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Existence and stability of bistable wavefronts in a nonlocal delayed reaction–diffusion epidemic system

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In this paper, we consider the monotone travelling wave solutions of a reaction–diffusion epidemic system with nonlocal delays. We obtain the existence of monotone travelling wave solutions by applying abstract existence results. By transforming the nonlocal delayed system to a non-delayed system and choosing suitable small positive constants to define a pair of new upper and lower solutions, we use the contraction technique to prove the asymptotic stability (up to translation) of monotone travelling waves. Furthermore, the uniqueness and Lyapunov stability of monotone travelling wave solutions will be established with the help of the upper and lower solution method and the exponential asymptotic stability.

Key words: Travelling wave front, nonlocal delay, contraction technique, upper and lower solutions

2010 Mathematics Subject Classification: 35R20, 35K57, 34K30

1 Introduction

The environmental pollution by an infective human population can lead to the spread of infectious diseases, which is regarded as one of the main factors of relevant epidemics, such as cholera and malaria [6]. In [8, 12], the authors proposed a model to describe the spread of cholera epidemic which happened in the European Mediterranean regions in 1973

$$\begin{cases} \dfrac{du_1(t)}{dt} = -a_{11}u_1(t) + a_{12}u_2(t), \\ \dfrac{du_2(t)}{dt} = -a_{22}u_2(t) + g(u_1(t)), \end{cases} \tag{1.1}$$

where $a_{11}, a_{12}, a_{22} > 0$, $u_1(t)$ and $u_2(t)$, respectively, are the densities of infectious agents and the infective human population at the time $t \geq 0$, a_{11} is the natural death rate of the agents, a_{22} is the natural diminishing rate of the infective human, a_{12} is the contribution of infectious population to the density of infectious agents and $g(x)$ denotes the infection rate of the human population due to the concentration of the agents.

If it only considers the mobility of the bacteria and neglect the mobility of the infectious population, Capasso and Maddalena [8] gave the following system (see also [13])

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + a_{12}u_2(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = -a_{22}u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{1.2}$$

where $u_i(x,t), i = 1, 2$, are the spatial densities of two species at the position x and at the time $t \geq 0$. The existence, uniqueness and regularity of (1.2) were considered in [7, 8]. In [36], the authors considered the minimal wave speed of System (1.2). In 2004, Xu and Zhao [41] considered the existence, uniqueness and global exponential stability of monotone travelling wave solutions of System (1.2) with bistable case.

To make the model be more realistic, taking into account the infective population too moving randomly, System (1.2) was modified as follows

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + a_{12}u_2(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2 \dfrac{\partial^2 u_2(x,t)}{\partial x^2} - a_{22}u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{1.3}$$

where $d_1 > 0$ and $d_2 \geq 0$. Under the homogeneous Neumann boundary conditions, the authors [9, 10] studied System (1.3) using the contracting rectangle technique [28] and obtained the same threshold results for (1.1). For $d_1, d_2 > 0$, the authors [8] considered the convergence problem of the equilibrium states of System (1.3).

Generally speaking, some infectious agents u_1, such as bacteria or viruses at position x, depend on u_2 at position x or neighbour position of x, and even all the position in space. For example, an important factor of the spread of typhoid fever, malaria, and so on, is the mobility of the infectious population, and in order to effectively control indirect transmission diseases, it should adopt different approach to control the production of the pollutants. Based on this idea, a possible model is the one proposed in [5]. For more details, we refer to [2]. Xu and Zhao [42] studied the spreading speed and monostable travelling wave solutions of the following system

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + \displaystyle\int_\Omega G(x-y)u_2(y,t)dy, \\ \dfrac{\partial u_2(x,t)}{\partial t} = -a_{22}u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}$, $G(x-y)$ is a kernel function. By taking into account the latent period of bacteria, the authors [34] investigated the asymptotic speed of spread and travelling waves of system with distributed delay

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + a_{12}u_2(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = -a_{22}u_2(x,t) + \displaystyle\int_0^\infty g(u_1(x,t-s))P(ds), \end{cases} \tag{1.5}$$

where P is a probability measure on $\mathbb{R}_+$. As its generalisation, Wu and Liu [39] considered the spreading speed and the minimal wave speed of the following system

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + \displaystyle\int_\Omega G(x-y)u_2(y,t)dy, \\ \dfrac{\partial u_2(x,t)}{\partial t} = -a_{22}u_2(x,t) + \displaystyle\int_0^\infty g(u_1(x,t-s))P(ds). \end{cases} \tag{1.6}$$

More generally, the infective human population and the concentration of the infectious agents in the environment have a direct effect on each other, which depends not simply on population density at one point in space and time, but on a weighted average involving values at all previous times and at all points in space. On one hand, changes in human population will result in a change of the bacteria population some time later, such as some recover population who is immune to bacteria in some time (immune period); on the other hand, due to the human and bacteria moving (by diffusion), they may not stay at the same position and at previous times. In order to describe this model reasonably, we introduce the spatiotemporal delays or nonlocal delays into System (1.3), which is modified as the following system

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - a_{11}u_1(x,t) + (g_1 * u_2)(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2 \dfrac{\partial^2 u_2(x,t)}{\partial x^2} - a_{22}u_2(x,t) + (g_2 * g(u_1))(x,t), \end{cases} \tag{1.7}$$

where $(g_1 * u_2)(x,t)$ and $(g_2 * g(u_1))(x,t)$ are defined by

$$\begin{cases} (g_1 * u_2)(x,t) = \displaystyle\int_{-\infty}^t \int_{-\infty}^\infty G_1(x-y,t-s)k_1(t-s)u_2(y,s)dyds, \\ (g_2 * g(u_1))(x,t) = \displaystyle\int_{-\infty}^t \int_{-\infty}^\infty G_2(x-y,t-s)k_2(t-s)g(u_1(y,s))dyds, \end{cases} \tag{1.8}$$

where G_1 and G_2 are chosen as

$$G_1(x,t) = \frac{1}{\sqrt{4\pi d_2 t}} e^{-\frac{x^2}{4d_2 t}} \quad \text{and} \quad G_2(x,t) = \frac{1}{\sqrt{4\pi d_1 t}} e^{-\frac{x^2}{4d_1 t}},$$

and the kernel functions are

$$k_1(s) = \frac{1}{\tau_1} e^{-\frac{1}{\tau_1}s} \quad \text{and} \quad k_2(s) = \frac{1}{\tau_2} e^{-\frac{1}{\tau_2}s},$$

τ_1 and τ_2 denote the immune period of the recover population and the latent period of the bacteria, respectively. Obviously, G_1 and G_2 satisfy

$$\frac{\partial G_1}{\partial t} = d_2 \frac{\partial^2 G_1}{\partial x^2} \quad \text{and} \quad \frac{\partial G_2}{\partial t} = d_1 \frac{\partial^2 G_2}{\partial x^2}, \quad G_i(x,0) = \delta(x), \quad i = 1, 2,$$

where $\delta(x)$ is the general Dirac function.

By variable transformation $\theta = t - s$ and $z = x - y$, we have

$$\begin{cases} (g_1 * u_2)(x,t) = \displaystyle\int_0^{\infty}\int_{-\infty}^{\infty} \frac{1}{\tau_1} e^{-\frac{1}{\tau_1}\theta} \frac{1}{\sqrt{4\pi d_2 \theta}} e^{-\frac{z^2}{4d_2\theta}} u_2(x-z, t-\theta) dz d\theta, \\ (g_2 * g(u_1))(x,t) = \displaystyle\int_0^{\infty}\int_{-\infty}^{\infty} \frac{1}{\tau_2} e^{-\frac{1}{\tau_2}\theta} \frac{1}{\sqrt{4\pi d_1 \theta}} e^{-\frac{z^2}{4d_1\theta}} g(u_1(x-z, t-\theta)) dz d\theta. \end{cases}$$

In this paper, we investigate the existence, uniqueness (up to translation) and global exponential stability (with phase shift) of travelling wave solutions of Systems (1.3) and (1.7). We note that many authors investigated the existence of travelling wave solutions of systems with discrete or nonlocal delays by Schauder's fixed point theorem and the iterative technique [18–22, 24, 33, 37, 40], and that the uniqueness of the wave speeds by usually using the upper and lower solution method [14, 23, 26, 31], and that the stability of travelling waves with the help of the contraction technique [14, 31, 38], spectral theory [4, 30, 35] and weighted energy method [44]. Since the delay in System (1.7) is infinite, the above methods are difficult in applying to (1.7) and the method in [41] does not apply to System (1.7) as well. By introducing new variables, Lin and Li [23] transformed the nonlocal delayed system to non-delayed system, and this method was also used in [17]. They established the existence, uniqueness of the wave speeds by means of the upper and lower solution method and the asymptotic stability of bistable travelling waves by spectral methods. In this paper, we will adopt these methods in [14, 23, 31] to deal with (1.3) and (1.7).

This paper is organised as follows. In Section 2, we first study ODE System (1.1). In Sections 3 and 4, we investigate the existence of bistable travelling waves and discuss the existence and regularity of mild solutions of Systems (1.3) and (1.7), respectively. In Section 5, motivated by [14, 31], by choosing suitable small positive constants, we will define a pair of new upper and lower solutions of the system without delays, which is different from that in [14, 31], and use the contraction technique to obtain the stable results. Moreover, the uniqueness and Lyapunov stability of travelling wave fronts of System (1.7) are also obtained in this section. In a similar argument, we also obtain similar properties for System (1.3).

2 ODE system

We first investigate ODE System (1.1). For simplicity, we study the following rescaled system of (1.1)

$$\begin{cases} \dfrac{du_1(t)}{dt} = -u_1(t) + \alpha u_2(t), \\ \dfrac{du_2(t)}{dt} = -\beta u_2(t) + g(u_1(t)), \end{cases} \tag{2.1}$$

where $u_1^* = u_1, u_2^* = a_{11}u_2, t^* = a_{11}t$, and drop the stars, $\alpha = \frac{a_{12}}{a_{11}^2}, \beta = \frac{a_{22}}{a_{11}}$.

We need the following hypothesis about g:

(A) $g \in C^2(\mathbb{R}^+, \mathbb{R}^+), g(0) = 0, g'(0) \geq 0, g'(x) > 0, \forall x > 0, \lim_{x\to\infty} g(x) = 1$, and there exists $x_0 > 0$ such that $g''(x) > 0$ for $0 < x < x_0$ and $g''(x) < 0$ for $x > x_0$.

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To analyse the globally asymptotical behavior of System (2.1), we need the well-known Bendixson criterion. For the reader's convenience, we list it as the following lemma.

Lemma 2.1 (Bendixson criterion) *For system*

$$\begin{cases} \dfrac{dx}{\partial t} = M(x,y), \\ \dfrac{dy}{\partial t} = N(x,y), \end{cases} \tag{2.2}$$

if

$$\frac{\partial M(x,y)}{\partial x} + \frac{\partial N(x,y)}{\partial y}$$

keeps the same sign in some simply connected domain D, then the domain D does not contain any closed orbits of System (2.2).

By Lemma 2.1, we get the following result.

Lemma 2.2 *System (2.1) has no any closed orbits in* $\mathbb{R}^2$.

We give some known results for the asymptotical behavior of the equilibria of System (2.1), see [6, 11, 13].

Lemma 2.3 *Let* $\gamma = \frac{\beta}{\alpha} = \frac{a_{11}a_{22}}{a_{12}}$ *and* $\gamma_{crit} := \sup\limits_{z\in[0,+\infty)} \dfrac{g(z)}{z} > 0$. *Then*

(i) *when* $\gamma > \gamma_{crit}$, *(2.1) has a unique equilibrium* $(0,0)$, *which is globally asymptotically stable in the first quadrant of* $\mathbb{R}^2$;

(ii) *when* $\gamma = \gamma_{crit}$ *or* $0 < \gamma \le g'(0)$, *(2.1) has a unique nontrivial equilibrium besides* $(0,0)$;

(iii) *when*

$$g'(0) < \gamma < \gamma_{crit}, \tag{2.3}$$

(2.1) admits three equilibria in the first quadrant of $\mathbb{R}^2: E^- = (0,0), E^0 = (a, \frac{a}{\alpha}), E^+ = (b, \frac{b}{\alpha})$, *where* $0 < a < b$ *are the three roots of* $g(x) = \frac{\beta}{\alpha}x$, E^0 *is a saddle point,* E^- *and* E^+ *are stable nodes. Moreover, the first quadrant of* $\mathbb{R}^2$ *is the union of the domain of attraction* E^- *and* E^+ *and the stable manifold of* E^0.

Proof This lemma may be proved by simple mathematical knowledge, and we only sketch an outline of the proof. The equilibrium of System (2.1) is the intersection of the line $L: u_2 = \frac{1}{\alpha}u_1$ and the curve $\Gamma: u_2 = \frac{1}{\beta}g(u_1)$. The number of the intersections is determined by the slope of the curve Γ. The maximum slope of the curve Γ is $\gamma_{crit} := \sup_{x\in[0,+\infty)} g(x)/x > 0$.

(i) By $g(0) = 0$ in (A), it is obvious that $(0,0)$ is an equilibrium of System (2.1). If $\gamma > \gamma_{crit}$, then the curve Γ is below the straight line L except $(0,0)$ in the first quadrant of $\mathbb{R}^2$. That is, $(0,0)$ is only equilibrium of System (2.1). The characteristic equation of System (2.1) at $(0,0)$ is

$$\begin{vmatrix} -1-\lambda & \alpha \\ g'(0) & -\beta-\lambda \end{vmatrix} = 0, \tag{2.4}$$

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then the eigenvalues $\lambda_{1,2} = \frac{-1-\beta\pm\sqrt{(1+\beta)^2-4(\beta-\alpha g'(0))}}{2} < 0$ since $\gamma > \gamma_{crit}$. Hence, $(0, 0)$ is locally asymptotical stable. By Lemma 2.2, System (2.1) has no any closed orbits in $\mathbb{R}^2$. Therefore, $(0, 0)$ is globally asymptotical stable.

