INDEX INSURANCE DESIGN

BY

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ABSTRACT

In this article, we study the problem of optimal index insurance design under an expected utility maximization framework. For general utility functions, we formally prove the existence and uniqueness of optimal contract and develop an effective numerical procedure to derive the optimal solution. For exponential utility and quadratic utility functions, we obtain analytical expression of the optimal indemnity function. Our results show that the indemnity can be a highly nonlinear and even non-monotonic function of the index variable in order to align with the actual loss variable so as to achieve the best reduction in basis risk. Due to the generality of model setup, our proposed method is readily applicable to a variety of insurance applications including index-linked mortality securities, weather index agriculture insurance, and index-based catastrophe insurance. Our method is illustrated by numerical examples where weather index insurance is designed for protection against the adverse rice yield using temperature and precipitation as the underlying indices. Numerical results show that our optimal index insurance significantly outperforms linear-type index insurance contracts in terms of basis risk reduction.

KEYWORDS

Index insurance, basis risk, utility maximization, insurance subsidiary, ODE method.

1. INTRODUCTION

This paper is concerned with a class of insurance known as the index-indemnifying insurance or simply the index insurance. As opposed to the traditional loss-indemnifying insurance for which its payout (indemnity payment) is a function of the actual loss incurred by the policyholder, the payout of an index insurance depends exclusively on a pre-determined index or

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some appropriately chosen indicators. Prominent applications of index insurance can be found in insurance coverage provided to agricultural producers. In fact, in recent years there is a surge of interest in piloting index insurance for agricultural households in developing economies. In these applications, an index may be an average county crop yield, the number of heating days, the amount of rainfall received by a particular area during the growing season, or based on remote sensing satellite vegetation data. For example, Barnett and Mahul (2007) discuss the use of weather index insurance for agriculture in rural areas of lower-income countries. Chantarat *et al.* (2007) demonstrate that an index insurance with payout linked to some weather variables can be effective in improving drought response for famine prevention. Chantarat *et al.* (2013) describe an index-based livestock insurance by exploiting remote sensing vegetation data. Bokushevaa *et al.* (2016) analyze the effectiveness of the indices constructed based on the satellite-based vegetation health indices for insuring against drought-related yield losses. See also International Fund for Agricultural Development World Food Program (2010), Conradt *et al.* (2015) and Carter *et al.* (2016) for recent advances in agricultural index insurance.

Besides hedging agricultural and livestock risks, index-based securities that are issued in the capital market can also be effective in securitizing catastrophic risks. See, for example, the catastrophic-loss index options for hedging hurricane risk (Cummins *et al.*, 2004) and the Swiss Re mortality bonds for hedging mortality risk.

The popularity of index insurance stems from a number of reasons. The first and foremost reason lies in its potential of reducing or even eliminating moral hazard and adverse selections since the indemnity payments are based on an index that is transparent, well defined, and cannot be manipulated by either the insured or the insurer. The second reason is its low operational cost (such as the cost associated with the underwriting, administration, and loss assessment). Because the indemnity payments are completely determined by an index, there is no need to assess the losses actually incurred by the agricultural producers. Hence the claim settlement can be processed more efficiently and more timely whenever there is a claim from an index insurance. In contrast, a loss-indemnifying insurance requires loss assessment for every single claim arising. This procedure can be expensive and prohibitive, especially in rural areas where accessibility can be problematic and/or to agricultural households with small operation.

Despite all the aforementioned advantages, the challenge with the index insurance is basis risk, which arises due to the discrepancy between the indemnity payments dictated by the index and the losses actually incurred by the insured. The imperfect correlation between the adopted index and the loss random variable casts doubt on the effectiveness of index insurance in hedging agricultural production risk and as such leads to low demand in some pilot index insurance programs. See, for example, Miranda and Farrin (2012) for a review of recent theoretical and empirical research on index insurance for developing countries and a summary of lessons learned from index insurance projects implemented in the developing countries since 2000. See also Elabeda *et al.* (2013) and Jensen *et al.* (2016) for additional discussion on the basis risk associated with agricultural and livestock productions, respectively.

The presence of basis risk implies that the index must be chosen meticulously. This is typically achieved by formulating it as an optimization problem and seeking an index that optimally reduces the basis risk. In other words, this leads to a line of research inquiry on the optimal design of index insurance. The optimal design of loss-indemnifying contract is a widely studied problem in the actuarial literature. See, for example, Borch (1960) and Arrow (1963). It is, however, important to point out some subtle differences between the formulation of optimal loss-indemnifying contract and optimal index-indemnifying contract. More specifically, the indemnity function in a loss-indemnifying insurance contract needs to be non-decreasing, bounded from above by the actual loss, and has a non-zero deductible, in order to reduce moral hazard. In contrast, the indemnity function of an index insurance can have a very flexible structure. The indemnity is not necessarily increasing in the underlying indices. The indemnity payment can even exceed the loss incurred by the insured.

While index insurance is prevalent in agricultural production, its indemnity function in most cases is relatively simple and is of linear type (e.g., Giné *et al.,* 2007; Okhrin *et al.,* 2013). While linear-type indemnity functions may work well in certain contexts, the basis risk is generally high in most cases. For example, in the context of agricultural insurance, the dependence structure between crop yields and weather indices such as temperature and precipitation are so complex that it cannot be accurately captured by a linear function. Thus, innovative weather index insurance products need to be developed for farmers to better protect against the decline in crop yields due to adverse weather conditions.

In this article, we adopt a utility maximization framework for the design of index insurance (Raviv, 1979) and define the optimal index insurance as the one that maximizes the insureds' expected utility. The variance minimization problem can be viewed as a special case in our general utility maximization framework when a quadratic utility function is adopted. Mahul (2000), Vercammen (2001) and Mahul and Wright (2003) are the three relevant references. The mathematical models in these three references share a similar structure as the present paper, and they considered the insurance design problem in the presence of background risk under a utility maximization framework. They derived some characterization results about the optimal solution and presented certain interpretations on the shape of the optimal indemnity function. None of these three articles or their follow-ups offered a constructive procedure for the derivation of the optimal indemnity function in a relatively general setup. For some special utility functions, closed-form optimal solutions are obtained in these three articles, but the solutions hold only when the loss variable and the index variable have a linear relationship. Therefore, their results are not sufficient for insurers to design effective index based insurance products.

We contribute to the literature in the following aspects. First, we provide a rigorous mathematical examination on the existence and uniqueness of the optimal index insurance arrangement, while none of those aforementioned literature and their follow-ups made any conclusion in this aspect. Second, the explicit form of the optimal index insurance is derived for utility functions commonly adopted in insurance economics including quadratic and exponential utility functions. For a general strictly concave utility function, the optimal solution is characterized by an implicit ordinary differential equation (ODE), for which the solution can be easily obtained numerically, for example, by the Runge–Kutta method (Burden and Faires, 2011). Third, our results allow a general relationship between the loss variable and the index variable, as opposed to a linear one assumed in those aforementioned literature. Our model is not exactly a generalization of Mahul and Wright (2003) because they studied crop revenue insurance based on bivariate indices, while our paper focuses on the case with a univariate index, but the potential of generalizing our results to bivariate indices is demonstrated in Section 4.6. Fourth, an empirical agricultural index insurance is conducted and it shows that the index based contract from our results significantly outperforms those existing index contracts from the literature. Choosing the average temperature as the underlying index, we find that the optimal indemnity function generally follows a “first decreasing and then increasing” pattern and its specific shape relies on the premium level charged by the insurance contract, the maximum indemnity paid and the form of utility function. For quadratic utility function, the design is equivalent to minimizing the variance of insured’s resulting position, and our numerical results show that the effectiveness in terms of variance deduction does not continue to improve with the premium level after the premium exceeds certain threshold. This observation provides important and useful insights for government agency in making agricultural insurance premium subsidiaries. Further, our results also show that the proposed optimal contract generally outperforms the linear-type insurance contracts, and that the multi-index contracts can further reduce basis risk, when compared to the single-index ones.

The rest of this article is organized as follows. Section 2 describes the problem formulation of index insurance we will study in the paper and discusses the existence and uniqueness of the optimal index insurance contract for our formulation. Section 3 provides an ODE-based method for the computation of the optimal solution and applies this method to derive explicit optimal solution for quadratic and exponential utility functions, respectively. Section 4 provides an empirical study on the viability of our proposed optimal index insurance to weather index insurance. Section 5 concludes the paper. Mathematical proofs to all theorems, lemmas, and propositions, as well as a numerical procedure for solving the ODE arising from Section 3 are provided in the appendix.

2. Problem Setup and Existence of Solution

2.1. Problem setup

Suppose that a potential loss, which can hardly be insured or well hedged by any existing insurance of financial program on the market, is modeled by a random variable Y. Throughout this paper, all the random variables are defined on a probability space $(\Omega, \mathscr{F}, \Pr)$. Our objective is to design an index based insurance which is linked to an index X to protect an insured from such a risk. Let $[c, d]$ and $[a, b]$ with $c < d$ and $a < b$ be the supports of X and Y, respectively. Further, we assume that X and Y have a joint probability density function $f(x, y)$ for $(x, y) \in [c, d] \times [a, b]$, so that $\int_c^d \int_a^b f(x, y)\, \mathrm{d}y\, \mathrm{d}x = 1$. In this article, we assume that $f(x, y)$ is continuous on $[c, d] \times [a, b]$ and we write the marginal density functions for X and Y, respectively, as follows:

$$h(x) := \int_a^b f(x, y)\, \mathrm{d}y \text{ for } x \in [c, d] \text{ and } g(y) := \int_c^d f(x, y)\, \mathrm{d}x \text{ for } y \in [a, b].$$

Obviously, $g(y)$ and $h(x)$ are continuous on $[a, b]$ and $[c, d]$, respectively. Additionally, we assume that $f(x, y) > 0$ on $[c, d] \times [a, b]$ a.e., and thus $h(x) > 0$ a.e. on $[c, d]$ and $g(y) > 0$ a.e. on $[a, b]$.

Let $I(X)$ be the indemnity function of the index insurance. This means that the actual payoff of the insurance is completely determined by the realization of the index X. We further assume that $0 \leq I(X) \leq M$ for a constant $M > 0$ which represents the maximum amount paid by the insurer. The maximum payout, M, is assumed to be exogenous. It is possible to have $M \geq b$ because the insured may want to over-insure its underlying for large losses in an incomplete market (Doherty and Schlesinger, 1983). Mathematically, we consider the following feasible set for the indemnity function in the design of index insurance:

$$\mathcal{I} := \{I \mid I : [c, d] \to [0, M] \text{ is measurable}\}.$$

For loss-indemnifying insurance where the payoff of the insurance contract depends on the actual loss occurred on the insured, the indemnity function is typically non-decreasing and bounded from above by the actual loss, and has a non-zero deductible, in order to preclude severe moral hazard from the insurance contract (e.g., Chi and Tan, 2011; Chi and Weng, 2013). For the design of index insurance, however, we do not need to impose these restrictions on the indemnity function because the index can hardly be manipulated by either the insured or the insurer, and thus no moral hazard is involved.

