ and $\text{tp}(b/N)$ is definable. If $Y \subseteq G(\mathbb{M})$ is definable over b then $\text{st}_N^{\mathbb{M}}(Y) \subseteq G(N)$ is definable and

$$\dim(\text{st}_N^{\mathbb{M}}(Y)) \leq \dim(Y).$$

LEMMA 6.4. There is an \mathcal{L} -formula ϕ such that $\dim(\text{stab}_\phi(\mu \cdot p)) \leq 1$.

PROOF. Take $\psi \in p$ such that $\dim(\psi(\mathbb{M})) = 1$. Then $\dim(\text{st}_N^{\mathbb{M}}(\psi(\mathbb{M})\beta^{-1})) \leq 1$ by Fact 6.3. By Remark 5.12 and Lemma 5.13,

$$\bigcap_{\phi \in \mathcal{L}} \text{stab}_\phi(\mu \cdot p)(N) = \text{stab}(\mu^N \cdot p^N) \subseteq \text{st}_N^{\mathbb{M}}(\psi(\mathbb{M})\beta^{-1}).$$

The intersection on the left is directed, and the set on the right is definable, so by $|M|^+$ -saturation of N there is some $\phi \in \mathcal{L}$ such that

$$\text{stab}_\phi(\mu \cdot p)(N) \subseteq \text{st}_N^{\mathbb{M}}(\psi(\mathbb{M})\beta^{-1}).$$

Then $\dim(\text{stab}_\phi(\mu \cdot p)) \leq \dim(\text{st}_N^{\mathbb{M}}(\psi(\mathbb{M})\beta^{-1})) \leq 1$. ⊣

To finish our main result, we now show that each $\text{stab}_\phi(\mu \cdot p)$ is not definably compact.

LEMMA 6.5. Assume G is nearly abelian (Definition 1.1). For any N -definable set I containing β , the set $\text{st}_N^{\mathbb{M}}(I(\mathbb{M})\beta^{-1})$ is unbounded.

PROOF. Suppose $\text{st}_N^{\mathbb{M}}(I(\mathbb{M})\beta^{-1})$ is bounded. Then $\text{st}_N^{\mathbb{M}}(I(\mathbb{M})\beta^{-1}) \subseteq O_t(N)$ for some $t \in \Gamma_N$. By Remark 6.1, I is unbounded. By Proposition 4.14, there is some $t' \in \Gamma_N$ such that the set $I^{-1} \diamond O_{t'} \setminus O_t$ is bounded. Then

$$\beta^{-1} \notin (I^{-1} \diamond O_{t'} \setminus O_t)(\mathbb{M})$$

by Remark 6.1. This means that $\beta^{-1}(O_{t'}(\mathbb{M}) \setminus O_t(\mathbb{M})) \cap I(\mathbb{M})^{-1} \neq \emptyset$. Therefore there is $a \in O_{t'}(\mathbb{M}) \setminus O_t(\mathbb{M})$ such that $a\beta \in I(\mathbb{M})$. By definable Skolem functions, we can take $a \in \text{dcl}(N\beta)$. Note $a \in I(\mathbb{M})\beta^{-1}$. By Lemma 2.23, $\text{st}_N^{\mathbb{M}}(a)$ exists. Because $O_{t'} \setminus O_t$ is closed, we see that $\text{st}_N^{\mathbb{M}}(a) \in O_{t'}(N) \setminus O_t(N)$. This contradicts the fact that

$$\text{st}_N^{\mathbb{M}}(a) \in \text{st}_N^{\mathbb{M}}(I(\mathbb{M})\beta^{-1}) \subseteq O_t(N). \quad \text{⊣}$$

LEMMA 6.6. If G is nearly abelian, then the type-definable group $\text{stab}(\mu^N \cdot p^N) \subseteq G(N)$ is one-dimensional and unbounded.