(ii) Using similar analysis as in (i), one easily obtain that (2.1) has a unique nontrivial equilibrium besides $(0, 0)$ when $0 < \gamma \le g'(0)$ or $\gamma = \gamma_{crit}$. By calculating directly the eigenvalues of the characteristic equation of System (2.1) at $(0, 0)$ and nontrivial equilibrium, respectively, we can obtain that $(0, 0)$ is a saddle point for $0 < \gamma < g'(0)$, and nontrivial equilibrium is an asymptotical stable node for $0 < \gamma \le g'(0)$. But when $\gamma = g'(0)$ and $\gamma = \gamma_{crit}$, there exists a zero eigenvalue of the characteristic equation of System (2.1) at both $(0, 0)$ and nontrivial equilibria, respectively. In this case, linear analysis may be invalid and the behavior of the equilibrium will become more complicated.

(iii) The eigenvalues of the characteristic equation of System (2.1) at both $(0, 0)$ and $(b, \frac{b}{\alpha})$ are both negative. Hence, they are stable nodes. But the characteristic equation of System (2.1) at $(a, \frac{a}{\alpha})$ has one negative eigenvalue and one positive eigenvalue. Hence, it is a saddle point. For the first quadrant of $\mathbb{R}^2$ being union of the domain of attraction E^- and E^+ and the stable manifold of E^0, one can refer to [6]. The proof is completed. □

3 Travelling wave fronts and mild solutions of System (1.3)

In this section, we investigate the existence of monotone travelling wave solutions and mild solutions of (1.3).

A travelling wave solution of (1.3) has the special form $(u_1(x, t), u_2(x, t)) = (\phi_1(\xi), \phi_2(\xi))$, $\xi = x + ct$, where c is the wave speed and $(\phi_1(\xi), \phi_2(\xi)$ is the wave profile. If $(\phi_1(\xi), \phi_2(\xi)$ is monotone in $\xi \in \mathbb{R}$, then it is called the travelling wave front.

For simplicity, we study the rescaled system of (1.3)

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_2(x,t), \\[2ex] \dfrac{\partial u_2(x,t)}{\partial t} = d_2 \dfrac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + g(u_1(x,t)), \end{cases} \tag{3.1}$$

where $u_1^* = u_1, u_2^* = a_{11}u_2, t^* = a_{11}t, x^* = \sqrt{a_{11}}x$, and drop the stars, $\alpha = \frac{a_{12}}{a_{11}^2}$, $\beta = \frac{a_{22}}{a_{11}}$.

We want to find the travelling wave fronts of (1.3) connecting E^- with E^+ provided that (2.3) holds. Then, denoting $x + ct$ by t, (3.1) has a travelling wave front $\Phi(t) = (\phi_1(t), \phi_2(t))$ which connects E^- with E^+ if and only if the wave system

$$\begin{cases} d_1\phi_1''(t) - c\phi_1'(t) - \phi_1(t) + \alpha\phi_2(t) = 0, \\[1ex] d_2\phi_2''(t) - c\phi_2'(t) - \beta\phi_2(t) + g(\phi_1(t)) = 0 \end{cases} \tag{3.2}$$

satisfying

$$\lim_{t\to-\infty}(\phi_1(t), \phi_2(t)) = (0, 0) := \Phi_-, \qquad \lim_{t\to+\infty}(\phi_1(t), \phi_2(t)) = \left(b, \frac{b}{\alpha}\right) := \Phi_+ \tag{3.3}$$

has a monotone solution $(\phi_1(t), \phi_2(t))$ on $\mathbb{R}$.

By Theorem 3.3.2 in [35], it easily follows the existence theorem.

Theorem 3.1 *Assume that (2.3) holds. Then there exists a monotone function* $(\phi_1(t), \phi_2(t)) \in C^2(\mathbb{R}, \mathbb{R}^2)$ *satisfying (3.2) and (3.3).*

Now we investigate the mild solutions of (3.1). Consider the Cauchy problem of (3.1) with the initial values

$$u_1(x, 0) = \psi_1(x), \quad u_2(x, 0) = \psi_2(x), \quad x \in \mathbb{R}, \tag{3.4}$$

where

$$(0, 0) \le (\psi_1(x), \psi_2(x)) \le \left(b, \frac{b}{\alpha}\right) \text{ with } \psi_i(x) \in C(\mathbb{R}, \mathbb{R}), i = 1, 2. \tag{3.5}$$

Let

$$\beta_1 = 1, \quad \beta_2 = \beta. \tag{3.6}$$

For $(0, 0) \le (u_1(x, t), u_2(x, t)) \le \left(b, \frac{b}{\alpha}\right), (x, t) \in \mathbb{R} \times \mathbb{R}^+$, define $F = (F_1, F_2)$ by

$$\begin{cases} F_1(u_1, u_2)(x, t) = \beta_1 u_1(x, t) - u_1(x, t) + \alpha u_2(x, t), \\ F_2(u_1, u_2)(x, t) = \beta_2 u_2(x, t) - \beta u_2(x, t) + g(u_1(x, t)). \end{cases} \tag{3.7}$$

Then, for any $(0, 0) \le (v_1(x, t), v_2(x, t)) \le (u_1(x, t), u_2(x, t)) \le \left(b, \frac{b}{\alpha}\right), x \in \mathbb{R}$, it is easy to see that

$$(0, 0) = F(0, 0) \le F(v_1, v_2)(x, t) \le F(u_1, u_2)(x, t) \le F\left(b, \frac{b}{\alpha}\right). \tag{3.8}$$

From (3.6) and (3.7), (3.1) can be rewritten as

$$\begin{cases} \dfrac{\partial u_1(x, t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x, t)}{\partial x^2} - \beta_1 u_1(x, t) + F_1(u_1, u_2)(x, t), \\ \dfrac{\partial u_2(x, t)}{\partial t} = d_2 \dfrac{\partial^2 u_2(x, t)}{\partial x^2} - \beta_2 u_2(x, t) + F_2(u_1, u_2)(x, t). \end{cases} \tag{3.9}$$

Let $X = BUC(\mathbb{R}, \mathbb{R}^2)$ be a Banach space of bounded and uniformly continuous vector-valued function from $\mathbb{R}$ to $\mathbb{R}^2$ with the general super norm $\|\cdot\|$ and

$$X_I = \left\{u(x) \in X : (0, 0) \le u(x) \le \left(b, \frac{b}{\alpha}\right) \text{ for } x \in \mathbb{R}\right\}.$$

Define

$$\begin{cases} u_1(x, t) = \dfrac{e^{-\beta_1 t}}{\sqrt{4\pi d_1 t}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_1 t}} u_1^0(y) dy := T_1(t) u_1^0(x), \\ u_2(x, t) = \dfrac{e^{-\beta_2 t}}{\sqrt{4\pi d_2 t}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_2 t}} u_2^0(y) dy := T_2(t) u_2^0(x), \end{cases}$$

and $T(t) = (T_1(t), T_2(t))$. It is easy to see that $T(t) : X \to X$ is a C_0 semigroup. Furthermore, by [15, 16, 27, 32], it is easy to see that $T(t)$ is a positive and analytic semigroup. Furthermore, with the help of upper and lower solution method and the theory for integral equations, see Theorems 1, 2 and Proposition 1 in Martin and Smith [25], and Theorems 4.1 and 5.1 in Ruan and Wu [29], we have the following result.

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Theorem 3.2 *Assume that* $(\psi_1(x), \psi_2(x))$ *satisfies (3.5). Then* $(u_1(x,t), u_2(x,t))$ *defined by*

$$\begin{cases} u_1(x,t) = T_1(t)\psi_1(x) + \displaystyle\int_0^t T_1(t-s)F_1(u_1,u_2)(x,s)ds, \\ u_2(x,t) = T_2(t)\psi_2(x) + \displaystyle\int_0^t T_2(t-s)F_2(u_1,u_2)(x,s)ds \end{cases} \tag{3.10}$$

for all $x \in \mathbb{R}, t > 0$ *is a unique mild solution of (3.9) and (3.4) (also (3.1) and (3.4)). Moreover,* $(u_1(\cdot,t), u_2(\cdot,t)) \in X_I$ *for all* $t > 0$.

It is clear that u_1 and u_2 are C^2 in $x \in \mathbb{R}$ and C^1 in $t > 0$. So the smoothness of $u_1(x,t)$ and $u_2(x,t)$ implies that the mild solution $(u_1(x,t), u_2(x,t))$ described by Theorem 3.2 is a classical solution of (3.1) with initial value (3.4) for all $(x,t) \in \mathbb{R} \times (0, +\infty)$. The asymptotic stability of bistable travelling waves and uniqueness of wave speed of (1.3) will be given in Section 5, which is based on a similar method used to System (1.7).

4 Travelling wave fronts and mild solutions of System (1.7)

As the above, we also study the rescaled system of (1.7)

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha(g_1 * u_2)(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2 \dfrac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + (g_2 * g(u_1))(x,t), \end{cases} \tag{4.1}$$

where $u_1^* = u_1, u_2^* = a_{11}u_2, t^* = a_{11}t, x^* = \sqrt{a_{11}}x, \theta^* = a_{11}\theta, z^* = \sqrt{a_{11}}z, \tau_i^* = a_{11}\tau_i, i = 1, 2$, and drop the stars, $\alpha = \frac{1}{a_{11}^2}, \beta = \frac{a_{22}}{a_{11}}$.

Let $\gamma = \frac{\beta}{\alpha} = a_{11}a_{22}$. Lemma 2.3 is still valid with α, γ replaced by new α, γ. In order to investigate the existence of monotone travelling wave solutions of (1.7) connecting E^- with E^+, we will assume that (2.3) holds. Then (4.1) has a travelling wave front $\Phi(t) = (\phi_1(t), \phi_2(t))$ connecting E^- with E^+ if and only if the wave system

$$\begin{cases} d_1\phi_1''(t) - c\phi_1'(t) - \phi_1(t) + \alpha(g_1 * \phi_2)(t) = 0, \\ d_2\phi_2''(t) - c\phi_2'(t) - \beta\phi_2(t) + (g_2 * g(\phi_1))(t) = 0 \end{cases} \tag{4.2}$$

with

$$\lim_{t\to-\infty}(\phi_1(t), \phi_2(t)) = (0,0), \qquad \lim_{t\to+\infty}(\phi_1(t), \phi_2(t)) = \left(b, \frac{b}{\alpha}\right) \tag{4.3}$$

has a monotone solution on $\mathbb{R}$, where $(g_1 * \phi_2)(t)$ and $(g_2 * g(\phi_1))(t)$ are defined by

$$\begin{cases} (g_1 * \phi_2)(t) = \displaystyle\int_0^\infty \int_{-\infty}^\infty \frac{1}{\tau_1} e^{-\frac{1}{\tau_1}\theta} \frac{1}{\sqrt{4\pi d_2\theta}} e^{-\frac{z^2}{4d_2\theta}} \phi_2(t - c\theta - z)dzd\theta, \\ (g_2 * g(\phi_1))(t) = \displaystyle\int_0^\infty \int_{-\infty}^\infty \frac{1}{\tau_2} e^{-\frac{1}{\tau_2}\theta} \frac{1}{\sqrt{4\pi d_1\theta}} e^{-\frac{z^2}{4d_1\theta}} g(\phi_1(t - c\theta - z))dzd\theta. \end{cases} \tag{4.4}$$

If we introduce other two functions as

$$u_3(x,t) = (g_1 * u_2)(x,t), \qquad u_4(x,t) = (g_2 * g(u_1))(x,t),$$

then (4.1) with the nonlocal delays is transformed into the non-delayed system

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_3(x,t), \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2 \dfrac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + u_4(x,t), \\ \dfrac{\partial u_3(x,t)}{\partial t} = d_2 \dfrac{\partial^2 u_3(x,t)}{\partial x^2} - \frac{1}{\tau_1} u_3(x,t) + \frac{1}{\tau_1} u_2(x,t), \\ \dfrac{\partial u_4(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_4(x,t)}{\partial x^2} - \frac{1}{\tau_2} u_4(x,t) + \frac{1}{\tau_2} g(u_1(x,t)). \end{cases} \tag{4.5}$$

This is a well-known method which has been often used, for example, in [17, 23]. We give the relation between (4.1) and (4.5). If $(u_1(x,t), u_2(x,t))$ is a solution of (4.1), then $(u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))$ is a solution of (4.5); conversely, if $(u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))$ is a solution of (4.5), then $(u_1(x,t), u_2(x,t))$ is a solution of (4.1).

Obviously, (4.5) has a travelling wave solution $\Phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$ connecting $\Phi_- = (0,0,0,0)$ with $\Phi_+ = \left(b, \frac{b}{\alpha}, \frac{b}{\alpha}, \frac{\beta b}{\alpha}\right)$ if and only if the wave system

$$\begin{cases} d_1\phi_1''(t) - c\phi_1'(t) - \phi_1(t) + \alpha\phi_3(t) = 0, \\ d_2\phi_2''(t) - c\phi_2'(t) - \beta\phi_2(t) + \phi_4(t) = 0, \\ d_2\phi_3''(t) - c\phi_3'(t) - \frac{1}{\tau_1}\phi_3(t) + \frac{1}{\tau_1}\phi_2(t) = 0, \\ d_1\phi_4''(t) - c\phi_4'(t) - \frac{1}{\tau_2}\phi_4(t) + \frac{1}{\tau_2}g(\phi_1(t)) = 0 \end{cases} \tag{4.6}$$

with

$$\lim_{t\to-\infty} \Phi(t) = \Phi_-, \qquad \lim_{t\to+\infty} \Phi(t) = \Phi_+ \tag{4.7}$$

has a solution on $\mathbb{R}$.

