Math. Struct. in Comp. Science (2005), *vol.* 15, *pp.* 1033–1088. © 2005 Cambridge University Press
doi:10.1017/S0960129505004858 Printed in the United Kingdom

The semantics of BI and resource tableaux

D. GALMICHE[†], D. MÉRY[†] and D. PYM[‡]

[†]*LORIA UMR 7503 – Université Henri Poincaré Nancy 1, Campus Scientifique, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France.*
Email: galmiche@loria.fr, dmery@loria.fr

[‡]*Department of Computer Science, University of Bath, Claverton Down, Bath BA2 7AY, England, U.K.*
Email: d.j.pym@bath.ac.uk

Received 7 June 2003; revised 31 March 2005

The logic of bunched implications, BI, provides a logical analysis of a basic notion of resource that is rich enough, for example, to form the logical basis for 'pointer logic' and 'separation logic' semantics for programs that manipulate mutable data structures. We develop a theory of semantic tableaux for BI, so providing an elegant basis for efficient theorem proving tools for BI. It is based on the use of an algebra of labels for BI's tableaux to solve the resource-distribution problem, the labels being the elements of resource models. For BI with inconsistency, $\perp$, the challenge consists in dealing with BI's Grothendieck topological models within such a proof-search method, based on labels. We prove soundness and completeness theorems for a resource tableaux method TBI with respect to this semantics and provide a way to build countermodels from so-called dependency graphs. Then, from these results, we can define a new resource semantics of BI, based on partially defined monoids, and prove that this semantics is complete. Such a semantics, based on partiality, is closely related to the semantics of BI's (intuitionistic) pointer and separation logics. Returning to the tableaux calculus, we propose a new version with liberalised rules for which the countermodels are closely related to the topological Kripke semantics of BI. As consequences of the relationships between semantics of BI and resource tableaux, we prove two new strong results for propositional BI: its decidability and the finite model property with respect to topological semantics.

1. Introduction

The notion of *resource* is a basic one in many fields, including economics, engineering and psychology, but it is perhaps most clearly illuminated in computer science. The location, ownership, access to and, indeed, consumption of, resources are central concerns in the design of systems (such as networks, within which processors must access devices such as file servers, disks and printers) and in the design programs, which access memory and manipulate data structures (such as pointers).

The development of a mathematical theory of resource is one of the objectives of the programme of study of BI, the logic of bunched implications, introduced by O'Hearn and Pym (O'Hearn and Pym 1999; Pym 1999; Pym 2002; Pym 2004). The basic idea is to model directly the observed properties of resources and then to give a logical axiomatisation. Initially, we require the following properties of resources, beginning with the simple assumption of a set R of elements of a resource: a *combination*, $\bullet$, of resources, together with a *zero* resource, e; and a *comparison*, $\sqsubseteq$, of resources. Mathematically, we model this set-up with a (for now, commutative) preordered† monoid $\mathscr{R} = (R, \bullet, e, \sqsubseteq)$, in which $\bullet$, with unit e, has the property $m \sqsubseteq n$ and $m' \sqsubseteq n'$ imply $m \bullet m' \sqsubseteq n \bullet n'$, for any m, n, m', n'. Taking such a structure as an algebra of *worlds*, we obtain a forcing semantics for (propositional) BI that freely combines multiplicative (intuitionistic linear $\otimes$ and $\multimap$) and additive (intuitionistic $\wedge$, $\rightarrow$ and $\vee$) structure. A significant variation takes classical additives instead. BI is described in the necessary detail in Section 2. For now, the key property of the semantics is the *sharing interpretation* (O'Hearn and Pym 1999; O'Hearn 1999).

The (elementary) semantics of the multiplicative conjunction, $m \models \phi_1 * \phi_2$ iff there are n_1 and n_2 such that $n_1 \bullet n_2 \sqsubseteq m$, $n_1 \models \phi_1$ and $n_2 \models \phi_2$, is interpreted as follows: the resource m is sufficient to support $\phi_1 * \phi_2$ just when it can be divided into resources n_1 and n_2 such that n_1 is sufficient to support ϕ_1 and n_2 is sufficient to support ϕ_2. The assertions ϕ_1 and ϕ_2 – think of them as expressing properties of programs – *do not share* resources. In contrast, in the semantics of the additive conjunction, $m \models \phi_1 \wedge \phi_2$ iff $m \models \phi_1$ and $m \models \phi_2$, the assertions ϕ_1 and ϕ_2 *share* the resource m. Similarly, the semantics of the multiplicative implication, $m \models \phi \mathrel{-\!\!*} \psi$ iff for all n such that $n \models \phi$, $m \bullet n \models \psi$, is interpreted as follows: the resource m is sufficient to support $\phi \mathrel{-\!\!*} \psi$ – think of the proposition as (the type of) a function – just when for any resource n that is sufficient to support ϕ – think of it as the argument to the function – the combination $m \bullet n$ is sufficient to support ψ. The function and its argument *do not share* resources. In contrast, in the semantics of additive implication, $m \models \phi \rightarrow \psi$ iff for all $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$, the function and its argument *share* the resource n. For a simple example of resource as *cost*, let the monoid be given by the natural numbers with addition and unit zero, ordered by less than or equals. A more substantial example, 'pointer logic', PL, and its spatial semantics, has been provided by Ishtiaq and O'Hearn (Ishtiaq and O'Hearn 2001). In fact, the semantics of pointer logic is based on *partial monoids*, in which the operation $\bullet$ is partially defined.

An elementary Kripke resource semantics, formulated in categories of *presheaves* on preordered monoids, has been defined for BI (O'Hearn and Pym 1999; Pym 1999; Pym 2002; Pym 2004) but it is sound and complete only for BI without inconsistency, $\perp$, the unit of the additive disjunction. This elementary forcing semantics handles inconsistency only by denying the existence of a world at which $\perp$ is forced. The completeness of BI *with* $\perp$ for a monoid-based forcing semantics is achieved, first, in categories of *sheaves* on *open topological monoids* and, second, in the more abstract topological setting of *Grothendieck sheaves* on preordered monoids (Pym 2002; Pym *et al.* 2004; Pym 2004). These different semantics of BI are sketched in Section 2. In each of these cases, inconsistency is internalised in the semantics. The semantics of (intuitionistic) pointer logic can be incorporated into the Kripke semantics based on Grothendieck sheaves (Pym 2002; Pym *et al.* 2004; Pym 2004), but it suggests partial monoids as a basis for a 'Kripke resource semantics'.

† The preorder $\sqsubseteq$ is the reverse of that taken in O'Hearn and Pym (1999). It corresponds to the one usually used in labelled deductive systems and thus allows us to relate directly the resources with labels in a traditional way.

BI provides a logical analysis of a basic notion of resource (Pym 2002; Pym 2004), quite different from linear logic's 'number-of-uses' reading (Girard 1987), which has proved rich enough to provide intuitionistic (that is, the additives) 'pointer logic' semantics for programs that manipulate mutable data structures (Ishtiaq and O'Hearn 2001; O'Hearn *et al.* 2001; Pym *et al.* 2004; Pym 2004). In this context, efficient and useful proof-search methods are required. For many logics, semantic tableaux have provided elegant and efficient bases for tools based on both proof-search and countermodel generation (Fitting 1990). We should like to have bases for such tools for BI and PL. The main difficulty to be overcome in giving such a system for BI is the presence of multiplicatives. We need a mechanism for calculating the distribution of 'resources' with multiplicative rules, which, in BI's sequent calculus, given in Section 2, is handled via side-formulae. A solution is a specific use of *labels* that allow the capture of the semantic relationships between connectives during proof-search or proof analysis (Balat and Galmiche 2000; Gabbay 1996; Harland and Pym 2003).

Recent work has proposed a tableaux calculus, with labels, for BI without $\perp$, which captures the elementary Kripke resource semantics (Galmiche and Méry 2001), but an open question until now has been whether a similar approach or calculus can be extended to full BI, including $\perp$, and its Grothendieck topological semantics. Such a calculus and its related tableaux method would provide a decision procedure for BI (decidability of BI has been conjectured, via a different method in Pym (2002; 2004), but not explicitly proved). A real difficulty lies in the treatment of a monoid-based forcing semantics, like Grothendieck topological semantics (Pym 2002; Pym 2004), with such a labelled calculus. In this paper, we are concerned mainly with the relationships and connections between semantics of BI and so-called 'resource tableaux', which lead to new results from the perspective of both proof-search and semantics.

In Section 2, we briefly review the BI logic, its sequent calculus and, mainly, its various semantics, namely the Kripke resource semantics and the Grothendieck topological semantics. We explain why problems for completeness with respect to the former arise from the presence of inconsistency $\perp$ and how topological semantics solve them.

In Section 3, we define a system of labelled semantic tableaux, TBI, in which the labels are drawn from BI's algebra of worlds and which use BI's forcing semantics, which is based on Grothendieck sheaves. The rules are similar to the ones of Galmiche and Mry (2001) with the introduction of label constraints (called assertions and requirements), but the specific way to deal with $\perp$ topologically involves delicate new closure and provability conditions. Moreover, we introduce a specific graph called dependency graph (or Kripke resource graph), which is built in parallel with the tableau expansion and reflects the information that can be derived from a given set of assertions. Two examples illustrate how resource tableaux deal with $\perp$ and how provability in BI can be analysed. In Section 4, we study the soundness of TBI, which can only be proved for so-called basic Grothendieck resource models. We need new results on semantics, developed in the next section, to be able to prove directly the soundness of TBI. We also show the completeness of TBI with respect to the Grothendieck topological semantics. We then use our completeness proof to show that in the case of a *failed* tableau, that is, non-provability, we can build a *a countermodel* from a *dependency graph*. Moreover, observing that a dependency graph only deals with the relevant resources needed to decide provability, it seems possible to propose a new resource semantics for BI, which corresponds to an alternative way of dealing with $\perp$ by considering partially defined monoids. In Section 5, we define such a semantics called PDM semantics, which was previously anticipated but not developed in Pym (2002), Pym *et al.* (2004) and Pym (2004). For that, we start by defining a new relational semantics of BI, based on a ternary-relation $Rxyz$ with particular properties, and then prove its soundness and completeness with respect to BI. This semantics is such that the class of PDM models is included in the class of the relational models and, therefore, the PDM semantics corresponding to this relational semantics with a specific relation defined by $Rxyz \equiv x \bullet y \sqsubseteq z$. Thus, we can solve the problem of soundness of TBI by showing that TBI is sound with respect to the relational semantics. Returning to the PDM semantics, considered as a specialisation of the relational semantics, we have now a new resource semantics that is naturally related to our study of resource tableaux, through labels and constraints, and is proved complete with respect to BI. Similar semantics can be defined for Affine BI, in which the multiplicative conjunction satisfies the structural rule of weakening. This illustrates the power of the partiality in this context, knowing that such fragments of BI are the logical bases of pointer logic (Ishtiaq and O'Hearn 2001) and separation logic (Reynolds 2000).

In Section 6, we study how, by a special treatment of the additive disjunction, we can propose liberalised rules for TBI, which yields an improvement of the initial version of the calculus. This new version is related to the topological Kripke semantics of BI (Pym *et al.* 2004; Pym 2004), which allows $\perp$ to be taken into account together with a non-indecomposable treatment of the disjunction. In fact, the topological semantics considers open sets, while the canonical interpretation of BI we define, considers sets that are closed under deduction. We show that a new semantic clause for the additive disjunction is required to achieve a suitable canonical forcing relation. However, the semantic changes made to $\vee$ have a syntactic counterpart and the corresponding initial expansion rules have to be accordingly modified. In Section 7, we give new expansion rules for TBI and thus a resulting tableaux system called TBI$'$. We prove its soundness and completeness, but the construction of countermodels is less direct than in TBI because of the extension of the label algebra. Moreover, those countermodels are related to BI-algebras, which are themselves closely related to the topological Kripke semantics, from which they can be viewed as an algebraic counterpart. Therefore TBI$'$ appears as the syntactic reflection of the forcing semantics in the category of sheaves over a topological monoid, and dependency graphs can be viewed as (partial) topological Kripke models.

In Section 8, we prove two *new* results for propositional BI, namely, the *finite model property* with respect to Grothendieck topological semantics, and the *decidability* of propositional BI, conjectured but not proved in Pym (2002; 2004). The relationships identified between resources, labels, dependency graphs, proof-search and resource semantics are central in this study. Moreover, dependency graphs can be seen directly as countermodels in this new semantics. We conclude, in Section 9, with a summary of our contribution and a brief discussion of future directions for this research.

2. The semantics and proof theory of BI

We review briefly the semantics and proof theory of BI, which freely combines linear conjunction, $*$ with unit I, and linear implication, $-\!*$, with intuitionistic conjunction, $\wedge$ with unit $\top$, disjunction, $\vee$ with unit $\bot$, and implication, $\rightarrow$. There is an elementary Kripke resource semantics, which, because of the interaction between $-\!*$ and $\bot$ (Pym 2002; Pym *et al.* 2004; Pym 2004), is complete only for BI without $\bot$. In order to have completeness with $\bot$, it is necessary to use the topological setting introduced in O'Hearn and Pym (1999), Pym (2002), Pym *et al.* (2004) and Pym (2004) and described below, which is a significant step over the elementary case.

Definition 2.1 (Propositions). The propositional language of BI consists of: a multiplicative unit I, the multiplicative connectives $*$, $-\!*$, the additive units $\top$, $\bot$, the additive connectives $\wedge$, $\rightarrow$, $\vee$, and a countable set $L = \mathrm{p}, \mathrm{q}, \ldots$ of propositional letters. $\mathscr{P}(L)$, the collection of BI propositions over L, is given by the following inductive definition:

$$\phi ::= \ \mathrm{p} \mid \mathrm{I} \mid \phi * \phi \mid \phi -\!* \phi \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \rightarrow \phi \mid \phi \vee \phi.$$

The additive connectives correspond to those of intuitionistic logic (IL), whereas the multiplicative connectives correspond to those of multiplicative intuitionistic linear logic (MILL). The antecedents of logical consequences are structured as bunches, in which there are two ways to combine operations that display additive and multiplicative behavior, respectively.

Definition 2.2 (Bunches). Bunches are given by the following grammar:

$$\Gamma ::= \phi \mid \varnothing_a \mid \Gamma \,;\, \Gamma \mid \varnothing_m \mid \Gamma \,,\, \Gamma.$$

Equivalence, $\equiv$, is given by commutative monoid equations for ',' and ';', whose units are $\varnothing_m$ and $\varnothing_a$ respectively, together with the evident substitution congruence for sub-bunches – we write $\Gamma(\Delta)$ to denote a sub-bunch Δ of Γ – determined by the grammar.

Judgements are expressions of the form $\Gamma \vdash \phi$, where Γ is a bunch and ϕ is a proposition. The LBI sequent calculus is given in Figure 1[†]. The following results hold (see Pym (2002; 2004) for the proofs):

Theorem 2.1 (Cut-elimination). If $\Gamma \vdash \phi$ is provable in LBI including Cut, then it is provable in LBI without Cut.

A proposition ϕ is a theorem if $\varnothing_a \vdash \phi$ or $\varnothing_m \vdash \phi$ is provable in LBI, but the following theorem provides a simpler definition (Pym 1999; Pym 2002):

[†] In the definition of LBI given in Pym (2002), the $\vee L$ rule is misstated: it is corrected in Pym (2004) and the corrected version is as given in Figure 1. This error was known prior to the publication of Pym (2002), but persisted because of an editing error by the author. There are no known consequences. The error was also propagated to Galmiche *et al.* (2002), but Harland and Pym (2003) is correct.

$$\frac{}{\phi \vdash \phi}\,ax \qquad \frac{\Gamma \vdash \phi}{\Delta \vdash \phi}\,\Delta \equiv \Gamma \qquad \frac{\Gamma(\Delta) \vdash \phi}{\Gamma(\Delta\,;\Delta') \vdash \phi}\,w \qquad \frac{\Gamma(\Delta\,;\Delta) \vdash \phi}{\Gamma(\Delta) \vdash \phi}\,c \qquad \frac{\Delta \vdash \phi \quad \Gamma(\phi) \vdash \psi}{\Gamma(\Delta) \vdash \psi}\,cut$$

$$\frac{}{\Gamma(\bot) \vdash \phi}\,\bot_L \qquad \frac{\Gamma(\emptyset_m) \vdash \phi}{\Gamma(\mathrm{I}) \vdash \phi}\,\mathrm{I}_L \qquad \frac{}{\emptyset_m \vdash \mathrm{I}}\,\mathrm{I}_R \qquad \frac{\Gamma(\emptyset_a) \vdash \phi}{\Gamma(\top) \vdash \phi}\,\top_L \qquad \frac{}{\emptyset_a \vdash \top}\,\top_R$$

$$\frac{\Gamma(\phi\,,\psi) \vdash \chi}{\Gamma(\phi * \psi) \vdash \chi}\,*_L \qquad \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma\,,\Delta \vdash \phi * \psi}\,*_R \qquad \frac{\Delta \vdash \phi \quad \Gamma(\psi\,,\Delta') \vdash \chi}{\Gamma(\Delta\,,\phi \mathrel{-\!\!*} \psi\,,\Delta') \vdash \chi}\,-\!\!*_L \qquad \frac{\Gamma\,,\phi \vdash \psi}{\Gamma \vdash \phi \mathrel{-\!\!*} \psi}\,-\!\!*_R$$

$$\frac{\Gamma(\phi\,;\psi) \vdash \chi}{\Gamma(\phi \wedge \psi) \vdash \chi}\,\wedge_L \qquad \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma\,;\Delta \vdash \phi \wedge \psi}\,\wedge_R \qquad \frac{\Delta \vdash \phi \quad \Gamma(\psi\,;\Delta') \vdash \chi}{\Gamma(\Delta\,;\phi \to \psi\,;\Delta') \vdash \chi}\,\to_L$$

$$\frac{\Gamma\,;\phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\,\to_R \qquad \frac{\Gamma(\phi) \vdash \chi \quad \Gamma(\psi) \vdash \chi}{\Gamma(\phi \vee \psi) \vdash \chi}\,\vee_L \qquad \frac{\Gamma \vdash \phi_{i\,(i=1,2)}}{\Gamma \vdash \phi_1 \vee \phi_2}\,\vee_{Ri}$$

Figure 1. The LBI Sequent Calculus.

Theorem 2.2. $\varnothing_a \vdash \phi$ (respectively, $\varnothing_m \vdash \phi$) is provable in LBI if and only if $\top \vdash \phi$ (respectively, $\mathrm{I} \vdash \phi$) is provable in LBI.

Corollary 2.1. A proposition ϕ is a theorem iff $\varnothing_m \vdash \phi$ is provable in LBI.

2.1. *Kripke resource models*

As explained in the introduction, BI has a simple and natural truth-functional semantics, presented as a Kripke-like forcing relation relative to a preordered commutative monoid or worlds. It may be seen as freely combining Kripke's semantics for intuitionistic logic (Kripke 1965) with Urquhart's semantics for the relevant connectives (Urquhart 1972), which comprise the multiplicative fragment of intuitionistic linear logic (MILL) (O'Hearn and Pym 1999; Pym 2002). As we have seen, the meaning of the formal semantics may be explained in terms of a *resource*, an appropriate generalisation of Urquhart's notion of *pieces of information*.

So our basic semantic structure is a preordered commutative monoid upon which we impose a bifunctoriality condition. This condition, though credible from the point of view of resource semantics, is motivated mathematically. Other properties that are well motivated by resource semantics, such as 'aggregation',

$$\text{(A): for all } m \text{ and } n,\ m \sqsubseteq m \bullet n \text{ and } n \sqsubseteq m \bullet n,$$

are not required for our mathematical development and are not adopted here.

Definition 2.3. A *Kripke resource monoid* (KRM) $\mathscr{M} = (M, \bullet, e, \sqsubseteq)$ is a preordered commutative monoid in which $\bullet$ is bifunctorial with respect to $\sqsubseteq$:

$$\text{(P): if } m \sqsubseteq n \text{ and } m' \sqsubseteq n', \text{ then } m \bullet m' \sqsubseteq n \bullet n'.$$

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

We frequently refer to the bifunctoriality condition by saying that $\bullet$ is order-preserving. Consequently, we call $\mathscr{M}$ an order-preserving preordered commutative monoid.