PROOF. The dimension of $\text{stab}(\mu^N \cdot p^N)$ is at most 1 by Lemma 6.4 and Remark 5.12. If $\text{stab}(\mu^N \cdot p^N)$ is bounded, then $\text{stab}(\mu^N \cdot p^N) \subseteq O_t(N)$ for some $t \in \Gamma_N$. By Lemma 6.2, we have

$$\bigcap_{\psi \in p} \text{st}_N^{\mathbb{M}}(\psi(N)\beta^{-1}) = \text{stab}(\mu^N \cdot p^N) \subseteq O_t(N).$$

The intersection on the left is a filtered intersection of definable sets. There are at most $|M|$ sets in the intersection, and N is $|M|^+$ -saturated. Therefore there is some $\psi \in p$ such that $\text{st}_N^{\mathbb{M}}(\psi(N)\beta^{-1}) \subseteq O_t(N)$, contradicting Lemma 6.5. Therefore $\text{stab}(\mu^N \cdot p^N)$ is unbounded. In particular, it is infinite, so it has dimension at least 1. ⊣

LEMMA 6.7. *Suppose G is nearly abelian. Then there is $\phi \in \mathcal{L}$ such that the M -definable group $\text{stab}_\phi(\mu \cdot p)$ is not definably compact and has dimension 1.*

PROOF. By Lemma 6.4 there is an \mathcal{L} -formula ϕ such that $\dim(\text{stab}_\phi(\mu \cdot p)) \leq 1$. By Remark 5.12, we have

$$\text{stab}_\phi(\mu \cdot p)(N) \supseteq \text{stab}(\mu^N \cdot p^N),$$

and therefore $\text{stab}_\phi(\mu \cdot p)$ is unbounded by Lemma 6.6. In particular, $\text{stab}_\phi(\mu \cdot p)$ is infinite, and $\dim(\text{stab}_\phi(\mu \cdot p)) \geq 1$. ⊣

THEOREM 6.8. *Let G be a definable group in a p -adically closed field M . Suppose G is nearly abelian, and not definably compact.*

1. *G has a one-dimensional definable subgroup which is not definably compact.*
2. *If $p \in S_G(M)$ is a definable unbounded one-dimensional type, then there is $\phi \in \mathcal{L}$ such that $\text{stab}_\phi(\mu \cdot p)$ is a one-dimensional definable subgroup of G which is not definably compact.*
3. *Suppose in addition that M is \aleph_1 -saturated. If $p \in S_G(M)$ is a definable unbounded one-dimensional type, then $\text{stab}^\mu(p)$ is a one-dimensional type-definable subgroup of G which is unbounded.*

PROOF. Part (2) is Lemma 6.7. Part (1) then follows because there is at least one unbounded one-dimensional definable type by Proposition 2.24. For Part (3), take a countable model $M_0 \preceq M$ such that G and p are M_0 -definable. Then apply Lemma 6.6 to M and M_0 in place of N and M (respectively), to see that $\text{stab}(\mu \cdot p)$ is one-dimensional and unbounded. Type-definability is by Fact 5.7. ⊣

Recall from [19] that in an NIP context, a global type $p \in S_G(\mathbb{M})$ is said to be a definable f -generic, abbreviated as *dfg*, if there is a small submodel M_0 such that every left G -translate of p is definable over M_0 . In [19], Pillay and the second author showed that:

FACT 6.9. *A group G definable over \mathbb{Q}_p has *dfg* iff there is a normal sequence of definable subgroups*

$$G_0 \triangleleft \cdots \triangleleft G_i \triangleleft G_{i+1} \cdots \triangleleft G_n$$

such that G_0 is finite, G_n is a finite index subgroup of G , and each G_{i+1}/G_i is definably isomorphic to either the additive group \mathbb{G}_a , or a finite index subgroup of the multiplicative group \mathbb{G}_m .

The intuition is that “*dfg*” means “totally non-compact” in the p -adic context.

LEMMA 6.10. *Let G be a one-dimensional definable group. If G is not definably compact, then G has *dfg*.*

PROOF. Recall from [7, Section 4] that G has *finitely satisfiable generics* (*fsg*) if there is a small model M_0 and a global type $p(x)$ in G such that every left G -translate of p is finitely satisfiable in M_0 . By [19, Lemma 2.9], G has *fsg* or *dfg*.¹ It suffices to show that G does not have *fsg*. Suppose otherwise, witnessed by p

¹In [19] this is stated only for groups definable over \mathbb{Q}_p . The assumption is used in order to apply [20, Theorem 2.4]. However, [20, Remark 2.5] shows that this assumption is unnecessary.

and M_0 . Recall that a definable set $X \subseteq G$ is *generic* if finitely many left translates cover G . By [7, Proposition 4.2], the complement of a non-generic set is generic, and every generic set intersects $G(M_0)$. Let $\{W_\gamma\}_\gamma$ be a Γ -exhaustion of G . By taking γ sufficiently large, we can arrange for $G(M_0) \subseteq W_\gamma$, because M_0 is small. Then $G \setminus W_\gamma$ does not intersect $G(M_0)$, so it is not generic. Therefore the complement W_γ is generic. A finite union of left translates of W_γ will be definably compact, so it cannot be all of G . We conclude that G does not have *fsg*, and instead has *dfg*. \dashv

The following is then a corollary of Theorem 6.8.