To obtain the existence result of (4.1), we need the well-known Hurwitz criterion applied to (4.6). For the reader's convenience, we list it as the following lemma.

Lemma 4.1 (Hurwitz criterion) *Consider the following polynomial equation*

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0, \tag{4.8}$$

all the roots of Equation (4.8) have negative real parts if and only if

$$H_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2k-1} \\ 1 & a_2 & a_4 & \cdots & a_{2k-2} \\ 0 & a_1 & a_3 & \cdots & a_{2k-3} \\ 0 & 1 & a_2 & \cdots & a_{2k-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{vmatrix} > 0,$$

where $k = 1, 2, \cdots, n$ and $a_j = 0$ for $j > n$.

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Theorem 4.1 *Assume that (2.3) holds. Then there exists a monotone function* $(\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)) \in C^2(\mathbb{R}, \mathbb{R}^4)$ *satisfying (4.6) and (4.7).*

Proof It is clear that

$$f'(\Lambda) = \begin{pmatrix} -1 & 0 & \alpha & 0 \\ 0 & -\beta & 0 & 1 \\ 0 & \frac{1}{\tau_1} & -\frac{1}{\tau_1} & 0 \\ \frac{1}{\tau_2} g'(\omega) & 0 & 0 & -\frac{1}{\tau_2} \end{pmatrix},$$

where $\Lambda = \Phi_-, \Phi^1, \Phi_+$ corresponding to $\omega = 0, a, b$, respectively, and $\Phi^1 = \left(a, \frac{a}{\alpha}, \frac{a}{\alpha}, \frac{\beta a}{\alpha}\right)$.

We first check that $f'(\Phi_-)$ and $f'(\Phi_+)$ only have eigenvalues with negative real parts. By direct calculation, we have

$$\begin{aligned} |\lambda I - f'(\Phi_-)| &= \begin{vmatrix} \lambda + 1 & 0 & -\alpha & 0 \\ 0 & \lambda + \beta & 0 & -1 \\ 0 & -\frac{1}{\tau_1} & \lambda + \frac{1}{\tau_1} & 0 \\ -\frac{1}{\tau_2} g'(0) & 0 & 0 & \lambda + \frac{1}{\tau_2} \end{vmatrix} \\ &= (\lambda + 1)(\lambda + \beta)\left(\lambda + \frac{1}{\tau_1}\right)\left(\lambda + \frac{1}{\tau_2}\right) - \frac{1}{\tau_1 \tau_2} \alpha g'(0) \\ &= \lambda^4 + \left(1 + \beta + \frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\lambda^3 + \left[\beta + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)(1 + \beta) + \frac{1}{\tau_1 \tau_2}\right]\lambda^2 \\ &\quad \left[\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\beta + \frac{1}{\tau_1 \tau_2}\beta\right]\lambda + \frac{1}{\tau_1 \tau_2}(\beta - \alpha g'(0)). \end{aligned}$$

By Lemma 4.1, we only need to verify that $H_k > 0, k = 1, 2, 3, 4$. It is obvious that

$$H_1 = 1 + \beta + \frac{1}{\tau_1} + \frac{1}{\tau_2} > 0,$$

$$H_2 = \begin{vmatrix} 1 + \beta + \frac{1}{\tau_1} + \frac{1}{\tau_2} & \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\beta + \frac{1}{\tau_1 \tau_2}\beta \\ 1 & \beta + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)(1 + \beta) + \frac{1}{\tau_1 \tau_2} \end{vmatrix} > 0.$$

By (A), we have $\beta > \alpha g'(0)$, hence

$$H_3 = \begin{vmatrix} 1 + \beta + \frac{1}{\tau_1} + \frac{1}{\tau_2} & \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\beta + \frac{1}{\tau_1 \tau_2}\beta & 0 \\ 1 & \beta + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)(1 + \beta) + \frac{1}{\tau_1 \tau_2} & \frac{1}{\tau_1 \tau_2}(\beta - \alpha g'(0)) \\ 0 & 1 + \beta + \frac{1}{\tau_1} + \frac{1}{\tau_2} & \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\beta + \frac{1}{\tau_1 \tau_2}\beta \end{vmatrix}$$

$$= \left(1+\beta+\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\Big\{\Big[\beta+\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)(1+\beta)+\frac{1}{\tau_1\tau_2}\Big]\Big[\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)\beta+\frac{1}{\tau_1\tau_2}\beta\Big]$$
$$-\frac{1}{\tau_1\tau_2}\Big(1+\beta+\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)(\beta-\alpha g'(0))\Big\}-\Big[\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)\beta+\frac{1}{\tau_1\tau_2}\beta\Big]^2$$
$$> \left(1+\beta+\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\Big\{\Big[\beta+\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)(1+\beta)+\frac{1}{\tau_1\tau_2}\Big]\Big[\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)\beta+\frac{1}{\tau_1\tau_2}\beta\Big]$$
$$-\frac{1}{\tau_1\tau_2}\Big(1+\beta+\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)\beta\Big\}-\Big[\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)\beta+\frac{1}{\tau_1\tau_2}\beta\Big]^2$$
$$> \left(1+\beta+\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\Big[\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)^2\beta^2+\frac{1}{\tau_1\tau_2}\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)\beta(1+\beta)+\Big(\frac{1}{\tau_1\tau_2}\Big)^2\beta\Big]$$
$$-\Big[\Big(\frac{1}{\tau_1}+\frac{1}{\tau_2}\Big)\beta+\frac{1}{\tau_1\tau_2}\beta\Big]^2>0.$$

Furthermore, we have

$$H_4 = \begin{vmatrix} 1+\beta+\frac{1}{\tau_1}+\frac{1}{\tau_2} & \left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\beta+\frac{1}{\tau_1\tau_2}\beta & 0 & 0 \\ 1 & \beta+\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(1+\beta)+\frac{1}{\tau_1\tau_2} & \frac{1}{\tau_1\tau_2}(\beta-\alpha g'(0)) & 0 \\ 0 & 1+\beta+\frac{1}{\tau_1}+\frac{1}{\tau_2} & \left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)\beta+\frac{1}{\tau_1\tau_2}\beta & 0 \\ 0 & 1 & \beta+\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)(1+\beta)+\frac{1}{\tau_1\tau_2} & \frac{1}{\tau_1\tau_2}(\beta-\alpha g'(0)) \end{vmatrix}$$

$$= \frac{1}{\tau_1\tau_2}(\beta-\alpha g'(0))H_3>0.$$

Similarly, by $\beta>\alpha g'(b)$, we can prove that $f'(\Phi_+)$ only have eigenvalues with negative real parts.

Next, we choose $\nu=(\nu_1,\nu_2,\nu_3,\nu_4)(\nu_i\geq 0, i=1,2,3,4)$ such that $\nu f'(\Phi^1)>0$. Notice that

$$\nu f'(\Phi^1)>0 \Longleftrightarrow \frac{1}{\tau_2}g'(a)\nu_4>\nu_1,\quad \frac{1}{\tau_1}\nu_3>\beta\nu_2,\quad \alpha\nu_1>\frac{1}{\tau_1}\nu_3,\quad \nu_2>\frac{1}{\tau_2}\nu_4. \tag{4.9}$$

Since $\beta<\alpha g'(a)$, it is easy to find $\nu_i>0, i=1,2,3,4$, such that (4.9) holds.

Hence, the conclusion is obtained by Theorem 3.3.2 in [35]. The proof is completed. □

Remark 4.1 *(4.1) can be rewritten as (4.5) because travelling wave fronts of (4.1) are twice continuous differentiable. By [1, 3, 17], the regularity of u_3 and u_4 is obvious.*

Now we consider the mild solutions of (4.1). Motivated by Lin and Li [23], in this subsection we adopt the same idea to talk about the mild solutions of (4.1).

Firstly, we study the existence and uniqueness of mild solutions of (4.1). Consider the Cauchy problem of (4.1):

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$$u_i(x,s)=\psi_i(x,s),\quad (x,s)\in\mathbb{R}\times(-\infty,0],\quad i=1,2, \tag{4.10}$$

where

$$(0,0)\le(\psi_1(x,s),\psi_2(x,s))\le\left(b,\frac{b}{\alpha}\right)\ \text{with}\ \ \psi_i(x,s)\in C(\mathbb{R}\times(-\infty,0],\mathbb{R}), i=1,2. \tag{4.11}$$

Let

$$\beta_1=1,\quad \beta_2=\beta. \tag{4.12}$$

For $(0,0)\le(u_1(x,t),u_2(x,t))\le\left(b,\frac{b}{\alpha}\right)$, $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, define $F=(F_1,F_2)$ by

$$\begin{cases} F_1(u_1,u_2)(x,t)=\beta_1u_1(x,t)-u_1(x,t)+\alpha(g_1*u_2)(x,t),\\ F_2(u_1,u_2)(x,t)=\beta_2u_2(x,t)-\beta u_2(x,t)+(g_2*g(u_1))(x,t). \end{cases} \tag{4.13}$$

Then, for any $(0,0)\le(u_1(x,t),u_2(x,t))\le(v_1(x,t),v_2(x,t))\le(b,\frac{b}{\alpha})$, $x\in\mathbb{R}$, $t>0$, we have

$$(0,0)=F(0,0)\le F(u_1,u_2)(x,t)\le F(v_1,v_2)(x,t)\le F\left(b,\frac{b}{\alpha}\right). \tag{4.14}$$

Together with (4.12) and (4.13), (4.1) can be rewritten as

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t}=d_1\dfrac{\partial^2u_1(x,t)}{\partial x^2}-\beta_1u_1(x,t)+F_1(u_1,u_2)(x,t),\\ \dfrac{\partial u_2(x,t)}{\partial t}=d_2\dfrac{\partial^2u_2(x,t)}{\partial x^2}-\beta_2u_2(x,t)+F_2(u_1,u_2)(x,t). \end{cases} \tag{4.15}$$

Using the same notations and discussion as in Section 3, we get the following theorem by [25,29].

Theorem 4.2 *Assume that $(\psi_1(\cdot,s),\psi_2(\cdot,s))$ satisfies (4.11). Then $(u_1(x,t),u_2(x,t))$ defined by*

$$\begin{cases} u_1(x,t)=T_1(t)\psi_1(x,0)+\displaystyle\int_0^t T_1(t-s)F_1(u_1,u_2)(x,s)ds,\\ u_2(x,t)=T_2(t)\psi_2(x,0)+\displaystyle\int_0^t T_2(t-s)F_2(u_1,u_2)(x,s)ds \end{cases} \tag{4.16}$$

for $(x,t)\in\mathbb{R}\times(0,\infty)$ is a unique mild solution of (4.15) and (4.10) (also (4.1) and (4.10)). Moreover, $(u_1(\cdot,t),u_2(\cdot,t))\in X_I$ for all $t>0$.

Secondly, we study the regularity of mild solution obtained in Theorem 4.2. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial v_1(x,t)}{\partial t}=d_1\dfrac{\partial^2v_1(x,t)}{\partial x^2}-v_1(x,t)+\alpha v_3(x,t),\\ \dfrac{\partial v_2(x,t)}{\partial t}=d_2\dfrac{\partial^2v_2(x,t)}{\partial x^2}-\beta v_2(x,t)+v_4(x,t),\\ \dfrac{\partial v_3(x,t)}{\partial t}=d_2\dfrac{\partial^2v_3(x,t)}{\partial x^2}-\frac{1}{\tau_1}v_3(x,t)+\frac{1}{\tau_1}v_2(x,t),\\ \dfrac{\partial v_4(x,t)}{\partial t}=d_1\dfrac{\partial^2v_4(x,t)}{\partial x^2}-\frac{1}{\tau_2}v_4(x,t)+\frac{1}{\tau_2}g(v_1(x,t)),\\ (v_1(x,0),v_2(x,0),v_3(x,0),v_4(x,0))=(v_1(x),v_2(x),v_3(x),v_4(x)), \end{cases} \tag{4.17}$$

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for $(u_3, u_4) \in X$. If we define $(T_3(t), T_4(t)) : X \to X$ by

$$\begin{cases} T_3(t)u_3^0(x) := \dfrac{e^{-\frac{1}{\tau_1}t}}{\sqrt{4\pi d_2 t}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_2 t}} u_3^0(y)dy, \\ T_4(t)u_4^0(x) := \dfrac{e^{-\frac{1}{\tau_2}t}}{\sqrt{4\pi d_1 t}} \displaystyle\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4d_1 t}} u_4^0(y)dy, \end{cases}$$

then (4.17) has a unique classical solution $(v_1(x, t), v_2(x, t), v_3(x, t), v_4(x, t))$ defined by

$$v_1(x, t) = T_1(t)v_1(x) + \int_0^t T_1(t-s)[\beta_1 v_1(x, s) - v_1(x, s) + \alpha v_3(x, s)]ds,$$

$$v_2(x, t) = T_2(t)v_2(x) + \int_0^t T_2(t-s)[\beta_2 v_2(x, s) - \beta v_2(x, s) + v_4(x, s)]ds,$$

$$v_3(x, t) = T_3(t)v_3(x) + \frac{1}{\tau_1}\int_0^t T_3(t-s)v_2(x, s)ds,$$

$$v_4(x, t) = T_4(t)v_4(x) + \frac{1}{\tau_2}\int_0^t T_4(t-s)g(v_1(x, s))ds.$$

Now we choose the initial values in (4.17)

$$v_i(x, 0) = \psi_i(x, 0), \quad i = 1, 2, 3, 4, \tag{4.18}$$

where $\psi_1(x, 0)$, $\psi_2(x, 0)$ are given by (4.10) and $\psi_3(x, 0)$, $\psi_4(x, 0)$ are defined by

$$\psi_3(x, 0) = \frac{1}{\tau_1}\int_0^\infty T_3(\theta)\psi_2(x, -\theta)d\theta, \qquad \psi_4(x, 0) = \frac{1}{\tau_2}\int_0^\infty T_4(\theta)g(\psi_1(x, -\theta))d\theta.$$

Then, by $T_3(t+s) = T_3(t)T_3(s)$ for any $t, s \geq 0$, we get

$$\begin{aligned} v_3(x, t) &= \frac{e^{-\frac{t}{\tau_1}}}{\sqrt{4\pi d_2 t}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4d_2 t}}\psi_3(x-y, 0)dy \\ &\quad + \int_0^t \frac{e^{-\frac{t-s}{\tau_1}}}{\tau_1\sqrt{4\pi d_2(t-s)}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{4d_2(t-s)}} v_2(x-y, s)dyds \\ &= \frac{1}{\tau_1}\int_0^\infty T_3(\theta)v_2(x, t-\theta)d\theta, \end{aligned}$$

and similarly,

$$v_4(x, t) = \frac{1}{\tau_2}\int_0^\infty T_4(\theta)g(v_1(x, t-\theta))d\theta.$$

By expressions of v_3, v_4 and the relation between (4.1) and (4.5), it is clear that v_1 and v_2 are independent of v_3 and v_4.