Having established the basic structure, we only require a notion of interpretation in order to be able to define a class of elementary models. At this stage, we require a condition (K) that will ensure that the hereditary property of Kripke models holds in our setting.

Definition 2.4. Let $\mathscr{M}$ be a KRM and $\mathscr{P}(L)$ be the collection of BI propositions over a language L of propositional letters. Then an *elementary Kripke resource interpretation* (EKRI) is a function $[\![-]\!] : L \to \mathscr{P}(M)$ that satisfies

$$\text{(K): for all } m, n \in M \text{ such that } m \sqsubseteq n, \text{ if } m \in [\![\mathrm{p}]\!], \text{ then } n \in [\![\mathrm{p}]\!].$$

All that remains for an elementary semantics is to give a forcing relation. The clauses for the additives ($\top$, $\wedge$, $\bot$, $\vee$, $\to$) just exploit the ordering on worlds, $\sqsubseteq$, and exactly those for intuitionistic Kripke models (Kripke 1965). Note, in particular, the clause for $\bot$: the model has no internal representative for inconsistency. The clauses for the multiplicatives (I, $\otimes$, $-\!\!*$) require the combination of worlds, $\bullet$, and follow Urquhart's semantics (Urquhart 1972).

Definition 2.5. An *elementary Kripke resource model* (EKRM), is a triple $\mathscr{K} = (\mathscr{M}, \models, [\![-]\!])$ in which $\mathscr{M}$ is a KRM, $[\![-]\!]$ is an EKRI, and $\models$ is a forcing relation on $M \times \mathscr{P}(L)$ satisfying the following conditions:

— $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
— $m \models \top$ iff always
— $m \models \bot$ iff never
— $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
— $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
— $m \models \phi \to \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
— $m \models \mathrm{I}$ iff $e \sqsubseteq m$
— $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \sqsubseteq m$, $n \models \phi$ and $n' \models \psi$
— $m \models \phi -\!\!* \psi$ iff for all $n \in M$ such that $n \models \phi$, $m \bullet n \models \psi$.

The semantics of propositions given by the relation $\models$ is parametrised by the interpretation $[\![-]\!]$ for which the property (K) of Definition 2.4 holds for atomic propositions. One can prove, by structural induction on propositions, that if (K) holds for atomic propositions, it also holds for any proposition.

Let $\mathscr{K}$ be an EKRM and Φ_Γ be the formula obtained from a bunch Γ by replacing each ';' by $\wedge$, each ',' by $*$, each $\varnothing_a$ by $\top$ and each $\varnothing_m$ by I, with association respecting the tree structure of Γ. The sequent $\Gamma \vdash \phi$ is *valid in* $\mathscr{K}$ (notation, $\Gamma \models_{\mathscr{K}} \phi$), if and only if, for all worlds $m \in M$, $m \models \Phi_\Gamma$ implies $m \models \phi$. The sequent $\Gamma \vdash \phi$ is *valid* (notation, $\Gamma \models \phi$), if and only if, for all EKRMs $\mathscr{K}$, $\Gamma \models_{\mathscr{K}} \phi$.

Theorem 2.3 (Soundness of BI). If $\Gamma \vdash \phi$ is provable in LBI, then $\Gamma \models \phi$.

The unit $\bot$ of the $\vee$ connective internalises inconsistency in BI, but the elementary Kripke resource semantics does not account for inconsistency ($\bot$ is nowhere forced).

Accordingly, $\perp$ must be excluded to obtain the completeness result with respect to this semantics and the completeness result is only proved for BI without $\perp$ (Pym 2002; Pym 2004).

Theorem 2.4 (Completeness of BI without $\perp$). If $\Gamma \models \phi$ in BI without $\perp$, then $\Gamma \vdash \phi$ is provable in LBI without $\perp$.

In fact, the incompleteness of BI (with $\perp$) arises from the interaction between multiplicative implication ($-\!\!*$) and the unit $\perp$, as illustrated by the following example:

For all ϕ and ψ, we have $(\phi -\!\!* \perp) \to \perp ; (\psi -\!\!* \perp) \to \perp \models ((\phi * \psi) -\!\!* \perp) \to \perp$ in the elementary Kripke resource semantics but $(\phi -\!\!* \perp) \to \perp ; (\psi -\!\!* \perp) \to \perp \vdash ((\phi * \psi) -\!\!* \perp) \to \perp$ is not provable in LBI (Pym 2002).

To understand how this incompleteness arises, it is necessary to take a slightly more abstract point of view. The elementary semantics can be formulated quite conveniently in presheaf categories $[\mathscr{M}^{op}, Set]$ (Lambek and Scott 1986). Here $\mathscr{M}$ is a KRM, considered as a category (of worlds). All of the connectives, except $\perp$, can be defined in this setting by exploiting the Cartesian closed (Lambek and Scott 1986) structure, carried by any functor category (we neglect concerns about size here), and the monoidal closed structure on $[\mathscr{M}^{op}, Set]$ induced by $\mathscr{M}$ via Day's constructions (Day 1970; O'Hearn and Pym 1999; Pym *et al.* 2004; Pym 2004). To see that the completeness argument fails in this setting, we must consider the interaction between $\perp$ and $-\!\!*$. It is easy to check that the sequent $\phi, \phi -\!\!* \perp \vdash \perp$ is provable in LBI. But, in the term model that must be constructed to establish completeness (Pym 2002; Pym 2004), the bunch $\phi, \phi -\!\!* \perp$ is equivalent to $\perp$. It then follows that we must have a *world* that represents, and so forces, $\perp$. Such a world is not present in the elementary semantics.

Completeness in the presence of $\perp$ can be recovered by adopting a semantics that has an internal representation of $\perp$. One such semantics is provided by moving from the presheaf-theoretic setting of the elementary semantics, to the topological setting of sheaves. By replacing the category of worlds with a kind of topological monoid, considered as category, we then obtain models in the category of sheaves over a topological space of worlds in which the empty set, which is an open set, provides a representative for $\perp$ (Pym 2002; Pym 2004). It is possible, and we would suggest desirable, to retain a direct connection with the simple algebraic structure of a pre-ordered commutative monoid by working with *Grothendieck sheaves.*

2.2. *Grothendieck sheaf-theoretic models*

BI's Kripke semantics may be adapted to take account of $\perp$ by moving from presheaves to sheaves on a topological monoid (Pym 2002; Pym *et al.* 2004; Pym 2004). We briefly review the topological semantics of BI, which allows $\perp$ to be taken into account together with a non-indecomposable treatment of the disjunction.

Definition 2.6 (TRM). A *topological resource monoid* (TRM) $(\mathscr{X}, \cdot, e)$ is a commutative monoid in the category **Top** of topological spaces and continuous maps between them, that is, a topological space $|\mathscr{X}|$, with open sets $\mathscr{O}(\mathscr{X})$, on which a monoidal product

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

$\cdot : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ of open sets is defined, together with its unit $e : 1 \to \mathscr{X}$, and such that $\cdot$ distributes over arbitrary unions of open sets:

$$V \cdot \Big(\bigcup_i U_i\Big) = \bigcup_i (V \cdot U_i).$$

The tensor product of two opens sets is not necessarily open, consequently, we must require that the monoidal structure be defined by *open maps*, that is, maps that map open sets to open sets. Thus, if $(|\mathscr{X}|, \mathscr{O}(\mathscr{X}))$ is a topological space and $(\cdot, e)$ are defined as above, we speak of the *topological monoid* $(\mathscr{X}, \cdot, e)$ on $(|\mathscr{X}|, \mathscr{O}(\mathscr{X}))$ and $(\cdot, e)$. If $(\cdot, e)$ are open, we speak of the *open topological monoid* $(\mathscr{X}, \cdot, e)$ on $(|\mathscr{X}|, \mathscr{O}(\mathscr{X}))$ and $(\cdot, e)$.

The symmetric monoidal structure of a commutative topological monoid gives rise, via Day's construction of a tensor product (Day 1970), to a symmetric monoidal closed structure on the category $\mathrm{Sh}(\mathscr{X})$ of sheaves on $\mathscr{X}$.

Definition 2.7 (TKM). Let $\mathscr{X} = (\mathscr{X}, \cdot, e)$ be a commutative open topological monoid. A *topological kripke model* (TKM) is a triple $\mathscr{T} = (\mathrm{Sh}(\mathscr{X}), \models, [\![-]\!])$, where $[\![-]\!] : \mathscr{P}(L) \rightharpoonup \mathrm{Sh}(\mathscr{X})$ is a partial function from the BI-propositions over a language L of propositional letters to the objects of $\mathrm{Sh}(\mathscr{X})$ such that

$$\text{Kripke monotonicity: if } V \subseteq U \text{ then } \forall\, \phi \in \mathscr{P}(L), U \models \phi \text{ implies } V \models \psi$$

and $\models \subseteq \mathscr{O}(\mathscr{X}) \times \mathscr{P}(L)$ satisfies

- — $U \models \mathrm{p}$ iff $[\![\mathrm{p}]\!](U) \neq \varnothing$, for $\mathrm{p} \in L$
- — $U \models \top$ for all $U \in \mathscr{O}(\mathscr{X})$
- — $U \models \bot$ iff $U = \varnothing$
- — $U \models \mathrm{I}$ iff $U \subseteq I$
- — $U \models \phi \wedge \psi$ iff $U \models \phi$ and $U \models \psi$
- — $U \models \phi \vee \psi$ iff, for some $V, V' \in \mathscr{O}(\mathscr{X})$ such that $U = V \cup V'$, $V \models \phi$ and $V' \models \psi$
- — $U \models \phi \to \psi$ iff, for all $V \subseteq U$, $V \models \phi$ implies $V \models \psi$
- — $U \models \phi * \psi$ iff, for some $V, V' \in \mathscr{O}(\mathscr{X})$, $U \subseteq V \cdot V'$ and $V \models \phi$ and $V' \models \psi$
- — $U \models \phi -\!\!* \psi$ iff, for all $V \in \mathscr{O}(\mathscr{X})$, $V \models \phi$ implies $U \cdot V \models \psi$.

Such a semantics considers an inconsistent world, at which $\bot$ is forced, together with the so-called non-indecomposable treatment of $\vee$: $U \models \phi \vee \psi$ iff, for some open sets V, V' such that $U = V \cup V'$, $V \models \phi$ and $V' \models \psi$. This semantics is shown to be sound and complete for BI (Pym 2002).

We now give an algebraic generalisation of the topological semantics, in a setting that recovers the simplicity of the previous elementary preordered monoid semantics and, also, the topological treatment of inconsistency. The basic idea in Grothendieck sheaves is to represent the essential topological structure in terms of the underlying preordered commutative monoid using a map J (the 'Grothendieck topology') to associate sets of sets of worlds with each world. Most of the necessary conditions, given in Definition 2.8, are quite standard (Mac Lane and Moerdijk 1992), and are required to handle the additive (intuitionistic) part of BI. We require on additional condition – continuity – to handle the monoid operation, $\bullet$, and so provide the additional structure required to interpret the

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

multiplicatives. This condition amounts to a requirement that J respects the composition induced by $\bullet$.

Definition 2.8 (GTM). A *Grothendieck topological monoid* (GTM) is given by a quintuple $\mathscr{M} = (M, \bullet, e, \sqsubseteq, J)$, where $(M, \bullet, e, \sqsubseteq)$ is a preordered commutative monoid, in which $\bullet$ is bifunctorial with respect to $\sqsubseteq$, and J is a map $J : M \to \wp(\wp(M))$ satisfying the following conditions:

— (Sieve): for all $m \in M$, $S \in J(m)$ and $n \in S$, $m \sqsubseteq n$;
— (Maximality): for all $m, n \in M$, $m = n$ implies $\{ n \} \in J(m)$ (where $m = n$ means $m \sqsubseteq n$ and $n \sqsubseteq m$);
— (Stability): for all $m, n \in M$ such that $m \sqsubseteq n$ and all $S \in J(m)$, there exists $T \in J(n)$ such that for all $t \in T$, there exists $s \in S$ such that $s \sqsubseteq t$;
— (Transitivity): for all $m \in M$, $S \in J(m)$ and $\{ S_n \in J(n) \}_{n \in S}$, $\bigcup_{n \in S} S_n \in J(m)$;
— (Continuity): for all $m, n \in M$ and $S \in J(m)$, $\{ k \bullet n \mid k \in S \} \in J(m \bullet n)$.

Such a J is usually called a *Grothendieck topology*.

Definition 2.9 (GTI). Let $\mathscr{M}$ be a GTM and $\mathscr{P}(L)$ be the collection of BI propositions over a language L of propositional letters, a *Grothendieck Topological Interpretation* is a function $[\![-]\!] : L \to \wp(M)$ satisfying:

— (K): for all $m, n \in M$ such that $m \sqsubseteq n$, $m \in [\![\mathrm{p}]\!]$ implies $n \in [\![\mathrm{p}]\!]$;
— (Sh): for all $m \in M$ and $S \in J(m)$, if, for all $n \in S$, $n \in [\![\mathrm{p}]\!]$, then $m \in [\![\mathrm{p}]\!]$.

It is shown in (Pym 2002; Pym *et al.* 2004; Pym 2004) that given an interpretation that makes (K) and (Sh) hold for atomic propositions, (K) and (Sh) also hold for any proposition of BI in that interpretation.

Definition 2.10 (GRM). A *Grothendieck resource model* (GRM) is a triple $\mathscr{G} = (\mathscr{M}, \models, [\![-]\!])$ in which $\mathscr{M} = (M, \bullet, e, \sqsubseteq, J)$ is a GTM, $[\![-]\!]$ is a GTI and $\models$ is a forcing relation on $M \times \mathscr{P}(L)$ satisfying the following conditions:

— $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
— $m \models \top$ iff always
— $m \models \bot$ iff $\varnothing \in J(m)$
— $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
— $m \models \phi \vee \psi$ iff there exists $S \in J(m)$ such that for all $m' \in S$, $m' \models \phi$ or $m' \models \psi$
— $m \models \phi \to \psi$ iff for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
— $m \models \mathrm{I}$ iff there exists $S \in J(m)$ such that for all $m' \in S$, $e \sqsubseteq m'$
— $m \models \phi * \psi$ iff there exists $S \in J(m)$ such that for all $m' \in S$, there exist $n, n' \in M$ such that $n \bullet n' \sqsubseteq m'$, $n \models \phi$ and $n' \models \psi$
— $m \models \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $n \models \phi$, $m \bullet n \models \psi$.

Let $\mathscr{G}$ be a GRM and Φ_Γ be the formula obtained from a bunch Γ by replacing each ';' by $\wedge$, each ',' by $*$, each $\varnothing_a$ by $\top$ and each $\varnothing_m$ by I, with association respecting the tree structure of Γ. A sequent $\Gamma \vdash \phi$ is *valid in* $\mathscr{G}$, written $\Gamma \models_{\mathscr{G}} \phi$, if and only if, for all worlds $m \in M$, we have $m \models \Phi_\Gamma$ implies $m \models \phi$. A sequent $\Gamma \vdash \phi$ is *valid*, written $\Gamma \models \phi$, iff, for all GRMs $\mathscr{G}$, it is valid in $\mathscr{G}$.

Theorem 2.5 (Soundness and completeness of BI). $\Gamma \vdash \phi$ is provable in LBI iff $\Gamma \models \phi$.

Proof. The proof is based on a term model construction for BI, with respect to GRMs (Pym *et al.* 2004; Pym 2002; Pym 2004). □

As a corollary, we obtain *validity*, that is, a proposition ϕ is valid iff for all GRMs $\mathscr{G}$, $e \models_{\mathscr{G}} \phi$.

Here we have summarised the results for BI semantics by focusing on the completeness results for BI with or without $\perp$. We will see that we will not use the Grothendieck models directly in our study of semantic proof-search methods for BI. In fact, a particular class of Grothendieck topological models, called *basic*, will appear as central in this study.

We now consider the related class of Grothendieck topological monoids called basic GTMs.

Lemma 2.1. Let $(M, \bullet, e, \sqsubseteq)$ be a KRM such that:

— (B1): M contains a greatest element π, that is, for all $m \in M$, $m \sqsubseteq \pi$;
— (B2): for all $m \in M$, $\pi \bullet m = \pi$.

The structure $\mathscr{M} = (M, \bullet, e, \sqsubseteq, J)$, where J is the map $J : M \to \wp(\wp(M))$ defined by the following condition is a *basic* GTM:

— (B3): $(\forall m \in M)(S \in J(m)$ iff $(S \neq \emptyset$ and $(\forall n \in S)(m = n))$ or $(S = \emptyset$ and $m = \pi))$

For partial orders, Condition (B3) corresponds to $J(m) = \{\{m\}\}$ if $m \neq \pi$ and $J(\pi) = \{\{\pi\}, \emptyset\}$.

Proof. Since $(M, \bullet, e, \sqsubseteq)$ is an order-preserving preordered commutative monoid, we only need to show that J satisfies the axioms required for a Grothendieck topology in Definition 2.8.

— (Sieve): We show $(\forall m \in M)(\forall S \in J(m))\ (\forall n \in S)(m \sqsubseteq n)$.
  - If $S = \emptyset$, the result is direct since there is no element in S.
  - If $S \neq \emptyset$, then $n \in S$ implies $n = m$ by the definition of J, which implies $m \sqsubseteq n$.

— (Maximality): We show $(\forall m, n \in M)(m = n \Rightarrow \{n\} \in J(m))$.
  - By the definition of J, we have $m = n$ implies $\{n\} \in J(m)$.

— (Stability): We show that
$(\forall m, n \in M)(m \sqsubseteq n \Rightarrow (\forall S \in J(m))(\exists T \in J(n))(\forall t \in T)(\exists s \in S)(s \sqsubseteq t)$.
  - If $m = \pi$, then $m \sqsubseteq n$ implies $n = \pi$ since π is a greatest element. Therefore, $S \in J(m)$ implies $S \in J(n)$ and we only need to choose $T = S$.
  - If $m \neq \pi$, we pick $S \in J(m)$. $m \neq \pi$ implies $S \neq \emptyset$ and we can choose $m' \in S$. By the definition of J, we have $m = m'$. Besides, by maximality, we have $\{n\} \in J(n)$. Thus, it is sufficient to set $T = \{n\}$ since $m \sqsubseteq n$ and $m = m'$ imply $m' \sqsubseteq n$.

— (Transitivity): We show

$$(\forall m \in M)(\forall S \in J(m))(\forall \{S_n \in J(n)\}_{n \in S})(\bigcup_{n \in S} S_n \in J(m)).$$

  - If $S = \emptyset$, then $\bigcup_{n \in S} S_n = \emptyset$ and we have $\emptyset \in J(m)$ by the definition of J.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

– If $S \neq \varnothing$, we let $\{ S_n \in J(n) \}_{n \in S}$ be a family of sets of worlds and let k be a world in S_n. We have $k = n$ by the definition of J since $S_n \in J(n)$. Moreover, $n \in S$ and $S \in J(m)$ imply $n = m$ by the definition of J. Therefore, we have $k = m$ for all $k \in \bigcup_{n \in S} S_n$, which implies $\bigcup_{n \in S} S_n \in J(m)$ by the definition of J.

— (Continuity): We show $(\forall\, m, n \in M)(\forall\, S \in J(m))(\{ k \bullet n \mid k \in S \} \in J(m \bullet n))$.

– If $m = \pi$,

(1) If $S = \varnothing$, then $\{ k \bullet n \mid k \in S \} = \varnothing$ and, by the definition of J, we have $\varnothing \in J(m)$.

(2) If $S \neq \varnothing$, then, by the definition of J, $k \in S$ implies $k = \pi$. Therefore, we get $k \bullet n = \pi$ and $m \bullet n = \pi$ because $\bullet$ is order-preserving and π satisfies Condition (B2) of Lemma 2.1. Therefore, we obtain $\{ k \bullet n \mid k \in S \} \in J(m \bullet n)$ by the definition of J.