In the article, we assume that the price of this insurance product is determined by the expected value premium principle:

$$P = \gamma \mathrm{E}[I(X)] = \gamma \int_c^d I(x) h(x)\, \mathrm{d}x,$$

where $\gamma - 1 \geq 0$ is the safety loading factor. For a given insurance premium level $P \in (0, \gamma M)$, the insurer aims to design an optimal insurance that maximizes its clients' expected utility. In other words, we are interested in solving the following optimization problem:

$$\begin{cases} \sup\limits_{I \in \mathcal{I}} J(I) := \mathrm{E}\{U(w + I(X) - Y - (1-\theta)P)\} \\ \text{s.t.}\ \ P = \gamma \int_c^d I(x)h(x)\,\mathrm{d}x, \end{cases} \tag{1}$$

where U is a strictly concave and non-decreasing utility function for the insured with $U'(x) \geq 0$ and $U''(x) < 0$ for x in the domain of the utility function U, $U''(x)$ is a continuous function, $0 \leq \theta \leq 1$ denotes any possible subsidy to the insured by a third party (which is usually a government agency in practice), w is the initial wealth of the insured, and thus, $w + I(X) - Y - (1-\theta)P$ denotes the terminal wealth of the insured in the presence of an index insurance. The constraint $P = \gamma \int_c^d I(x)h(x)\,\mathrm{d}x$ may also be interpreted as the participation constraint for risk-neutral insurers when the insurance costs are proportional to the insurance payments (Raviv, 1979). In this paper, we assume that P is exogenously given because, in practice, the amount of insurance in transaction is usually determined by the buyer (policyholder) instead of the seller (insurer). For the case in which the constraint has an inequality form $P \geq \gamma \int_c^d I(x)h(x)\,\mathrm{d}x$, the problem can be viewed as a two-step optimization problem with an additional step to select the optimal premium level P. We also assume $0 < P < \gamma M$ to ensure that the problem is well defined, and to exclude the trivial cases of $P = 0$ or $P = \gamma M$, where the optimal indemnity is either zero or the upper bound M. We note that it is very common among most countries for a government to subsidize farmers for purchasing agricultural insurance. The inclusion of θ in model (1) is to reflect such a practice. In the special case of $\theta = 0$, no subsidy is assumed for the insured in the above model.

2.2. Uniqueness and Existence of the optimal solution

Before discussing how to solve for the optimal solution, we first investigate the existence and uniqueness of the optimal solution. In fact, due to the strict convexity of the utility function, we have the following proposition regarding the uniqueness of optimal solution to the insurance design problem (1).

Proposition 1 (Uniqueness of optimal solution). *The optimal solution to problem* (1) *is unique up to the equality almost everywhere if it exists.*

Then, in order to discuss the existence of optimal solution as well as to solve problem (1), we introduce the Lagrange multiplier λ and define:

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$$
\begin{aligned}
&K(I,\lambda)\\
&:= J(I) + \lambda\left(P - \gamma\int_c^d I(x)h(x)\,\mathrm{d}x\right)\\
&= \mathrm{E}\left[U\left(w + I(X) - Y - (1-\theta)P\right)\right] + \lambda\int_c^d [P - \gamma I(x)]h(x)\,\mathrm{d}x\\
&= \int_c^d\int_a^b U\left(w + I(x) - y - (1-\theta)P\right) f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_c^d \lambda(P - \gamma I(x))h(x)\,\mathrm{d}x\\
&= \int_c^d \left\{\int_a^b U\left(w + I(x) - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y + \lambda(P - \gamma I(x))\right\} h(x)\,\mathrm{d}x,
\end{aligned}
\tag{2}
$$

where $f(y|x) = f(x,y)/h(x)$ is the conditional density function of Y given $X = x$. By the continuity and positiveness of $f(x,y)$ and $h(x)$, $f(y|x)$ is also continuous and positive for $x \in [c,d]$ and $y \in [a,b]$. The optimal solution to problem (1) can be recovered by the maximizer of $K(I,\lambda)$ defined in (2), as stated in the following lemma.

Lemma 1. *Let I_λ denote the maximizer of $K(I,\lambda)$ defined by Equation (2) for every $\lambda \in \mathbb{R}$. If there exists λ^* such that $\mathrm{E}[I_{\lambda^*}] = P/\gamma$, then $I^* := I_{\lambda^*}$ solves problem (1).*

By virtue of Lemma 1, we first investigate the maximizer of the function $K(I,\lambda)$ with respect to I for a given $\lambda \in \mathbb{R}$. In view of Equation (2), a sufficient condition is to pointwise maximize its integrand:

$$
\begin{aligned}
H(I(x),x,\lambda) := &\int_a^b U\left(w + I(x) - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y\\
&+ \lambda(P - \gamma I(x)),\ \ x \in [c,d].
\end{aligned}
\tag{3}
$$

The derivative of $H(I(x),x,\lambda)$ with respect to $I(x)$ is given by

$$
\dot{H}(I(x),x,\lambda) := G(I(x),x) - \lambda\gamma, \tag{4}
$$

where

$$
\begin{aligned}
G(\xi,x) &:= \int_a^b U'\left(w + \xi - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y\\
&= \mathrm{E}\left[U'\left(w + \xi - Y - (1-\theta)P\right)|X = x\right].
\end{aligned}
\tag{5}
$$

We note that $G(\xi,x)$ is strictly decreasing in ξ for any fixed x, since U is strictly concave. Accordingly, $G(\xi,x)$ attains its maximum value at $\xi = 0$ and its minimum value at $\xi = M$ for a given x. Based on this fact, we define the following three sets:

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$$S_1^\lambda := \left\{ x \in [c, d] \middle| G(0, x) < \lambda\gamma \right\}, \tag{6}$$

$$S_2^\lambda := \left\{ x \in [c, d] \middle| G(M, x) > \lambda\gamma \right\}, \tag{7}$$

$$S_3^\lambda := \left\{ x \in [c, d] \middle| G(M, x) \le \lambda\gamma \le G(0, x) \right\}. \tag{8}$$

Since $G(\xi, x)$ is strictly decreasing in ξ for a fixed x, we must have $S_1^\lambda \cap S_2^\lambda = \emptyset$, and thus S_1^λ, S_2^λ and S_3^λ constitute a partition of the interval $[c, d]$. Consequently, it is obvious to have

$$I_\lambda(x) := \underset{I(x)\in[0,M]}{\operatorname{argmax}} H(I(x), x, \lambda) = \begin{cases} 0, & \text{for } x \in S_1^\lambda, \\ M, & \text{for } x \in S_2^\lambda, \\ \widehat{I}_\lambda(x), & \text{for } x \in S_3^\lambda, \end{cases} \tag{9}$$

where $\widehat{I}_\lambda(x)$ satisfies $\dot{H}(\widehat{I}_\lambda(x), x, \lambda) = 0$, that is,

$$G(\widehat{I}_\lambda(x), x) = \lambda\gamma. \tag{10}$$

Obviously, $\dot{H}(0, x, \lambda) \ge 0$ and $\dot{H}(M, x, \lambda) \le 0$ for $x \in S_3^\lambda$. Thus, the continuity and strictly increasing property of $\dot{H}(I(x), x, \lambda)$ as a function of $I(x)$ implies that there exists a unique solution $\widehat{I}_\lambda(x) \in [0, M]$ to Equation (10) for every $x \in S_3^\lambda$.

Remark 1. *The partition by the three sets S_1^λ, S_2^λ, and S_3^λ for the index X represents different levels of insurance coverage for the insured. The expression of $H(I(x), x, \lambda)$ given in (3) implies that its maximizer strives to keep a balance between the marginal utility gained and the marginal expense on insurance premium from an increase of insurance coverage. When the index value lies in the set S_1^λ, the marginal utility gained from each unit of insurance coverage is less than the marginal cost of premium, and thus a zero insurance coverage is optimal. For the index value on S_2^λ, the marginal utility for each unit of insurance coverage is larger than the marginal cost of insurance premium, and thus the maximum coverage is optimal. On S_3^λ, the optimal coverage makes the marginal benefit of utility equal to the marginal cost of insurance premium. The insurance coverage for index on the set S_3^λ is between 0 and M, and thus S_3^λ represents the relatively medium coverage region.*

In the rest of this section, we use Lemma 1 to show the existence of a solution to problem (1). We need to verify the existence of λ^* such that $\mathrm{E}[I_{\lambda^*}] = P/\gamma$ for I_λ in Equation (9). To this end, we impose the following technical conditions:

H1: $\mu\left(\{x \in [c, d] | G(0, x) = k_1\}\right) = \mu\left(\{x \in [c, d] | G(M, x) = k_2\}\right) = 0$ for any k_1, $k_2 \in \mathbb{R}$,

where $\mu(\cdot)$ denotes the Lebesgue measure.

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The above condition means that the level sets have a zero Lebesgue measure at any level for both functions $G(0, x)$ and $G(M, x)$. This condition is quite mild from a practical point of view. For example, when the function $G(M, x)$ is piecewise strictly monotonic over $[c, d]$, then condition **H1** is satisfied. In the context of optimal insurance or risk sharing with background risk, the concept of stochastic monotonicity is commonly used to describe the dependence structure between two random variables (e.g., Dana and Scarsini, 2007). A random variable Z_1 is (strictly) stochastically monotonic in Z_2, if the map $z \mapsto \mathrm{E}[f(Z_1)|Z_2 = z]$ is (strictly) monotonic for every (strictly) monotonic function f. Obviously, condition **H1** is satisfied when the actual loss variable Y is strictly stochastically monotonic in the index variable X. For index insurance design, the stochastic monotonicity is generally too strong to apply, but condition **H1** is general enough for most applications.

Finally, armed with condition **H1**, we are ready to establish the existence of the optimal solution to problem (1). This is given by the following proposition.

Proposition 2 (Existence of optimal solution). *Assume that condition* **H1** *holds and* $P \in (0, \gamma M)$. *Then, there exists* λ^* *to satisfy* $\mathrm{E}[I_{\lambda^*}] = P/\gamma$ *for* I_λ *defined by Equations* (9) *and* (10). *In this case,* I_{λ^*} *is the optimal solution to problem* (1).

3. Computing the Optimal Solution

In the previous section, we have demonstrated the existence and uniqueness of the optimal insurance contract for problem (1). In order to derive a closed-form expression for the optimal solution, one may invoke Proposition 2. However, this involves the determination of the specific forms of the sets $S_2^{\lambda^*}$ and $S_3^{\lambda^*}$, as well as solving Equation (10) for $\widehat{I}_{\lambda^*}(x)$, where λ^* is given in Proposition 2. Recall that $\widehat{I}_{\lambda^*}(x)$ is defined on the set $S_3^{\lambda^*}$ only and it is solved from Equation (10) as the unique solution. In this section, we consider the case where the analytical form of $\widehat{I}_{\lambda^*}(x)$ derived from Equation (10) can be extended to the whole interval $[c, d]$, that is, $\widehat{I}_{\lambda^*}(x)$ is well defined for $x \in [c, d]$. As to be discussed in Section 3.1, we develop an ODE method which is more convenient for the derivation of the optimal solution. This ODE method will be demonstrated for quadratic and exponential utility functions in Sections 3.2 and 3.3, respectively. For other strictly concave utility functions, a numerical procedure is attached in the appendix for the derivation of optimal contract.