COROLLARY 6.11. *Let $G \subseteq M^k$ be a nearly abelian definable group. If G is not definably compact, then G has a one-dimensional *dfg* subgroup.*

Next, we consider the general non-abelian case. Let $G, M, N, \mathbb{M}, p, \beta$ be as in the start of this section.

LEMMA 6.12. *For any N -definable set I containing β , and any finite N -definable subgroup $F \subseteq G$, the set $\text{st}_N^{\mathbb{M}}(I(\mathbb{M})\beta^{-1})$ contains a point outside of F .*

PROOF. As in Lemma 6.5, I is unbounded. Let J be the unbounded set I^{-1} . By Proposition 4.15, there are $s, t \in \Gamma_N$ such that $J \diamond O_t \setminus FO_s$ is bounded. Then $\beta^{-1} \notin J \diamond O_t \setminus FO_s$. Therefore

$$\beta^{-1}(O_t \setminus FO_s) \cap J \neq \emptyset,$$

or equivalently

$$(O_t \setminus O_s F)\beta \cap I \neq \emptyset.$$

Therefore there is $a \in O_t(\mathbb{M}) \setminus O_s(\mathbb{M})F(\mathbb{M})$ such that $a\beta \in I(\mathbb{M})$. By definable Skolem functions, we can take $a \in \text{dcl}(N\beta)$. Note $a \in I(\mathbb{M})\beta^{-1}$. Because $O_t \setminus O_s F$ is definably compact, Lemma 2.23 implies that $\text{st}_N^{\mathbb{M}}(a)$ exists and is in $O_t \setminus O_s F$. Then $\text{st}_N^{\mathbb{M}}(a)$ is not in F , since $F \subseteq O_s F$. \dashv

LEMMA 6.13. *The group $\text{stab}(\mu^N \cdot p^N)$ is one-dimensional. In particular, it is infinite.*

PROOF. As in Lemma 6.6, the dimension is at most 1. If $\dim(\text{stab}(\mu^N \cdot p^N)) = 0$, then $\text{stab}(\mu^N \cdot p^N)$ is a finite subgroup F . By Lemma 6.2, we have

$$\bigcap_{\phi \in p} \text{st}_N^{\mathbb{M}}(\phi(N)\beta^{-1}) = \text{stab}(\mu^N \cdot p^N) = F.$$

As in Lemma 6.6, there is some $\phi \in p$ such that $\text{st}_N^{\mathbb{M}}(\phi(N)\beta^{-1}) \subseteq F$. This contradicts Lemma 6.12 \dashv

By Lemma 6.4 and Remark 5.12, we obtain the following conclusion.

LEMMA 6.14. *There is $\phi \in \mathcal{L}$ such that the M -definable group $\text{stab}_\phi(\mu \cdot p)$ has dimension 1.*

We summarize the non-abelian case in the following theorem.

THEOREM 6.15. *Let G be a definable group in a p -adically closed field M . Suppose G is not definably compact. Let $p \in S_G(M)$ be a definable unbounded one-dimensional type.*

1. There is $\phi \in \mathcal{L}$ such that $\text{stab}_\phi(\mu \cdot p)$ has dimension 1.
2. If M is \aleph_1 -saturated, then $\text{stab}^\mu(p)$ is a one-dimensional type-definable subgroup of G .

PROOF. Part (1) is Lemma 6.14. For Part (2), take a countable submodel $M_0 \preceq M$ such that G and p are M_0 -definable. Then apply Lemma 6.13 with M and M_0 in place of N and M . \dashv

Acknowledgments. The first author was supported by the National Natural Science Foundation of China (Grant No. 12101131). The second author was supported by the National Social Science Fund of China (Grant No. 20CZX050).