– If $m \neq \pi$, then, by the definition of J, we have $k \in S$ implies $k = m$. Consequently, as $\bullet$ is order-preserving, we get $k \bullet n = m \bullet n$, from which it follows that $\{ k \bullet n \mid k \in S \} \in J(m \bullet n)$ by the definition of J. □

A map $J : M \to \wp(\wp(M))$ that satisfies Condition (B3) of Lemma 2.1 is called a *basic Grothendieck topology*. We can now proceed with the definition of a basic GRM.

Definition 2.11 (Basic GRM). A GRM $\mathscr{G} = (\mathscr{M}, \models, [\![-]\!])$, where $\mathscr{M} = (M, \bullet, e, \sqsubseteq, J)$, is *basic* if and only if $\mathscr{M}$ is basic, that is, $\mathscr{M}$ satisfies Conditions (B1), (B2) and (B3) of Lemma 2.1.

As we said earlier, this restriction on the BI models will be at the centre of our study. Bearing in mind these completeness results for BI, with or without $\perp$, we aim to study now the proof-theoretic foundations of BI and to propose proof-search methods that build proofs or countermodels for BI. The key idea is to define labels, in the spirit of labelled deductive systems (Gabbay 1996), in order to capture the semantics of the logic, and then to provide labelled calculi for BI and related proof-search methods. A main concern is the generation of countermodels, and then of based-on semantic explanations in the case of non-validity. In the case of BI without $\perp$, we have already provided a labelling algebra that syntactically reflects the Kripke resource semantics (Galmiche and Méry 2001) and use it to define a proof-search procedure with countermodel generation. For BI with $\perp$, such an approach is much more delicate because of the Grothendieck topological semantics, which, it seems, cannot be directly captured by labels. We shall see that a key step of this semantic analysis is the use of *dependency graphs*, which are explained in Section 3.4.

3. Resource tableaux for BI

In this section, we set up the theory of labelled semantic tableaux for BI. We assume a basic knowledge of tableaux systems (Fitting 1990). We begin with algebras of labels, which provide the connection between the underlying syntactic tableaux and the semantics of the connectives used to regulate the multiplicative structure.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

3.1. *A labelling algebra*

We define a set of labels and constraints and a corresponding labelling algebra, that is, a preordered monoid whose elements are denoted by labels.

Definition 3.1. A *labelling language* consists of a unit symbol 1, a binary function symbol $\circ$, a binary relation symbol $\leqslant$, and a countable set of constants $c_1, c_2, \ldots$ *Labels* are inductively defined from the unit 1 and the constants as expressions of the form $x \circ y$ in which x and y are labels. *Atomic labels* are labels that do not contain any $\circ$, while *compound labels* contain at least one $\circ$. *Label constraints* are expressions of the form $x \leqslant y$, where x and y are labels.

Definition 3.2. Labels and constraints are interpreted in an order-preserving preordered commutative monoid of labels, or *labelling algebra* $\mathscr{L} = (L, \circ, 1, \leqslant)$, more precisely:

1. L is a set of labels.
2. $\leqslant$ is a preordering relation on L.
3. Equality on labels is defined by: $x = y$ iff $x \leqslant y$ and $y \leqslant x$.
4. $\circ$ is a binary operation on L such that:
   - (Associativity): $(x \circ y) \circ z = x \circ (y \circ z)$,
   - (Commutativity): $x \circ y = y \circ x$,
   - (Identity): $x \circ 1 = 1 \circ x = x$, and
   - (Bifunctoriality): if $x \leqslant y$ then $x \circ z \leqslant y \circ z$.

The length of a label x (notation, $|x|$), is given inductively by $|1| = 0$, $|c_i| = 1$ and $|x \circ y| = |x| + |y|$. A label y is a *sub-label* of the label x (notation, $y \preceq x$), if there exists a label z such that $z \circ y = x$. The sub-label y is said to be *strict* (notation, $y \prec x$), if $|y| < |x|$.

For notational simplicity, we can omit the binary symbol $\circ$ when writing labels, so xy represents $x \circ y$. We deal with *partially defined* labelling algebras, obtained from sets of constraints by means of a *closure operator*.

Definition 3.3 ($\overline{(\cdot)}$-closure). The *domain* of a set K of label constraints is the set of all the sub-labels occurring in some constraints of K, more formally, $\mathscr{D}(K) = \bigcup_{x \leqslant y \in K} (\mathscr{S}(x) \cup \mathscr{S}(y))$. The *closure* $\overline{K}$ of K is defined as the smallest set such that:

— (Extension) $K \subseteq \overline{K}$;
— (Reflexivity) if $x \in \mathscr{D}(\overline{K})$, then $x \leqslant x \in \overline{K}$;
— (Transitivity) if $x \leqslant y \in \overline{K}$ and $y \leqslant z \in \overline{K}$, then $x \leqslant z \in \overline{K}$; and
— (Compatibility) if $y \circ z \in \mathscr{D}(\overline{K})$ and $x \leqslant y \in \overline{K}$, then $x \circ z \leqslant y \circ z \in \overline{K}$.

Note that $x \circ z \leqslant y \circ z \in \overline{K}$ implies $x \circ z \in \mathscr{D}(\overline{K})$ by the definition of $\mathscr{D}(\overline{K})$.

We do not distinguish between the closure of a set of label constraints and the (partially defined) labelling algebra it generates.

3.2. *Expansion rules*

We can now define the expansion rules for TBI.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

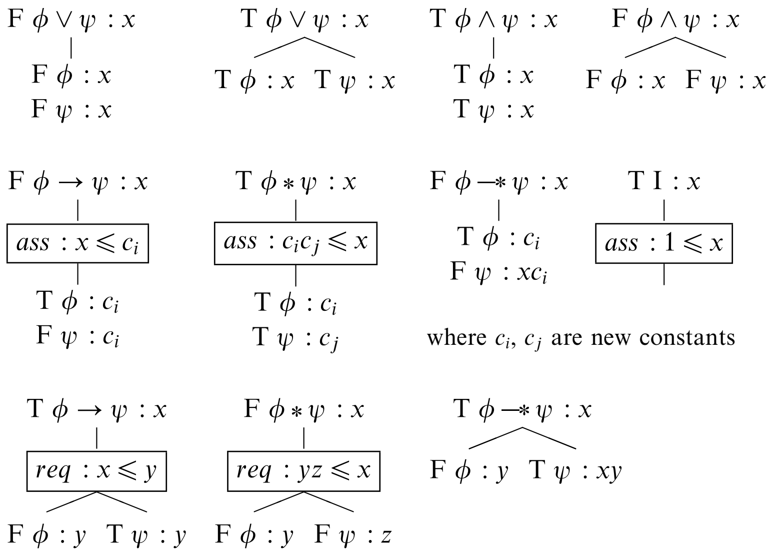


Figure 2. Expansion rules for TBI-tableaux.

Definition 3.4. A *signed formula* is a triple (S, ϕ, x), denoted $\mathrm{S}\ \phi : x$, $\mathrm{S}\ (\in \{\mathrm{F}, \mathrm{T}\})$ being the sign of the formula $\phi\ (\in \mathscr{P}(L))$ and $x\ (\in \mathscr{L})$ its label.

Definition 3.5 (TBI-tableau). Le χ be a BI-proposition. A TBI-*tableau for* χ is a binary tree $\mathscr{T}$ whose root node is labelled with the signed formula $\mathrm{F}\ \chi : 1$, all other nodes being either labelled with a signed formula, or with a label-constraint, and is built (respecting the structure of χ) according to the expansion rules of Figure 2.

In Figure 2, the rules of the first line are the standard α and β rules (Fitting 1990). The rules of the second line are called $\pi\alpha$ rules, and they introduce constraints, called *assertions*, with new (label) constants. The rules of the third line are called $\pi\beta$rules, and they introduce constraints, called *requirements*, the variables of which are instantiated with existing labels. (The precise meaning of 'existing' is given below.)

In fact, the assertions behave as known facts (or hypothesis), while the requirements express goals that must be satisfied (using assertions if necessary). Whilst the additive units are handled implicitly by the calculus (as in intuitionistic logic) a specific rule for the multiplicative unit I is required. It introduces an assertion of the form $1 \leqslant x$. We implicitly assume the reflexive assertion $x \leqslant x$ for any atomic label x (constant c_i or unit 1) occuring in a tableau branch. For example, the assertion $c_i \leqslant c_i$ is implicitly assumed for the expansion rule $\mathrm{F} \mathrel{-\!\!*}$.

To gain a better intuition about labels and constraints, note, for each connective, the relationship between the expansion rules and the clauses of the elementary Kripke semantics (see Definition 2.5), bearing in mind that labels represent worlds, $\pi\alpha$ rules (with the introduction of new labels) correspond to existential quantification on worlds and $\pi\beta$ rules (with variables that must be instantiated by known labels) correspond to universal quantification on worlds. This understanding includes the specific rule for I.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

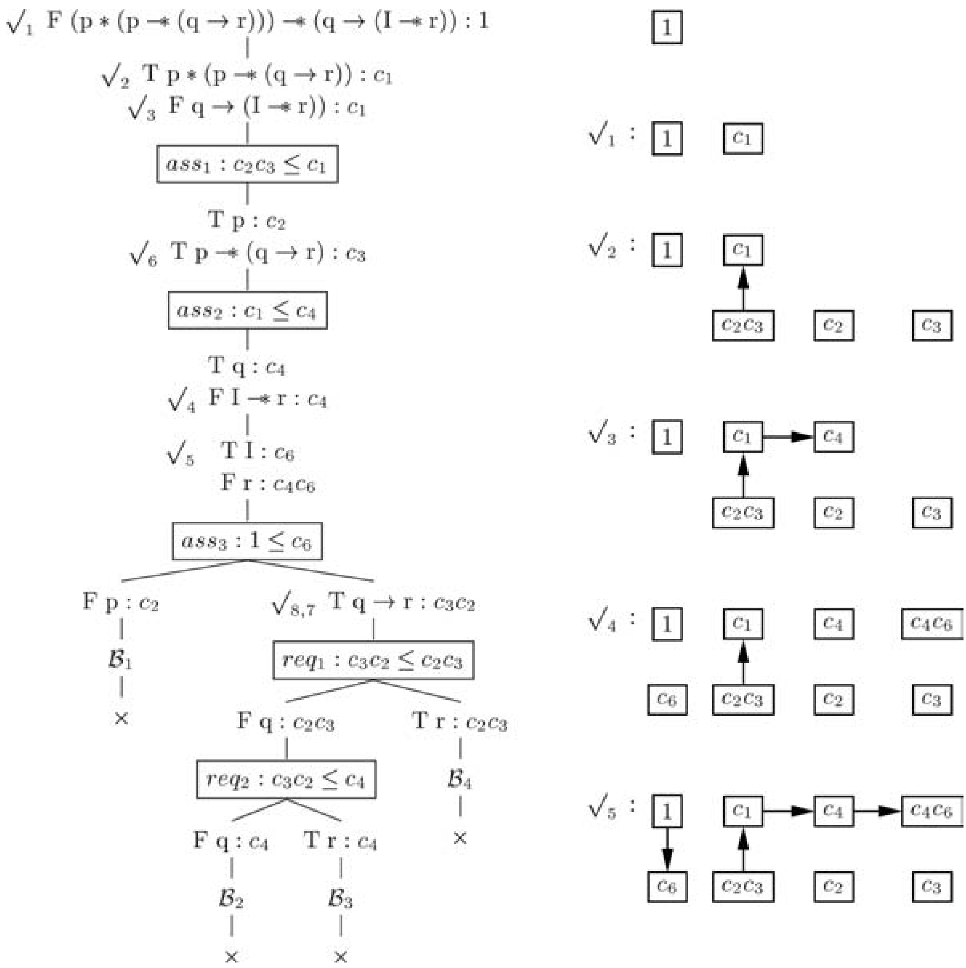


Figure 3. Tableau for $(p * (p \mathrel{-\!\!*} (q \to r))) \mathrel{-\!\!*} (q \to (I \mathrel{-\!\!*} r))$.

3.3. *Tableaux and dependency graphs*

Building a labelled tableau for an initial formula ϕ, following such expansion rules, the key problem is to define so-called closure conditions such that either the tableau is closed, and then ϕ is valid, or there exists an open branch, and then ϕ is not valid (Fitting 1990). Moreover, in the latter case, we aim to use the open branch in order to build a countermodel for ϕ. Tableaux methods have been studied for various logics (classical, intuitionistic, linear, modal, and so on) and in each case, the particular definitions of *complementary formulæ* and *closure conditions* allow us to capture the semantics of the logic in order to analyse the provability. Other problems, like the termination of tableau construction (or loop detection), are also studied and solved in different ways depending on the logic.

We begin by illustrating the key notions (and the related problems) in the case of BI without $\perp$, using the example of Figure 3.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

We start with the formula $\mathrm{F}\ (\mathrm{p} * (\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r}))) \mathrel{-\!\!*} (\mathrm{q} \to (\mathrm{I} \mathrel{-\!\!*} \mathrm{r}))\ :\ 1$ and apply the expansion rules. The first expansion step, marked $\surd_1$, introduces a new constant c_1. Steps 2 and 3 then introduce the new constants c_2, c_3, c_4 and the assertions $c_2c_3 \leqslant c_1$ and $c_1 \leqslant c_4$.

The first key point to notice is that we first expand the $\pi\alpha$ formulæ that introduce new constants c_i, c_j and the related assertions before expanding $\pi\beta$ formulæ that reuse existing labels. Therefore, although the signed formula $\mathrm{T}\ \mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\ :\ c_3$ precedes the signed formula $\mathrm{F}\ \mathrm{I} \mathrel{-\!\!*} \mathrm{r}\ :\ c_4$ in the tableau, Step 4 proceeds with the latter, which results in the introduction of the new constant c_6. Step 5 then expands $\mathrm{T}\ \mathrm{I}\ :\ c_6$, which leads to the introduction of the assertion $1 \leqslant c_6$.

All the assertions of a branch $\mathscr{B}$, including implicit reflexive assertions on atomic labels, are gathered in a specific set, denoted $Ass(\mathscr{B})$. The *domain* $\mathscr{D}(\mathscr{B})$ of a branch $\mathscr{B}$ is then defined as the set of all sub-labels occurring in the $\overline{(\cdot)}$-closure of its assertions, that is, $\mathscr{D}(\mathscr{B}) = \mathscr{D}(\overline{Ass}(\mathscr{B}))$. The second key point to notice is that we build a specific graph, called a *dependency graph* or *Kripke resource graph*, in parallel with the expansions of $\pi\alpha$ formulæ. This graph is designed to reflect the $\overline{(\cdot)}$-closure of the set of assertions occuring in a branch. More formally, the dependency graph $DG(\mathscr{B}) = [N(\mathscr{B}), A(\mathscr{B})]$ associated to a branch $\mathscr{B}$ is defined as the directed graph the nodes of which are labelled with labels of $\mathscr{D}(\mathscr{B})$, the arrows $A(\mathscr{B})$ deriving from $\overline{Ass}(\mathscr{B})$ as follows: there is an arrow $x \to y$ in $A(\mathscr{B})$ if and only if there is an assertion $x \leqslant y$ in $\overline{Ass}(\mathscr{B})$. For notational simplicity, we do not explicitly represent reflexive and transitive arrows. Therefore, each time a new assertion gets introduced in a branch $\mathscr{B}$, we must recalculate the set $\overline{Ass}(\mathscr{B})$ and update the corresponding dependency graph.

In our example, the introduction of the assertion $1 \leqslant c_6$ at Step 5 requires the addition of the assertion $c_4 \leqslant c_4c_6$ to meet the (Compatibility) condition of Definition 3.3. Accordingly, the dependency graph corresponding to Step 5 is obtained from the one of Step 4 by adding the arrows $1 \to c_6$ and $c_4 \to c_4c_6$. We mention that is possible to define formally a procedure that builds, in parallel with tableau expansions, the dependency graph $DG(\mathscr{B})$ of a branch $\mathscr{B}$ and so, the closure $\overline{Ass}(\mathscr{B})$. This procedure is such that a dependency graph gets updated only when $\pi\alpha$ rules are expanded, all the other rules, introducing neither new constants, nor new assertions, simply leave it unchanged.

After Step 5, there is no $\pi\alpha$ rule left to expand and we can start expanding the $\pi\beta$ rules, which introduce requirements. All the requirements of a branch $\mathscr{B}$ are gathered in a specific set, denoted $Req(\mathscr{B})$. At Step 6, we must expand the signed formula $\mathrm{T}\ \mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \to \mathrm{r})\ :\ c_3$. For that, we must find two labels x and y such that the label xy already exists in the dependency graph. Here, we choose $xy = c_3c_2$ (another possibility would have been $xy = c_31$). Therefore, the third key point to notice is that the expansion of a $\pi\beta$ rule in a branch $\mathscr{B}$ requires the reuse of labels that already exist in the dependency graph associated to $\mathscr{B}$.

Step 7, where the signed formula $\mathrm{T}\ \mathrm{q} \to \mathrm{r}\ :\ c_3c_2$ must be expanded, is when we reach our next key point. This time, we must not only find a label x such that x already occurs in the dependency graph, we are also required to perform an *admissible* expansion step, that is, the constraint $c_3c_2 \leqslant x$ must hold with respect to the assertions of the branch, which formally means that $c_2c_3 \leqslant x \in \overline{Ass}(\mathscr{B})$. On a dependency graph $DG(\mathscr{B})$, the fact that a requirement $x \leqslant y$ holds with respect to $\overline{Ass}(\mathscr{B})$ corresponds to the existence of a path from the node x to the node y. Here, we choose $x = c_2c_3$, using the implicit reflexive arrow $c_2c_3 \rightarrow c_2c_3$ and knowing that labels are considered modulo commutativity. Another solution is to choose $x = c_4$, using the arrow $c_2c_3 \rightarrow c_4$, which is exactly what Step 8 does. Therefore, our last key point is that a signed formula may be expanded several times by a $\pi\beta$ rule since there may be several distinct admissible expansions.

Before continuing with the example, we properly define the *admissibility* condition and proceed with the closure conditions for the TBI calculus.

Definition 3.6. A requirement $x \leqslant y$ occuring in a branch $\mathscr{B}$ of a tableau $\mathscr{T}$ is *admissible* in $\mathscr{T}$ if it holds with respect to the $\overline{(\cdot)}$-closure of the assertions that were introduced in $\mathscr{B}$ before the requirement $x \leqslant y$. A branch $\mathscr{B}$ is admissible if all of its requirements are admissible, and a tableau $\mathscr{T}$ is admissible if all of its branches are admissible.

3.4. *Resource tableaux for BI*

We emphasise that the labelling algebra is defined in order to capture the semantics inside the tableaux calculus. If we consider BI without $\perp$, the labels and constraints of TBI clearly reflect the elementary Kripke semantics at the syntactic level and thus provide resource tableaux with soundness and completeness properties (Galmiche and Méry 2001). But our aim is to consider BI with $\perp$ and its complete Grothendieck topological semantics. Thus, we have to give an appropriate definition of closed tableaux that takes the specificity of $\perp$ into account.

If we consider the problem of inconsistency from the point of view of the elementary Kripke semantics, which is not complete for BI with $\perp$, a branch must be closed (contradictory) when it contains a signed formula $\mathrm{T} \perp : x$. Indeed, such a branch cannot have a model in the elementary semantics, for then it would be in contradiction with the fact that $\perp$ should never be forced by any world. In the Grothendieck topological semantics, however, we can have worlds at which $\perp$ is forced. We will denote such worlds *inconsistent worlds*. Therefore, a branch cannot be considered as closed just because it contains a signed formula $\mathrm{T} \perp : x$. Additional conditions have to be defined in order to allow a branch to contain such a signed formula while still being realisable in some Grothendieck resource model. For that, first recall that if a world m is inconsistent, it forces all propositions ϕ because LBI is sound and complete with respect to Grothendieck resource models and $\perp \vdash \phi$ is an axiom of LBI. Moreover, the (Continuity) condition of Grothendieck topologies (see Definition 2.8) implies that if $m \models \perp$, then, for all worlds n, we have $m \bullet n \models \perp$. In other words, any world that is obtained by composition with an inconsistent world is itself inconsistent. We can now introduce the notion of *inconsistent label* in a tableau branch, which is designed to reflect the behaviour of inconsistent worlds in Grothendieck resource models.