3.1. The ODE method

Lemma 2 below provides an equivalent but more computationally friendly way for deriving I_λ to maximize $K(I, \lambda)$ in (2) when $\widehat{I}_\lambda(x)$ derived from Equation (10) can be extended to the interval $[c, d]$. Note that $\widehat{I}_\lambda(x)$ may no longer confine to the interval $[0, M]$ for x outside the set S_3^λ.

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Lemma 2. *Let λ be a constant such that $S_3^\lambda \neq \emptyset$, and assume that $\widehat{I}_\lambda(x)$ solved from (10) exists on $[c, d]$. Then, the optimal solution to maximize $K(I, \lambda)$ in (2) is given by*

$$I_\lambda(x) = \left[\left(\widehat{I}_\lambda(x)\right) \vee 0\right] \wedge M, \tag{11}$$

where "$\vee$" and "$\wedge$" denote $\max(\cdot,\cdot)$ *and* $\min(\cdot,\cdot)$*, respectively.*

The advantage of Lemma 2 lies in the fact that we do not need to determine the sets S_2^λ and S_3^λ for the determination of the optimal solution $I_\lambda(x)$. Once we derive an analytical form of $\widehat{I}_\lambda(x)$ by solving Equation (10) for $x \in [c, d]$, the optimal solution $I_\lambda(x)$ can be derived via Equation (11).

We can apply Lemma 2 to transform problem (1) into an ODE problem under certain smoothness conditions for $f(x, y)$ as shown in Theorem 1 in the sequel. The ODE method relies on the analytical continuation of $\widehat{I}_{\lambda^*}(x)$ from $S_3^{\lambda^*}$ to $[c, d]$, and thus, we need to make sure that $S_3^{\lambda^*}$ is non-empty so as to make the ODE method valid, where λ^* is given in Proposition 2. The following proposition confirms the non-emptiness of $S_3^{\lambda^*}$.

Proposition 3. *Assume that condition* **H1** *is satisfied. Then, $S_3^{\lambda^*}$ is a non-empty subset of $[c, d]$, where λ^* is any constant such that $\mathrm{E}[I_{\lambda^*}(X)] = P/\gamma$ with the existence guaranteed by Proposition 2.*

The following theorem states that the index insurance design problem (1) can be solved by the ODE approach:

Theorem 1. *Suppose that the derivative $\frac{\partial}{\partial x} f(y|x)$ exists and is continuous on $[c, d] \times [a, b]$, and a function $\widehat{L} : [c, d] \mapsto \mathbb{R}$ solves the following ODE problem:*

$$\begin{cases} \dfrac{\mathrm{d}L}{\mathrm{d}x} = F(x, L), \\ P = \gamma \mathrm{E}\left[(L(X) \vee 0) \wedge M\right], \end{cases} \tag{12}$$

where the function $F : [c, d] \times \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$F(x, L) := -\frac{\int_a^b U'\left(w + L - y - (1-\theta)P\right) \dfrac{\partial}{\partial x} f(y|x)\,\mathrm{d}y}{\int_a^b U''\left(w + L - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y}.$$

Then, $L^* := \left(\widehat{L}(x) \vee 0\right) \wedge M$ is the optimal solution to problem (1).

Theorem 1 provides us a sufficient condition to find the optimal indemnity function. The system of Equation (12) is an ODE with a general boundary condition, with the first equation determining the shape of the optimal solution and the second equation controlling the initial value of the optimal solution within its domain $[c, d]$. The optimal indemnity, L^*, which is non-negative and subject to a maximum coverage M, can be viewed as the truncated version of function $\widehat{L}$ by two straight lines $I=0$ and $I=M$. In the next two sections, we will demonstrate the applications of Theorem 1 for the derivation of optimal index insurance solutions for quadratic and exponential utility functions, respectively.

Remark 2. *It is interesting to consider the special case when X and Y are independent. Independence between X and Y implies that $f(y|x)=f(y)$ and further $F(x,L)=0$ by Theorem 1. Therefore, for any utility function, the optimal indemnity function $I^*(x)$ is always a constant which is independent of x. Interpretation for this is that index insurance runs on the idea that the insurer could use the information of Y conveyed in X to design the indemnity scheme. When X conveys more information of Y, that is, X and Y are more statistically correlated, the performance of the index insurance is generally better; on the contrary, when X and Y are independent, the selected index X is completely useless to predict Y, and thus we would expect that the contract will be highly ineffective.*

3.2. Quadratic utility

We suppose that the insured's utility function has a quadratic form, that is, $U(x)=\alpha x-\beta x^2$, $x\le\frac{\alpha}{2\beta}$, where the parameters $\alpha>0$ and $\beta>0$. We also assume that $w+M-a-(1-\theta)P\le\frac{\alpha}{2\beta}$, so that the insured's maximum possible wealth will not exceed the domain of the utility function, and $U'(x)\ge 0$, $U''(x)<0$ and continuity of $U''(x)$ hold for every x in its domain. Using Theorem 1, we can derive a closed-form solution of optimal index insurance as shown in the following proposition.

Proposition 4. *Suppose that $\frac{\partial}{\partial x}f(y|x)$ exists and is continuous on $[c,d]\times[a,b]$. If the policyholder's utility function $U(x)=\alpha x-\beta x^2$, $x\le\frac{\alpha}{2\beta}$, where the parameters $\alpha>0$ and $\beta>0$, then the optimal index insurance is given by*

$$I^*(x)=[(\mathrm{E}\,[Y|X=x]+\eta^*)\vee 0]\wedge M, \tag{13}$$

where η^* is determined by the equation

$$\mathrm{E}[I^*(X)]=\mathrm{E}\,\{[(\mathrm{E}\,[Y|X]+\eta^*)\vee 0]\wedge M\}=\frac{P}{\gamma}.$$

Under the quadratic utility assumption, we can see from Proposition 4 that the shape of optimal solution is determined by the conditional expectation $\mathrm{E}\,[Y|X=x]$. This result is consistent with Equations (11) and (12) in Mahul and Wright (2003), which means $\frac{d}{dx}I^*(x)=\frac{d}{dx}\mathrm{E}[Y|X=x]$ using the notation of the present paper. Moreover, Proposition 4 also indicates that how much coverage the policyholders get from the insurance contract is represented by η^*, which is a real number determined by the premium constraint $\mathrm{E}[I^*(X)]=P/\gamma$.

Obviously, η^* is non-decreasing with premium P: the higher the insurance premium paid by the policyholders, the higher the indemnity payoffs back to them in general. Under quadratic utility, the value of η^* only affects the ground level of optimal indemnity but not the shape of optimal indemnity before truncation. In other words, $\widehat{L}$ defined by Equation (12) with different premium levels are parallel curves on the x-I-plane. We will empirically illustrate this in Section 4.2.

Remark 3. *It is well known that the one-period quadratic utility maximization problem is equivalent to the one-period mean-variance problem. It is trivial to show that the optimal indemnity function is still given by Proposition 4 if the insurer aims at minimizing the variance of the insured's terminal wealth.*

Remark 4.

(a) As one can infer from Proposition 4, the optimal contract is independent of the parameters α and β under the quadratic utility.
(b) Proposition 4 also shows that, under quadratic utility function, the optimal indemnity function is irrelevant to both the insured's initial wealth w and the subsidy level θ.

Remark 5. *When the quadratic utility is adopted as the criterion, computation of the optimal index insurance is substantially simplified because it does not involve estimating the joint density function $f(x, y)$, $(x, y) \in [c, d] \times [a, b]$, but only the conditional expectation function $\mathrm{E}[Y|X = x]$, $x \in [c, d]$. In practice, it is sometimes possible to obtain a much quicker and more convenient estimation on $\mathrm{E}[Y|X = x]$ directly without estimating $f(x, y)$.*

3.3. Exponential utility

In this section, we consider the case when U is an exponential utility function, that is, $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where the parameter $\alpha > 0$. It is easy to verify that $U'(x) \geq 0$ and $U''(x) < 0$ for all $x \in \mathbb{R}$. We use Theorem 1 to derive a closed-form solution of optimal index insurance as shown in the following proposition.

Proposition 5. *Suppose that $\frac{\partial}{\partial x} f(y|x)$ exists and is continuous on $[c, d] \times [a, b]$. If $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, $x \in \mathbb{R}$, with the utility parameter $\alpha > 0$, then the optimal indemnity function is given by*

$$I^*(x) = \left[\left(\frac{1}{\alpha}\ln\left(\mathrm{E}\left[e^{\alpha Y}|X = x\right]\right) + \eta^*\right) \vee 0\right] \wedge M,$$

where η^* is a constant determined by $P = \gamma\,\mathrm{E}[I^*(X)]$.

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Under the exponential utility assumption, the shape of optimal indemnity function is determined by conditional expectation $\mathrm{E}\left[e^{\alpha Y}|X=x\right]$. Similar to the case under quadratic utility, the overall size of indemnity payoffs is controlled by a constant η^* which is determined by the premium constraint $\mathrm{E}[I^*(X)]=P/\gamma$ and is non-decreasing with premium P. Under both exponential utility and quadratic utility, η^* reflects the premium level P, and it only affects the ground level of optimal indemnity, but not the shape of optimal indemnity before truncation. This is not necessarily true for other utility functions. From Equation (12), we can see that the shape of $\widehat{L}$ generally depends on the value of $\widehat{L}$ itself, and thus optimal solutions with different premium levels may not be obtained by such a parallel shift η^*.

Remark 6. *Similar to the quadratic utility case as we commented in Remark 5, we only need to estimate the conditional expectation* $\mathrm{E}\left[e^{\alpha Y}|X=x\right]$*,* $x\in[c,d]$*, in order to determine the optimal index insurance under the exponential utility function. We do not have to estimate the joint density function* $f(x,y)$ *in practical applications.*

Remark 7. *Proposition 5 also indicates that the optimal indemnity function under the exponential utility is irrelevant to both the insureds' initial wealth* w *and the subsidy level* θ*. This phenomenon is similarly observed for optimal index insurance under the quadratic utility function (recall Remark 4).*

4. Applications in Weather Index Insurance Design

In this section, we apply our theoretical results to an example of weather index insurance contract design, where basis risk is a primary concern for policyholders. For Sections 4.1–4.5, we choose the temperature as the underlying index to protect insured's position from adverse weather conditions. We investigate the optimal index insurance under the quadratic, exponential, and logarithmic utility functions, respectively. We also benchmark our optimal index design against linear-type contracts. Finally, in Section 4.6, we extend the optimal index design to the bivariate case where the indemnity depends on two indices, the temperature and the precipitation.