REFERENCES

- [1] L. BELAIR, *Panorama of p -adic model theory*. *Annales des Sciences mathématiques du Québec*, vol. 36 (2012), no. 1, pp. 43–75.
- [2] A. DOLICH, J. GOODRICK, and D. LIPPEL, *Dp -minimality: Basic facts and examples*. *Notre Dame Journal of Formal Logic*, vol. 52 (2011), no. 3, pp. 267–288.
- [3] L. VAN DEN DRIES, *Algebraic theories with definable Skolem functions*, this JOURNAL, vol. 49 (1984), pp. 625–629.
- [4] A. FORNASIERO, *Definable compactness for topological structures*, in preparation, 2015.
- [5] A. GÜNAYDIN and P. HIERONYMI, *Dependent pairs*, this JOURNAL, vol. 76 (2011), no. 2, pp. 377–390.
- [6] Y. HALEVI, A. HASSON, and F. JAHNKE, *Definable V -topologies, Henselianity and NIP*. *Journal of Mathematical Logic*, vol. 20 (2020), no. 2, p. 2050008.
- [7] E. HRUSHOVSKI, Y. PETERZIL, and A. PILLAY, *Groups, measures, and the NIP*. *Journal of the American Mathematical Society*, vol. 21 (2008), no. 2, pp. 563–596.
- [8] E. HRUSHOVSKI and A. PILLAY, *Groups definable in local and pseudofinite fields*. *Israel Journal of Mathematics*, vol. 85 (1994), pp. 203–262.
- [9] W. JOHNSON, *Interpretable sets in dense o -minimal structures*, this JOURNAL, vol. 83 (2018), pp. 1477–1500.
- [10] I. KAPLAN, A. ONSHUUS, and A. USVYATSOV, *Additivity of the dp -rank*. *Transactions of the American Mathematical Society*, vol. 365 (2013), no. 11, pp. 5783–5804.
- [11] A. MACINTYRE, *On definable subsets of p -adic fields*, this JOURNAL, vol. 41 (1976), pp. 605–610.
- [12] D. MARKER, *Model Theory: An Introduction*, Springer, New York, 2002.
- [13] A. ONSHUUS and A. PILLAY, *Definable groups and compact p -adic lie groups*. *Journal of the London Mathematical Society*, vol. 78 (2008), no. 1, pp. 233–247.
- [14] D. PENAZZI, A. PILLAY, and N. YAO, *Some model theory and topological dynamics of p -adic algebraic groups*. *Fundamenta Mathematicae*, vol. 247 (2019), pp. 191–216.
- [15] Y. PETERZIL and S. STARCHENKO, *Topological groups, μ -types and their stabilizers*. *Journal of the European Mathematical Society (JEMS)*, vol. 19 (2017), no. 10, pp. 2965–2995.
- [16] Y. PETERZIL and C. STEINHORN, *Definable compactness and definable subgroups of o -minimal groups*. *Journal of the London Mathematical Society*, vol. 59 (1999), no. 3, pp. 769–786.
- [17] A. PILLAY, *Groups and fields definable in o -minimal structures*. *Journal of Pure and Applied Algebra*, vol. 53 (1988), pp. 233–255.
- [18] ———, *On fields definable in \mathbb{Q}_p* . *Archive for Mathematical Logic*, vol. 29 (1989), pp. 1–7.
- [19] A. PILLAY and N. YAO, *Definable f -generic groups over p -adic numbers*, preprint, 2019, [arXiv:1911.01833](https://arxiv.org/abs/1911.01833).
- [20] ———, *A note on groups definable in the p -adic field*. *Archive for Mathematical Logic*, vol. 58 (2019), pp. 1029–1034.
- [21] B. POIZAT, *A Course in Model Theory*, Springer, New York, 2000.
- [22] P. SIMON, *A Guide to NIP Theories*, Lecture Notes in Logic, vol. 44, Cambridge University Press, Cambridge, 2015.
- [23] N. YAO, *On dimensions, standard part maps, and p -adically closed fields*. *Studies in Logic*, vol. 13 (2020), no. 6, pp. 41–62.

SCHOOL OF PHILOSOPHY
FUDAN UNIVERSITY
220 HANDAN ROAD, GUANGHUA WEST BUILDING, ROOM 2503
SHANGHAI 20043, CHINA

E-mail: willjohnson@fudan.edu.cn

E-mail: yaony@fudan.edu.cn