Definition 3.7. Let $\mathscr{B}$ be a branch. A label x is *inconsistent in* $\mathscr{B}$ if there exists a label y such that $y \leqslant x \in \overline{Ass}(\mathscr{B})$ and a label z in $\mathscr{S}(y)$ (set of sub-labels of y) such that $\mathrm{T} \perp : z$ occurs in $\mathscr{B}$.
A label x is *consistent* in $\mathscr{B}$ if it is not inconsistent.

Definition 3.8. A tableau branch $\mathscr{B}$ is *closed*, or *contradictory*, if and only if it satisfies at least one of the following conditions:

— (CL1): $\mathscr{B}$ contains two signed formulæ $\mathrm{T}\ \phi : x$ and $\mathrm{F}\ \phi : y$ that are *complementary* in $\mathscr{B}$, that is, that are such that $x \leqslant y \in \overline{Ass}(\mathscr{B})$.
— (CL2): $\mathscr{B}$ contains a signed formula $\mathrm{F}\ \mathrm{I} : x$ and $1 \leqslant x \in \overline{Ass}(\mathscr{B})$.
— (CL3): $\mathscr{B}$ contains a signed formula $\mathrm{F}\ \top : x$.
— (CL4): $\mathscr{B}$ contains a signed formula $\mathrm{F}\ \phi : x$ with x inconsistent in $\mathscr{B}$.

A tableau branch that is not closed is said to be *open*. A tableau is *closed* if and only if all its branches are closed, otherwise, it is *open*.

If, in the above definition, we suppress the condition (CL4), we have the closure conditions that fit well with BI without $\perp$ and its elementary semantics (Galmiche and Méry 2003).

Definition 3.9 (TBI-proof). Let ϕ be a BI-proposition. A tableau $\mathscr{T}$ is a TBI-proof of ϕ if and only if there exists a finite sequence of tableaux $(\mathscr{T}_i)_{1 \leqslant i \leqslant n}$ such that:

— $\mathscr{T}_1$ is the tableau with only one node (the root) labelled with the signed formula $\mathrm{F}\ \phi : 1$;
— $\mathscr{T}_{i+1}$ is obtained from $\mathscr{T}_i$ by one of the expansion rules described in Definition 3.5; and
— $\mathscr{T}_n = \mathscr{T}$, $\mathscr{T}$ is closed and admissible.

The formula ϕ is TBI-*provable* if and only if there exists a TBI-proof of ϕ.

We shall return to our example of Figure 3 after Step 6. The left-hand branch is then closed by a standard complementarity between $\mathrm{T}\ \mathrm{p} : c_2$ and $\mathrm{F}\ \mathrm{p} : c_2$. Step 7 corresponds to the expansion of $\mathrm{T}\ \mathrm{q} \rightarrow \mathrm{r} : c_3c_2$ with y such that $c_3c_2 \leqslant y$. If we consider $y = c_2c_3$, knowing that we consider label composition modulo commutativity, we can close a branch with $\mathrm{T}\ \mathrm{r} : c_2c_3$ and $\mathrm{F}\ \mathrm{r} : c_4c_6$ because $c_3c_2 \leqslant c_4c_6$ holds (see the dependency graph). Step 8 corresponds to a new expansion of $\mathrm{T}\ \mathrm{q} \rightarrow \mathrm{r} : c_3c_2$ with $y = c_4$, the requirement $c_3c_2 \leqslant c_4$ being satisfied (see the dependency graph). Consequently, the third branch is closed because we have $\mathrm{T}\ \mathrm{r} : c_4$ and $\mathrm{F}\ \mathrm{r} : c_4c_6$, and $c_4 \leqslant c_4c_6$ holds (see the dependency graph).

The point now is to prove why and how the Condition (CL4) allows us to handle BI with $\perp$. Before we study the properties of the TBI calculus, we aim to illustrate the treatment of $\perp$ with two examples: one with a provable formula and the other with an unprovable formula.

Example 1. First, we consider the provable formula $((\mathrm{p} \mathrel{-\!\!*} \perp) * \mathrm{p}) \rightarrow \mathrm{q}$. Its closed tableau is given in Figure 4. The first two steps generate two assertions and the associated dependency graph. After Step 3, we have a tableau with two branches. The first branch is closed since it contains complementary formulae, namely, $(\mathrm{T}\ \mathrm{p} : c_3, \mathrm{F}\ \mathrm{p} : c_3)$. The second, however, contains no complementary formulae. We notice that the branch contains the formula $\mathrm{T}\ \perp\ : c_2c_3$. Thus, c_2c_3 is what we have called an inconsistent label and, by assertion $ass_2 : c_2c_3 \leqslant c_1$, we have c_1 is also inconsistent. Therefore, the branch is closed

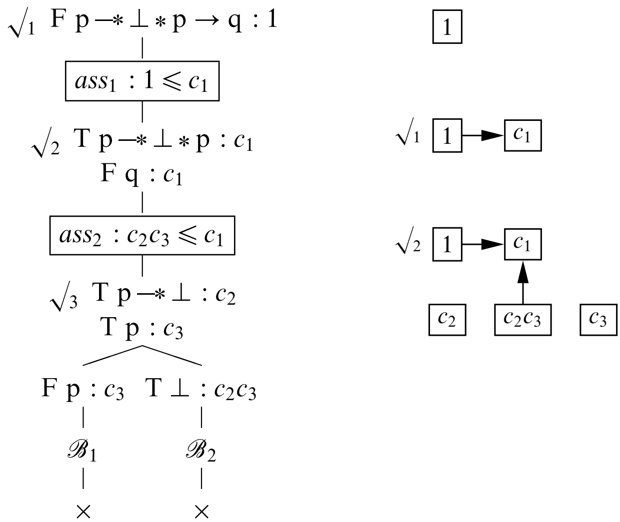


Figure 4. Tableau and Dependency Graph for $((p \mathrel{-\!\!*} \bot) * p) \to q$.

because it contains the formula F q : c_1 with label c_1 being inconsistent. Then we can deduce that the formula is TBI-provable.

Example 2. Consider another example with the formula $((p \mathrel{-\!\!*} \bot) \to \bot) \mathrel{-\!\!*} (((p * p) \mathrel{-\!\!*} \bot) \to \bot)$ that leads to an unclosed tableau (see Figure 5). The first steps are similar to the other examples, and after Step 6, the tableau has only so-called complete branches meaning that all signed formula have been completely analysed (this notion will be formally defined in Definition 4.5). The second branch is closed with (T p : c_3, F p : c_3), the third is closed with (T ⊥ : c_2c_3, F ⊥ : c_2c_3) and the fourth is closed with (T ⊥ : c_2, F ⊥ : c_2). The first branch, however, remains open since the only way to close it would be to have (T p : c_3, F p : 1), but $c_3 \leqslant 1$ cannot be deduced from the (closure of the) assertions of the branch. We will see in a following section how to build a countermodel from such an open branch.

Now we must show that this labelled calculus, whose restriction to BI without ⊥ is sound and complete for the elementary semantics, is also sound and complete for BI with respect to the Grothendieck topological semantics.

4. Properties of the TBI calculus

We aim to show the soundness and completeness of TBI with respect to GRMs, but the soundness can only be proved with respect to so-called basic GRMs. This deductive framework allows, not only a proof procedure, but also, in the case of non-provability, the systematic generation of countermodels.

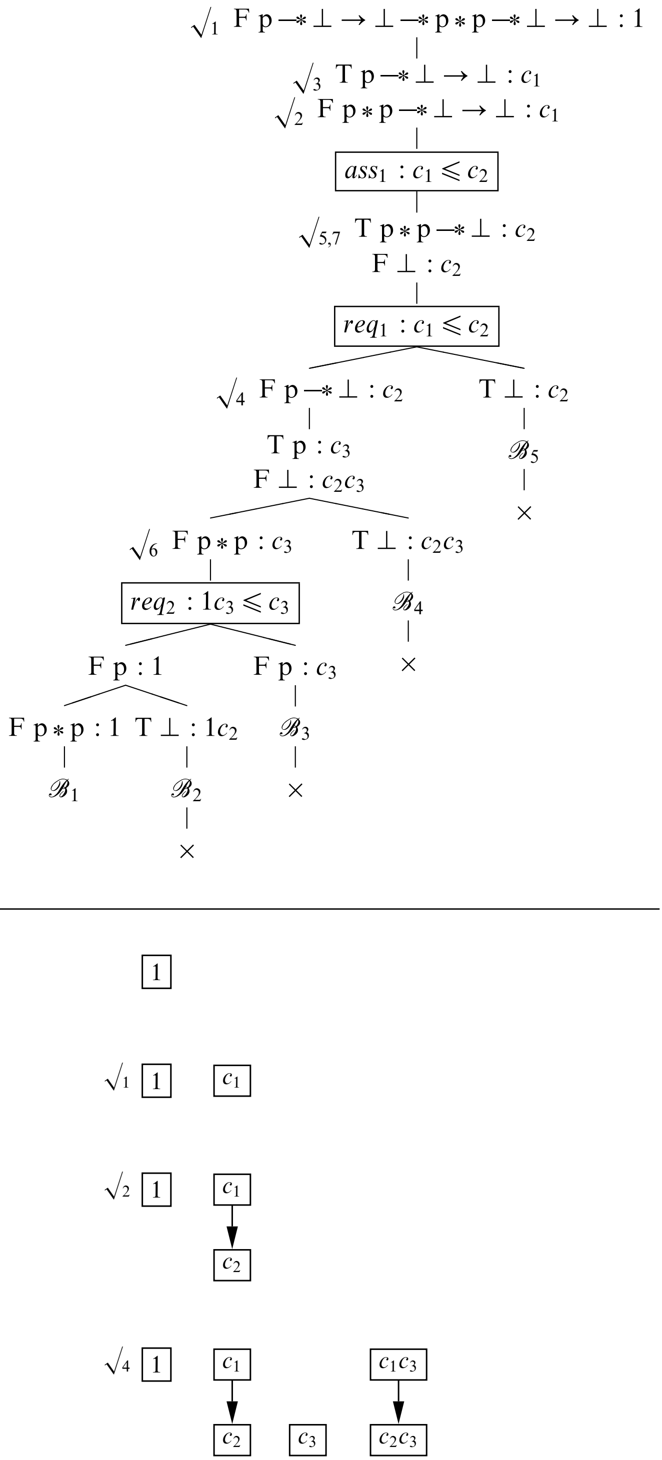


Figure 5. Tableau and Dependency Graph for $((\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot) \mathrel{-\!\!*} (((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot)$.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

4.1. *Soundness*

We prove here the soundness of TBI with respect to particular GRMs called basic GRMs, following a classical development, subject to the usual adaptations to BI from a notion of *realisability* that is preserved by the expansion rules (Galmiche and Méry 2001). But we cannot prove it with respect to GRMs and, consequently, with respect to the Grothendieck topological semantics.

Taking $\perp$ into account, the proof of the soundness of TBI cannot be a simple extension of the one in Galmiche and Méry (2003). It becomes more delicate because we have to deal with Grothendieck topological semantics. The best way to solve the problem is first to restrict the initial proof to so-called basic GRMs, and then to prove the soundness of TBI with respect to a new relational semantics that is complete and closely related to the TBI calculus.

Definition 4.1. Let $\mathscr{G} = (\mathscr{M}, \models, [\![-]\!])$ be a GRM with $\mathscr{M} = (M, \bullet, e, \sqsubseteq, J)$, and $\mathscr{B}$ be a tableau branch, a *realisation of* $\mathscr{B}$ *in* $\mathscr{G}$ is a mapping $\|-\| : \mathscr{D}(\mathscr{B}) \to M$ from the domain of $\mathscr{B}$ to the worlds of M that satisfies:

1. $\|1\| = e$,
2. $\|x \circ y\| = \|x\| \bullet \|y\|$,
3. for any $\mathrm{T}\ \phi : x$ in $\mathscr{B}$, $\|x\| \models \phi$,
4. for any $\mathrm{F}\ \phi : x$ in $\mathscr{B}$, $\|x\| \not\models \phi$, and
5. for any $x \leqslant y$ in $Ass(\mathscr{B})$, $\|x\| \sqsubseteq \|y\|$.

Lemma 4.1. Let $\mathscr{T}$ be a tableau, $\mathscr{B}$ be a branch of $\mathscr{T}$ and $\|-\|$ be a realisation of $\mathscr{B}$ in a GRM $\mathscr{G}$. Then, for any $x \leqslant y \in \overline{Ass}(\mathscr{B})$, $\|x\| \sqsubseteq \|y\|$ holds in $\mathscr{G}$.

Proof. The proof is by a straightforward induction. □

Definition 4.2. A tableau branch $\mathscr{B}$ is *realisable* if there exists a realisation of $\mathscr{B}$ in some GRM $\mathscr{G}$. A tableau $\mathscr{T}$ is *realisable* if it contains a realisable branch.

Lemma 4.2. A closed tableau is not realisable.

Proof. Let $\mathscr{T}$ be a closed tableau that is also realisable. Then, $\mathscr{T}$ contains a branch $\mathscr{B}$ that is realisable in some GRM $\mathscr{G} = (\mathscr{M}, \models, [\![-]\!])$. If the branch is closed because of complementary formulae $(\mathrm{T}\ \phi : x, \mathrm{F}\ \phi : y)$, then, by definition, we have $x \leqslant y \in \overline{Ass}(\mathscr{B})$, which, by Lemma 4.1, implies $\|x\| \sqsubseteq \|y\|$. But, since $\|-\|$ realises $\mathscr{B}$, we also have $\|x\| \models \phi$ and $\|y\| \not\models \phi$. Therefore, we reach a contradiction because, by property (K), we should have $\|y\| \models \phi$. If the branch is closed because of a formula $\mathrm{F}\ \phi : x$ whose label x is inconsistent in $\mathscr{B}$, then, by definition, there exists a label y such that $y \leqslant x \in \overline{Ass}(\mathscr{B})$ and a label z in $\wp(y)$ such that $\mathrm{T} \perp : z \in \mathscr{B}$. Since $\|-\|$ realises $\mathscr{B}$, we have $x \not\models \phi$ and $z \models \perp$. Since z is a sublabel of y, the continuity axiom of J implies that $y \models \perp$. Therefore, as Lemma 4.1 implies $\|y\| \sqsubseteq \|x\|$, (K) yields $x \models \perp$ and, once again, we reach a contradiction because, if $x \models \perp$, then, for any ϕ, we should have $x \models \phi$. Other cases are similar. □

Compared with the soundness proof for BI without $\perp$ and its elementary Kripke models (EKRMs) (Galmiche and Méry 2003), we must now consider a restriction on the GRMs

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

to the basic GRMs in order to have a soundness result for the TBI calculus. All the previous lemmas that hold for GRMs also hold for basic GRMs. Now, we consider a lemma that only holds for the restricted models. We say that a tableau branch $\mathscr{B}$ is *b-realisable* if there exists a realisation of $\mathscr{B}$ in some basic GRM $\mathscr{G}$ and a tableau $\mathscr{T}$ is *b-realisable* if it contains a b-realisable branch.

Lemma 4.3. If $\mathscr{T}'$ is a tableau obtained from a tableau $\mathscr{T}$ by application of an expansion rule of TBI, then if $\mathscr{T}$ is b-realisable, $\mathscr{T}'$ is also b-realisable.

Proof. Since $\mathscr{T}$ is realisable, it contains a branch $\mathscr{B}$ that is realisable in some basic GRM for some realisation $\|-\|$. If the signed formula $\mathrm{S}\ \mathrm{X} : x$ that has been expanded to obtain $\mathscr{T}'$ does not belong to $\mathscr{B}$, then $\mathscr{T}'$ is realisable since it still contains $\mathscr{B}$. Otherwise, we show by case analysis on $\mathrm{S}\ \mathrm{X} : x$ that the corresponding expansion rule preserves realisability. $\mathscr{F}(\mathscr{B})$ denotes the set of all the signed formulae of branch $\mathscr{B}$.

— Case $\mathrm{T}\ \phi * \psi : x$.
$\mathscr{B}$ is expanded into $\mathscr{B}'$ with $\mathscr{F}(\mathscr{B}') = \mathscr{F}(\mathscr{B}) \cup \{\mathrm{T}\ \phi : c_i, \mathrm{T}\ \psi : c_j\}$ and $Ass(\mathscr{B}') = Ass(\mathscr{B}) \cup \{c_i c_j \leqslant x\}$, c_i and c_j being new constants. Since $\|-\|$ realises $\mathscr{B}$, we have $\|x\| \models \phi * \psi$. Therefore, there exists $S \in J(\|x\|)$ such that for any $m' \in S$ there exist $n_1, n_2 \in M$ such that $n_1 \bullet n_2 \sqsubseteq m'$, $n_1 \models \phi$ and $n_2 \models \psi$. As we consider a basic GRM (*cf.* Definition 2.11), we have $J(\|x\|) = \{\{\|x\|\}\}$. We simply extend $\|-\|$ to c_i, c_j by $\|c_i\| = n_1$ and $\|c_j\| = n_2$ and consider $m' = \|x\|$. Thus we directly deduce that $\|c_i\| \bullet \|c_j\| \sqsubseteq \|x\|$. Therefore, $\mathscr{B}'$ is realisable and, consequently, $\mathscr{T}'$ is realisable.
— Case $\mathrm{F}\ \phi * \psi : x$.
$\mathscr{B}$ splits into $\mathscr{B}'$ and $\mathscr{B}''$ such that $\mathscr{F}(\mathscr{B}') = \mathscr{F}(\mathscr{B}) \cup \{\mathrm{F}\ \phi : y\}$, $\mathscr{F}(\mathscr{B}'') = \mathscr{F}(\mathscr{B}) \cup \{\mathrm{F}\ \psi : z\}$ and $Req(\mathscr{B}') = Req(\mathscr{B}'') = Req(\mathscr{B}) \cup \{yz \leqslant x\}$. An admissible application of the $\mathrm{F}\ *$ rule requires that $yz \leqslant x$ should be in $\overline{Ass}(\mathscr{B})$. Thus, by Lemma 4.1, we have $\|y\| \bullet \|z\| \sqsubseteq \|x\|$. Since $\|-\|$ realises $\mathscr{B}$, we have $\|x\| \not\models \phi * \psi$. Therefore, for any m, $n \in M$ such that $m \bullet n \sqsubseteq \|x\|$, either $m \not\models \phi$, or $n \not\models \psi$, which implies that either $\|y\| \not\models \phi$, or $\|z\| \not\models \psi$. Then, either $\mathscr{B}'$, or $\mathscr{B}''$ is realisable and, consequently, $\mathscr{T}'$ is realisable.
— Case $\mathrm{T}\ \phi \vee \psi : x$.
$\mathscr{B}$ splits into $\mathscr{B}'$ with $\mathscr{F}(\mathscr{B}') = \mathscr{F}(\mathscr{B}) \cup \{\mathrm{T}\ \phi : x\}$ and $\mathscr{B}''$ with $\mathscr{F}(\mathscr{B}'') = \mathscr{F}(\mathscr{B}) \cup \{\mathrm{T}\ \psi : x\}$. Moreover, $Ass(\mathscr{B}') = Ass(\mathscr{B}'') = Ass(\mathscr{B})$. Since $\|-\|$ realises $\mathscr{B}$, we have $\|x\| \models \phi \vee \psi$. Therefore, there exist $S \in J(\|x\|)$ such that for any $m' \in S$, either $m' \models \phi$ or $m' \models \psi$. As we consider a basic GRM, we obtain either $\|x\| \models \phi$ or $\|x\| \models \psi$. Therefore, either $\mathscr{B}'$ or $\mathscr{B}''$ is realisable and, consequently, $\mathscr{T}'$ is realisable.
— Other cases are similar. □

It is important to notice that we cannot, at this step, prove this lemma without the restriction to the basic GRMs. If we consider the above $*$ and $\vee$ cases with general GRMs, it appears that we cannot get the required result following this approach.