4.1. Dependence modeling

We choose the average temperature of the whole product growing cycle as the underlying index to protect insured's position from adverse weather conditions. For a certain kind of agricultural product, both too high and too low temperatures would normally have an adverse impact on the product yield. As a result, the indemnity function of a well-designed contract should take larger values at both ends of the interval at temperature axis but smaller values in the middle area. In our study, we use county-level data of rice yield and

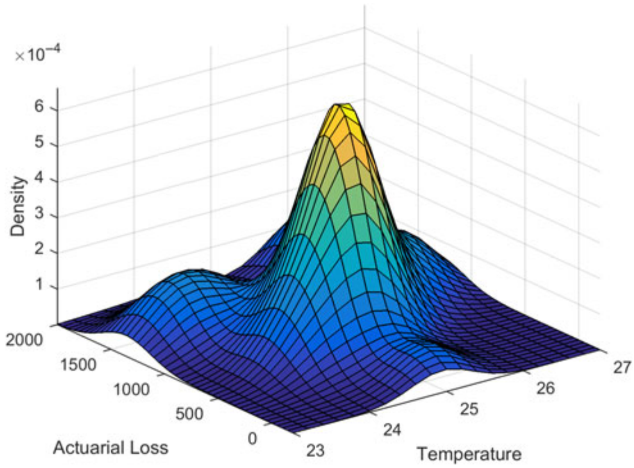


FIGURE 1: Calibrated joint density $f(x, y)$ of the actual loss and the average temperature.

temperature data in Jiangsu Province, China for the period from 1992 to 2011. The same data set has been studied by Shi and Jiang (2016). We first remove the trend in the historical yield which represents production improvement factors over time such as technology, and then define the actual loss variable to the insured Y as the highest detrended yield during the last 20 years less the detrended yield. We apply kernel smoothing method to calibrate the joint density between the average temperature and the actual loss variables. The graph of the joint density function $f(x, y)$ is illustrated in Figure 1.

4.2. Quadratic utility

We begin with investigation into the shape of the optimal index insurance contract when the insured's risk preference is represented by a quadratic utility function. As discussed in Remark 3, the optimal indemnity function is independent of the parameter values under a quadratic utility function, and therefore it is unnecessary to specify the parameter values for the investigation of the optimal index insurance. There are two major exogenous factors that determine the shape and the scale of the optimal contract: the premium level P and the maximum indemnity level M. Weather index agricultural insurance is often subsidized by the government, and the premium level P also reflects the subsidy level because the latter is usually proportional to P. The maximum indemnity level M is also very important factor for index insurance design, because the insurer can prevent itself from extreme large losses by imposing such an upper limit.

4.2.1. *Optimal contracts with different premium levels*

We fix the maximum indemnity level at $M = 300$ and use Proposition 4 to construct the optimal indemnity functions for four different premium levels

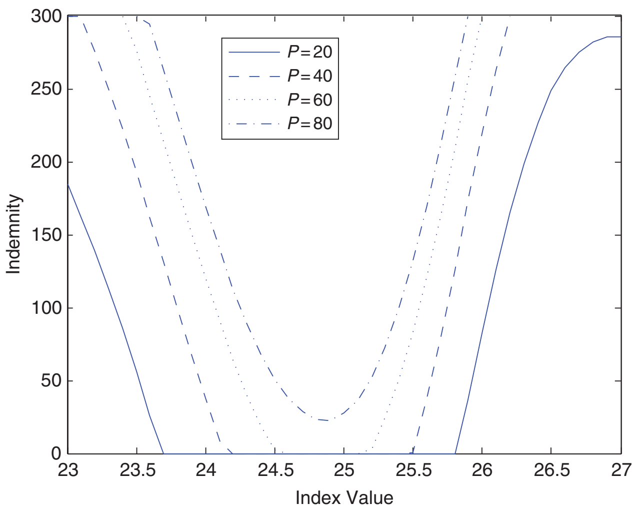


FIGURE 2: Optimal indemnity functions for different premium levels and under quadratic utility.

P varying from 20 to 80 in multiples of 20. The resulting optimal indemnity functions are illustrated in Figure 2. The optimal indemnity function follows a "first decreasing and then increasing" pattern, confirming that the farmers are more concerned with extreme weathers, and hence higher indemnities are to be expected for the extremes. However, the risk exposure triggered by the high extreme weather or low extreme weather is not necessarily symmetric. The slope of right half of the indemnity function is steeper than the left half, indicating high temperatures have a more severe adverse impact on the crop yield than low temperatures do. Finally, as we discussed in Section 3.2, curves shown in Figure 2 representing optimal indemnity functions under different premium levels are actually parallel to each other before being truncated by straight lines $I = 0$ and $I = M$.

Recall from Equation (9) that the indemnity from the optimal index insurance is zero over the set $S_1^{\lambda^*}$, attains the maximum amount M over $S_2^{\lambda^*}$, and lies between 0 and M over $S_3^{\lambda^*}$. Figure 2 indicates that, as the premium level P goes larger, simultaneously, the set $S_1^{\lambda^*}$ diminishes and $S_2^{\lambda^*}$ expands. When $P = 20$, $S_2^{\lambda^*} = \emptyset$. This means that the premium level is too small to cover the maximum indemnity level $M = 300$ over any region. When $P = 80$, $S_1^{\lambda^*} = \emptyset$ which means that the premium level is large enough in this case to cover the whole range of the index variable.

4.2.2. *The impact of maximum indemnity on the optimal contracts*

In this subsection, we fix the premium level $P = 50$ and investigate the impact of maximum indemnity on the optimal optimal indemnity function. Figure 3 depicts the optimal indemnity functions for different levels of maximum indemnity M. The graph similarly shows that the optimal indemnity function exhibits

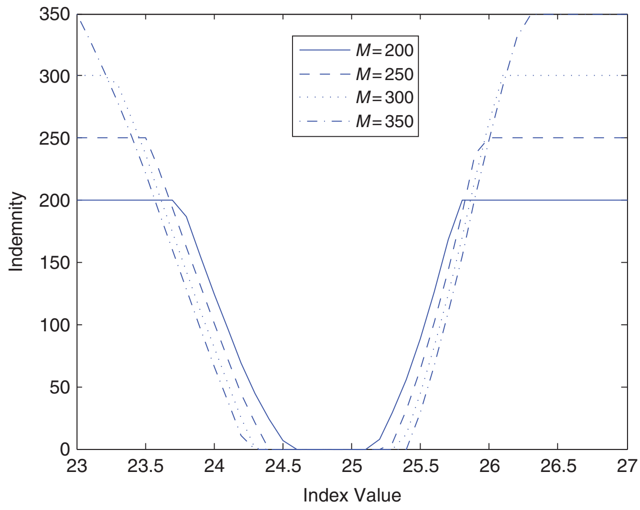


FIGURE 3: Optimal indemnity for different maximum indemnity levels and under quadratic utility.

the "first decreasing and then increasing" pattern for all the different maximum indemnity levels we considered. As the maximum indemnity M increases, the more coverage is provided to both extremes, and thus the set $S_1^{\lambda^*}$ becomes larger. In the meanwhile, the set $S_2^{\lambda^*}$ becomes smaller as the indemnity payment on this set has been increased. From the perspective of the insured, when M is too small, the indemnity function does not sufficiently reflect the impact of the weather index on the insured's actual loss, and thus the index insurance contract is ineffective in this case; on the other hand, from the perspective of the insurance company, an increase in M also increases its own tail risk and thus potentially high capital cost. Therefore, in practice, the choice of M should be determined by the bargaining power between these two parties.

4.2.3. *Risk mitigation performance*

In this section, we are interested in the effectiveness of our proposed index insurance in reducing basis risk, which is measured by the standard deviation of the residual risk after the indemnity payment, that is, square root of the variance of the residual risk $[Y - I^*(X)]$. If the standard deviation of the residual risk is large, it means that policyholders' risk is not effectively mitigated and the basis risk is high, and vice versa. Figure 4 reports the standard deviation of the residual risk for P and M vary over intervals [0, 100] and [200, 600], respectively.

The straight line intersected by the surface and plane $P = 0$ in Figure 4 represents the uninsured position of the policyholder and the standard deviation of the actual loss variable is about 443 in this case. The whole surface in the figure is lower than 443, which means a positive impact from our proposed index insurance on reducing the basis risk. The shape of the surface also indicates

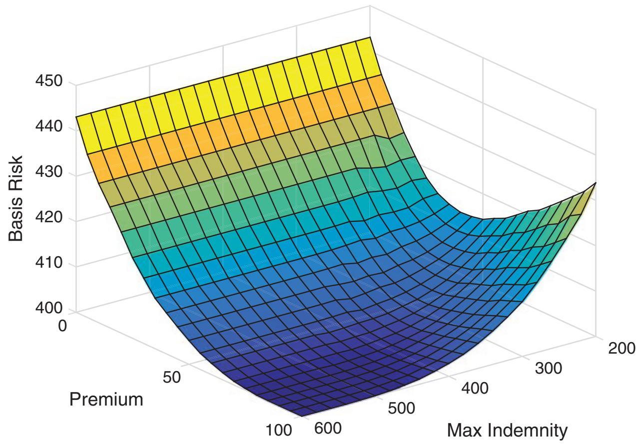


FIGURE 4: Basis risk (i.e., the standard deviation of residual risk) for different levels of premium and maximum indemnity.

that the basis risk can be reduced by increasing premium level P and maximum indemnity level M. The exception occurs when P becomes too large relatively to M. This phenomenon can be explained by the curve on the top (corresponding to $P = 80$) in Figure 2, which says that, when P is large relatively to M, any additional premium goes to cover the actual losses in the middle area, in other words, the coverage for small losses increases while coverage for large ones remains unchanged; as a result, basis risk increases rather than decreases.

Since the premium level also indicates how much subsidy the government is paying for the policyholders, Figure 4 also provides some suggestions for the government in determining the subsidy amount according to M. This example shows that a wise choice for the subsidy amount needs to comply with the maximum indemnity level M.

4.3. Exponential utility

Risk preference of the insured is essential in index insurance contract design, and one advantage of our method is its capability to take into account the insured's utility function. In the present and the next subsection, we will investigate the optimal index insurance designs under the exponential and the logarithmic utility functions.

First we consider the shape of the optimal indemnity function under an exponential utility in the form of $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where the parameter $\alpha > 0$. By Equation (14), the optimal indemnity function depends on the parameter α, which measures the degree of risk preference of the policyholder. Risk averse policyholders always have $\alpha > 0$, and a higher α means a higher degree of risk aversion. In order to see how α affects the shape of the optimal index insurance contract, we illustrate the optimal indemnity functions for four different values of α in Figure 5. In this example, we fix $P = 50$ and $M = 300$.

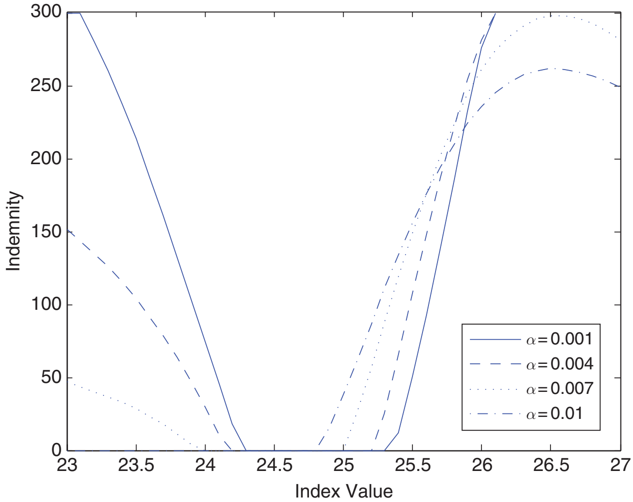


FIGURE 5: Optimal indemnity under exponential utility.