Motivated by Lin and Li [23], and together with the relation between (4.1) and (4.5), we have the following result.

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Lemma 4.2 *Assume that (4.17) with the initial values given by (4.18), then* $(u_1(x,t),u_2(x,t)) = (v_1(x,t),v_2(x,t))$ *holds for* $(x,t)\in\mathbb{R}\times(0,\infty)$.

Proof On one hand, from the expression of v_3, we have that for all $t>0$,

$$v_1(x,t)=T_1(t)v_1(x,0)+\int_0^t T_1(t-s)\Big[\beta_1 v_1(x,s)-v_1(x,s)+\alpha\int_0^\infty \frac{1}{\tau_1}T_3(\theta)v_2(x,s-\theta)d\theta\Big]ds.$$

On the other hand, it follows from Theorem 4.2 that

$$u_1(x,t)=T_1(t)u_1(x,0)+\int_0^t T_1(t-s)\Big[\beta_1 u_1(x,s)-u_1(x,s)+\alpha\int_0^\infty \frac{1}{\tau_1}T_3(\theta)u_2(x,s-\theta)d\theta\Big]ds.$$

Let $p(t)=p_1(t)+p_2(t)$, where $p_i(t):=\sup_{x\in\mathbb{R}}|v_i(x,t)-u_i(x,t)|$, $i=1,2$. Since $\| T(t)\|\le 1$ for $t>0$, then

$$p_1(t)\le p_1(0)+J\int_0^t\Big[\sup_{\theta\le s}p_1(\theta)+\sup_{\theta\le s}p_2(\theta)\Big]ds,$$

where

$$J=\beta_1+\beta_2+(1+\alpha)+(\beta+\varpi)\ \text{ with }\ \varpi=\max\{g'(x)|x\in[0,b]\}>0 \text{ by (A).}$$

By a similar argument as above, we have that for $t>0$,

$$p_2(t)\le p_2(0)+J\int_0^t\Big[\sup_{\theta\le s}p_1(\theta)+\sup_{\theta\le s}p_2(\theta)\Big]ds.$$

So it follows from the above two inequalities that

$$p(t)\le p(0)+2J\int_0^t\Big[\sup_{\theta\le s}p_1(\theta)+\sup_{\theta\le s}p_2(\theta)\Big]ds\le p(0)+4J\int_0^t\sup_{\theta\le s}p(\theta)ds.$$

It is easy to see that for all $t>0$,

$$\begin{aligned}p(t)+\sup_{\theta\le 0}p(\theta)&\le p(0)+\sup_{\theta\le 0}p(\theta)+4J\int_0^t\sup_{\theta\le s}p(\theta)ds\\&\le p(0)+\sup_{\theta\le 0}p(\theta)+4J\int_0^t\Big[\sup_{0\le\theta\le s}p(\theta)+\sup_{\theta\le 0}p(\theta)\Big]ds.\end{aligned}$$

Define $q(t):=\sup_{0\le\theta\le t}\{p(\theta)+\sup_{r\le 0}p(r)\}$ for $t>0$. Note that $\int_0^t[\sup_{0\le\theta\le s}p(\theta)+\sup_{\theta\le 0}p(\theta)]ds$ is increasing in $t>0$, it follows from the above inequality that

$$q(t)\le q(0)+4J\int_0^t q(s)ds.$$

Therefore, together with the Gronwall's inequality, we obtain that $q(t)\equiv 0$ for $t\ge 0$ when $q(0)=0$. The proof is completed. □

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It follows from Lemma 4.2 and the smoothness of $v_i(x,t)$ that $(u_1(x,t), u_2(x,t))$ is a classical solution of (4.1). Define

$$\begin{cases} u_3(x,t) = \frac{1}{\tau_1}\int_0^{\infty} T_3(\theta)u_2(x,t-\theta)d\theta, \\ u_4(x,t) = \frac{1}{\tau_2}\int_0^{\infty} T_4(\theta)g(u_1(x,t-\theta))d\theta. \end{cases} \tag{4.19}$$

Obviously, u_3 and u_4 are C^2 in $x \in \mathbb{R}$ and C^1 in $t > 0$. So from the smoothness of u_1 and u_2, we obtain the following result.

Theorem 4.3 *$(u_1(x,t), u_2(x,t))$ given in Theorem 4.2 is a classical solution of (4.1) and (4.10) for $(x,t) \in \mathbb{R} \times (0,\infty)$. Furthermore, $u_i(x,t), i = 1,2,3,4$, satisfy (4.5), where u_3 and u_4 are given in (4.19).*

Hence, to obtain the asymptotic stability of solutions of (4.1) and (4.10), we only need to investigate the corresponding non-delayed System (4.5).

5 Asymptotic stability and uniqueness of travelling wave fronts

In order to give the comparison principle, we first give the definition of upper and lower solutions of (4.5) with the initial values $(\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))$ as follows.

Definition 5.1 Assume that $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))$ is C^2 in $x \in \mathbb{R}$ and C^1 in $t > 0$ and $\Phi_- \le u(x,t) \le \Phi_+$. Then $u(x,t)$ is called an upper (a lower) solution of (4.5) if it satisfies

$$\begin{cases} \frac{\partial u_1(x,t)}{\partial t} \ge (\le)\, d_1 \frac{\partial^2 u_1(x,t)}{\partial x^2} - u_1(x,t) + \alpha u_3(x,t), \\ \frac{\partial u_2(x,t)}{\partial t} \ge (\le)\, d_2 \frac{\partial^2 u_2(x,t)}{\partial x^2} - \beta u_2(x,t) + u_4(x,t), \\ \frac{\partial u_3(x,t)}{\partial t} \ge (\le)\, d_2 \frac{\partial^2 u_3(x,t)}{\partial x^2} - \frac{1}{\tau_1}u_3(x,t) + \frac{1}{\tau_1}u_2(x,t), \\ \frac{\partial u_4(x,t)}{\partial t} \ge (\le)\, d_1 \frac{\partial^2 u_4(x,t)}{\partial x^2} - \frac{1}{\tau_2}u_4(x,t) + \frac{1}{\tau_2}g(u_1(x,t)), \\ (u_1(x,0), u_2(x,0), u_3(x,0), u_4(x,0)) \ge (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)). \end{cases} \tag{5.1}$$

Lemma 5.1 (Comparison principle) *Let*

$$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t)) \text{ and } v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t), v_4(x,t))$$

be two solutions of (4.5) with $u(x,0) = \Psi_1$ and $v(x,0) = \Psi_2$, respectively, where

$$\Psi_1(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)) \text{ and } \Psi_2(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)) \in C(\mathbb{R}, \mathbb{R}^4)$$

with $\Phi_- \le \Psi_2(x) \le \Psi_1(x) \le \Phi_+, x \in \mathbb{R}$. Then for any $(x,t) \in \mathbb{R} \times (0,\infty)$,

$$\Phi_- \le v(x,t) \le u(x,t) \le \Phi_+$$

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and

$$u_i(x,t) - v_i(x,t) \geq J_i(L, t-t_0) \int_y^{y+1} (u_i(z,t_0) - v_i(z,t_0))dz \geq 0 \tag{5.2}$$

for $L \geq 0, x, y \in \mathbb{R}$ *satisfying* $|x-y| \leq L$ *and* $t > t_0 \geq 0, i = 1, 2, 3, 4,$ *where*

$$J_1(L, t-t_0) = \frac{e^{-\beta_1(t-t_0)}}{\sqrt{4\pi d_1(t-t_0)}} e^{-\frac{(L+1)^2}{4d_1(t-t_0)}}, \quad J_2(L, t-t_0) = \frac{e^{-\beta_2(t-t_0)}}{\sqrt{4\pi d_2(t-t_0)}} e^{-\frac{(L+1)^2}{4d_2(t-t_0)}},$$

$$J_3(L, t-t_0) = \frac{e^{-\frac{1}{\tau_1}(t-t_0)}}{\sqrt{4\pi d_2(t-t_0)}} e^{-\frac{(L+1)^2}{4d_2(t-t_0)}}, \quad J_4(L, t-t_0) = \frac{e^{-\frac{1}{\tau_2}(t-t_0)}}{\sqrt{4\pi d_1(t-t_0)}} e^{-\frac{(L+1)^2}{4d_1(t-t_0)}}.$$

Proof For the proof of $\Phi_- \leq v(x,t) \leq u(x,t) \leq \Phi_+$, it is very similar to these of Theorem 14.16 in Smoller [32], Theorem 5.5.5 in Volpert et al. [35] and Theorem 5.2.9 in Ye and Li [43], we omit the details.

We only prove that (5.2) holds. Since the semigroup $(T_1(t), T_2(t), T_3(t), T_4(t))$ is positive, then any solution $\Phi_- \leq (u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t)) \leq \Phi_+$ of (4.5) satisfies

$$\begin{cases} u_1(x,t) = T_1(t-r)u_1(x,r) + \displaystyle\int_r^t T_1(t-s)[\beta_1 u_1(x,s) - u_1(x,s) + \alpha u_3(x,s)]ds, \\ u_2(x,t) = T_2(t-r)u_2(x,r) + \displaystyle\int_r^t T_2(t-s)[\beta_2 u_2(x,s) - \beta u_2(x,s) + u_4(x,s)]ds, \\ u_3(x,t) = T_3(t-r)u_3(x,r) + \dfrac{1}{\tau_1}\displaystyle\int_r^t T_3(t-s)u_2(x,s)ds, \\ u_4(x,t) = T_4(t-r)u_4(x,r) + \dfrac{1}{\tau_2}\displaystyle\int_r^t T_4(t-s)g(u_1(x,s))ds \end{cases}$$

for all $0 \leq r < t < a (a > 0)$. For any two solutions $\Phi_- \leq v(x,t) \leq u(x,t) \leq \Phi_+$ of (4.5) with $u(x,0) = \Psi_1$ and $v(x,0) = \Psi_2$, respectively, we only prove $u_1(x,t) \geq v_1(x,t)$ since the others are similar. Let $w(x,t) = u_1(x,t) - v_1(x,t)$. For any given $0 \leq t_0 < t$ and $x, y \in \mathbb{R}$ satisfying $|x-y| \leq L$, it easily follows that

$$\begin{aligned} w(x,t) &= T_1(t-t_0)w(x,t_0) + \int_{t_0}^t T_1(t-t_0)[\beta_1 w(x,s) - w(x,s) + \alpha w(x,s)]ds \\ &\geq T_1(t-t_0)w(x,t_0) \\ &= \frac{e^{-\beta_1(t-t_0)}}{\sqrt{4\pi d_1(t-t_0)}} \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4d_1(t-t_0)}} w(z,t_0)dz \\ &\geq \frac{e^{-\beta_1(t-t_0)}}{\sqrt{4\pi d_1(t-t_0)}} \int_y^{y+1} e^{-\frac{(x-z)^2}{4d_1(t-t_0)}} w(z,t_0)dz \\ &\geq \frac{e^{-\beta_1(t-t_0)}}{\sqrt{4\pi d_1(t-t_0)}} e^{-\frac{(L+1)^2}{4d_1(t-t_0)}} \int_y^{y+1} w(z,t_0)dz. \end{aligned}$$

The proof is completed. □

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Remark 5.1 *From (5.2) we know that if $u_i(x,0) \not\equiv v_i(x,0)$, then for every $t > 0$,*

$$u_i(x,t) - v_i(x,t) \geq J_i(L,t) \int_y^{y+1} (u_i(z,0) - v_i(z,0))dz > 0, \quad i = 1,2,3,4.$$

Hence, every nontrivial travelling wave front of (4.5) is strictly monotone. Therefore, the bistable travelling wave fronts of (4.1) are also strictly monotone.

In fact, Remark 5.1 is obvious. If the strict inequalities do not hold, then, by $u_i(x,0) \geq v_i(x,0)$ and the arbitrariness of x and $L > 0$, it yields that $u_i(x,0) \equiv v_i(x,0), x \in \mathbb{R}$, which is a contradiction. For every nontrivial travelling wave front $(u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t))$ of (4.5), where $u_i(x,t) = u_i(\xi), \xi = x + ct, i = 1,2,3,4$, since the travelling wave fronts of (4.5) are translation invariant solutions, $(u_1(\xi+h), u_2(\xi+h), u_3(\xi+h), u_4(\xi+h))$ is also a travelling wave front of (4.5) for any $h \in \mathbb{R}$. Note that $u_i(\xi)$ is monotone in $\xi \in \mathbb{R}$ by the definition of travelling wave front, so $u_i(x+h,0) \geq u_i(x,0)$ for any $h > 0$. It follows from the strict inequalities that $u_i(\xi+h) > u_i(\xi)$ for any $h > 0$, which implies that $u_i(\xi), i = 1,2,3,4$, are strictly monotone.