Corollary 4.1. Let $\mathscr{T}_1, \mathscr{T}_2, \ldots$ be a tableaux sequence, if $\mathscr{T}_i$ is b-realisable, then, for $j > i$, $\mathscr{T}_j$ is b-realisable.

Proof. The statement follows directly from the previous lemma. □

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Theorem 4.1 (Soundness of BI with respect to basic GRMs). Let ϕ be a proposition of BI. If there exists a closed tableaux sequence $\mathscr{T}$ for ϕ, then ϕ is valid in basic Grothendieck topological resource models.

Proof. Let $\mathscr{T} = \mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_n$ be a closed tableaux sequence. Suppose that ϕ does not hold in basic Grothendieck resource semantics. Then, there exists a basic GRM $\mathscr{G}$ for which $e \not\models \phi$. Then, the initial tableau $\mathscr{T}_1$ is trivially b-realisable and Lemma 4.3 implies that all $\mathscr{T}_i$ such that $i > 1$ are also b-realisable. It follows from Lemma 4.2 that none of the $\mathscr{T}_i$ can be closed and, consequently, $\mathscr{T}$ cannot be closed. □

We observe that we are not in position to prove the soundness of TBI, but we show, in the next section, how to solve this problem by analysing BI's semantics and by defining a new relational semantics of BI that is naturally related to the TBI calculus and reflects in a better way the semantical interactions between connectives. First, we study the completeness of TBI, which needs no such restrictions on models.

4.2. *Countermodel construction*

We describe how to construct a countermodel of ϕ from an open branch in a tableau for ϕ. The proof of the finite model property, in a following section, relies critically on the introduction of a special element, here called π, which is used to collect the inessential (and possibly infinite) parts of the model.

Definition 4.3. A signed formula $\mathrm{S}\ \phi : x$ is *analysed* in a tableau branch $\mathscr{B}$, which is denoted $\mathscr{B} \succ \mathrm{S}\ \phi : x$, if and only if:

— $\mathrm{S} = \mathrm{F}$ and $(\exists\, \mathrm{F}\ \phi : y \in \mathscr{B})(x \leqslant y \in \overline{Ass}(\mathscr{B}))$; or
— $\mathrm{S} = \mathrm{T}$ and $(\exists\, \mathrm{T}\ \phi : y \in \mathscr{B})(y \leqslant x \in \overline{Ass}(\mathscr{B}))$.

Definition 4.4. We define the relation $\mathscr{B} \Vdash \mathrm{S}\ \phi : x$, which means that the signed formula $\mathrm{S}\ \phi : x$ is *completely analysed* or *fulfilled* in a tableau branch $\mathscr{B}$, by case analysis as follows:

— $\mathscr{B} \Vdash \mathrm{F}\ \mathrm{p} : x$ iff $\mathscr{B} \succ \mathrm{F}\ \mathrm{p} : x$
— $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x$ iff $\mathscr{B} \succ \mathrm{T}\ \mathrm{p} : x$
— $\mathscr{B} \Vdash \mathrm{F}\ \top : x$ iff $\mathscr{B} \succ \mathrm{F}\ \top : x$
— $\mathscr{B} \Vdash \mathrm{T}\ \top : x$ iff $\mathscr{B} \succ \mathrm{T}\ \top : x$
— $\mathscr{B} \Vdash \mathrm{F}\ \mathrm{I} : x$ iff $\mathscr{B} \succ \mathrm{F}\ \mathrm{I} : x$ and $1 \leqslant x \notin \overline{Ass}(\mathscr{B})$
— $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{I} : x$ iff $\mathscr{B} \succ \mathrm{T}\ \mathrm{I} : x$ and $1 \leqslant x \in \overline{Ass}(\mathscr{B})$
— $\mathscr{B} \Vdash \mathrm{F}\ \bot : x$ iff $\mathscr{B} \succ \mathrm{F}\ \bot : x$
— $\mathscr{B} \Vdash \mathrm{T}\ \bot : x$ iff $\mathscr{B} \succ \mathrm{T}\ \bot : x$
— $\mathscr{B} \Vdash \mathrm{F}\ \psi \wedge \chi : x$ iff $\mathscr{B} \succ \mathrm{F}\ \psi : x$ or $\mathscr{B} \succ \mathrm{F}\ \chi : x$
— $\mathscr{B} \Vdash \mathrm{T}\ \psi \wedge \chi : x$ iff $\mathscr{B} \succ \mathrm{T}\ \psi : x$ and $\mathscr{B} \succ \mathrm{T}\ \chi : x$
— $\mathscr{B} \Vdash \mathrm{F}\ \psi \vee \chi : x$ iff $\mathscr{B} \succ \mathrm{F}\ \psi : x$ and $\mathscr{B} \succ \mathrm{F}\ \chi : x$
— $\mathscr{B} \Vdash \mathrm{T}\ \psi \vee \chi : x$ iff $\mathscr{B} \succ \mathrm{T}\ \psi : x$ or $\mathscr{B} \succ \mathrm{T}\ \chi : x$
— $\mathscr{B} \Vdash \mathrm{F}\ \psi \to \chi : x$ iff $(\exists\, y \in \mathscr{D}(\mathscr{B}))(x \leqslant y \in \overline{Ass}(\mathscr{B})$ and $\mathscr{B} \succ \mathrm{T}\ \psi : y$ and $\mathscr{B} \succ \mathrm{F}\ \chi : y)$
— $\mathscr{B} \Vdash \mathrm{T}\ \psi \to \chi : x$ iff $(\forall\, y \in \mathscr{D}(\mathscr{B}))(x \leqslant y \in \overline{Ass}(\mathscr{B}) \Rightarrow (\mathscr{B} \succ \mathrm{F}\ \psi : y$ or $\mathscr{B} \succ \mathrm{T}\ \chi : y))$

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

— $\mathscr{B} \Vdash \mathrm{F}\ \psi * \chi : x$ iff $(\forall\, y, z \in \mathscr{D}(\mathscr{B}))(yz \leqslant x \in \overline{Ass}(\mathscr{B}) \Rightarrow (\mathscr{B} \succ \mathrm{F}\ \psi : y$ or $\mathscr{B} \succ \mathrm{F}\ \chi : z))$
— $\mathscr{B} \Vdash \mathrm{T}\ \psi * \chi : x$ iff $(\exists\, y, z \in \mathscr{D}(\mathscr{B}))(yz \leqslant x \in \overline{Ass}(\mathscr{B})$ and $\mathscr{B} \succ \mathrm{T}\ \psi : y$ and $\mathscr{B} \succ \mathrm{T}\ \chi : z)$
— $\mathscr{B} \Vdash \mathrm{F}\ \psi \mathrel{-\!\!*} \chi : x$ iff $(\exists\, y \in \mathscr{D}(\mathscr{B}))(xy \in \mathscr{D}(\mathscr{B})$ and $\mathscr{B} \succ \mathrm{T}\ \psi : y$ and $\mathscr{B} \succ \mathrm{F}\ \chi : xy)$
— $\mathscr{B} \Vdash \mathrm{T}\ \psi \mathrel{-\!\!*} \chi : x$ iff $(\forall\, y \in \mathscr{D}(\mathscr{B}))(xy \in \mathscr{D}(\mathscr{B}) \Rightarrow (\mathscr{B} \succ \mathrm{F}\ \psi : y$ or $\mathscr{B} \succ \mathrm{T}\ \chi : xy))$.

Lemma 4.4. Let $\mathscr{B}$ be a tableau branch, then:

(a) $\mathscr{B} \succ \mathrm{F}\ \phi : x$ and $y \leqslant x \in \overline{Ass}(\mathscr{B}) \Rightarrow \mathscr{B} \succ \mathrm{F}\ \phi : y$,
(b) $\mathscr{B} \succ \mathrm{T}\ \phi : x$ and $x \leqslant y \in \overline{Ass}(\mathscr{B}) \Rightarrow \mathscr{B} \succ \mathrm{T}\ \phi : y$,
(c) $\mathscr{B} \Vdash \mathrm{F}\ \phi : x$ and $y \leqslant x \in \overline{Ass}(\mathscr{B}) \Rightarrow \mathscr{B} \Vdash \mathrm{F}\ \phi : y$, and
(d) $\mathscr{B} \Vdash \mathrm{T}\ \phi : x$ and $x \leqslant y \in \overline{Ass}(\mathscr{B}) \Rightarrow \mathscr{B} \Vdash \mathrm{T}\ \phi : y$.

Proof. The proof is by structural induction on $\mathrm{S}\ \phi : x$. □

Definition 4.5. A tableau branch $\mathscr{B}$ is a *complete* if and only if it is open and all of its signed formulæ $\mathrm{S}\ \phi : x$ are fulfilled. A tableau $\mathscr{T}$ is *complete* if and only if it contains at least one complete branch.

Lemma 4.5. If a tableau branch $\mathscr{B}$ is complete, then:

(a) $\mathscr{B} \succ \mathrm{S}\ \phi : x \Rightarrow \mathscr{B} \Vdash \mathrm{S}\ \phi : x$,
(b) $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x \Rightarrow \mathscr{B} \nVdash \mathrm{F}\ \mathrm{p} : x$, and
(c) $\mathscr{B} \Vdash \mathrm{F}\ \mathrm{p} : x \Rightarrow \mathscr{B} \nVdash \mathrm{T}\ \mathrm{p} : x$.

Proof. For property (a), we consider the case where $\mathrm{S} = \mathrm{F}$, the other case being similar. If $\mathscr{B} \succ \mathrm{F}\ \phi : x$, then, by the definition of $\succ$, $(\exists\, \mathrm{F}\ \phi : y \in \mathscr{B})(x \leqslant y \in \overline{Ass}(\mathscr{B}))$. Since $\mathscr{B}$ is assumed to be complete, $\mathrm{F}\ \phi : y \in \mathscr{B}$ implies $\mathscr{B} \Vdash \mathrm{F}\ \phi : y$, so $x \leqslant y \in \overline{Ass}(\mathscr{B})$ finally leads to $\mathscr{B} \Vdash \mathrm{F}\ \phi : x$ by Lemma 4.4.

For properties (b) and (c), we show that it cannot be the case that $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x$ and $\mathscr{B} \Vdash \mathrm{F}\ \mathrm{p} : x$ both hold at the same time†. Suppose we have both $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x$ and $\mathscr{B} \Vdash \mathrm{F}\ \mathrm{p} : x$. Then we have $\mathrm{T}\ \mathrm{p} : y \in \mathscr{B}$ for some label y such that $y \leqslant x \in \overline{Ass}(\mathscr{B})$, and we also have $\mathrm{F}\ \mathrm{p} : z \in \mathscr{B}$ for some label z such that $x \leqslant z \in \overline{Ass}(\mathscr{B})$. By the transitivity of the closure $\overline{(\cdot)}$, we get $y \leqslant z \in \overline{Ass}(\mathscr{B})$, which implies that the branch $\mathscr{B}$ is closed by condition (CL1) of Definition 3.8. This is a contradiction since, by definition, a complete branch is open. □

The dependency graph related to a formula ϕ during the resource tableau construction represents the closure of the assertions in the sense of Definition 3.3 and thus captures the computational content of ϕ. Therefore, if a formula ϕ happens to be unprovable, we should have enough information in its dependency graph to extract a countermodel for ϕ. For that, we must provide a preordered commutative monoid together with a Grothendieck topology and a forcing relation that falsifies ϕ in some world. The idea behind the countermodel construction is to regard the dependency graph itself as the desired countermodel, thereby considering it as a central semantic structure. For that, we

† Lemma 4.5 does not imply that $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x$ or $\mathscr{B} \Vdash \mathrm{F}\ \mathrm{p} : x$ for all propositional variables p since p might not appear in any signed formula of $\mathscr{B}$, for example, if p does not occur in the initial signed formula that labels the root of $\mathscr{B}$.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

take the nodes (labels) of the graph as the elements of a monoid whose multiplication is given by the composition of the labels. The preordering relation is then given by the arrows and the forcing relation simply reflects the property of being fulfilled.

The key problem is that, since the closure operator induces a *partially defined* labelling algebra, the dependency graph only deals with those pieces of information (resources) that are relevant for deciding provability. Therefore, the monoidal product should be completed with suitable values for those compositions that are undefined. The problem of undefinedness is solved in Definition 4.6 by the introduction of a particular element, denoted π, to which all undefined compositions are mapped and for which the equation $(\forall x)(x \bullet \pi = \pi \bullet x = \pi)$, meaning that any composition with something undefined is itself undefined, is assumed.

We must, however, be careful because introducing a new element may affect the property of a formula $\phi \mathrel{-\!\!*} \psi$ of being realised in a world x although the signed formula $\mathrm{T}\ \phi \mathrel{-\!\!*} \psi : x$ was fulfilled in the dependency graph. Indeed, if π forces ϕ, then, since $x \bullet \pi = \pi$, we also need π to force ψ. But, if π forces any formula ψ, then everything works as it should. On the other hand, we know that an inconsistent world necessarily forces any formula ψ because $\bot \vdash \psi$ is an axiom. Therefore, making π an inconsistent world by setting $\emptyset \in J(\pi)$ just solves the problem.

Definition 4.6 (M-structure). Let $\mathscr{B}$ be a complete branch and $\mathscr{D}c(\mathscr{B})$ be the restriction of $\mathscr{D}(\mathscr{B})$ to the labels that are consistent in $\mathscr{B}$. We define $\overline{Assc}(\mathscr{B})$ as the restriction of $\overline{Ass}(\mathscr{B})$ to consistent constraints, that is, $\overline{Assc}(\mathscr{B}) = \{\, x \leqslant y \mid x \leqslant y \in \overline{Ass}(\mathscr{B}) \text{ and } x, y \in \mathscr{D}c(\mathscr{B}) \,\}$. The M-structure $\mathscr{M}(\mathscr{B}) = (M, \bullet, 1, \sqsubseteq, J)$ associated to $\mathscr{B}$ is defined as follows:

1. $M = \mathscr{D}c(\mathscr{B}) \cup \{\, \pi \,\}$, where $\pi \notin \mathscr{D}(\mathscr{B})$.
2. The product $\bullet$ is given by
$$\forall\, x, y \in M \begin{cases} x \bullet 1 = 1 \bullet x = x & \\ x \bullet y = y \bullet x = x \circ y & \text{if } x \circ y \in M \\ x \bullet y = y \bullet x = \pi & \text{otherwise.} \end{cases}$$
3. The relation $\sqsubseteq$ between elements of M is defined by
$$x \sqsubseteq y \quad \text{iff} \quad y \equiv \pi \ \text{ or } \ x \leqslant y \in \overline{Assc}(\mathscr{B}).$$
4. The map $J : M \to \wp(\wp(M))$, called the J-map of $\mathscr{B}$, is given by
$$(\forall\, x \in M)(S \in J(x) \text{ iff } (S \neq \emptyset \text{ and } (\forall\, y \in S)(x = y)) \text{ or } (S = \emptyset \text{ and } x = \pi).$$

Lemma 4.6. Let $\mathscr{B}$ be a complete branch. The M-structure $\mathscr{M}(\mathscr{B}) = (M, \bullet, 1, \sqsubseteq, J)$ is a GTM.

Proof. A routine calculation shows that $(M, \bullet, 1, \sqsubseteq)$ is an order-preserving preordered monoid. The commutativity of $\bullet$ is by definition; the associativity of $\bullet$ comes from that of $\circ$; and the compatibility condition of the $\overline{(\cdot)}$-closure implies order-preservation. Finally, Lemma 2.1 ensures that J is a Grothendieck topology. □

Definition 4.7. Let $\mathscr{M}(\mathscr{B}) = (M, \bullet, 1, \sqsubseteq, J)$ be the M-structure of a complete branch $\mathscr{B}$, and $\mathscr{P}(L)$ denote the collection of BI propositions over a language L of propositional

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

letters. Then the interpretation $[\![-]\!]_{\mathscr{B}} : L \to \wp(M)$ is given, for all atomic propositions p, by

$$[\![\mathrm{p}]\!]_{\mathscr{B}} = \{\,\pi\,\} \cup \{\,x \mid \mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x\,\}.$$

Lemma 4.7. $[\![-]\!]_{\mathscr{B}}$ is a GTI, that is, it satisfies properties (K) and (Sh) of Definition 2.9.

Proof. For (K), we have to prove that $(\forall m, n \in M)((m \sqsubseteq n \text{ and } m \in [\![\mathrm{p}]\!]_{\mathscr{B}}) \Rightarrow n \in [\![\mathrm{p}]\!]_{\mathscr{B}})$. If $n \equiv \pi$, then, $\pi \in [\![\mathrm{p}]\!]_{\mathscr{B}}$ by definition. Otherwise, $n \equiv x$ for some $x \in \mathscr{D}c(\mathscr{B})$, and then $m \sqsubseteq n$ implies that $m \equiv y$ for some $y \in \mathscr{D}c(\mathscr{B})$ such that $y \leqslant x \in \overline{Assc}(\mathscr{B})$. Moreover, $m \in [\![\mathrm{p}]\!]_{\mathscr{B}}$ implies that $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : y$, which, by Lemma 4.4, yields $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x$, that is, $m \in [\![\mathrm{p}]\!]_{\mathscr{B}}$.

For (Sh), we have to prove that $(\forall m \in M)(\forall S \in J(m))((\forall n \in S)(n \in [\![\mathrm{p}]\!]_{\mathscr{B}}) \Rightarrow m \in [\![\mathrm{p}]\!]_{\mathscr{B}})$. If $m \equiv \pi$, then $\pi \in [\![\mathrm{p}]\!]_{\mathscr{B}}$, by definition. Otherwise, $m \equiv x$ for some $x \in \mathscr{D}c(\mathscr{B})$, so $n \equiv y$ for some $y \in \mathscr{D}c(\mathscr{B})$ such that $y = x$, and since $y \in [\![\mathrm{p}]\!]_{\mathscr{B}}$, condition (K) implies $x \in [\![\mathrm{p}]\!]_{\mathscr{B}}$. □

Theorem 4.2. Let $\mathscr{B}$ be a complete branch. Then $(\mathscr{M}(\mathscr{B}), \models, [\![-]\!]_{\mathscr{B}})$ is a Grothendieck resource model of $\mathscr{B}$, that is, for all propositions ϕ, we have:

(a) $\pi \models \phi$.
(b) If $\mathscr{B} \succ \mathrm{T}\ \phi : x$ and x is consistent in $\mathscr{B}$, then $x \models \phi$.
(c) If $\mathscr{B} \succ \mathrm{F}\ \phi : x$ and x is consistent in $\mathscr{B}$, then $x \not\models \phi$.

Proof. Property (a) follows directly from Condition (Sh) since $\varnothing \in J(\pi)$ by the definition of J. Properties (b) and (c) can be proved simultaneously by induction on $\mathrm{S}\ \phi : x$ knowing that, on the one hand, $\mathscr{B} \succ \mathrm{S}\ \phi : x$ implies $\mathscr{B} \Vdash \mathrm{S}\ \phi : x$ by Lemma 4.5 because $\mathscr{B}$ is complete, and, on the other hand, $x \neq \pi$ because $\pi \notin \overline{Assc}(\mathscr{B})$ by definition. We give just a few illustrative cases, the others being similar.

— *Case* $\mathrm{T}\ \mathrm{p} : x$
$\mathscr{B} \succ \mathrm{T}\ \mathrm{p} : x$ implies $\mathscr{B} \Vdash \mathrm{T}\ \mathrm{p} : x$, hence $x \models \mathrm{p}$ by the definition of $\models$.