Figure 5 shows that the shape of the optimal indemnity function depends heavily on the choice of α. For an α as small as 0.001, the indemnity function also shows the "first decreasing and then increasing" pattern which has been previously observed under the quadratic utility. As α becomes larger, that is, the insured becomes more risk averse, the coverage on the low-temperature region decreases and more premium is spent on the high-temperature region. When the parameter α is as large as 0.01, the optimal index insurance contract only indemnifies losses occurred in the high-temperature region, but not those in the low-temperature region. This phenomenon can be explained from two perspectives. First, policyholders with larger α are generally more risk averse than those using "quadratic utility" as their risk preferences. As a result, they would like to have more coverage on the most severe losses, which occur in the high-temperature region. Second, it can be explained by the asymmetric effects of temperature on the loss in rice yield. The adverse effect is more severe from the high temperatures than the low temperatures.

4.4. Logarithmic utility

The logarithmic utility function takes a form of $U(x) = \ln x$ for $x > 0$. There is no closed-form for the optimal index insurance contract under the logarithmic utility. We use Theorem 1 and apply a numerical scheme to solve the ODE in (12) for the optimal solution. The numerical scheme is specified in the appendix. We fix the maximum indemnity level $M = 300$ and compute the optimal indemnity functions for a set of different premium levels P. The resulting optimal indemnity functions are illustrated in Figure 6. The figure shows that the optimal indemnity function takes a similar shape as the one under

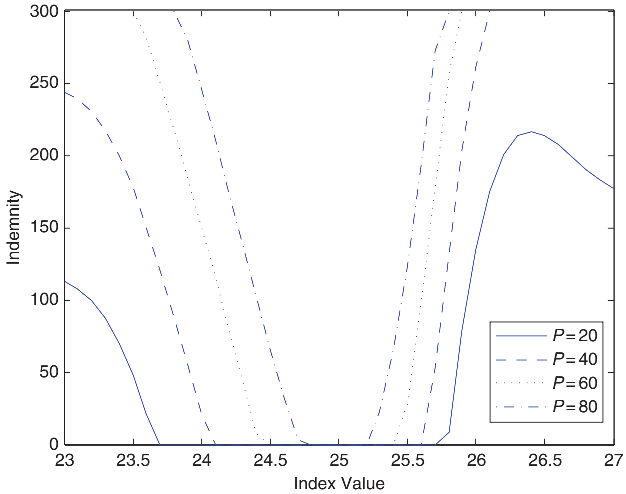


FIGURE 6: Optimal indemnity under logarithmic utility.

the quadratic utility function. It also generally follows the "first decreasing and then increasing" pattern and the coverage increases throughout the whole region as the premium increases. Additionally, as we mentioned earlier in Section 3.3, indemnity functions under different premium levels are not parallel to each other, and this is quite different from what we have under quadratic and exponential utilities.

In order to make a close comparison of the optimal index insurance among the three utility functions (i.e., quadratic, exponential, and logarithmic), we fix $P=50$ and $M=300$ and demonstrate the resulting optimal indemnity functions in Figure 7, where $\alpha=0.005$ is set for the exponential utility function. Clearly, the contract under the logarithmic utility is quite similar to the one under the quadratic utility. The left parts of the two curves almost coincide with each other and the right part of optimal indemnity function under the logarithmic utility is slightly steeper. In contrast, the optimal indemnity function under the exponential utility function is quite different from the other two, with less coverage on the left half but more coverage on the right half.

4.5. Comparison with linear contracts

In this section, we compare the effectiveness of our optimally designed index insurance contracts with the linear-type contract (e.g., Giné *et al.*,, 2007; Okhrin *et al.*,, 2013), which is based on a linear regression procedure and widely applied in both practice and academia as a benchmark. The effectiveness is measured by the standard deviation of the residual risk after the indemnity payment for the policyholders, which we also call basis risk. The comparison is conducted for a set of premium levels and two maximum indemnity payments at $M=250$ and $M=300$, respectively. The results are demonstrated in Figure 8.

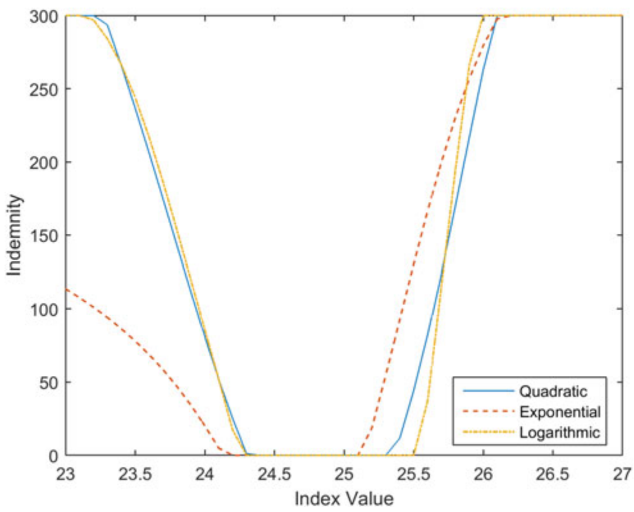
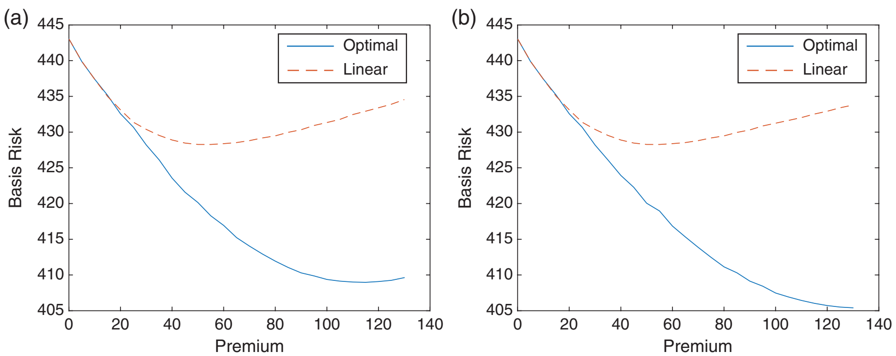


FIGURE 7: Comparison of optimal indemnity functions under three different utilities.



$M = 250$ $M = 300$

FIGURE 8: Effectiveness: our optimal index insurance versus the linear-type contract.

Maximizing the expected quadratic utility is the same as minimizing the variance of the residual risk for the insured in our index insurance design model. In theory, our proposed indemnity function achieves the smallest standard deviation reduction of the residual risk for the insured, as guaranteed by Proposition 4. Figure 8 shows the superiority of our proposed insurance contract compared with the linear-type contract. The basis risk measured by the standard deviation of the insured's residual risk is smaller under our optimally design index insurance than the linear-type contract. Our index insurance performs equally well as the linear-type contract for small premium level (say, $P < 40$). It substantially outperforms the linear-type contract for larger premium levels, and the advantage becomes more obvious as P increases.

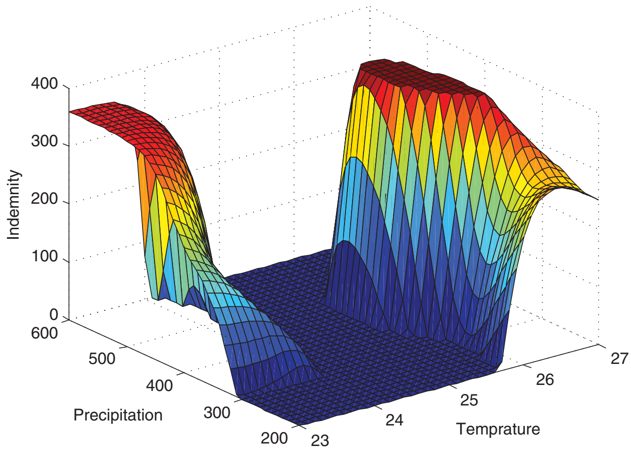


FIGURE 9: Indemnity function of a bivariate-index insurance: temperature and precipitation.

While the linear-type contract suffers from not being able to benefit from an increase in P and M, we can generally enhance the performance of our proposed contract by increasing the premium level P and enlarging the maximum coverage level M.

4.6. An example of bivariate-index insurance

In the previous sections, we focus on the single-index case, where the insurance indemnity is determined by a single index. In this section, we incorporate a second index to the design of the insurance contract and derive an optimal bivariate-index insurance. In principle, introducing more indices to the insurance payoff function is always helpful in reducing basis risk. We will illustrate this empirically based on an bivariate-index insurance involving two weather indices: temperature and precipitation. We use the average temperature and the total amount of precipitation during the whole growing season as the underlying indices to construct the index insurance contract. In this example, we assume that the policyholders' utility function has a quadratic form, and results under other utility functions can be analyzed similarly. Mathematically, it is straightforward to show that Proposition 4 still holds for the multi-dimensional case and the optimal indemnity function takes a similar form as $I^*(x)$ in (13). Let X_1 and X_2, respectively, denote the two indices under our consideration. Then the optimal indemnity function is given by

$$I^*(x_1, x_2) = [(\mathrm{E}\,[Y|X_1 = x_1, X_2 = x_2] + \eta^*) \vee 0] \wedge M,$$

where η^* is determined by $\mathrm{E}[I^*(X)] = P/\gamma$. The optimal indemnity function of this bivariate-index insurance contract is illustrated in Figure 9.

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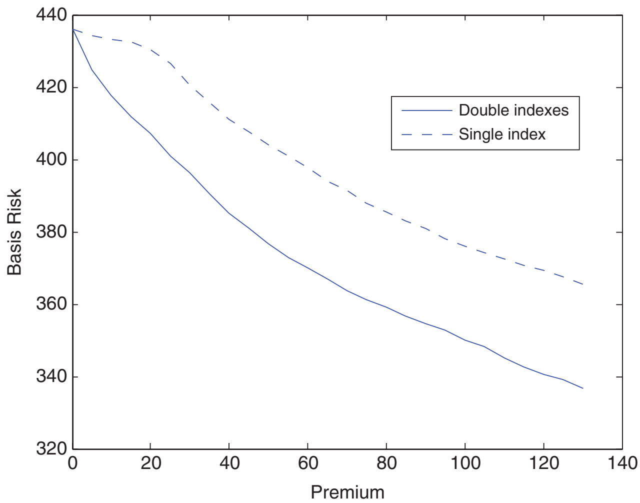

FIGURE 10: Effectiveness improvement of additional index. Basis risk is measured by standard deviation of residual risk.

The indemnity function similarly shows the "first decreasing and then increasing" pattern with respect to the increase in the temperature index. Further, the precipitation index also plays an important role in the optimal indemnity function. Indemnity amount is higher in the "low-temperature and high-precipitation" region than in the "low-temperature and low-precipitation" region. Within the "high-temperature" region, a "medium precipitation" corresponds to the largest amount of indemnity and a "low precipitation" leads to a larger indemnity amount than a "high precipitation." Further, there is no indemnity payment when both temperature and precipitation are moderate, which corresponds to a good harvest and negligible actual loss.