In what follows, we always denote $\Phi_+ = (b, \frac{b}{\alpha}, \frac{b}{\alpha}, \frac{\beta b}{\alpha}) := (k_1,k_2,k_3,k_4)$. By (A) and the continuity of $g'(x) > 0$, we can find sufficiently small constants $p_i > 0, i = 1,2,3,4$, such that

$$p_1 > \alpha p_3, \quad \beta p_2 > p_4, \quad p_3 > p_2, \quad p_4 > \varrho p_1, \tag{5.3}$$

where $\varrho = \max\{g'(x) | x \in [0,p_1] \cup [k_1 - p_1, k_1]\} > 0$.

To use the contraction technique to prove the asymptotic stability, we give a pair of upper and lower solutions.

Lemma 5.2 *Assume that (A) holds and $\Phi(x+ct) = (\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct), \phi_4(x+ct))$ is a travelling wave front of (4.5). Define $w^{\pm}(x,t) = (w_1^{\pm}(x,t), w_2^{\pm}(x,t), w_3^{\pm}(x,t), w_4^{\pm}(x,t))$ by*

$$w_i^+(x,t) = \min\left\{\phi_i(\eta^+(x,t)) + \delta p_i e^{-\beta_0 t}, k_i\right\}, \quad w_i^-(x,t) = \max\left\{\phi_i(\eta^-(x,t)) - \delta p_i e^{-\beta_0 t}, 0\right\},$$
$$i = 1,2,3,4,$$

where $\eta^{\pm}(x,t) = x + ct + \xi_0 \pm \sigma_0 \delta(1 - e^{-\beta_0 t})$. Then there exist $\sigma_0 > 0, \beta_0 > 0, \delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ and any ξ_0, $w^+(x,t)$ and $w^-(x,t)$ are an upper solution and a lower solution of (4.5) on $\mathbb{R}^+$, respectively.

Proof We only check that $w^+(x,t)$ is an upper solution of (4.5) because the proof of a lower solution of (4.5) is similar. Since $w_i^+(x,t) = k_i$ is an upper solution of (4.5), we only consider the case $w_i^+(x,t) < k_i, i = 1,2,3,4$.

For simplicity, denote $\eta^+(x,t)$ by η. Fix $\beta_0 \in (0,\mu)$ and $\delta^* \in (0,p_1)$, then there exists $M = M(\Phi, \beta_0, \delta^*)$ large enough such that

$$\phi_1(\eta) + \delta p_1 \geq k_1 - \delta^* \text{ for all } \delta \in (0,\delta^*] \text{ and for all } \eta \geq M,$$
$$\phi_1(\eta) - \delta p_1 \leq \delta^* \text{ for all } \delta \in (0,\delta^*] \text{ and for all } \eta \leq -M,$$

where

$$\mu := \min\left\{\frac{p_1 - \alpha p_3}{p_1}, \frac{\beta p_2 - p_4}{p_2}, \frac{p_3 - p_2}{\tau_1 p_3}, \frac{p_4 - \varrho p_1}{\tau_2 p_4}\right\} > 0.$$

Since $\phi_i'(\eta) > 0, |\eta| \leq M, i = 1, 2, 3, 4$, we can take

$$\sigma_0 := \frac{p_0(c_0 + \beta_0)}{\beta_0 m_0} > 0, \qquad \delta_0 := \min\left\{\delta^*, \frac{1}{\sigma_0}\right\},$$

where

$$m_0 := \min\{\phi_1'(\eta) \| \eta| \leq M\} > 0,\ p_0 := \max\{p_1, p_4\},\ c_0 := \max\left\{\frac{1}{\tau_2}|g'(\xi)| | \xi \in [0, k_1]\right\}.$$

By direct calculation, we have

$$\frac{\partial w_i^+(x,t)}{\partial t} = c\phi_i'(\eta) + \beta_0\sigma_0\delta e^{-\beta_0 t}\phi_i'(\eta) - \beta_0\delta p_i e^{-\beta_0 t} \quad \text{and} \quad \frac{\partial^2 w_i^+(x,t)}{\partial x^2} = \phi_i''(\eta).$$

For $w_i^+(x,t) < k_i, i = 1, 2, 3$, we only need to prove that

$$\begin{aligned}
\sigma_0\beta_0\phi_1'(\eta) - \beta_0 p_1 &\geq -p_1 + \alpha p_3,\\
\sigma_0\beta_0\phi_2'(\eta) - \beta_0 p_2 &\geq -\beta p_2 + p_4,\\
\sigma_0\beta_0\phi_3'(\eta) - \beta_0 p_3 &\geq -\frac{1}{\tau_1}p_3 + \frac{1}{\tau_1}p_2,
\end{aligned}$$

respectively. Obviously, the above three inequalities hold by $\phi_i'(\eta) \geq 0, i = 1, 2, 3$, and the choice of $\beta_0 > 0$.

For $w_4^+(x,t) < k_4$, we only need to prove that

$$\sigma_0\beta_0\phi_4'(\eta) - \beta_0 p_4 \geq -\frac{1}{\tau_2}p_4 + \frac{1}{\tau_2}\delta^{-1}e^{\beta_0 t}[g(w_1^+(x,t)) - g(\phi_1(\eta))] = -\frac{1}{\tau_2}p_4 + \frac{1}{\tau_2}g'(\theta)p_1, \quad (5.4)$$

where $\theta \in [\phi_1(\eta), w_1^+(x,t)]$. For $|\eta| > M$, by the choice of M, it is sufficient to show

$$\sigma_0\beta_0\phi_4'(\eta) - \beta_0 p_4 \geq -\frac{1}{\tau_2}p_4 + \frac{1}{\tau_2}\varrho p_1. \quad (5.5)$$

For $|\eta| \leq M$, by the choice of σ_0, we have

$$\begin{aligned}
&\sigma_0\beta_0\phi_4'(\eta) - \beta_0 p_4 + \frac{1}{\tau_2}p_4 - \frac{1}{\tau_2}\max\{|g'(\eta)| | \eta \in [0, k_1]\}p_1\\
&\geq m_0\sigma_0\beta_0 - p_0\left(\beta_0 + \frac{1}{\tau_2}\max\{|g'(\eta)| | \eta \in [0, k_1]\}\right) \geq 0.
\end{aligned}$$

For $|\eta| > M$, (5.5) holds by $\phi_4'(\eta) \geq 0$ and the choice of $\beta_0 > 0$. This completes the proof. □

Now we define another pair of upper and lower solutions. Fix a function $\zeta(\cdot) \in C^\infty(\mathbb{R})$ with the following properties:

$$\zeta(x) = 0 \text{ on } (-\infty, 0]; \quad \zeta(x) = 1 \text{ on } [4, \infty); \quad \zeta'(x) \in (0, 1); \quad |\zeta''(x)| \leq 1 \text{ on } (0, 4).$$

Lemma 5.3 *Assume that (A) holds. Then, for every $\delta \in (0, \frac{1}{2}]$, there exist $\epsilon = \epsilon(\delta) > 0$ and $C = C(\delta) > 0$ such that, for any $\xi \in \mathbb{R}$, $v^+(x,t)$ and $v^-(x,t)$ are an upper solution and a lower solution of (4.5) on $\mathbb{R}^+$, respectively, where $v^\pm(x,t) = (v_1^\pm(x,t), v_2^\pm(x,t), v_3^\pm(x,t), v_4^\pm(x,t))$ defined by*

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$$v_i^+(x,t) = \min\{k_i + \delta p_i - [k_i - (1-2\delta)p_i e^{-\epsilon t}]\zeta(\varsigma_{\epsilon,C}^+(x,t)), k_i\},$$
$$v_i^-(x,t) = \max\{-\delta p_i + [k_i - (1-2\delta)p_i e^{-\epsilon t}]\zeta(\varsigma_{\epsilon,C}^-(x,t)), 0\},$$
$$\varsigma_{\epsilon,C}^{\pm}(x,t) = \mp\epsilon(x - \xi \pm Ct), \quad i = 1,2,3,4.$$

Proof It suffices to prove $v^+(x,t)$ is an upper solution of (4.5) because the proof of a lower solution of (4.5) is analogous. Since $w_i^+(x,t) = k_i$ is an upper solution of (4.5), we only consider the case $v_i^+(x,t) < k_i, i = 1,2,3,4$.

We can directly calculate that for $i = 1,2,3,4$,

$$\frac{\partial v_i^+(x,t)}{\partial t} = \epsilon C[k_i - (1-2\delta)p_i e^{-\epsilon t}]\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - \epsilon(1-2\delta)p_i e^{-\epsilon t}\zeta(\varsigma_{\epsilon,C}^+(x,t)),$$
$$\geq \epsilon C(k_i - p_i)\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - k_i\epsilon,$$
$$\frac{\partial^2 v_i^+(x,t)}{\partial x^2} = -\epsilon^2[k_i - (1-2\delta)p_i e^{-\epsilon t}]\zeta''(\varsigma_{\epsilon,C}^+(x,t)) \leq k_i\epsilon^2.$$

Choose $\epsilon = \epsilon(\delta)$ sufficiently small satisfying

$$\begin{aligned} &-k_1\epsilon - d_1k_1\epsilon^2 + \delta(p_1 - \alpha p_3) > 0, \quad -k_2\epsilon - d_2k_2\epsilon^2 + \delta(\beta p_2 - p_4) > 0, \\ &-k_3\epsilon - d_2k_3\epsilon^2 + \frac{1}{\tau_1}\delta(p_3 - p_2) > 0, \quad -k_4\epsilon - d_1k_4\epsilon^2 + \frac{1}{\tau_2}\delta(p_4 - \varrho p_1) > 0. \end{aligned} \tag{5.6}$$

By p_1, p_4 small enough $(\frac{\delta p_1}{2k_1} + \frac{\delta p_4}{2k_4} < 1)$ and $\zeta'(s) > 0$ for $\zeta(s) \in (0,1)$, we can take $C = C(\delta)$ satisfying

$$\begin{aligned} \min\Big\{&\epsilon C(k_4 - p_4)\zeta'(s) - k_4\epsilon - d_1k_4\epsilon^2 + \frac{1}{\tau_2}v_4^+(x,t) \\ &-\frac{1}{\tau_2}g(v_1^+(x,t)) \Big| \frac{\delta p_4}{2k_4} \leq \zeta(s) \leq 1 - \frac{\delta p_1}{2k_1}, v_1^+ \in [\delta p_1, k_1], v_4^+ \in [\delta p_4, k_4]\Big\} > 0. \end{aligned} \tag{5.7}$$

For $v_1^+(x,t) < k_1$, by $k_1 = \alpha k_3$ and (5.6),

$$\begin{aligned} &\frac{\partial v_1^+(x,t)}{\partial t} - d_1\frac{\partial^2 v_1^+(x,t)}{\partial x^2} + v_1^+(x,t) - \alpha v_3^+(x,t) \\ &\geq \epsilon C(k_1 - p_1)\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - k_1\epsilon - d_1k_1\epsilon^2 \\ &\quad + \delta(p_1 - \alpha p_3) + (1-2\delta)(p_1 - \alpha p_3)e^{-\epsilon t}\zeta(\varsigma_{\epsilon,C}^+(x,t)) \\ &\geq -k_1\epsilon - d_1k_1\epsilon^2 + \delta(p_1 - \alpha p_3) > 0. \end{aligned}$$

For $v_2^+(x,t) < k_2$, by $\beta k_2 = k_4$ and (5.6),

$$\begin{aligned} &\frac{\partial v_2^+(x,t)}{\partial t} - d_2\frac{\partial^2 v_2^+(x,t)}{\partial x^2} + \beta v_2^+(x,t) - v_4^+(x,t) \\ &\geq \epsilon C(k_2 - p_2)\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - k_2\epsilon - d_2k_2\epsilon^2 \\ &\quad + \delta(\beta p_2 - p_4) + (1-2\delta)(\beta p_2 - p_4)e^{-\epsilon t}\zeta(\varsigma_{\epsilon,C}^+(x,t)) \\ &\geq -k_2\epsilon - d_2k_2\epsilon^2 + \delta(\beta p_2 - p_4) > 0. \end{aligned}$$

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For $v_3^+(x,t) < k_3$, by $k_2 = k_3$ and (5.6),

$$\frac{\partial v_3^+(x,t)}{\partial t} - d_2 \frac{\partial^2 v_3^+(x,t)}{\partial x^2} + \frac{1}{\tau_1} v_3^+(x,t) - \frac{1}{\tau_1} v_2^+(x,t)$$
$$\geq \epsilon C(k_3 - p_3)\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - k_3\epsilon - d_2 k_3 \epsilon^2$$
$$+ \frac{1}{\tau_1}[\delta(p_3 - p_2) + (1-2\delta)(p_3 - p_2)e^{-\epsilon t}\zeta(\varsigma_{\epsilon,C}^+(x,t))]$$
$$\geq -k_3\epsilon - d_2 k_3\epsilon^2 + \frac{1}{\tau_1}\delta(p_3 - p_2) > 0.$$

For $v_4^+(x,t) < k_4$, we have two cases (since $v_4^+(x,t) = k_4$ for $\zeta(\varsigma_{\epsilon,C}^+(x,t)) \leq \frac{\delta p_4}{2k_4}$):

Case (i): when $\zeta(\varsigma_{\epsilon,C}^+(x,t)) > 1 - \frac{\delta p_1}{2k_1}$, we have $\delta p_1 < v_1^+(x,t) < p_1 - \frac{\delta p_1}{2}$. Hence, by the mean value theorem, $k_4 - \varrho k_1 = \frac{b}{\alpha}(\beta - \alpha\varrho) > 0$ and (5.6),

$$\frac{\partial v_4^+(x,t)}{\partial t} - d_1 \frac{\partial^2 v_4^+(x,t)}{\partial x^2} + \frac{1}{\tau_2} v_4^+(x,t) - \frac{1}{\tau_2} g(v_1^+(x,t))$$
$$\geq \epsilon C(k_4 - p_4)\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - k_4\epsilon - d_1 k_4\epsilon^2 + \frac{1}{\tau_2} v_4^+(x,t) - \frac{1}{\tau_2} g'(\theta) v_1^+(x,t)$$
$$\geq \epsilon C(k_4 - p_4)\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - k_4\epsilon - d_1 k_4\epsilon^2 + \frac{1}{\tau_2}(k_4 - \varrho k_1)[1 - \zeta(\varsigma_{\epsilon,C}^+(x,t))]$$
$$+ \frac{1}{\tau_2}\delta(p_4 - \varrho p_1) + \frac{1}{\tau_2}(1-2\delta)(p_4 - \varrho p_1)e^{-\epsilon t}\zeta(\varsigma_{\epsilon,C}^+(x,t))$$
$$\geq -k_4\epsilon - d_1 k_4\epsilon^2 + \frac{1}{\tau_2}\delta(p_4 - \varrho p_1) > 0,$$

where $\theta \in [0, v_1^+(x,t)]$.