— *Case* $\mathrm{F}\ \mathrm{p} : x$
By Lemma 4.5, $\mathscr{B} \succ \mathrm{F}\ \mathrm{p} : x$ implies $\mathscr{B} \Vdash \mathrm{F}\ \mathrm{p} : x$ and $\mathscr{B} \nVdash \mathrm{T}\ \mathrm{p} : x$, hence $x \not\models \mathrm{p}$ by the definition of $\models$ since $x \neq \pi$.

— *Case* $\mathrm{T}\ \mathrm{I} : x$
$\mathscr{B} \Vdash \mathrm{T}\ \mathrm{I} : x$ implies $1 \leqslant x \in \overline{Assc}(\mathscr{B})$, hence $1 \sqsubseteq x$ by the definition of $\sqsubseteq$.

— *Case* $\mathrm{F}\ \mathrm{I} : x$
$\mathscr{B} \Vdash \mathrm{F}\ \mathrm{I} : x$ implies $1 \leqslant x \notin \overline{Assc}(\mathscr{B})$, hence $1 \not\sqsubseteq x$ because $x \neq \pi$.

— *Case* $\mathrm{T}\ \bot : x$
In this case, x is inconsistent in $\mathscr{B}$, so the implication is trivially verified.

— *Case* $\mathrm{F}\ \bot : x$
Suppose that $x \models \bot$. Then $\varnothing \in J(x)$, which implies $x = \pi$, which is a contradiction since $\pi \notin \mathscr{D}(\mathscr{B})$ by definition.

— *Case* $\mathrm{T}\ \psi * \chi : x$
By Lemma 4.5, $\mathscr{B} \succ \mathrm{T}\ \psi * \chi : x$ implies $\mathscr{B} \Vdash \mathrm{T}\ \psi * \chi : x$. Therefore, there are labels y,

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

$z \in \mathscr{D}(\mathscr{B})$ such that $yz \leqslant x \in \overline{Ass}(\mathscr{B})$, $\mathscr{B} \succ \mathrm{T}\ \psi : y$ and $\mathscr{B} \succ \mathrm{T}\ \chi : z$. Since x is consistent in $\mathscr{B}$, $yz \leqslant x \in \overline{Ass}(\mathscr{B})$ implies that yz, y and z are consistent in $\mathscr{B}$, so $yz, y, z \in \mathscr{D}c(\mathscr{B})$. Thus, as $x \not\equiv \pi$, $yz \leqslant x \in \overline{Ass}(\mathscr{B})$ implies $y \bullet z \sqsubseteq x$ by the definition of $\bullet$ and $\sqsubseteq$. Moreover, we get $y \models \psi$ and $z \models \chi$ from $\mathscr{B} \succ \mathrm{T}\ \psi : y$ and $\mathscr{B} \succ \mathrm{T}\ \chi : z$ by the induction hypothesis. Finally, since $\{x\} \in J(x)$ by the definition of J, we can conclude $x \models \psi * \chi$.

— *Case* $\mathrm{F}\ \psi * \chi : x$

Let $S \in J(x)$. We have $S \neq \varnothing$ by the definition of J since $x \not\equiv \pi$. Let $m \in S$ and n, $n' \in M$ be such that $n \bullet n' \sqsubseteq m$. We have $m = x$ by the definition of J, which implies $m \not\equiv \pi$. Thus, $n \bullet n' \not\equiv \pi$ because π is the greatest element in M by the definition of $\sqsubseteq$. In turn, $n \bullet n' \not\equiv \pi$ implies $nn' \in \mathscr{D}c(\mathscr{B})$ by the definition of $\bullet$, and then n, $n' \in \mathscr{D}c(\mathscr{B})$ by the definition of $\mathscr{D}c(\mathscr{B})$. By the definition of $\sqsubseteq$, $n \bullet n' \sqsubseteq m$ then leads to $nn' \leqslant x \in \overline{Assc}(\mathscr{B})$. Moreover, by Lemma 4.5, $\mathscr{B} \succ \mathrm{F}\ \psi * \chi : x$ implies $\mathscr{B} \Vdash \mathrm{F}\ \psi * \chi : x$, from which we get $\mathscr{B} \succ \mathrm{F}\ \psi : n$ or $\mathscr{B} \succ \mathrm{F}\ \chi : n'$ by the definition of $\Vdash$, and we can finally conclude $n \not\models \psi$ or $n' \not\models \chi$ by the induction hypothesis. □

Returning to the example of Figure 5, we now show how to build a countermodel from the open branch. As the reader can check, all formulae in the open branch are fulfilled and $\mathscr{B}$ is therefore what we have called a complete branch.

First, following the steps of Definition 4.6, we build from $\mathscr{B}$ a GTM $\mathscr{M}(\mathscr{B}) = (M, \bullet, 1, \sqsubseteq, J)$.

1. M is the subset of labels of $\mathscr{D}(\mathscr{B})$ that are consistent, to which we add the element π, that is,

$$M = \{1, c_1, c_2, c_3, c_1c_3, c_2c_3, \pi\}.$$

Notice that, because of the presence in $\mathscr{B}$ of both the assertion $ass_1 : c_1 \leqslant c_2$ and of the label c_2c_3, the label c_1c_3, although not initially present in $\mathscr{B}$, is added by the closure operation in order to respect the compatibility requirement.

2. The multiplication $\bullet$ is

$\bullet$	1	c_1	c_2	c_3	c_1c_3	c_2c_3	π
1	1	c_1	c_2	c_3	c_1c_3	c_2c_3	π
c_1	c_1	π	π	c_1c_3	π	π	π
c_2	c_2	π	π	c_2c_3	π	π	π
c_3	c_3	c_1c_3	c_2c_3	π	π	π	π
c_1c_3	c_1c_3	π	π	π	π	π	π
c_2c_3	c_2c_3	π	π	π	π	π	π
π	π	π	π	π	π	π	π

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

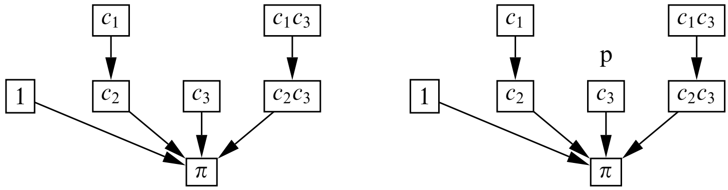


Figure 6. Countermodel for $((\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot) \mathrel{-\!\!*} (((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot)$.

3. The preordering relation $\sqsubseteq$ reflects the structure of the assertions $\overline{Assc}(\mathscr{B})$. If we omit reflexive and π relations, we have two non-trivial relations, namely, $c_1 \sqsubseteq c_2$ and $c_1c_3 \sqsubseteq c_2c_3$. The corresponding diagram is shown on the left-hand side of Figure 6.
4. The Grothendieck topology J is given by the following table:

x	1	c_1	c_2	c_3	c_1c_3	c_2c_3	π
$J(x)$	$\{\{1\}\}$	$\{\{c_1\}\}$	$\{\{c_2\}\}$	$\{\{c_3\}\}$	$\{\{c_1c_3\}\}$	$\{\{c_2c_3\}\}$	$\{\{\pi\},\varnothing\}$

The second step is to apply Definition 4.7 to the only atomic proposition p occurring in the branch $\mathscr{B}$, which leads to the GTI $[\![\mathrm{p}]\!]_{\mathscr{B}} = \{\pi, c_3\}$. This, in turn, finally gives rise to the GRM $\mathscr{G} = (\mathscr{M}(\mathscr{B}), \models, [\![-]\!]_{\mathscr{B}})$, the desired countermodel shown on the right-hand side of Figure 6.

Now we check that:

(i) $c_1 \models (\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot$; and
(ii) $c_1 \not\models ((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot$.

For (i), we have $c_3 \models \mathrm{p}$ because $c_3 \in [\![\mathrm{p}]\!]_{\mathscr{B}}$ and $c_2c_3 \not\models \bot$ because $\varnothing \notin J(c_2c_3)$. Thus, we have $c_2 \not\models \mathrm{p} \mathrel{-\!\!*} \bot$ and, since $c_1 \sqsubseteq c_2$, we obtain, by (K), $c_1 \not\models \mathrm{p} \mathrel{-\!\!*} \bot$. Therefore, we have $c_1 \models (\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot$.

For (ii), we note that π is the only world that forces $\mathrm{p} * \mathrm{p}$. Thus, we have $c_2 \models (\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot$ only if $c_2 \bullet \pi \models \bot$, which is the case because $c_2 \bullet \pi = \pi$ and $\pi \models \bot$. Note that it would not be the case in the elementary semantics for which no world can force $\bot$. On the other hand, $c_2 \not\models \bot$ because $\varnothing \notin J(c_2)$. Therefore, $c_1 \not\models ((\mathrm{p} * \mathrm{p}) \mathrel{-\!\!*} \bot) \to \bot$. Then the initial formula, although valid in the elementary semantics, is not provable in BI.

4.3. *Tableau construction and completeness*

In the previous section, we explained how to build a model from a complete branch. To show the completeness theorem, we now need a tableaux construction procedure that, given a formula ϕ, builds a tableaux sequence $\mathscr{T}_1, \mathscr{T}_2, \ldots$ until there exists a tableau $\mathscr{T}_i$ that is either closed or contains a (possibly infinite) complete branch.

BI has such a procedure, with F $\phi : 1$ as initial formula. Until $\mathscr{T}$ is closed or completed, choose an open branch $\mathscr{B}$: if there is an unfulfilled α or $\pi\alpha$ formula (S $\phi : x$) in $\mathscr{B}$, then apply the related expansion rule; otherwise, if there is an unfulfilled β or $\pi\beta$ formula

(S ϕ : x) in $\mathscr{B}$, apply the corresponding expansion rule with all labels for which the formula is not fulfilled.

Note that in the case of T I : x, although there is no explicit expansion rule in TBI, the fulfilled condition requires the addition of the constraint $1 \leqslant x$ to the set of assertions $Ass(\mathscr{B})$.

Theorem 4.3 (Completeness of TBI). If I $\models \phi$, there is a closed tableau sequence for ϕ.

Proof. Suppose there is no closed tableaux sequence for ϕ. Then the above tableau construction procedure yields a tableau in which there is a completed branch $\mathscr{B}$. Since $\mathscr{B}$ contains the initial formula F ϕ : 1, Theorem 4.2 implies that we can build a Grothendieck resource model $(\mathscr{M}(\mathscr{B}), \models, [\![-]\!]_{\mathscr{B}})$ of $\mathscr{B}$ such that $1 \not\models \phi$, which means that ϕ is not valid in the Grothendieck resource semantics. □

In this section we have proved the completeness of TBI with respect to Grothendieck resource models but its soundness is only proved, at this point, with respect to basic Grothendieck resource models. In the next section, we revisit BI's semantics from the point of view of resource tableaux. The resulting results will lead to a proof of soundness for the general models.

5. BI's semantics revisited

As discussed in the introduction, the initial semantics of BI, based on pre-ordered commutative monoids, may be motivated by modelling units of resource as entities that may be zero, combined and compared. In Pym (2002), Pym *et al.* (2004) and Pym (2004), and in the preceding sections, it has been shown that a great deal of logical theory may be developed quite naturally, and that this simple model of resource quite naturally encompasses a wide range of examples of resource, including ambients, Petri nets, memory allocation and deallocation, logic programming, and money (Ishtiaq and O'Hearn 2001). It may readily be seen, however, that models based not on monoids with total combinations but rather on monoids with partial combination operations, $\bullet : M \times M \rightharpoonup M$, would not only encompass these examples more naturally, but also may be motivated abstractly by a desire to capture the notion of *separation* (Reynolds 2000; Ishtiaq and O'Hearn 2001). The key idea here is that two units of resource may be combined only if they are disjoint, or separated, or non-interfering. An excellent example arises quite simply in the 'pointer logic' model of BI given in Reynolds (2000), Ishtiaq and O'Hearn (2001) and O'Hearn *et al.* (2001), in which we may illustrate the composition by taking 'resource' to mean 'portion of computer memory'. The pointer logic has, in addition to $*$, as a form of assertion, the 'points-to' relation, $\mapsto$, which is used to make statements about the contents of heap cells. For example, $(x \mapsto 3, y) * (y \mapsto 4, x)$ says that x and y denote distinct binary cells in memory, where the second part of x is a pointer to y, the second part of y is a pointer to x, and where the first parts contain 3 and 4.

In this context, an open question arises: Is it possible to propose a metatheoretically satisfactory, general semantics of BI that is based on partial monoids, as taken in, for example, the pointer logic model of BI? In this section, we provide, in the intuitionistic setting, a positive answer via the definition of a new semantics for BI, which is based on partially defined pre-ordered commutative monoids ('PDM semantics'), that is intermediate between the elementary semantics and the Grothendieck topological semantics. This semantics arises from our study of resource tableaux and their specific relationships with BI's semantics.

5.1. *A new relational semantics for BI*

We first define a relational semantics of BI based on specific ternary relations, such that the PDM semantics will be a particular case (or instantiation) of this relational semantics that we prove sound and complete for BI.

Definition 5.1 (BI frame). A BI *frame* is a structure $\mathscr{F} = (M, e, R, \sqsubseteq)$ in which M is a set of resources with two distinguished elements, e and π, and R is a ternary relation on $M \times M \times M$ that satisfies the following conditions, in which $x \sqsubseteq y$ is defined as $x \sqsubseteq y \equiv Rexy$:

— $\forall x.\ Rexx$ (reflexivity);
— $\forall x \forall y \forall z.\ (Rxyz \to Ryxz)$ (commutativity);
— $\forall x \forall y \forall z \forall t.\ \exists u (Rxyu \wedge Ruzv) \leftrightarrow \exists t (Ryzt \wedge Rxtv)$ (associativity);
— $\forall x \forall y \forall z \forall x'.\ (Rxyz \wedge x \sqsubseteq x') \to Rx'yz$ (compatibility);
— $\forall x \forall y \forall z \forall z'.\ (Rxyz \wedge z \sqsubseteq z') \to Rxyz'$ (transitivity);
— $\forall x \forall y . Rxy\pi$ (π-max);
— $\forall x \forall y . (R\pi xy \to\ \pi \sqsubseteq y)$ (π-abs).

We observe that e is neutral for the ternary relation R. Moreover, the π-max condition entails that $\forall x . x \sqsubseteq \pi$, that is, π is the greatest element for the preorder induced by the R relation. Consequently, in the (π-abs) condition, $\pi \sqsubseteq y$ can be replaced by $\pi = y$.

Definition 5.2 (Relational interpretation). Let M be a set of resources with a greatest element π (with respect to a preorder $\sqsubseteq$) and $\mathscr{P}(L)$ be the collection of BI propositions over a language L of propositional letters.
A *relational interpretation* (RI) is a function $[\![-]\!] : L \to \mathscr{P}(M)$ that satisfies:

(K) For any $m, n \in M$ such that $m \sqsubseteq n$, $m \in [\![\mathrm{p}]\!]$ implies $n \in [\![\mathrm{p}]\!]$.
(B) For any m such that $\pi \sqsubseteq m$, we have $m \in [\![\mathrm{p}]\!]$.

Definition 5.3 (Relational model). Let $\mathscr{P}(L)$ be the collection of BI propositions over a language L of propositional letters. Then a *relational model* (RM) is defined to be a structure $\mathscr{R} = (M, e, R, \sqsubseteq, [\![-]\!], \models)$ in which $(M, e, R, \sqsubseteq)$ is a *BI frame*, $[\![-]\!]$ is a relational interpretation, and $\models$ is a forcing relation on $M \times \mathscr{P}(L)$, satisfying the following conditions:

— $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
— $m \models \top$ iff always
— $m \models \bot$ iff $m = \pi$
— $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
— $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
— $m \models \phi \to \psi$ iff, for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

— $m \models \mathrm{I}$ iff $e \sqsubseteq m$
— $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $Rnn'm$, $n \models \phi$ and $n' \models \psi$
— $m \models \phi -\!\!* \psi$ iff, for all $n, n' \in M$, $n \models \phi$ and $Rmnn'$ entails $n' \models \psi$.

Given a relational model $\mathscr{R}$, the validity is defined as follows:

$$\phi \models_{\mathscr{R}} \psi \text{ iff } \mathscr{R}, e \models \phi -\!\!* \psi \text{ and } \phi \models \psi \text{ iff } \forall \mathscr{R}, \phi \models_{\mathscr{M}} \psi.$$

Theorem 5.1 (Soundness of BI). BI is sound with respect to the relational semantics.

Proof. We show, by case analysis, that every LBI-rule preserves validity.

— $(*_R)$ We assume $\Gamma \models \phi$ and $\Delta \models \psi$, and then show that $\Gamma, \Delta \models \phi * \psi$. Let $\mathscr{R}$ be a relational model and m be a world in $\mathscr{R}$ such that $m \models \Phi_{\Gamma,\Delta}$, we have to show that $m \models \phi * \psi$. Since $\Phi_{\Gamma,\Delta} = \Phi_\Gamma * \Phi_\Delta$, we have $m \models \Phi_\Gamma * \Phi_\Delta$. Therefore, there exist n and n' in $\mathscr{R}$ such that $Rnn'm$, $n \models \Phi_\Gamma$ and $n' \models \Phi_\Delta$. From $\Gamma \models \phi$ and $n \models \Phi_\Gamma$, we deduce $n \models \phi$. Similarly, $\Delta \models \psi$ and $n' \models \Phi_\Delta$ imply $n' \models \psi$. Consequently, we get $Rnn'm$, $n \models \phi$ and $n' \models \psi$. Hence, $m \models \phi * \psi$.
— $(*_L)$ This case is immediate since $\Phi_{\Gamma(\phi,\psi)} = \Phi_{\Gamma(\phi * \psi)}$.
— $(-\!\!*_R)$ We assume $\Gamma, \phi \models \psi$, and then show that $\Gamma \models \phi -\!\!* \psi$. Let $\mathscr{R}$ be a relational model and m be a world in $\mathscr{R}$ such that $m \models \Phi_\Gamma$. We have to show that $m \models \phi -\!\!* \psi$. Suppose n and n' are worlds in $\mathscr{R}$ such that $Rmnn'$ and $n \models \phi$. Then $n' \models \Phi_\Gamma * \phi$. Since $\Phi_\Gamma * \phi = \Phi_{\Gamma,\phi}$, we have $n' \models \Phi_{\Gamma,\phi}$, which, using the assumption $\Gamma, \phi \models \psi$, entails that $n' \models \psi$. Therefore, $m \models \phi -\!\!* \psi$.
— $(-\!\!*_L)$ We show that $\Delta \models \phi$ and $\Gamma(\psi, \Delta') \models \chi$ imply $\Gamma(\Delta, \phi -\!\!* \psi, \Delta') \models \chi$ by induction on the structure of Γ.

(a) *Base case.* Assuming $\Delta \models \phi$ and $\psi, \Delta' \models \chi$, we have to show that $\Delta, \phi -\!\!* \psi, \Delta' \models \chi$. Let $\mathscr{R}$ be a relational model and m be a world in $\mathscr{R}$ such that $m \models \Phi_{\Delta,\phi -\!* \psi,\Delta'}$. Since $\Phi_{\Delta,\phi -\!* \psi,\Delta'} = \Phi_\Delta * (\phi -\!\!* \psi) * \Phi_{\Delta'}$, there exist two worlds n and n' in $\mathscr{R}$ such that $Rnn'm$, $n \models \Phi_\Delta$ and $n' \models (\phi -\!\!* \psi) * \Phi_{\Delta'}$. Similarly, there are two worlds k and k' in $\mathscr{R}$ such that $Rkk'n'$, $k \models \phi -\!\!* \psi$ and $k' \models \Phi_{\Delta'}$. Since we have a world n' such that $Rkk'n'$ and $Rnn'm$, by the associativity axiom (of Definition 5.1) for R, there exists a world t such that $Rnkt$ and $Rtk'm$. From $n \models \Phi_\Delta$ and $\Delta \models \phi$, we get $n \models \phi$. From $Rnkt$, $n \models \phi$ and $k \models \phi -\!\!* \psi$, we get $t \models \psi$. Since we now have $Rtk'm$, $t \models \psi$ and $k' \models \Phi_{\Delta'}$, we can deduce $m \models \psi * \Phi_{\Delta'}$, which, using the assumption that $\psi, \Delta' \models \chi$, entails that $m \models \chi$.