To demonstrate the benefit of including the precipitation variable as the additional index in the optimal insurance contract, we compare the basis risk (measured by the standard deviation of residual risk) between the bivariate-index contract and the single-index contract under a set of different premium levels. The maximum indemnity amount payment is fixed at $M = 400$. The comparison results are illustrated in Figure 10. Obviously, the inclusion of the precipitation index significantly reduces basis risk. The residual risk is consistently lower for the bivariate-index contract than the single-index contract. As the premium level P increases, the gap in residual risk between the two contracts becomes larger. In particular, the basis risk is reduced from 370 down to 340, which means a reduction rate of 8.1%, when the premium level $P = 120$. This suggests that more relevant indices should be included into the optimal insurance design if estimation of the joint distribution between the actual loss variable and indices is not an issue.

5. Conclusion

In this article, we investigate the optimal index insurance design problem under a utility maximization framework. Under quite general and practical assumptions, we show that the optimal index insurance contract exists and is uniquely determined by the policyholder's utility function, the premium level, and the maximum indemnity covered by the insurance contract. The optimal index insurance contract is obtained by solving an implicit ODE problem. Additionally, when the insured has a quadratic utility or an exponential utility, the optimal indemnity functions have explicit forms which are computationally friendly for real applications.

Our theoretical results are applied to a real data example, in which the temperature and precipitation variables are used as the underlying indices of the insurance contract to protect rice yield in Jiangsu, China. The shape of the optimal indemnity functions under different utility functions, premium levels, and maximum indemnity amounts generally follow the "first decreasing and then increasing" pattern. The risk mitigation performance measured by the standard deviation reduction of the insured's residual risk is also discussed. Our results confirm that our optimally designed index insurance significantly outperforms the linear-type contract, which is a popular solution applied both in practice and in the literature for reducing farmers' basis risk. Finally, an example of a bivariate-index insurance contract based on the temperature and precipitation variables is introduced to show the benefit of incorporating multiple indices into the insurance contract design for mitigating basis risk.

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References

Arrow, K.J. (1963) Uncertainty and the welfare economics of medical care. *American Economic Review,* **53**(5), 941–973.

Barnett, B. J. and Mahul, O. (2007) Weather index insurance for agriculture and rural areas in lower-income countries. *American Journal of Agricultural Economics,* **89**(5), 1241–1247.

Bokushevaa, B., Koganb, F., Vitkovskayac, I., Conradta, S. and Batyrbayevacaa M. (2016) Satellite-based vegetation health indices as a criteria for insuring against drought-related yield losses. *Agricultural and Forest Meteorology,* **220**, 200-206.

BORCH, K. (1960) An attempt to determine the optimum amount of stop loss reinsurance. *Transactions of the 16th International Congress of Actuaries,* **I**, 597-610.

BURDEN, R.L. and FAIRES, J.D. (2011) *Numerical Analysis.* 9th edition. Brooks/Cole, Boston, USA.

CARTER, M.R., CHENG, L. and SARRIS, A. (2016) Where and how index insurance can boost the adoption of improved agricultural technologies. *Journal of Development Economics,* **118**, 59-71.

CARTER, M.R., DE JANVRY, A., SADOULET, E. and SARRIS, A. (2014) *Index-based weather insurance for developing countries: A review of evidence and a set of propositions for up-scaling.* FERDI Working Paper 112.

CHANTARAT, S., BARRETT, C.B., MUDE, A.G. and TURVEY, C.G. (2007) Using weather index insurance to improve drought response for famine prevention. *American Journal of Agricultural Economics,* **89**(5), 1262–1268.

CHANTARAT, S., MUDE, A.G., BARRETT, C.B. and CARTER, M.R. (2013) Designing indexbased livestock insurance for managing asset risk in northern Kenya. *Journal of Risk and Insurance,* **80**(1), 205–237.

CHI, Y. and TAN, K.S. (2011) Optimal reinsurance under VaR and CVaR risk measures: A simplified approach. *ASTIN Bulletin,* **41**(2), 487–509.

CHI, Y. and WENG, C. (2013) Optimal reinsurance subject to Vajda condition. *Insurance: Mathematics and Economics,* **53**(1), 179–189.

CONRADT, S., FINGER, R. and SPORRI, M. (2015) Flexible weather index-based insurance design. *Climate Risk Management,* **10**, 106–117.

CUMMINS, J.D., LALONDE, D. and PHILLIPS, R.D. (2004) The basis risk of catastrophic-loss index securities. *Journal of Financial Economics,* **71**(1), 77-111.

DANA, R.A. and SCARSINI, M. (2007) Optimal risk sharing with background risk. *Journal of Economic Theory,* **133**(1), 152–176.

DOHERTY, N.A. and SCHLESINGER, H. (1983) Optimal insurance in incomplete markets. *Journal of Political Economy,* **91**(6), 1045–1054.

ELABEDA, G., BELLEMAREB, M.F., CARTER, M.R. and Guirkinger, C. (2013) Managing basis risk with multiscale index insurance. *Agricultural Economics,* **44**, 419–431.

GINÉ, X., TOWNSEND, R. and VICKERY, J. (2007) Statistical analysis of rainfall insurance payouts in Southern India. *American Journal of Agricultural Economics,* **89**(5), 1248–1254.

INTERNATIONAL FUND FOR AGRICULTURAL DEVELOPMENT WORLD FOOD PROGRAM (2010) *The potential for scale and sustainability in weather index insurance for agriculture and rural livelihoods.* Technical Report.

JENSEN, N.D., BARRETT, C.B. and MUDE, A.G. (2016) Index insurance quality and basis risk: Evidence from northern Kenya. *American Journal of Agricultural Economics,* **98**(5), 1450–1469.

LIN, Y. and COX, S.H. (2008) Securitization of catastrophe mortality risks. *Insurance: Mathematics and Economics,* **42**(2), 628–637.

MAHUL, O. (2000) Optimum crop insurance under joint yield and price risk. *Journal of Risk and Insurance,* **67**(1), 109–122.

MAHUL, O. and WRIGHT, B.D. (2003) Designing optimal crop revenue insurance. *American Journal of Agricultural Economics*, **85**(3), 580–589.

MIRANDA, M.J. and FARRIN, K.M. (2012) Index insurance for developing countries. *Applied Economic Perspectives and Policy,* **34**(3), 391–427.

OKHRIN, O., ODENING, M. and XU, W. (2013) Systemic weather risk and crop insurance: The case of China. *Journal of Risk and Insurance,* **80**(2), 351–372.

RAVIV, A. (1979) The design of an optimal insurance policy. *The American Economic Review,* **69**(1), 84–96.

SHI, H. and JIANG, Z. (2016) The efficiency of composite weather index insurance in hedging rice yield risk: Evidence from China. *Agricultural Economics,* **47**(3), 319–328.

VERCAMMEN, J. (2001) Optimal insurance with nonseparable background risk. *Journal of Risk and Insurance,* **68**(3), 437–447.

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APPENDIX

A.1. Proof of Proposition 1

Proof. Let I_1 and I_2 be two optimal solutions to problem (1) with $\mu(D) > 0$ where $D := \{x \in [c, d] \,|\, I_1(x) \neq I_2(x)\}$ and $\mu(D)$ denotes the Lebesgue measure of the set D. Denote $I_\lambda(x) := \lambda I_1(x) + (1-\lambda)I_2(x)$, $x \in [c, d]$ for a constant $\lambda \in (0, 1)$. Obviously, I_λ is a feasible indemnity function for problem (1) because $I_\lambda \in \mathcal{I}$ and it satisfies the constraint in problem (1). We also note that X has a positive density function on the interval $[c, d]$. Consequently, $\mu(D) > 0$ implies $\Pr(A) > 0$ where $A := \{\omega \in \Omega \,|\, I_1(X) \neq I_2(X)\}$.

Let $v(P)$ denote the supremum value for problem (1). We must have $v(P) < \infty$ because both $I(X)$ and Y are bounded random variables. Thus, using the strict concavity, we obtain

$$\begin{aligned} J(I_\lambda) &= \mathrm{E}\big[U\big(w + \lambda I_1(X) + (1-\lambda)I_2(X) - Y - (1-\theta)P\big)\big] \\ &> \lambda \mathrm{E}\big[U\big(w + I_1(X) - Y - (1-\theta)P\big)\big] \\ &\quad + (1-\lambda)\mathrm{E}\big[U\big(w + I_2(X) - Y - (1-\theta)P\big)\big] \\ &= \lambda v(P) + (1-\lambda)v(P) \\ &= v(P), \end{aligned}$$

which contradicts to the optimality of I_1 and I_2. Thus, the optimal solution to problem (1) is unique up to the equality almost everywhere if it exists. ■

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A.2. Proof of Lemma 1

Proof. Recall that $v(P)$ denotes the supremum value of problem (1). Therefore,

$$\begin{aligned} v(P) &= \sup_{I\in\mathcal{I} \text{ s.t. } \gamma \mathrm{E}[I]=P} \mathrm{E}[J(I)] \\ &= \sup_{I\in\mathcal{I} \text{ s.t. } \gamma \mathrm{E}[I]=P} \{\mathrm{E}[J(I)] + \lambda^*(P-\gamma\mathrm{E}[I])\} \\ &\le \sup_{I\in\mathcal{I}} \{\mathrm{E}[J(I)] + \lambda^*(P-\gamma\mathrm{E}[I])\} \\ &= \mathrm{E}[J(I^*)] + \lambda^*(P-\gamma\mathrm{E}[I^*]) \\ &= \mathrm{E}[J(I^*)] \\ &\le \sup_{I\in\mathcal{I} \text{ s.t. } \gamma \mathrm{E}[I]=P} \mathrm{E}[J(I)] \\ &= v(P), \end{aligned}$$

which implies that I^* is the solution of problem (1). ■

A.3. Proof of Proposition 2

Proof. We only need to show the existence of λ^* to satisfy $\mathrm{E}[I_{\lambda^*}] = P/\gamma$, because this combined with Lemma 1 implies the optimality of I_{λ^*} for problem (1).

For $x \in [c, d]$, define $\lambda_U := \max_{x\in[c,d]} \frac{1}{\gamma} G(0, x)$ and $\lambda_L := \min_{x\in[c,d]} \frac{1}{\gamma} G(M, x)$. Condition **H1** implies that both $S_1^{\lambda_U}$ and $S_2^{\lambda_L}$ differ from the set $[c, d]$ by only a μ-null set. Thus, by (9), $\mathrm{E}[I_{\lambda_U}(X)] = 0$ and $\mathrm{E}[I_{\lambda_L}(X)] = M$. As a result, it is sufficient to show that $\mathrm{E}[I_\lambda(X)]$ is continuous on $[\lambda_L, \lambda_U]$.