Case (ii): when $\frac{\delta p_4}{2k_4} \leq \zeta(\varsigma_{\epsilon,C}^+(x,t)) \leq 1 - \frac{\delta p_1}{2k_1}$, it follows from (5.7) that

$$\frac{\partial v_4^+(x,t)}{\partial t} - d_1 \frac{\partial^2 v_4^+(x,t)}{\partial x^2} + \frac{1}{\tau_2} v_4^+(x,t) - \frac{1}{\tau_2} g(v_1^+(x,t))$$
$$\geq \min\{\epsilon C(k_4 - p_4)\zeta'(\varsigma_{\epsilon,C}^+(x,t)) - k_4\epsilon - d_1 k_4\epsilon^2$$
$$+ \frac{1}{\tau_2} v_4^+(x,t) - \frac{1}{\tau_2} g(v_1^+(x,t)) \mid v_1^+ \in [\delta p_1, k_1], v_4^+ \in [\delta p_4, k_4]\} > 0.$$

The proof is completed. □

Remark 5.2 *Obviously, $v_i^+(x,t)$ and $v_i^-(x,t), i = 1, 2, 3, 4$ in Lemma 5.3, imply that:*

(P1) $v_i^+(x,0) = k_i$ *on* $[\xi, \infty)$; $v_i^+(x,0) \geq (1-\delta)p_i$ *on* $(-\infty, \infty)$; $v_i^+(x,t) \leq \delta p_i + (1-2\delta)p_i e^{-\epsilon t}$ *on* $(-\infty, \xi - Ct - 4\epsilon^{-1}] \times \mathbb{R}^+$.

(P2) $v_i^-(x,0) = 0$ *on* $(-\infty, \xi]$; $v_i^-(x,0) \leq k_i - (1-\delta)p_i$ *on* $(-\infty, \infty)$; $v_i^-(x,t) \geq k_i - \delta p_i - (1-2\delta)p_i e^{-\epsilon t}$ *on* $[\xi + Ct + 4\epsilon^{-1}, \infty) \times \mathbb{R}^+$.

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The following result is similar to Lemma 2.5 in [31] and we omit the proof here.

Lemma 5.4 *Let* $\Phi(\xi)=(\phi_1(\xi),\phi_2(\xi),\phi_3(\xi),\phi_4(\xi))$ *of (4.5) be any travelling wave front satisfying* $0\le\phi_i(\xi)\le k_i, \xi=x+ct\in\mathbb{R}$, *then* $\lim_{|\xi|\to\infty}\Phi'(\xi)=\mathbf{0}$.

Next, we prove the global asymptotic stability and uniqueness of travelling wave front. To do this, we first give the following two lemmas.

Let $\Phi(x+ct)=(\phi_1(x+ct),\phi_2(x+ct),\phi_3(x+ct),\phi_4(x+ct))$ be a travelling wave front of (4.5). From Lemma 5.2, we define $w^{\pm}(x,t,\xi_0,\delta)=(w_1^{\pm}(x,t,\xi_0,\delta),w_2^{\pm}(x,t,\xi_0,\delta),w_3^{\pm}(x,t,\xi_0,\delta),w_4^{\pm}(x,t,\xi_0,\delta))$ by

$$\begin{aligned}
&w_i^+(x,t,\xi_0,\delta):=\min\{\phi_i(x+ct+\xi_0+\sigma_0\delta(1-e^{-\beta_0 t}))+\delta p_i e^{-\beta_0 t},k_i\},\\
&w_i^-(x,t,\xi_0,\delta):=\max\{\phi_i(x+ct+\xi_0-\sigma_0\delta(1-e^{-\beta_0 t}))-\delta p_i e^{-\beta_0 t},0\},\\
&x\in\mathbb{R},t\in[0,\infty),\xi_0\in\mathbb{R},\ \text{and}\ \delta\in[0,\infty),i=1,2,3,4,
\end{aligned}$$

and ξ_0 and β_0 are as in Lemma 5.2.

Lemma 5.5 *Assume that* $\Phi(x+ct)=(\phi_1(x+ct),\phi_2(x+ct),\phi_3(x+ct),\phi_4(x+ct))$ *is a travelling wave front of (4.5). Then there exists* $\epsilon^*>0$ *such that, if* $u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t),u_4(x,t))$ *is a solution of (4.5) on* $[0,\infty)$ *with the initial data* $u(x,0), 0\le u_i(x,0)\le k_i$ *for all* $x\in\mathbb{R}, i=1,2,3,4$, *and the following is true:*

$$w^-(x,0,cT+\xi,\delta)\le u(x,T)\le w^+(x,0,cT+\xi+h,\delta)$$

on $\mathbb{R}$ *provided that for some* $\xi\in\mathbb{R},T\ge 0,h>0$ *and* $\delta\in(0,\min\{\frac{\delta_0}{2},\frac{1}{\sigma_0}\})$, *then for every* $t\ge T+1$, *there exist* $\hat\xi(t),\hat\delta(t)$ *and* $\hat h(t)$ *satisfying*

$$w^-(x,0,ct+\hat\xi(t),\hat\delta(t))\le u(x,t)\le w^+(x,0,ct+\hat\xi(t)+\hat h(t),\hat\delta(t)),$$

where $\hat\xi(t),\hat h(t)$ *and* $\hat\delta(t)$ *are as follows*

$$\begin{aligned}
&\hat\xi(t)\in[\xi-\sigma_0\delta,\xi+h+\sigma_0\delta],\\
&\hat h(t)\in[0,h-\sigma_0\epsilon^*\min\{h,1\}+2\sigma_0\delta],\\
&\hat\delta(t)=(\delta e^{-\beta_0}+\epsilon^*\min\{h,1\})e^{-\beta_0(t-(T+1))}.
\end{aligned}$$

Proof The result of Lemma 5.2 shows that $w^+(x,t,cT+\xi+h,\delta)$ and $w^-(x,t,cT+\xi,\delta)$, respectively, are upper and lower solutions of (4.5). Obviously, $\tilde u(x,t)=u(x,T+t)(t\ge 0)$ is a solution of (4.5) with initial value $\tilde u(x,0)=u(x,T)$ for $x\in\mathbb{R}$. From comparison principle, we have

$$w^-(x,t,cT+\xi,\delta)\le u(x,T+t)\le w^+(x,t,cT+\xi+h,\delta)\ \text{for}\ (x,t)\in\mathbb{R}\times\mathbb{R}^+.$$

That is, for $i=1,2,3,4$,

$$\max\{\phi_i(\eta^-(x,t,T))-\delta p_i e^{-\beta_0 t},0\}\le u_i(x,T+t)\le\min\{\phi_i(\eta^+(x,t,T)+h)+\delta p_i e^{-\beta_0 t},k_i\} \tag{5.8}$$

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for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, where $\eta^{\pm}(x,t,T)=x+c(T+t)+\xi\pm\sigma_0\delta(1-e^{-\beta_0 t})$. Set $y=-cT-\xi$. By comparison principle, it follows that for every nonnegative constant L, any $x\in\mathbb{R}$ satisfying $|x-y|\le L$ and every $t>0$, $i=1,2,3,4$,

$$u_i(x,T+t)-w_i^-(x,t,cT+\xi,\delta)\ge J_i(L,t)\int_y^{y+1}(u_i(z,T)-w_i^-(z,0,cT+\xi,\delta))dz. \tag{5.9}$$

By Lemma 5.4, $\lim_{|x|\to\infty}\phi_i'(x)=0, i=1,2,3,4$. Fix $M>0$ such that $\phi_i'(x)\le\frac{\min_{1\le i\le 4}\{p_i\}}{2\sigma_0}$ for all $|x|\ge M, i=1,2,3,4$. Let

$$L=M+|c|+1,\quad \bar h=\min\{h,1\}\quad\text{and}\quad \epsilon_1=\frac{1}{2}\min_{1\le i\le 4}\{\phi_i'(x):|x|\le 2\}>0.$$

Since

$$w_i^-(z,0,-y,\delta)<\phi_i(z-y),\quad w_i^+(z,0,-y+\bar h,\delta)>\phi_i(z-y+\bar h),\quad i=1,2,3,4,$$

it follows that

$$\begin{aligned}&\int_y^{y+1}[w_i^+(z,0,cT+\xi+\bar h,\delta)-w_i^-(z,0,cT+\xi,\delta)]dz\\&\quad>\int_y^{y+1}[\phi_i(z+cT+\xi+\bar h)-\phi_i(z+cT+\xi)]dz=\int_y^{y+1}[\phi_i(z+\bar h)-\phi_i(z)]dz\ge 2\epsilon_1\bar h.\end{aligned}$$

Therefore, either (i) or (ii) is true, where (i) and (ii) are as follows:

(i) $\int_y^{y+1}[u_i(z,T)-w_i^-(z,0,cT+\xi,\delta)]dz\ge\epsilon_1\bar h$;

(ii) $\int_y^{y+1}[w_i^+(z,0,cT+\xi+\bar h,\delta)-u_i(z,T)]dz\ge\epsilon_1\bar h$.

We only need to consider the case (i) since the other is similar. For any $|x-y|\le L$, letting $t=1$ in (5.9), it holds

$$\begin{aligned}u_i(x,T+1)&\ge w_i^-(x,1,cT+\xi,\delta)+J_i(L)\epsilon_1\bar h\\&\ge\phi_i(x-y+c-\sigma_0\delta(1-e^{-\beta_0}))-\delta p_i e^{-\beta_0}+J_0(L)\epsilon_1\bar h,\quad i=1,2,3,4,\end{aligned}$$

where $J_0(L)=\min_{1\le i\le 4}\{J_i(L,1)\}$. Let

$$L_1=L+|c|+2,\quad \epsilon^*=\min_{1\le i\le 4}\left\{\min_{|x|\le L_1}\frac{J_0(L)\epsilon_1}{2\sigma_0\phi_i'(x)},\frac{1}{2\sigma_0},\frac{\delta_0}{2}\right\}.$$

Using the mean value theorem, we have that for all $|x-y|\le L$,

$$\begin{aligned}&\phi_i(x-y+c+2\sigma_0\epsilon^*\bar h-\sigma_0\delta(1-e^{-\beta_0}))-\phi_i(x-y+c-\sigma_0\delta(1-e^{-\beta_0}))\\&=\phi_i'(x-y+c+2\theta_i\sigma_0\epsilon^*\bar h-\sigma_0\delta(1-e^{-\beta_0}))2\sigma_0\epsilon^*\bar h\le J_0(L)\epsilon_1\bar h,\quad\theta_i\in(0,1),\quad i=1,2,3,4.\end{aligned}$$

Hence,

$$u_i(x,T+1)\ge\phi_i(\eta^-(x,1,T)+2\sigma_0\epsilon^*\bar h)-\delta p_i e^{-\beta_0}, i=1,2,3,4. \tag{5.10}$$

Together with the mean value theorem and the definitions of M, L, we have that for any $|x-y| \geq L$,

$$\phi_i(\eta^-(x,1,T)) - \phi_i(\eta^-(x,1,T) + 2\sigma_0\epsilon^*\bar{h}) = \phi_i'(\eta^-(x,1,T) \\ - 2\theta_i\sigma_0\epsilon^*\bar{h})(-2\sigma_0\epsilon^*\bar{h}) \geq -\epsilon^*\bar{h}p_i, \quad \theta_i \in (0,1), \tag{5.11}$$

$i = 1, 2, 3, 4$. That is, for all $|x-y| \geq L$,

$$\phi_i(\eta^-(x,1,T)) \geq \phi_i(\eta^-(x,1,T) + 2\sigma_0\epsilon^*\bar{h}) - \epsilon^*\bar{h}p_i, \quad i = 1,2,3,4, \tag{5.12}$$

and therefore, by (5.8) with $t = 1$, it holds

$$u_i(x, T+1) \geq \max\{\phi_i(\eta^-(x,1,T) + 2\sigma_0\epsilon^*\bar{h}) - \epsilon^*\bar{h}p_i - \delta p_i e^{-\beta_0}, 0\} \tag{5.13}$$

for all $|x-y| \geq L$, $i = 1, 2, 3, 4$. By (5.10) and (5.13), it follows that for all $x \in \mathbb{R}$, $i = 1, 2, 3, 4$,

$$\begin{aligned} u_i(x, T+1) &\geq \max\{\phi_i(\eta^-(x,1,T) + 2\sigma_0\epsilon^*\bar{h}) - (\delta e^{-\beta_0} + \epsilon^*\bar{h})p_i, 0\} \\ &= \max\{\phi_i(x+\iota) - (\delta e^{-\beta_0} + \epsilon^*\bar{h})p_i, 0\}, \end{aligned} \tag{5.14}$$

where

$$\iota = c(T+1) + 2\sigma_0\epsilon^*\bar{h} + \xi + \bar{\xi}, \quad \bar{\xi} = \sigma_0\delta(e^{-\beta_0} - 1). \tag{5.15}$$