(b) *Case* $\Gamma = \Gamma'(\psi, \Delta'), \Gamma''$. Assuming we have $\Delta \models \phi$ and $\Gamma'(\psi, \Delta'), \Gamma'' \models \chi$, we get $\Gamma'(\psi, \Delta') \models \Phi_{\Gamma''} -\!\!* \chi$. Then, we obtain $\Gamma'(\Delta, \phi -\!\!* \psi, \Delta') \models \Phi_{\Gamma''} -\!\!* \chi$, by the induction hypothesis, and finally we get $\Gamma'(\Delta, \phi -\!\!* \psi, \Delta'), \Gamma'' \models \chi$.

(c) *Case* $\Gamma = \Gamma'(\psi, \Delta'); \Gamma''$. This case is similar to the previous one.

— The other cases are similar. □

We now consider the question of completeness. We aim to build a term model such that if $\Gamma \nvdash \phi$, there exists a world m in the model such that $m \models \Phi_\Gamma$ and $m \not\models \phi$. For that, we consider the idea of *prime theory* in order to have the structure required by the semantic clauses for the connectives, and more particularly for $\vee$ and $*$ (Pym 2002; Pym 2004).

Here we need some simple definitions. A bunch Γ is said to be *prime* if it verifies that $\Gamma \vdash \phi \vee \psi$ implies $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$. A *prime extension* of a bunch Γ is a bunch Γ' such that Γ' is prime and $\Gamma' \models \Phi_\Gamma$. Moreover, we extend to bunches the definition of BI connectives in the following way: $\Delta \odot \Theta$ is defined as $\Phi_\Delta \odot \Phi_\Theta$, where $\odot \in \{*, \wedge, \vee, -\!*, \rightarrow\}$. Similarly, the notations $\Gamma \vdash \Delta$ and $\Gamma \models \Delta$ stand for $\Gamma \vdash \Phi_\Delta$ and $\Gamma \models \Phi_\Delta$, respectively.

Definition 5.4 (Term model). The term model is defined as $\mathscr{T} = (P, \varnothing_m, R, \sqsubseteq, [\![-]\!], \models)$ in which:

1. P is the set $B/\dashv\vdash$ where B is the set of all the prime bunches and $\dashv\vdash$ is the equality generated by derivability.
2. $\varnothing_m$ is the multiplicative unit of bunches (trivially prime).
3. $R\Gamma\Delta\Theta$ is defined as $R\Gamma\Delta\Theta$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$.
4. $\sqsubseteq$ is defined by $\Gamma \sqsubseteq \Delta$ iff $R\varnothing_m\Gamma\Delta^\dagger$.
5. $[\![p]\!] = \{\Gamma \in P \mid \Gamma \vdash p\}$.
6. $\models$ is defined by $\Gamma \models \phi$ iff $\Gamma \vdash \phi$ for any $\Gamma \in P$.

First we mention two results concerning the relation R of the term model $\mathscr{T}$.

Lemma 5.1. If we consider R the relation of the term model $\mathscr{T}$, then $R\Gamma\Delta\Theta$ if and only if $\Theta \vdash \Gamma * \Delta$.

Proof. If $R\Gamma\Delta\Theta$, then $\forall\phi, \psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$. In particular, as $\Gamma \vdash \Gamma$ and $\Delta \vdash \Delta$, we deduce $\Theta \vdash \Gamma * \Delta$. If $\Theta \vdash \Gamma * \Delta$, then suppose $\Gamma \vdash \phi$ and $\Delta \vdash \psi$. By bifunctoriality of $*$, we get $\Gamma * \Delta \vdash \phi * \psi$. Since $\Theta \vdash \Gamma * \Delta$, we finally have $\Theta \vdash \phi * \psi$. $\square$

Corollary 5.1. If we consider $\sqsubseteq$, the preorder of the term model $\mathscr{T}$, then $\Gamma \sqsubseteq \Delta$ if and only if $\Delta \vdash \Gamma$.

Proof. We have $\Gamma \sqsubseteq \Delta$ iff $R\varnothing_m\Gamma\Delta$ iff $\Delta \vdash \varnothing_m * \Gamma$ (by Lemma 5.1) iff $\Delta \vdash \Gamma$. $\square$

Moreover, we can easily deduce that $\perp$ and $\top$ are, respectively, the greatest and least elements of the term model with respect to the preorder, $\sqsubseteq$.

Having defined the term model, the next step consists of verifying that the relation R of Definition 5.4 satisfies the conditions of Definition 5.1 and that the forcing relation $\models$ satisfies the conditions of Definition 5.3. The proofs of these results rely on the following two fundamental lemmas.

Lemma 5.2 (Extension lemma). If $\Gamma \not\vdash \chi$, then there exists a prime extension Γ' of Γ such that $\Gamma' \not\vdash \chi$.

Proof. The proof is similar to the corresponding proof in Pym (2002; 2004). Given a fair enumeration of BI propositions, Γ' is obtained as the limit of the following inductive

† This condition is mis-stated in the corresponding constructions in Pym (2002): it is corrected in Pym (2004) and the corrected statement is as in Corollary 5.1, below. This error was known prior to the publication of Pym (2002) but persisted because of an editing error by the author. There are no known consequences. Similarly, Pym *et al.* (2004) requires the following correction: page 285, line -12: ‘, for some $P', Q \equiv P; P'$ ’ should be ‘$P \vdash Q$’.

construction. For the base case, we set $\Gamma_0 = \Gamma$. For the induction step, we set $\Gamma_{i+1} = \Gamma_i$ if Γ_i is prime. If Γ_i is not prime, we pick the first formula $\phi \vee \psi$ in the enumeration such that $\Gamma_i \vdash \phi \vee \psi$ and neither $\Gamma_i \vdash \phi$, nor $\Gamma_i \vdash \psi$. We then set $\Gamma_{i+1} = \Gamma_i ; \phi$ if $\Gamma_i ; \phi \nvdash \chi$, and $\Gamma_{i+1} = \Gamma_i ; \psi$, otherwise.

We need to show that, for any i, we have $\Gamma_i \nvdash \chi$. For the base case $i = 0$, the result is immediate since $\Gamma \nvdash \chi$ by hypothesis. For the induction step, suppose that $\Gamma_i \nvdash \chi$, we show that $\Gamma_{i+1} \nvdash \chi$ by showing that it cannot be the case that $\Gamma_i ; \phi \vdash \chi$ and $\Gamma_i ; \psi \vdash \chi$ both hold at the same time: if this were not the case, then, by the $\vee_L$ rule of LBI, we would get $\Gamma_i ; \phi \vee \psi \vdash \chi$, and since $\Gamma_i \vdash \phi \vee \psi$ by hypothesis, an application of the *cut* rule immediately followed by a contraction on Γ_i would lead to $\Gamma_i \vdash \chi$, which is a contradiction of the induction hypothesis.

Finally, Γ' is obtained as the limit, in the evident notation, $\wedge_i \Gamma_i$, $\Gamma \wedge \Delta$ being equivalent to $\Gamma ; \Delta$. □

Lemma 5.3 (Primeness lemma). If Γ is prime and $\Gamma \vdash \Delta * \Theta$, there exists a prime extension Δ' of Δ such that $\Gamma \vdash \Delta' * \Theta$.

Proof. The proof is similar to the corresponding proofs given in Dunn (1986) or Routley and Meyer (1972). Given a fair enumeration of BI propositions, Δ' is obtained as the limit of the following inductive construction. For the base case, we set $\Delta_0 = \Delta$. For the induction step, we set $\Delta_{i+1} = \Delta_i$ if Δ_i is prime. If Δ_i is not prime, we pick the first formula $\phi \vee \psi$ in the enumeration such that $\Delta_i \vdash \phi \vee \psi$ and neither $\Delta_i \vdash \phi$, nor $\Delta_i \vdash \psi$. We then set $\Delta_{i+1} = \Delta_i ; \phi$ if $\Gamma \vdash (\Delta_i ; \phi) * \Theta$, and $\Delta_{i+1} = \Delta_i ; \psi$, otherwise.

We need to show that, for any i, we have $\Gamma \vdash \Delta_i * \Theta$. For the base case $i = 0$, the result is immediate since $\Gamma \vdash \Delta * \Theta$ by hypothesis. For the induction step, suppose that $\Gamma \vdash \Delta_i * \Theta$. We show that $\Gamma \vdash \Delta_{i+1} * \Theta$ by showing that either $\Gamma \vdash (\Delta_i ; \phi) * \Theta$ or $\Gamma \vdash (\Delta_i ; \psi) * \Theta$. Indeed, by the induction hypothesis, we have $\Gamma \vdash \Delta_i * \Theta$. Since $\Delta_i \vdash \phi \vee \psi$, we get $\Gamma \vdash (\Delta_i ; \phi \vee \psi) * \Theta$. By distribution of ';' over $\vee$ (recall that ';' represents $\wedge$), we have $\Gamma \vdash ((\Delta_i ; \phi) \vee (\Delta_i ; \psi)) * \Theta$, from which we get $\Gamma \vdash ((\Delta_i ; \phi) * \Theta) \vee ((\Delta_i ; \psi) * \Theta)$ by distribution of $*$ over $\vee$. Since Γ is assumed to be prime, we conclude that either $\Gamma \vdash (\Delta_i ; \phi) * \Theta$ or $\Gamma \vdash (\Delta_i ; \psi) * \Theta$.

Finally, as in Lemma 5.2, Δ' is obtained as the limit $\wedge_i \Delta_i$. □

Now we can show the completeness using the next two lemmas.

Lemma 5.4. Let $\mathscr{T} = (P, \varnothing_m, R, \sqsubseteq, [\![-]\!], \models)$ be the term model, $(P, \varnothing_m, R, \sqsubseteq)$ is a BI frame:

Proof. R is defined as $R\Gamma\Delta\Theta$ iff $\forall \phi, \forall \psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$. Now, we verify each condition of Definition 5.1.

— Reflexivity: $\forall \phi, \forall \psi, \varnothing_m \vdash \phi$ and $\Delta \vdash \psi$ entails $\Delta \vdash \phi * \psi$. Therefore, we have $R\varnothing_m\Delta\Delta$.
— Commutativity: This is trivial.
— Associativity: We show that $\exists \Theta (R\Gamma\Delta\Theta$ and $R\Theta\Psi\Sigma)$ iff $\exists \Phi (R\Delta\Psi\Phi$ and $R\Gamma\Phi\Sigma)$. We prove here the implication from left to right, the other being analogous. By hypothesis:

(i) $R\Gamma\Delta\Theta$ iff $\forall \phi, \forall \psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$; and

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

(ii) $R\Delta\Psi\Sigma$ iff $\forall\phi, \forall\psi, \Delta \vdash \phi$ and $\Psi \vdash \psi$ entails $\Sigma \vdash \phi * \psi$.

We consider $\Phi \equiv (\Delta * \Psi)'$, which is a prime bunch built by two applications of Lemma 5.3.

We have $R\Delta\Psi(\Delta * \Psi)'$, which is trivial. We first show that we have $\Sigma \vdash (\Gamma * \Delta) * \Psi$. As $\Theta \vdash \Theta$ and $\Psi \vdash \Psi$, by (ii) we can deduce that $\Sigma \vdash \Theta * \Psi$. Moreover, as $\Gamma \vdash \Gamma$ and $\Delta \vdash \Delta$, by (i) we can deduce that $\Theta \vdash \Gamma * \Delta$, and therefore $\Theta * \Psi \vdash (\Gamma * \Delta) * \Psi$. Finally, we have $\Sigma \vdash (\Gamma * \Delta) * \Psi$. Therefore, we have $\Sigma \vdash (\Gamma * \Delta) * \Psi$ iff $\Sigma \vdash \Gamma * (\Delta * \Psi)$ iff $R\Gamma(\Delta * \Psi)\Sigma$.

— Transitivity: We show that $R\Gamma\Delta\Theta$ and $\Gamma' \sqsubseteq \Gamma$ entails $R\Gamma'\Delta\Theta'$. By hypothesis:

(i) $R\Gamma\Delta\Theta$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\Theta \vdash \phi * \psi$; and

(ii) $\Gamma' \sqsubseteq \Gamma$ iff $R\varnothing_m\Gamma'\Gamma'$ iff $\forall\phi, \forall\psi, \varnothing_m \vdash \phi$ and $\Gamma' \vdash \psi$ entails $\Gamma \vdash \phi * \psi$.

If $\Gamma' \vdash \phi$, then, since $\varnothing_m \vdash \mathrm{I}$, by (ii) we have $\Gamma \vdash \mathrm{I} * \phi$, and then $\Gamma \vdash \phi$. Moreover, if $\Delta \vdash \psi$, by (i) we get $\Theta \vdash \phi * \psi$, and therefore $R\Gamma'\Delta\Theta'$.

— π-max: We have to show that $R\Gamma\Delta\bot$ iff $\forall\phi, \forall\psi, \Gamma \vdash \phi$ and $\Delta \vdash \psi$ entails $\bot \vdash \phi * \psi$, which is trivial because it is an axiom.

— The other cases are similar. □

Lemma 5.5. The term model $\mathscr{T} = (P, \varnothing_m, R, \sqsubseteq, [\![-]\!], \models)$ is a relational model.

Proof. The proof is by induction on the formula ϕ.

— (Monotonicity) Suppose $\Gamma \models \phi$ and $\Gamma \sqsubseteq \Delta$, and prove that $\Delta \models \phi$. By definition, we have $\Gamma \vdash \phi$. As $\Gamma \sqsubseteq \Delta$ iff $\Delta \vdash \Gamma$, by Corollary 5.1, we deduce $\Delta \vdash \phi$ by transitivity of $\vdash$.

— ($\wedge$) $\Gamma \models \phi \wedge \psi$ iff $\Gamma \vdash \phi \wedge \psi$ iff $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$.

— ($\vee$) $\Gamma \models \phi \vee \psi$ iff $\Gamma \vdash \phi \vee \psi$ iff $\Gamma \vdash \phi$ or $\Gamma \vdash \psi$ (because Γ is prime).

— ($\rightarrow$) $\Gamma \models \phi \rightarrow \psi$ iff, for all Δ such that $\Gamma \sqsubseteq \Delta$, if $\Delta \models \phi$, then $\Delta \models \psi$. Therefore, suppose $\Gamma \not\models \phi \rightarrow \psi$. Then $\Gamma; \phi \nvdash \psi$, so there exists, by Lemma 5.2, $(\Gamma; \phi)'$ such that $(\Gamma; \phi)' \nvdash \psi$. Moreover, we have $(\Gamma; \phi)' \vdash (\Gamma; \phi)$, and thus $\Gamma \sqsubseteq (\Gamma; \phi)'$ and $(\Gamma; \phi)' \vdash \phi$. Therefore, there exists $\Delta \equiv (\Gamma; \phi)$ such that $\Gamma \sqsubseteq \Delta$, $\Delta \models \phi$ and $\Delta \not\models \psi$. Suppose there exists Δ such that $\Delta \models \phi$, $\Delta \not\models \psi$ and $\Gamma \sqsubseteq \Delta$. Thus, we have $\Delta \vdash \phi$ and $\Delta \nvdash \psi$ and also $\Delta \vdash \Gamma$ by Corollary 5.1. Therefore, we have $\Gamma \nvdash \phi \rightarrow \psi$ and then $\Gamma \not\models \phi \rightarrow \psi$.

— ($*$) $\Gamma \models \phi * \psi$ iff $\exists\Delta, \Psi / R\Delta\Psi\Gamma$ and $\Delta \models \phi$ and $\Psi \models \psi$. Suppose $\Gamma \models \phi * \psi$. Therefore we have $\Gamma \vdash \phi * \psi$, and thus there exist α, β such that $\alpha \vdash \phi$ and $\beta \vdash \psi$ and $\Gamma \vdash \alpha * \beta$. Note that α and β are not necessarily prime. By Lemma 5.3, applied twice, we can extend α (respectively, β) into a prime bunch Δ (respectively, Ψ) such that $\Gamma \vdash \Delta * \Psi$. Therefore, by Lemma 5.1, we have $R\Delta\Psi\Gamma$ and $\Delta \vdash \phi$ (respectively, $\Psi \vdash \psi$) implies $\Delta \models \phi$ (respectively, $\Psi \models \psi$). Suppose that $\exists\Delta, \Psi / R\Delta\Psi\Gamma$ and $\Delta \models \phi$ and $\Psi \models \psi$. Then, by Lemma 5.1, we have $\Gamma \vdash \Delta * \Psi$ and $\Delta \models \phi$ (respectively, $\Psi \models \psi$) implies $\Delta \vdash \phi$ (respectively, $\Psi \vdash \psi$). Then, we have $\Gamma \vdash \phi * \psi$, and thus $\Gamma \models \phi * \psi$.

— ($-\!\!*$) $\Gamma \models \phi -\!\!* \psi$ iff $\forall\Delta, \Psi$ $\Delta \models \phi$ and $R\Gamma\Delta\Psi$ entails $\Psi \models \psi$. Suppose $\exists\Delta, \Psi / R\Gamma\Delta\Psi$ and $\Delta \models \phi$ entails $\Psi \models \psi$. $\Delta \models \phi$ entails $\Delta \vdash \phi$, so $\Gamma * \Delta \vdash \Gamma * \phi$. Moreover, $R\Gamma\Delta\Psi$ entails $\Psi \vdash \Gamma * \Delta$ by Lemma 5.1. By transitivity of $\vdash$, we obtain $\Psi \vdash \Gamma * \phi$; if $\Gamma \vdash \phi -\!\!* \psi$, then $\Psi \vdash (\phi -\!\!* \psi) * \phi$. We can deduce $\Psi \vdash \psi$, which is a contradiction. Consequently, we have $\Gamma \nvdash \phi -\!\!* \psi$, which entails $\Gamma \not\models \phi -\!\!* \psi$. Suppose

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

that $\Gamma \not\models \phi \multimap \psi$. Then $\Gamma \nvdash \phi \multimap \psi$, so $\Gamma, \phi \nvdash \psi$, which is equivalent to $\phi \nvdash \Gamma \multimap \psi$. As $\phi \nvdash \Gamma \multimap \psi$, by Lemma 5.2, there exists ϕ' such that $\phi' \nvdash \Gamma \multimap \psi$, which is equivalent to $\Gamma, \phi' \nvdash \psi$. If we consider $\Delta \equiv \phi'$ and $\Psi \equiv (\Gamma, \phi')'$, we get $\Gamma \vdash \phi$, and thus $\Delta \models \phi$. Moreover, we have $R\Gamma\Delta\Psi$ and $\Psi \nvdash \psi$ entails $\Psi \not\models \psi$.

— ($\top$) This is immediate since $\Gamma \vdash \top$ is an axiom.

— ($\bot$) $\Gamma \models \bot$ iff $\Gamma = \bot$. $\Gamma \models \bot$ iff $\Gamma \vdash \bot$ iff $\Gamma \dashv\vdash \bot$ since $\bot \vdash \Gamma$ is an axiom. Therefore, we have $\Gamma \sqsubseteq \bot$ and $\bot \sqsubseteq \Gamma$.

— (I) $\Gamma \models \mathrm{I}$ iff $\mathrm{I} \sqsubseteq \Gamma$. $\Gamma \models \mathrm{I}$ iff $\Gamma \vdash \mathrm{I}$ iff $\mathrm{I} \sqsubseteq \Gamma$ by Corollary 5.1. □

Theorem 5.2 (Completeness of BI). BI is complete with respect to the relational semantics.

Proof. The statement follows from Definition 5.4 (of the term model) and by Lemmas 5.4 and 5.5. □

We have defined a new semantics for BI, but the key points of this proposal are that we can relate it to a new semantics based on partial monoids and also prove the soundness of the TBI calculus with respect to such a semantics (which has not been achieved directly for the Grothendieck topological semantics).