Below we only show the right continuity of $\mathrm{E}[I_\lambda(X)]$ on $[\lambda_L, \lambda_U]$, as its left continuity follows in the same fashion. Define $\Delta_\epsilon^\lambda := |\mathrm{E}[I_{\lambda+\epsilon}(X)] - \mathrm{E}[I_\lambda(X)]|$ for $\lambda \in [\lambda_L, \lambda_U - \epsilon]$ and $\epsilon > 0$. Then,

$$\begin{aligned} \Delta_\epsilon^\lambda &= \Big| M\cdot \Pr\big(X\in S_2^{\lambda+\epsilon}\big) + \int_{S_3^{\lambda+\epsilon}} I_{\lambda+\epsilon}(x)h(x)\,\mathrm{d}x \\ &\qquad - M\cdot\Pr\big(X\in S_2^\lambda\big) - \int_{S_3^\lambda} I_\lambda(x)h(x)\,\mathrm{d}x \Big| \\ &\le M\cdot \left|\Pr\big(X\in S_2^{\lambda+\epsilon}\big) - \Pr\big(X\in S_2^\lambda\big)\right| + \left|\int_{S_3^{\lambda+\epsilon}} I_{\lambda+\epsilon}(x)h(x)\,\mathrm{d}x - \int_{S_3^\lambda} I_\lambda(x)h(x)\,\mathrm{d}x\right| \\ &=: M|J_2^\epsilon| + |J_3^\epsilon|. \end{aligned}$$

By definition of S_2^λ in (7), $J_2^\epsilon = -\Pr(\lambda\gamma < G(M, X) \le (\lambda+\epsilon)\gamma) = \Pr(G(M, X) \le \lambda\gamma) - \Pr(G(M, X) \le (\lambda+\epsilon)\gamma) \to 0$ as $\epsilon \to 0^+$.

It remains to show $J_3^\epsilon \to 0$ as $\epsilon \to 0^+$. Indeed,

$$J_3^\epsilon = \int_{S_3^{\lambda+\epsilon} \cap \overline{S_3^\lambda}} I_{\lambda+\epsilon}(x)h(x)\,\mathrm{d}x - \int_{S_3^\lambda \cap \overline{S_3^{\lambda+\epsilon}}} I_\lambda(x)h(x)\,\mathrm{d}x + \int_{S_3^{\lambda+\epsilon} \cap S_3^\lambda} \left(I_{\lambda+\epsilon}(x) - I_\lambda(x)\right) h(x)\,\mathrm{d}x, \tag{A.1}$$

where $\bar{A}$ denotes the complement of a set A. It is easy to verify

$$S_3^{\lambda+\epsilon} \cap \overline{S_3^\lambda} = \{x \in [c,d]\,|\,\lambda\gamma < G(M,x) \le (\lambda+\epsilon)\gamma \le G(0,x)\}$$

and

$$S_3^\lambda \cap \overline{S_3^{\lambda+\epsilon}} = \{x \in [c,d]\,|\,G(M,x) \le \lambda\gamma \le G(0,x) < (\lambda+\epsilon)\gamma\}.$$

Condition **H1** implies both $\mu\left(S_3^{\lambda+\epsilon} \cap \overline{S_3^\lambda}\right) \to 0$ and $\mu\left(S_3^\lambda \cap \overline{S_3^{\lambda+\epsilon}}\right) \to 0$ as $\epsilon \to 0^+$. Further noting $I_{\lambda+\eta}(x) \in [0, M]$ for any $\eta \ge 0$ and $x \in [c,d]$, the first two items in (A.1) converge to 0 as $\epsilon \to 0^+$.

For $x \in S_3^{\lambda+\epsilon} \cap S_3^\lambda$, we have

$$G(I_{\lambda+\epsilon}(x), x) = (\lambda+\epsilon)\gamma \quad \text{and} \quad G(I_\lambda(x), x) = \lambda\gamma.$$

Further, $G(\xi, x)$ is a continuous and differentiable function of ξ for any $x \in [c,d]$, and thus, we apply the mean value theorem to obtain

$$\epsilon\gamma = G(I_{\lambda+\epsilon}(x), x) - G(I_\lambda(x), x) = G'_\xi(\xi, x)\left(I_{\lambda+\epsilon}(x) - I_\lambda(x)\right)$$

for some constant $\xi := \xi_{x,\lambda}$ valued between $I_{\lambda+\epsilon}(x)$ and $I_\lambda(x)$, where $G'_\xi(\xi,x) := \frac{\partial}{\partial\xi}G(\xi,x)$. This implies

$$|I_{\lambda+\epsilon}(x) - I_\lambda(x)| = \frac{\epsilon\gamma}{\left|G'_\xi(\xi,x)\right|} \le \frac{\epsilon\gamma}{\inf_{\xi\in[0,M],\ x\in[c,d]}\left|G'_\xi(\xi,x)\right|}. \tag{A.2}$$

From (5), we have $G'_\xi(\xi,x) = \int_a^b U''\left(w + \xi - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y$. Since $U''(\cdot)$ is a continuous function over its domain, there exits a constant $\delta > 0$ such that

$$\left|G'_\xi(\xi,x)\right| \ge \delta \int_a^b f(y|x)\,\mathrm{d}y = \delta, \quad \forall\ \xi \in [0,M] \text{ and } x \in [c,d].$$

Consequently, it follows from (A.2) that

$$\int_{S_3^{\lambda+\epsilon} \cap S_3^\lambda} \left(I_{\lambda+\epsilon}(x) - I_\lambda(x)\right) h(x)\,\mathrm{d}x \le \int_{S_3^{\lambda+\epsilon} \cap S_3^\lambda} \frac{\epsilon\gamma}{\delta} h(x)\,\mathrm{d}x \le \epsilon\frac{\gamma(d-c)}{\delta} \to 0, \quad \text{as } \epsilon \to 0^+.$$

Therefore, from (A.1), we have $J_3^\epsilon \to 0$ as $\epsilon \to 0^+$, by which we complete the proof. ■

A.4. Proof of Lemma 2

Proof. Without loss of generality, we assume that both S_1^λ and S_2^λ are non-empty. One can see from the rest of the proof that it is actually easier to show the desired result when either or both of the two sets are empty.

By Equation (9), it is sufficient to show $\widehat{I}_\lambda(x) \le 0$ for $x \in S_1^\lambda$ and $\widehat{I}_\lambda(x) \ge M$ for $x \in S_2^\lambda$. In fact, if $\widehat{I}_\lambda(x_1) > 0$ for some $x_1 \in S_1$, then

$$\begin{aligned} 0 &= G\left(\widehat{I}_\lambda(x_1), x_1\right) - \lambda\gamma \\ &= \int_a^b U'\left(w + \widehat{I}_\lambda(x_1) - y - (1-\theta)P\right) f(y|x_1)\,\mathrm{d}y - \lambda\gamma \\ &< \int_a^b U'\left(w - y - (1-\theta)P\right) f(y|x_1)\,\mathrm{d}y - \lambda\gamma \\ &= G(0, x_1) - \lambda\gamma \\ &< 0, \end{aligned}$$

where the first inequality is due to the strict convexity of U and the second one follows from the fact $x_1 \in S_1^\lambda$. The last display means $0 < 0$, a contradiction. Thus, we must have $\widehat{I}_\lambda(x) \le 0$ for $x \in S_1^\lambda$. We can use the same contradiction argument to show $\widehat{I}_\lambda(x) \ge M$ for $x \in S_2^\lambda$. ■

A.5. Proof of Proposition 3

Proof. We prove the proposition by contradiction. Suppose $S_3^{\lambda^*} = \emptyset$. Then, it must be one of the following three scenarios:

Case 1: $S_1^{\lambda^*} = [c, d]$,

Case 2: $S_2^{\lambda^*} = [c, d]$,

Case 3: $S_1^{\lambda^*} \cup S_2^{\lambda^*} = [c, d]$, $S_1^{\lambda^*} \neq \emptyset$ and $S_2^{\lambda^*} \neq \emptyset$.

For Case 1, it follows from (9) that $I^*(x) = 0$, $\forall x \in [c, d]$, and thus $P = \gamma \mathrm{E}[I_{\lambda^*}(X)] = 0$. Similarly, for Case 2, $I^*(x) = M$, $\forall x \in [c, d]$, and thus $P = \gamma \mathrm{E}[I_{\lambda^*}(X)] = \gamma M$. Since the insurance premium budget $P \in (0, \gamma M)$, both Cases 1 and 2 are impossible. Consider Case 3 and take $x_1 \in S_1^{\lambda^*}$ and $x_2 \in S_2^{\lambda^*}$. By Equations (6) and (7) and the fact that $G(\xi, x)$ is strictly decreasing in ξ, we have

$$G(0, x_1) < \lambda\gamma < G(M, x_2) < G(0, x_2). \tag{A.3}$$

Further, since $f(y|x)$ is continuous on $(x, y) \in [c, d] \times [a, b]$, it must be uniformly continuous on $[a, b]$. Therefore, for $\epsilon > 0$,

$$
\begin{aligned}
&|G(0, x+\epsilon) - G(0, x)| \\
&= \left| \int_a^b U'\left(w - y - (1-\theta)P\right) f(y|x+\epsilon)\,\mathrm{d}y \right. \\
&\qquad \left. - \int_a^b U'\left(w - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y \right| \\
&\le \max_{y\in[a,b]} \left| U'\left(w - y - (1-\theta)P\right) \right| \cdot \int_a^b |f(y|x+\epsilon) - f(y|x)|\,\mathrm{d}y \\
&\to 0, \;\text{ as } \epsilon \to 0^+, \;\; \forall x \in [c, d],
\end{aligned}
$$

which implies the continuity of $G(0, x)$ as a function of x on $[c, d]$. Thus, those inequalities in (A.3) imply the existence of a constant x_3 between x_1 and x_2 to satisfy $G(0, x_3) = \lambda\gamma$. Again, by the strictly increasing property of $G(\xi, x)$ in ξ, we have $G(M, x_3) < G(0, x_3) = \lambda\gamma$, which means that $x_3 \in S_3^{\lambda^*}$. This contradicts to the assumption of $S_3^{\lambda^*} = \emptyset$, and thus the proof is complete. ■

A.6. Proof of Theorem 1

Proof. By our assumption of $U''(x) < 0$ and $f(y|x) \equiv \frac{f(x,y)}{h(x)} > 0$, a.e., for $(x, y) \in [c, d] \times [a, b]$, we have

$$\int_a^b U''\left(w + \widehat{I}(x) - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y < 0, \;\; \forall \; x \in [c, d],$$

and thus $F(x, L)$ is well defined.

By Proposition 3, $S_3^{\lambda^*} \neq \emptyset$ for any constant λ^* such that $\mathrm{E}[I_{\lambda^*}(X)] = P/\gamma$, where $I_{\lambda^*}(x)$ is defined in (9). If we could find a constant λ^* to satisfy $\mathrm{E}[I_{\lambda^*}(X)] = P/\gamma$ and show that $\widehat{L}(x) = \widehat{I}_{\lambda^*}(x)$, a.e., on $S_3^{\lambda^*}$, then Lemma 2, along with the fact that $\widehat{L}(x)$ is well defined on $[c, d]$, implies that $L^* = \left(\widehat{L}(x) \vee 0\right) \wedge M$ is the optimal solution to problem (1).