Then

$$u(x, T+1) \geq w^-(x, 0, \iota, \bar{\mu}), \quad x \in \mathbb{R}, \tag{5.16}$$

where $\bar{\mu} = \delta e^{-\beta_0} + \epsilon^*\bar{h} \leq \delta_0$, then, by comparison principle and the choice of ϵ^*, it yields

$$w^-(x, \tilde{t}, \iota, \bar{\mu}) \leq u(x, T+1+\tilde{t}) \text{ for } \tilde{t} \geq 0. \tag{5.17}$$

Then for all $t \geq T+1$, letting $\tilde{t} = t - (T+1)$ in (5.17), we have

$$\begin{aligned} u_i(x,t) &\geq w_i^-(x, t-(T+1), \iota, \bar{\mu}) \\ &= \phi_i(x + ct - c(T+1) + \iota - \sigma_0\bar{\mu}(1 - e^{-\beta_0(t-(T+1))})) - \bar{\mu}p_i e^{-\beta_0(t-(T+1))} \\ &\geq \phi_i(x + ct - c(T+1) + \iota - \sigma_0\bar{\mu}) - \hat{\delta}(t)p_i, \quad i = 1,2,3,4, \end{aligned} \tag{5.18}$$

where $\hat{\delta}(t) = \bar{\mu}e^{-\beta_0(t-(T+1))}$. Since $\phi_i(\cdot)$ is monotone, together with the choice of η and (5.15), it holds

$$u_i(x,t) \geq w_i^-(x, 0, ct + \hat{\xi}(t), \hat{\delta}(t)), \quad x \in \mathbb{R}, i = 1,2,3,4, \tag{5.19}$$

where

$$\hat{\xi}(t) = 2\sigma_0\epsilon^*\bar{h} + \xi - \sigma_0\delta(1 - e^{-\beta_0}) - \sigma_0\bar{\mu} = \sigma_0\epsilon^*\bar{h} + \xi - \sigma_0\delta.$$

Hence, we have

$$\hat{\xi}(t) \geq \xi - \sigma_0\delta, \tag{5.20}$$

and, from the definition of ϵ^*,

$$\hat{\xi}(t) \leq \xi + \sigma_0\epsilon^*\bar{h} \leq \xi + h + \sigma_0\delta. \tag{5.21}$$

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For every $t \geq T$, by the first inequality of (5.8), it follows that

$$\begin{aligned} u_i(x,t) &\leq \min\{\phi_i(\eta^+(x,t-T,T)+h)+\delta p_i e^{-\beta_0(t-T)}, k_i\} \\ &\leq \min\{\phi_i(x+ct+\xi+h+\sigma_0\delta)+\hat{\delta}(t)p_i, k_i\}, \quad x\in\mathbb{R}, i=1,2,3,4. \end{aligned} \tag{5.22}$$

Hence, for any $t \geq T+1$, we have

$$u_i(x,t) \leq w_i^+(x,0,ct+\hat{\xi}(t)+\hat{h}(t),\hat{\delta}(t)), \quad x\in\mathbb{R}, i=1,2,3,4,$$

that is, for $x \in \mathbb{R}$,

$$u(x,t) \leq w^+(x,0,ct+\hat{\xi}(t)+\hat{h}(t),\hat{\delta}(t)), \tag{5.23}$$

where

$$\hat{h}(t) = \xi + h + \sigma_0\delta - \hat{\xi}(t) = h - \sigma_0\epsilon^*\bar{h} + 2\sigma_0\delta. \tag{5.24}$$

From the definition of ϵ^*, it holds $h - \sigma_0\epsilon^*\bar{h} \geq h - \sigma_0\epsilon^* h > 0$, and so

$$\hat{h}(t) \in (0, h - \sigma_0\epsilon^*\bar{h} + 2\sigma_0\delta]. \tag{5.25}$$

Combining (5.19) and (5.23), now we complete the proof. □

Lemma 5.6 *Let $\Phi(x+ct) = (\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct), \phi_4(x+ct))$ be a travelling wave front of (4.5), and $\Psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))$ with $\psi_i \in [0, k_i]$ be such that*

$$\lim_{x\to\infty} \psi_i(x) > k_i - p_i, \qquad \lim_{x\to-\infty} \psi_i(x) < p_i, \quad i=1,2,3,4.$$

Then, for every $\delta > 0$, there exist $T = T(\Psi,\delta) > 0$, $\xi = \xi(\Psi,\delta) \in \mathbb{R}$ and $h = h(\Psi,\delta) > 0$ such that

$$w^-(x,0,cT+\xi,\delta) \leq u(x,T,\Psi) \leq w^+(x,0,cT+\xi+h,\delta), \quad x\in\mathbb{R}.$$

Proof By comparison principle, $u(x,t,\Psi) = (u_1(x,t,\psi_1), u_2(x,t,\psi_2), u_3(x,t,\psi_3), u_4(x,t,\psi_4))$ exists on $\mathbb{R}^+$ and $0 \leq u_i(x,t,\psi_i) \leq k_i$, $(x,t) \in \mathbb{R}\times\mathbb{R}^+$, $i=1,2,3,4$. For every $\delta > 0$, one can take $\delta_1 = \delta_1(\delta,\Psi) \in (0, \min\{\delta,\delta_0\})$ satisfying

$$\lim_{x\to\infty} \psi_i(x) > k_i - (1-\delta_1)p_i, \qquad \lim_{x\to-\infty} \psi_i(x) < (1-\delta_1)p_i, \quad i=1,2,3,4.$$

So we can choose $M = M(\Psi,\delta_1) > 0$ such that, for $i=1,2,3,4$,

$$\psi_i(x) \leq (1-\delta_1)p_i \text{ for all } x \leq -M, \quad \psi_i(x) \geq k_i - (1-\delta_1)p_i \text{ for all } x \geq M. \tag{5.26}$$

Let $\epsilon = \epsilon(\delta_1)$, $C = C(\delta_1)$ and $v^\pm(x,t)$ be described by Lemma 5.3 with δ replaced by δ_1 and $\xi = \xi^\pm$, where $\xi^\pm = \mp M$. Together with (5.26) and Remark 5.2, we have that for $i=1,2,3,4$,

$$\begin{aligned} &\psi_i(x) \leq (1-\delta_1)p_i \leq v_i^+(x,0) \quad \text{for } x \leq -M, \\ &\psi_i(x) \leq k_i = v_i^+(x,0) \quad \text{for } x \geq \xi^+ = -M \end{aligned}$$

and

$$\psi_i(x) \geq k_i - (1-\delta_1)p_i \geq v_i^-(x,0) \quad \text{for } x \geq M, \quad \psi_i(x) \geq 0 = v_i^-(x,0) \quad \text{for } x \leq M.$$

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Then

$$v^-(x,0) \le \Psi(x) \le v^+(x,0), \quad x \in \mathbb{R}. \tag{5.27}$$

By Lemma 5.3 and comparison principle, we have

$$v^-(x,t) \le u(x,t,\Psi) \le v^+(x,t) \quad \text{for all } x \in \mathbb{R}, t \ge 0. \tag{5.28}$$

In view of $\delta_1 < \delta$, we take $T > 0$ sufficiently large such that, for any $t \ge T$,

$$\delta_1 p_i + (1-2\delta_1)p_i e^{-\epsilon t} < \delta p_i \text{ and } k_i - \delta_1 p_i + (1-2\delta_1)p_i e^{-\epsilon t} > k_i - \delta p_i, i = 1,2,3,4$$

and hence, again by Remark 5.2, for $i = 1,2,3,4$,

$$u_i(x,t,\psi_i) \le v_i^+(x,t) < \delta p_i \quad \text{when } \ x \le x^-(t) \tag{5.29}$$

and

$$u_i(x,t,\psi_i) \ge v_i^-(x,t) > k_i - \delta p_i \quad \text{when } \ x \ge x^+(t), \tag{5.30}$$

where $x^{\pm}(t) = \xi^{\mp} \pm Ct \pm 4\epsilon^{-1}$. Together with (5.29) and (5.30), we have

$$u_i(x,T,\psi_i) < \delta p_i \quad \text{for any } x \le x^-(T), \quad u_i(x,T,\psi_i) > k_i - \delta p_i \quad \text{for any } x \ge x^+(T), \quad i = 1,2,3,4. \tag{5.31}$$

By $\lim_{x\to-\infty} \phi_i(x) = 0$ and $\lim_{x\to\infty} \phi_i(x) = k_i$, $i = 1,2,3,4$, we can choose $H > 0$ large enough such that $\frac{H}{2} > x^+(T)$, $-\frac{H}{2} < x^-(T)$, and

$$\phi_i(x) + \delta p_i > k_i \quad \text{for} \quad x \ge \frac{H}{2} \text{ and } \phi_i(x) - \delta p_i < 0 \quad \text{for} \quad x \le -\frac{H}{2}. \tag{5.32}$$

Since $0 \le \phi_i(x) \le k_i$ and $0 \le u_i(x,t,\psi_i) \le k_i$ for any $x \in \mathbb{R}$ and $t \in [0,\infty)$, and together with (5.31) and (5.32), we have that for $i = 1,2,3,4$,

$$\max\{\phi_i(-H+x) - \delta p_i, 0\} \le u_i(x,T,\psi_i) \le \min\{\phi_i(H+x) + \delta p_i, k_i\} \text{ for } x \in \mathbb{R}. \tag{5.33}$$

Let $\xi_0 = -H - cT$, $h_0 = 2H > 0$. It is clear that (5.33) implies that, for $i = 1,2,3,4$,

$$\max\{\phi_i(x+cT+\xi_0) - \delta p_i, 0\} \le u_i(x,T,\psi_i) \le \min\{\phi_i(x+cT+\xi_0+h_0) + \delta p_i, k_i\}, \quad x \in \mathbb{R}. \tag{5.34}$$

Let $\xi = \xi_0$ and $h = h_0 > 0$. Then it follows from (5.34) that for any $x \in \mathbb{R}$,

$$\begin{aligned} w_i^-(x,0,cT+\xi,\delta) &= w_i^-(x,0,cT+\xi_0,\delta) \le u_i(x,T,\psi_i) \\ w_i^+(x,0,cT+\xi+h,\delta) &= w_i^+(x,0,cT+\xi_0+h_0,\delta) \ge u_i(x,T,\psi_i), \quad i = 1,2,3,4. \end{aligned}$$

Hence, we prove the conclusion of this lemma. This completes the proof. □

Theorem 5.1 *Assume that (A) holds and (4.5) is a travelling wave front* $\Phi(x+ct) = (\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct), \phi_4(x+ct))$. *Then* $\Phi(x+ct)$ *is globally asymptotically stable with phase shift in the sense that there exists a positive constant* k *such that for every* $\psi_i \in [0,k_i]$ *satisfying*

$$\lim_{x\to\infty} \psi_i(x) > k_i - p_i, \qquad \lim_{x\to-\infty} \psi_i(x) < p_i, \quad i = 1,2,3,4,$$

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the solution $u(x,t,\Psi)=(u_1(x,t,\psi_1),u_2(x,t,\psi_2),u_3(x,t,\psi_3),u_4(x,t,\psi_4))$ *of (4.5) satisfies*

$$\| u(\cdot,t,\Psi)-\Phi(\cdot+ct+\xi)\| \le Ke^{-kt},\quad t\ge 0$$

for some $K=K(\Psi)>0$ *and* $\xi=\xi(\Psi)\in\mathbb{R}$, $\Psi(x)=(\psi_1(x),\psi_2(x),\psi_3(x),\psi_4(x))$*, where* $\|\cdot\|$ *is the general super norm in* $\mathbb{R}^4$.

Proof Let $\beta_0,\sigma_0,\delta_0$ be described by Lemma 5.2, and then let ϵ^* be described by Lemma 5.5 with ϵ^* satisfying $\sigma_0\epsilon^*<1$. Take $0<\delta^*<\min\{\frac{\delta_0}{2},\frac{1}{2\sigma_0}\}$ such that

$$1>k^*:=\sigma_0\epsilon^*-2\sigma_0\delta^*>0,$$

and fix $t^*\ge 1$ satisfying

$$e^{-\beta_0(t^*-1)}\Big(1+\frac{\epsilon^*}{\delta^*}\Big)<1-k^*.$$

We need to prove two conclusions.

Conclusion 1. There exist two constants $T^*=T^*(\Psi)>0,\xi^*=\xi^*(\Psi)\in\mathbb{R}$ satisfying

$$w^-(x,0,cT^*+\xi^*,\delta^*)\le u(x,T^*,\Psi)\le w^+(x,0,cT^*+\xi^*+1,\delta^*) \text{ for any } x\in\mathbb{R}. \tag{5.35}$$

In fact, by Lemma 5.6, there exist three constants $T=T(\Psi)>0,\xi=\xi(\Psi)\in\mathbb{R}$ and $h=h(\Psi)>0$ satisfying

$$w^-(x,0,cT+\xi,\delta^*)\le u(x,T,\Psi)\le w^+(x,0,cT+\xi+h,\delta^*),\quad x\in\mathbb{R}. \tag{5.36}$$

When $h\le 1$, (5.35) holds since $\phi_i(\cdot), i=1,2,3,4$, are monotone. Then when $h>1$, denote

$$N=\max\{m|\ m\in\mathbb{Z}^+ \text{ and } mk^*<h\}.$$

In view of $k^*\in(0,1)$ and $h>1$, then $N\ge 1, h\in(Nk^*,(N+1)k^*]$, and furthermore, $h-Nk^*\in(0,1)$. Note that $\bar{h}:=\min\{1,h\}=1$. Together with (5.36), Lemma 5.5 and the definitions of t^* and k^*, it holds

$$\begin{aligned}&w^-(x,0,c(T+t^*)+\hat{\xi}(T+t^*),\hat{\delta}(T+t^*))\\&\quad\le u(x,T+t^*,\Psi)\\&\quad\le w^+(x,0,c(T+t^*)+\hat{\xi}(T+t^*)+\hat{h}(T+t^*),\hat{\delta}(T+t^*)),\quad x\in\mathbb{R},\end{aligned} \tag{5.37}$$

where

$$\begin{aligned}&\hat{\xi}(T+t^*)\in[\xi-\sigma_0\delta^*,\xi+h+\sigma_0\delta^*],\\&0\le\hat{h}(T+t^*)\le h-\sigma_0\epsilon^*+2\sigma_0\delta^*,\\&\hat{\delta}(T+t^*)=(\delta^*e^{-\beta_0}+\epsilon^*)e^{-\beta_0(t^*-1)}\le(1-k^*)\delta^*<\delta^*.\end{aligned}$$

Using the similar argument N times, then for some $\xi^*=\hat{\xi}\in\mathbb{R},\hat{\delta}\in(0,\delta^*],0\le\hat{h}\le h-Nk^*<1$, (5.37) still holds when $T+t^*$ is replaced by $T^*=T+Nt^*$ Since $\Phi(\cdot)$ is monotone, it follows that (5.35) holds.

Conclusion 2. Let $p=2\sigma_0\delta^*+1, T_n=T^*+nt^*,\delta_n^*=(1-k^*)^n\delta^*$ and $h_n=(1-k^*)^n, n\ge 0$. So we can choose a sequence $\{\xi_n\}_{n=0}^{\infty}\subset\mathbb{R}$ with $\xi_0=\xi^*$ satisfying

$$|\xi_{n+1}-\xi_n|\le ph_n \quad \text{for any } n\ge 0 \tag{5.38}$$

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and

$$w^-(x,0,cT_n+\xi_n,\delta_n^*) \le u(x,T_n,\Psi) \le w^+(x,0,cT_n+\xi_n+h_n,\delta_n^*), \quad \text{for any } x\in\mathbb{R}, n\ge 0. \quad (5.39)$$

Indeed, Conclusion 1 implies that (5.39) holds when $n=0$. Now we assume that (5.39) holds for some $n=m\ge 0$. From Lemma 5.5 with $T=T_m, \xi=\xi_m, h=h_m, \delta=\delta_m^*$, and $t=T_m+t^*=T_{m+1}$ (since $t\ge 1$), it follows that

$$w^-(x,0,cT_{m+1}+\hat{\xi},\hat{\delta}) \le u(x,T_{m+1},\Psi) \le w^+(x,0,cT_{m+1}+\hat{\xi}+\hat{h},\hat{\delta}), x\in\mathbb{R}, \quad (5.40)$$

where

$$\begin{aligned}
\hat{\xi} &\in [\xi_m-\sigma_0\delta_m^*, \xi_m+h_m+\sigma_0\delta_m^*],\\
\hat{\delta} &= (\delta_m e^{-\beta_0}+\epsilon^* h_m)e^{-\beta_0(T_{m+1}-T_m-1)}\\
&\le (1-k^*)^m\delta^*\left[\left(1+\frac{\epsilon^*}{\delta^*}\right)e^{-\beta_0(t^*-1)}\right] \le (1-k^*)^m\delta^*(1-k^*)=\delta_{m+1}^*,\\
\hat{h} &\le h_m-\sigma_0\epsilon^* h_m+2\sigma_0\delta_m=(1-k^*)^m[1-\sigma_0\epsilon^*+2\sigma_0\delta^*]=h_{m+1}.
\end{aligned}$$

Take $\xi_{m+1}=\hat{\xi}$. We have

$$|\xi_{m+1}-\xi_m| \le |\xi_m+h_m+\sigma_0\delta_m^*-(\xi_m-\sigma_0\delta_m^*)| = ph_m.$$

So (5.38) holds when $n=m$ and (5.39) holds when $n=m+1$. (5.38) and (5.39) hold for all $n\ge 0$ by means of induction.

For each $n\ge 0$, by (5.39) and comparison principle, we have that for any $t\ge T_n$ and $x\in\mathbb{R}$,

$$\max\{\phi_1(\eta_n^-(x,t))-\delta_n^* p_i e^{-\beta_0(t-T_n)},0\} \le u(x,t,\psi_i) \le \min\{\phi_i(\eta_n^+(x,t)+h_n)+\delta_n^* p_i e^{-\beta_0(t-T_n)},k_i\}, \quad (5.41)$$

$i=1,2,3,4$, where $\eta_n^{\pm}(x,t)=x+ct+\xi_n\pm\sigma_0\delta_n^*(1-e^{-\beta_0(t-T_n)})$. For every $t\ge T^*$, set $n=\left[\frac{t-T^*}{t^*}\right]\ge 0$ and denote $\delta(t)=\delta_n^*, \xi(t)=\xi_n-\sigma_0\delta_n^*$, and $h(t)=h_n+2\sigma_0\delta_n^*$, we have $t\in[T_n,T_{n+1}), T_n=T+nt^*$. Together with (5.41), we obtain that for every $t\ge T^*, x\in\mathbb{R}$, $i=1,2,3,4$,

$$\phi_i(x+ct+\xi(t))-p_i\delta(t) \le u_i(x,t,\psi_i) \le \phi_i(x+ct+\xi(t)+h(t))+p_i\delta(t). \quad (5.42)$$

Furthermore, for $t\ge T^*$,

$$\delta(t)=\delta_n^* \le \delta^* q(t), \quad (5.43)$$

$$h(t)=(2\sigma_0\delta^*+1)(1-k^*)^n \le (2\sigma_0\delta^*+1)q(t), \quad (5.44)$$

where $q(t):=e^{(\frac{t-T^*}{t^*}-1)\ln(1-k^*)}$, and by (5.38), we have that for every $s\ge t\ge T^*$,

$$\begin{aligned}
|\xi(s)-\xi(t)| &= |\xi_m-\sigma_0\delta_m^*-(\xi_n-\sigma_0\delta_n^*)|\\
&\le \sum_{l=n}^{m-1}|\xi_{l+1}-\xi_l|+2\sigma_0\delta_n^* \le \sum_{l=n}^{m-1}ph_l+2\sigma_0\delta_n^* \quad (5.45)\\
&\le \frac{ph_n}{1-(1-k^*)}+2\sigma_0\delta_n^*=\nu\delta(t),
\end{aligned}$$

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where $m = \left[\frac{r-T^*}{t^*}\right] \geq n$ and $\nu = \frac{p}{k^*\delta^*} + 2\sigma_0$. Clearly, (5.45) implies that $\xi(t)$ is finite at positive infinity and

$$|\xi(\infty) - \xi(t) \leq \nu\delta(t), \quad t \geq T^*.$$

Then, we have

$$|\xi(\infty) - \xi(t) \leq \nu\delta^* q(t), \quad t \geq T^*. \tag{5.46}$$

Hence, by letting $k = -\frac{1}{t^*}\ln(1-k^*) > 0$ and together with (5.42), (5.43), (5.44) and (5.46), we prove the conclusion of this theorem. The proof is completed. □

Together with Lemma 5.2 and comparison principle, we can get the Lyapunov stability of travelling wave front of (4.5).

Theorem 5.2 *Each travelling wave front of (4.5) is Lyapunov stable.*

Proof Since the uniform continuity of $\phi_i(\cdot)$ on $\mathbb{R}$, $i = 1, 2, 3, 4$, we have that for every positive constant ϵ, there exists a positive constant $\delta_1 = \delta_1(\epsilon)$ satisfying

$$|\phi_i(\cdot + y) - \phi_i(\cdot)| < \frac{\epsilon}{8}, \tag{5.47}$$

for any $|y| \leq \delta_1$. We can further take $\delta = \delta(\epsilon) \in (0, \min\{\frac{\epsilon}{8(1+\max_{1\leq i\leq 4}\{p_i\})}, \frac{\delta_1}{\sigma_0}, \delta_0\})$, β_0, σ_0 and δ_0 are described by Lemma 5.2. Then for every Ψ with $\| \Psi - \Phi \| < \delta$, we have that for $i = 1, 2, 3, 4$,

$$\max\{\phi_i(x) - \delta p_i, 0\} \leq \psi_i(x) \leq \min\{\phi_i(x) + \delta p_i, k_i\}, \quad x \in \mathbb{R}. \tag{5.48}$$

Combining Lemma 5.2 with comparison principle, we have

$$\max\{\phi_i(\eta^-(x,t)) - \delta p_i e^{-\beta_0 t}, 0\} \leq u_i(x,t,\psi_i) \leq \min\{\phi_i(\eta^+(x,t)) + \delta p_i e^{-\beta_0 t}, k_i\}, \quad x \in \mathbb{R}, \tag{5.49}$$

$i = 1, 2, 3, 4$, where $\eta^{\pm}(x,t) := x + ct \pm \sigma_0\delta(1 - e^{-\beta_0 t})$. Hence, for any $t \geq 0$,

$$|\pm\sigma_0\delta(1 - e^{-\beta_0 t})| \leq \sigma_0\delta < \delta_1(\epsilon).$$

Together with (5.47) and (5.49), we have $\phi_i(x+ct) - \frac{\epsilon}{4} \leq u_i(x,t,\psi_i) \leq \phi_i(x+ct) + \frac{\epsilon}{4}, (x,t) \in \mathbb{R} \times \mathbb{R}^+, i = 1, 2, 3, 4$, that is, $\| u(\cdot,t,\Psi) - \Phi(\cdot + ct) \| < \epsilon, t \in \mathbb{R}^+$. The proof is completed. □

By Theorem 5.1, we can obtain the uniqueness result of travelling wave fronts of system (4.5).

Theorem 5.3 *Let $\Phi(x+ct) = (\phi_1(x+ct), \phi_2(x+ct), \phi_3(x+ct), \phi_4(x+ct))$ be a travelling wave front of (4.5). Then for every travelling wave front $\bar{\Phi}(x+\bar{c}t) = (\bar{\phi}_1(x+\bar{c}t), \bar{\phi}_2(x+\bar{c}t), \bar{\phi}_3(x+\bar{c}t), \bar{\phi}_4(x+\bar{c}t))$ with $0 \leq \bar{\phi}_i(x+\bar{c}t) \leq k_i$, it must be $\bar{c} = c$ and there exists $\xi_0 = \xi_0(\bar{\Phi}) \in \mathbb{R}$ such that $\bar{\Phi}(\cdot) = \Phi(\cdot + \xi_0)$.*

Proof Note that

$$\lim_{x\to\infty} \bar{\phi}_i(x) > k_i - p_i \quad \text{and} \quad \lim_{x\to-\infty} \bar{\phi}_i(x) < p_i, \quad i = 1, 2, 3, 4. \tag{5.50}$$

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Using the results of Theorem 5.1, it follows that there exist $K_0 = K_0(\bar{\Phi}) > 0$ and $\xi_0 = \xi_0(\bar{\Phi}) \in \mathbb{R}$ satisfying

$$\| \Phi(\cdot + ct + \xi_0) - \bar{\Phi}(\cdot + \bar{c}t) \| \le K_0 e^{-kt} \text{ for any } t \ge 0. \tag{5.51}$$

Set $\bar{\xi} \in \mathbb{R}$ satisfying $0 < \bar{\phi}_i(\bar{\xi}) < k_i$, $i = 1, 2, 3, 4$, and denote $I(\bar{\xi}) := \{(x, t) \in \mathbb{R} \times \mathbb{R}^+; x + ct = \bar{\xi}\}$. From (5.51), we have that for any $(x, t) \in I(\bar{\xi})$,

$$\phi_i(\bar{\eta}) - K_0 e^{-kt} \le \bar{\phi}_i(\bar{\xi}) \le \phi_i(\bar{\eta}) + K_0 e^{-kt}, \quad i = 1, 2, 3, 4. \tag{5.52}$$

where $\bar{\eta} := \bar{\xi} + \xi_0 + (c - \bar{c})t$. Since $\lim_{x\to\infty} \phi_i(x) = k_i$ and $\lim_{x\to-\infty} \phi_i(x) = 0$, $i = 1, 2, 3, 4$, by letting $t \to \infty$ in (5.52), it follows that $\bar{c} \ge c$ and $\bar{c} \le c$ by the left- and right-hand side inequalities, respectively. So $\bar{c} = c$. By (5.51), it follows that for any $(x, t) \in I(\xi)$,

$$\| \Phi(\cdot + \xi_0) - \bar{\Phi}(\cdot) \| \le K_0 e^{-kt}. \tag{5.53}$$

Hence, it follows from (5.53) that $\bar{\Phi}(\cdot) = \Phi(\cdot + \xi_0)$ as $t \to \infty$. The proof is completed. □

Remark 5.3 *By adopting a similar method, with only some slight modifications, we also obtain the stable results of bistable travelling waves for (1.3).*

Remark 5.4 *Motivated by [23, 30, 35], we can also apply the idea of spectral analysis to obtain the exponential stability of bistable travelling waves for (1.3) and (1.7). For more details, one can refer to [23].*

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Conflicts of interest

None.

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