Since $\widehat{L}(x)$ satisfies Equation (12), we have

$$
\begin{aligned}
\int_a^b \Bigg\{ & U''\left(w + \widehat{L}(x) - y - (1-\theta)P\right) f(y|x) \frac{\mathrm{d}\widehat{L}(x)}{\mathrm{d}x} \\
& + U'\left(w + \widehat{L}(x) - y - (1-\theta)P\right) \frac{\partial}{\partial x} f(y|x) \Bigg\}\,\mathrm{d}y = 0,
\end{aligned}
$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^b U'\left(w + \widehat{L}(x) - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y = 0, \;\; x \in [c, d].$$

This implies

$$G(\widehat{L}(x), x) = \int_a^b U'\left(w + \widehat{L}(x) - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y = \lambda_0\gamma, \;\; x \in [c, d], \quad \text{(A.4)}$$

where the constant λ_0 is defined as

$$\lambda_0 := \frac{1}{\gamma}\int_a^b U'\left(w + \widehat{L}(c) - y - (1-\theta)P\right) f(y|c)\,\mathrm{d}y.$$

The last two displays, together with the fact that Equation (10) has a unique solution $\widehat{I}_\lambda(x)$ for every $x \in S_3^{\lambda_0}$, imply that $\widehat{L}(x) = \widehat{I}_{\lambda^*}(x)$ on $S_3^{\lambda^*}$ for $\lambda^* = \lambda_0$. Comparing (10) and (A.4), we see $\widehat{L}$ and $\widehat{I}_{\lambda^*}$ satisfy the same equation. Thus, from the proof of Lemma 2, $\widehat{L}(x) \le 0$ for $x \in S_1^{\lambda^*}$ and $\widehat{L}(x) \ge M$ for $x \in S_2^{\lambda^*}$. Further, the second equation in (12) obviously implies $\mathrm{E}[I_{\lambda^*}(X)] = P/\gamma$, and thus, the proof is complete. ■

A.7. Proof of Proposition 4

Proof. With the given utility, the function F in Equation (12) becomes

$$\begin{aligned}
&F(x, L)\\
&= -\frac{\int_a^b U'\left(w + L(x) - y - (1-\theta)P\right)\frac{\partial}{\partial x} f(y|x)\,\mathrm{d}y}{\int_a^b U''\left(w + L(x) - y - (1-\theta)P\right) f(y|x)\,\mathrm{d}y}\\
&= -\frac{\int_a^b (\alpha - 2\beta w - 2\beta L(x) + 2\beta(1-\theta)P)\frac{\partial}{\partial x} f(y|x)\,\mathrm{d}y + \int_a^b 2\beta y \frac{\partial}{\partial x} f(y|x)\,\mathrm{d}y}{-2\beta}\\
&= \frac{(\alpha - 2\beta w - 2\beta L(x) + 2\beta(1-\theta)P)\frac{\partial}{\partial x}\int_a^b f(y|x)\,\mathrm{d}y + 2\beta\frac{\partial}{\partial x}\int_a^b y f(y|x)\,\mathrm{d}y}{2\beta}\\
&= \frac{0 + 2\beta\frac{\partial}{\partial x}\mathrm{E}[Y|X=x]}{2\beta}\\
&= \frac{\partial}{\partial x}\mathrm{E}[Y|X=x],
\end{aligned}$$

where we apply the fact that $\int_a^b f(y|x)\,\mathrm{d}y = 1$, and thus $\frac{\partial}{\partial x}\int_a^b f(y|x)\,\mathrm{d}y = 0$.

Due to the existence and continuity of $\frac{\partial}{\partial x} f(y|x)$, $\frac{\partial}{\partial x}\mathrm{E}[Y|X=x] = \int_a^b y\frac{\partial}{\partial x} f(y|x)\,\mathrm{d}y$ exists for every $x \in [c, d]$. Therefore, a direct application of Proposition 1 implies the following optimal index insurance:

$$I^*(x) = [(\mathrm{E}\,[Y|X=x] + \eta^*) \vee 0] \wedge M,$$

given that a constant η^* exists to satisfy

$$\mathrm{E}[I^*(X)] = \mathrm{E}\,\{[(\mathrm{E}\,[Y|X] + \eta^*) \vee 0] \wedge M\} = \frac{P}{\gamma}.$$

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In fact, $\mathrm{E}[I^*(X)]$ is apparently continuous and non-decreasing in η^*, and thus, a solution η^* must exist for the above equation. Therefore, the proof is complete. ■

A.8. Proof of Proposition 5

Proof. From the given utility, the function F in Equation (12) becomes

$$\begin{aligned}
F(x, L) &= -\frac{\int_a^b U'\left(w+L(x)-y-(1-\theta)P\right)\frac{\partial}{\partial x}f(y|x)\,\mathrm{d}y}{\int_a^b U''\left(w+L(x)-y-(1-\theta)P\right)f(y|x)\,\mathrm{d}y}\\
&= -\frac{\int_a^b e^{-\alpha(w+L(x)-y-(1-\theta)P)}\frac{\partial}{\partial x}f(y|x)\,\mathrm{d}y}{\int_a^b(-\alpha)e^{-\alpha(w+L(x)-y-(1-\theta)P)}f(y|x)\,\mathrm{d}y}\\
&= \frac{1}{\alpha}\cdot\frac{e^{-\alpha(w+L(x)-(1-\theta)P)}\int_a^b e^{\alpha y}\frac{\partial}{\partial x}f(y|x)\,\mathrm{d}y}{e^{-\alpha(w+L(x)-(1-\theta)P)}\int_a^b e^{\alpha y}f(y|x)\,\mathrm{d}y}\\
&= \frac{1}{\alpha}\frac{\frac{\partial}{\partial x}\mathrm{E}\left[e^{\alpha Y}|X=x\right]}{\mathrm{E}\left[e^{\alpha Y}|X=x\right]}\\
&= \frac{\partial}{\partial x}\left\{\frac{1}{\alpha}\ln\left(\mathrm{E}\left[e^{\alpha Y}|X=x\right]\right)\right\}.
\end{aligned}$$

Similar to the proof of Proposition 4, the continuity of $\frac{\partial}{\partial x}f(y|x)$ and $f(y|x)$ implies that both $\frac{\partial}{\partial x}\mathrm{E}\left[e^{\alpha Y}|X=x\right]$ and $\mathrm{E}\left[e^{\alpha Y}|X=x\right]$ exist for every $x\in[c,d]$, and thus

$$\frac{\partial}{\partial x}\left\{\frac{1}{\alpha}\ln\left(\mathrm{E}\left[e^{\alpha Y}|X=x\right]\right)\right\}$$

also exists for every $x\in[c,d]$. Then, by invoking Proposition 1, we derive the optimal index insurance as follows:

$$I^*(x)=\left[\left(\frac{1}{\alpha}\ln\left(\mathrm{E}\left[e^{\alpha Y}|X=x\right]\right)+\eta^*\right)\vee 0\right]\wedge M,$$

given the existence of a constant η^* to satisfy $P=\gamma\,\mathrm{E}[I^*(X)]$. Apparently, $\mathrm{E}[I^*(X)]$ is continuous and non-decreasing in η^*, which indicates the existence of η^*. Hence, the proof is complete. ■

A.9. A Numerical Scheme for Solving ODE (12)

For utility functions other than the quadratic and exponential ones, closed-form solutions for the ODE (12) are generally unavailable, and thus a numerical scheme is needed. The general boundary ODE problem (12) can be viewed as an initial value problem (A.5) along with an algebraic Equation (A.6):

$$\begin{cases} \dfrac{\mathrm{d}I}{\mathrm{d}x} = F(x, I), \quad x \in [c, d], \\ I(c) = I_c, \end{cases} \tag{A.5}$$

with initial value I_c determined by

$$P = \gamma \mathrm{E}\,[I^*(X)] = \gamma \mathrm{E}\left[\left(\widehat{I}_{\lambda^*}(X) \vee 0\right) \wedge M\right]. \tag{A.6}$$

For any fixed I_c, the initial value problem (A.5) is a standard ODE problem. If Equation (A.5) yields a unique solution for a given I_c, then Equation (A.6) becomes an algebraic equation of I_c. By Theorems 5.4 and 5.6 in Burden and Faires (2011), a sufficient condition for existence and uniqueness of the solution, and the well-posedness of problem (A.5) is given by

$$\textbf{H2:} \quad \begin{cases} \left|\dfrac{\partial F(x, I)}{\partial I}\right| \le L, \forall\, (x, I) \in [c, d] \times \mathbb{R} \text{ for some constant } L > 0, \\ F(x, I) \text{ is continuous on } [c, d] \times \mathbb{R}. \end{cases}$$

If condition **H2** holds, then the implicit ODE (A.5)–(A.6) are well posted and it can be solved using a numerical procedure. We recommend the 4th order Runge–Kutta (RK4) method combined with a binary search to numerically compute $I^*(x)$, $x \in [c, d]$. The specific numerical scheme is summarized in six steps below.

Step 1: Find a large enough interval $[L_c, U_c]$ such that $I(c) \in [L_c, U_c]$. Check that $(P_{L_c} - P)(P_{U_c} - P) < 0$, where P_{I_c} denotes the premium calculated by Equation (A.6) for the contract starting at I_c. Suppose $P_{L_c} - P < 0$, $P_{U_c} - P > 0$, and define $I_0(c) = \frac{1}{2}(L_c + U_c)$.

Step 2: Apply RK4 with a step-size $\delta > 0$ to the initial value problem $\frac{\mathrm{d}I}{\mathrm{d}x} = F(x, I)$, $x \in [c, d]$, with $I(c) = I_0(c)$: For $n = 0, 1, 2, \ldots, \lfloor \frac{d-c}{\delta} - 1 \rfloor$, define

1. $k_1 = F(x_n, I_n)$,
2. $k_2 = F\left(x_n + \dfrac{\delta}{2}, I_n + \dfrac{\delta}{2} k_1\right)$,
3. $k_3 = F\left(x_n + \dfrac{\delta}{2}, I_n + \dfrac{\delta}{2} k_2\right)$,
4. $k_4 = F\left(x_n + \delta, I_n + \delta k_3\right)$,
5. $x_{n+1} = x_n + \delta$,
6. $I_{n+1} = I_n + \dfrac{\delta}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

Step 3: Define $I_n^* = (I_n \vee 0) \wedge M$, $n = 0, 1, 2, ..., \frac{d-c}{\delta}$, where I_n is obtained from the previous step.

Step 4: Approximate the premium constraint $P = \gamma \mathrm{E}[I^*]$ numerically using

$$P_0 := \frac{\delta\gamma}{2}\left[2\left(\sum_{n=0}^{\frac{d-c}{\delta}} I_n^* h(c+n\delta)\right) - I_0^* h(c) - I_{\frac{d-c}{\delta}}^* h(d)\right].$$

Step 5: Verify whether $|P_0 - P| < \epsilon$ is satisfied by the given tolerance ϵ. If yes, I^* is already an accurate approximation to the solution of ODE (A.5) and (A.6), and we stop the algorithm; otherwise, we go to Step 6.

Step 6: If $P_0 < P$, then define $I_1(c) = \frac{1}{2}(I_0(c) + U_c)$; if $P_0 > P$, then define $I_1(c) = \frac{1}{2}(L_c + I_0(c))$. Go back to Step 2, replace the initial condition with $I(c) = I_1(c)$, and repeat Steps 2–6.

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