5.2. *A new Kripke resource semantics for BI*

In Section 4, we showed how countermodels can be built from dependency graphs. We now observe that those models are very closely related to the ones recently proposed in the semantics of (intuitionistic) 'pointer logic' (Ishtiaq and O'Hearn 2001; Pym *et al.* 2004; Pym 2004). Indeed, the Grothendieck topology used to characterise the pointer logic model corresponds exactly to our definition of the J-map for basic GRMs. Moreover, in our models, a special element called π is used to capture undefinedness as the image of all undefined compositions and is the only element to force $\bot$ (because $\emptyset$ only belongs to $J(\pi)$).

We now define what we call Kripke resource models and show that they correspond to a particular class of relational resource models, taking the relation $R_{\sqsubseteq}$ defined by

$$R_{\sqsubseteq}xyz \equiv x \bullet y \sqsubseteq z.$$

We now reconstruct the definitions of Kripke resource monoids, interpretations and Kripke resource models in the partially defined setting. Since no confusion is likely, we reuse their names.

Definition 5.5. A *Kripke resource monoid* (KRM) is a preordered commutative monoid $\mathscr{M} = (M, \bullet, e, \sqsubseteq)$ containing a greatest element, denoted π, such that $\pi \bullet m = \pi$ for any $m \in M$, and in which $\bullet$ is bifunctorial with respect to $\sqsubseteq$.

We consider such a collection of resources, a preordered commutative monoid, from the relational semantics perspective. In fact, we consider the relation $R_{\sqsubseteq}$ that naturally verifies the first three conditions of Definition 5.1. The conditions on π correspond to the satisfaction, for $R_{\sqsubseteq}$, of the (π-max) and (π-abs) conditions. The fact that $\bullet$ is bifunctorial with respect to $\sqsubseteq$ is captured by the (*compatibility*) and (*transitivity*) conditions for $R_{\sqsubseteq}$. This implies that this set of resources corresponds to a BI frame.

Definition 5.6. Let $\mathscr{M}$ be a KRM and $\mathscr{P}(L)$ be a language of BI propositions over a language L of propositional letters. Then, a *Kripke resource interpretation*, or KRI, is a function $[\![-]\!] : L \to \mathscr{P}(M)$ satisfying Kripke monotonicity (K) and such that for any $\mathrm{p} \in L$, $\pi \in [\![\mathrm{p}]\!]$.

Again, such an interpretation can be seen, from a relational semantics perspective, as a relational interpretation (see Definition 5.2).

Definition 5.7. A *Kripke resource model* is a triple $\mathscr{K} = (\mathscr{M}, \models, [\![-]\!])$ in which $\mathscr{M}$ is a KRM, $[\![-]\!]$ is a KRI and $\models$ is a forcing relation on $M \times \mathscr{P}(L)$ satisfying the following conditions:

— $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
— $m \models \top$ iff always
— $m \models \bot$ iff $m = \pi$
— $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
— $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
— $m \models \phi \to \psi$ iff, for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
— $m \models \mathrm{I}$ iff $e \sqsubseteq m$
— $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \sqsubseteq m$, $n \models \phi$ and $n' \models \psi$
— $m \models \phi \mathrel{-\!\!*} \psi$ iff, for all $n \in M$ such that $n \models \phi$, $m \bullet n \models \psi$.

Soundness

We observe that the above definition corresponds to a particular relational model (see Definition 5.3) in which we consider the relation $R_{\sqsubseteq}xyz \equiv x \bullet y \sqsubseteq z$. Therefore, it is clear that the class of Kripke resource models is included in the class of relational resource models.

Theorem 5.3 (Soundness of BI). BI is sound with respect to Kripke resource models.

Proof. The statement is obvious from the proof of soundness with respect to the relational semantics (see Theorem 5.1) since relational models include Kripke resource models. □

We now return to the question of completeness. We observe that it is difficult to obtain a direct proof that BI is complete for Kripke resource models. The proof in Pym (2002; 2004) requires a delicate construction of sets of choices of 'evaluated prime bunches' in order to ensure a consistent definition and, in the presence of $\bot$, the topological nature of the models considered therein is, as we have seen, essential. In particular, it is necessary to have a world that forces $\bot$. The idea with the partially defined semantics, based on monoids with the element π, is to have an elementary semantics that does not require the topological structure. It seems that such a requirement is not compatible with obtaining a completeness theorem unless we move to the relational construction based on BI frames.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Another advantage of this move is that we are able to work with our simpler notion of prime bunch, which greatly reduces the technical and conceptual complexity of the completeness argument.

First, we relate a Kripke resource model with a basic GRM of Definition 2.11.

Lemma 5.6. The class of Kripke resource models coincides with the class of basic Grothendieck resource models.

Proof. Let $\mathscr{G}{=}((M, \bullet, e, \sqsubseteq, J), \models_{\mathscr{G}}, [\![-]\!])$ be a basic GRM. We establish that it is a Kripke model. Since $\mathscr{G}$ is basic, we simply show that $\models_{\mathscr{G}}$ satisfies the conditions of Definition 2.5. In the case of $\perp$, since $\varnothing$ only belongs to $J(\pi)$, the condition $\varnothing \in J(m)$ is equivalent to $m = \pi$. Now, for any world $m \neq \pi$, we have $J(m) = \{\{m\}\}$. Thus, in the case of I, the condition $(\exists S \in J(m))$ $(\forall m' \in S)$ $(e \sqsubseteq m')$ simplifies to $(\forall m' \in \{m\})$ $(e \sqsubseteq m')$, which is equivalent to $e \sqsubseteq m$. The cases of $\vee$ and $*$ are similar. Conversely, endowing a Kripke model $((M, \bullet, e, \sqsubseteq), \models_{\mathscr{K}}, [\![-]\!])$ with the basic topology turns it into a basic Grothendieck model We can easily show that, for such a J, that Kripke monotonicity for $[\![-]\!]$ implies (Sh). □

Consequences for TBI

Now we return to the TBI calculus and its relationships with the Kripke resource semantics.

Theorem 5.4. TBI is sound with respect to Kripke resource models, that is, if there exists a closed tableaux sequence for a BI formula ϕ, then ϕ is valid in Kripke resource models.

Proof. TBI is sound with respect to the basic GRMs. From Lemma 5.6 we deduce that TBI is sound with respect to Kripke resource models. □

Theorem 5.5. TBI is complete with respect to Kripke resource semantics, that is, if there is no closed tableau sequence for a BI formula ϕ, then ϕ is not valid in the Kripke resource semantics.

Proof. TBI is complete with respect to the basic GRMs. From Lemma 5.6 we deduce that TBI is complete with respect to Kripke resource models. □

Therefore, we can obtain the soundness of TBI from the previous results to give the counterpart to the Theorem 4.3.

Theorem 5.6 (Soundness of TBI). TBI is sound with respect to BI, that is, if there exists a closed tableaux sequence for a BI formula ϕ, then ϕ is valid in BI.

Proof. TBI is sound with respect to Kripke resource models by Theorem 5.3. Moreover, the Kripke resource semantics is sound and complete with respect to BI by Theorems 5.4 and 5.5. Then we can conclude the soundness of TBI. □

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Completeness

Finally, we are able to give a completeness theorem for BI with respect to the partially defined semantics, thereby establishing completeness for a class of models that includes pointer logic, thereby demonstrating the strength of our resource semantics.

Theorem 5.7 (Completeness of BI). BI is complete with respect to Kripke resource models.

Proof. Suppose $\mathrm{I} \not\vdash \phi$. Then, by Theorem 4.3, there exists a tableau containing a complete branch from which one can build, as explained in Definition 4.6, a basic GRM that is a countermodel of ϕ. From Lemma 5.6 it is also a Kripke resource model that is a countermodel of ϕ. Moreover, the Kripke resource semantics is a particular case of the relational semantics that has been proved complete (Theorem 5.2). □

5.3. *A partially defined semantics for BI*

BI has been proved sound and complete for the reconstruction of Kripke resource semantics that makes explicit use of the element π. In this section, we revisit this semantics yet again in order to show how to handle the necessary undefinedness implicitly.

An alternative (and equivalent) way of dealing with $\perp$ is to handle undefinedness implicitly via a partially defined monoid (PDM), that is, a monoid in which the product $\bullet$ is a partial operation, with no requirement other than that $(x \bullet y) \bullet z$ is defined whenever $x \bullet (y \bullet z)$ is defined, and *vice versa*. This gives rise to the PDM semantics, which is obtained from the Kripke resource semantics by making some minor changes to the forcing relation.

Definition 5.8. A *PDM model* is a triple $\mathscr{K} = (\mathscr{M}, \models, [\![-]\!])$ in which $\mathscr{M}$ is a KRM, $[\![-]\!]$ is a KRI and $\models$ is a forcing relation on $M \times \mathscr{P}(L)$ satisfying the following conditions:

— $m \models \mathrm{p}$ iff $m \in [\![\mathrm{p}]\!]$
— $m \models \top$ iff always
— $m \models \perp$ iff never
— $m \models \phi \wedge \psi$ iff $m \models \phi$ and $m \models \psi$
— $m \models \phi \vee \psi$ iff $m \models \phi$ or $m \models \psi$
— $m \models \phi \to \psi$ iff, for all $n \in M$ such that $m \sqsubseteq n$, if $n \models \phi$, then $n \models \psi$
— $m \models \mathrm{I}$ iff $e \sqsubseteq m$
— $m \models \phi * \psi$ iff there exist $n, n' \in M$ such that $n \bullet n' \downarrow$, $n \bullet n' \sqsubseteq m$, $n \models \phi$ and $n' \models \psi$
— $m \models \phi \mathrel{-\!\!*} \psi$ iff for all $n \in M$ such that $n \models \phi$, $m \bullet n \downarrow$ implies $m \bullet n \models \psi$

where $\downarrow$ denotes definedness.

The PDM semantics is easily seen to be equivalent to the previous Kripke resource semantics. For example, in the case of $\perp$, given that, on one hand, π is the only element to force $\perp$ and, on the other hand, that π means undefinedness, no defined world should force $\perp$, that is, $\perp$ is nowhere forced. Similar reasoning applies to $*$ and $\mathrel{-\!\!*}$. Moreover, the term model construction required to demonstrate the completeness of the PDM directly would require that if $\Gamma, \Delta \vdash \perp$, then the corresponding $\Gamma \bullet \Delta$ is undefined.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

An immediate consequence of moving to the PDM semantics is that dependency graphs can be considered straightforwardly as countermodels in this semantics. Soundness and completeness of BI with respect to the PDM semantics are consequences of the previous results.

Theorem 5.8 (Soundness of BI). BI is sound with respect to PDM models.

Proof. The proof follows from the proof of soundness with respect to Kripke resource models (see Theorem 5.3). $\square$

Theorem 5.9 (Completeness of BI). BI is complete with respect to PDM models.

Proof. The proof follows from the proof of completeness with respect to Kripke resource models (see Theorem 5.7). $\square$

A question naturally arises from these results: Can we define such a new semantics for some variants of BI, like for instance Affine BI or Boolean BI? In Affine BI, the comma, or $*$, admits weakening. Boolean BI, with the affine comma, has been used as the basis for the program logics 'pointer logic', introduced by Ishtiaq and O'Hearn (Ishtiaq and O'Hearn 2001) and 'separation logic' (Reynolds 2000). It has also been used as a basis of the type systems used to provide unified accounts (O'Hearn 1999; O'Hearn and Pym 1999; Pym 2002; Pym 2004) of Reynolds' Syntactic Control of Interference and Idealised Algol languages. In fact, both pointer logic and separation logic have both intuitionistic and classical versions, that is, the additives may be intuitionistic or classical. The system with classical additives is known as Boolean BI.

In Affine BI, the multiplicative conjunction satisfies the structural rule of weakening, that is, $\phi * \psi \vdash \psi$ for any ϕ and ψ[†]. Compared with Definition 2.3, Kripke resource monoids of affine BI are characterised by monoidal products that satisfy the (weakening) condition that for any worlds m and n, $m \sqsubseteq m \bullet n$. Intuitively, the weakening condition demands that the composition of two resources should result in something bigger (with respect to the preordering) than the two components. Such a condition is met by many (if not most) natural resource compositions.

An immediate consequence is that e becomes the least element, and so, since any world m is now greater than e, the condition $m \models \mathrm{I}$ iff $e \sqsubseteq m$ simplifies to $m \models \mathrm{I}$ iff always, implying that $\mathrm{I} = \top$. Conversely, if e is the least element, the functoriality of $\bullet$ implies the weakening condition. Therefore, affine Kripke resource monoids can be viewed equivalently as Kripke resource monoids having e as their least element. Another interesting consequence of the weakening for $\bullet$ is that the condition $n \bullet n' \sqsubseteq m$ in the forcing clause for $*$ can be simplified to an equality, namely, $n \bullet n' = m$, even in the presence of $\sqsubseteq$. Therefore, we can modify Definition 2.5 in order to define what an *affine Kripke resource model* is, and then provide an affine PDM semantics that can be proved sound and complete for affine BI.

We can derive a similar result directly for Boolean BI without the unit I, but not for full Boolean BI, seemingly, because of particular interactions between this unit and the

[†] Equivalently, $\top = \mathrm{I}$.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

additive conjunction[†]. A deeper analysis of this problem is beyond the scope of this paper and will be provided in further work, perhaps addressing in detail the question of how pointer and separation logics, which may usefully be understood as specific models of BI, fit into our semantic framework.

6. BI's semantics and liberalised rules

In this section, we show how, by a special treatment of the additive disjunction, we can give liberalised rules for TBI, which can be seen as an improvement of the initial version of the calculus. Liberalised rules have been proposed to improve free variable tableau methods dedicated to classical logic, for instance, by giving a new so-called δ rule (Hähnle and Schmitt 1994). In previous work (Galmiche and Méry 2003), we have given, starting from our calculus and its restrictions to intuitionistic logic (IL) and multiplicative intuitionistic linear logic (MILL), liberalised rules that improve the efficiency of the proof-search (by reducing the number of new constants to deal with) and that also provide easy arguments for termination.

As an illustration, we consider for MILL the following liberalised rules,

$$
\begin{array}{c}
\mathrm{F}\ \phi \mathrel{-\!\!*} \psi : x \\
| \\
\mathrm{T}\ \phi : a \\
\mathrm{F}\ \psi : xa
\end{array}
\qquad\qquad
\begin{array}{c}
\mathrm{T}\ \phi * \psi : x^{1} \\
| \\
\boxed{ass : ab \leqslant x} \\
| \\
\mathrm{T}\ \phi : a \\
\mathrm{T}\ \psi : b
\end{array}
$$

where a (respectively, b) need not be new if $\mathrm{T}\ \phi : a$ (respectively, $\mathrm{T}\ \psi : b$) has already been introduced in $\mathscr{B}$ by a previous $\pi\alpha$ tableau expansion.

These rules have been proved admissible in the restriction of TBI to MILL (Galmiche and Méry 2003) and it would be rather interesting if they could be extended to BI.

Unfortunately, the previous liberalised rules are not sound for TBI, as can be seen with the formula $(\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r})) \mathrel{-\!\!*} ((\mathrm{p} \mathrel{-\!\!*} \mathrm{q}) \vee (\mathrm{p} \mathrel{-\!\!*} \mathrm{r}))$, for which Figure 7 gives an open tableau. As the tableau contains an open branch that is also completed, we can, as previously explained, build a countermodel out of this complete branch. Therefore, the formula is not provable in BI.

However, if we use liberalised rules, the tableau of Figure 7 turns out to be closed and we eventually end up with a closed tableau for a non-provable formula. Indeed, using the liberalised version of $\mathrm{F} \mathrel{-\!\!*}$ we can reuse at Step 4 the constant c_2 introduced at Step 3, instead of creating the constant c_3. The corresponding tableau is obtained from Figure 7 by replacing each occurrence of c_3 by c_2, which leads to the closure of the previously open fourth branch due to $(\mathrm{T}\ \mathrm{q} : c_1c_2, \mathrm{F}\ \mathrm{q} : c_1c_2)$.

[†] We are grateful to Hongseok Yang for his observations on this point.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

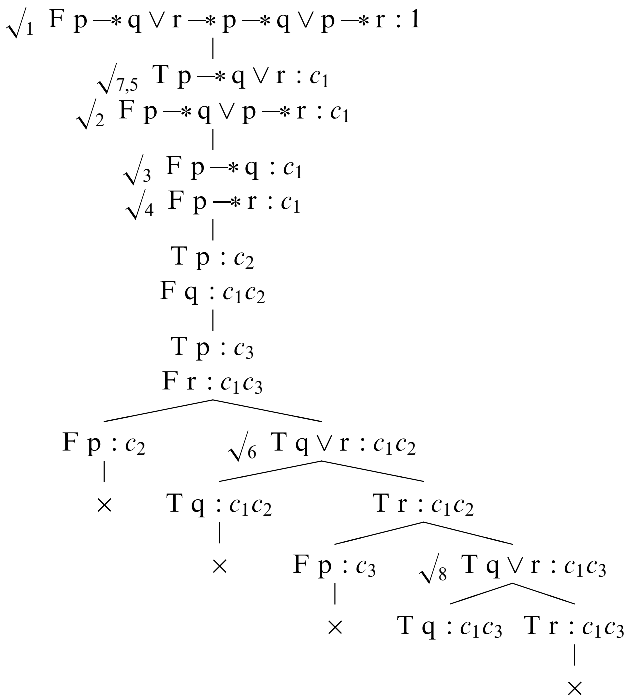


Figure 7. Tableau for $(\mathrm{p} \multimap* (\mathrm{q} \vee \mathrm{r})) \multimap* ((\mathrm{p} \multimap* \mathrm{q}) \vee (\mathrm{p} \multimap* \mathrm{r}))$.

6.1. *The canonical interpretation*

To understand why the liberalised rules for MILL cannot be extended to TBI, we must recall their justification in the case of MILL, that is, the completeness of the logic with respect to *regular* Kripke resource models.

Definition 6.1. Let $(\mathscr{M}, \models, [\![-]\!])$ be a Kripke resource model. A world m is ϕ-*characteristic* for $\models$ in $\mathscr{M}$ if $m \models \phi$ and, for any world n such that $n \models \phi$, $m \sqsubseteq n$. The forcing relation $\models$ is *regular* on $\mathscr{M}$ if whenever there exists a world m such that $m \models \phi$, there also exists a ϕ-characteristic world. Finally, we say that $(\mathscr{M}, \models, [\![-]\!])$ is *regular* if its forcing relation $\models$ is regular on $\mathscr{M}$.

Theorem 6.1. BI is not complete with respect to regular Kripke resource models.

Proof. We prove that there exists no regular Kripke resource countermodel for the sequent $\mathrm{p} \multimap* (\mathrm{q} \vee \mathrm{r}) \vdash (\mathrm{p} \multimap* \mathrm{q}) \vee (\mathrm{p} \multimap* \mathrm{r})$. We suppose there is one such countermodel $(\mathscr{M}, \models, [\![-]\!])$, then there is a world m such that:

(1) $m \models \mathrm{p} \multimap* (\mathrm{q} \vee \mathrm{r})$; and
(2) $m \not\models (\mathrm{p} \multimap* \mathrm{q}) \vee (\mathrm{p} \multimap* \mathrm{r})$.

As (2) implies $m \not\models \mathrm{p} \multimap* \mathrm{q}$, there is a world n such that $n \models \mathrm{p}$ and $m \bullet n \not\models \mathrm{q}$. Since $\mathscr{K}$ is regular, $n \models \mathrm{p}$ implies that there exists a ϕ-characteristic world c_ϕ. Then $c_\phi \models \phi$ and $c_\phi \sqsubseteq n$, which, by order preservation, yields $m \bullet c_\phi \sqsubseteq m \bullet n$. It then follows from

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Kripke monotonicity and $m \bullet n \not\models \mathrm{q}$ that $m \bullet c_\phi \not\models \mathrm{q}$. As (2) also implies $m \not\models \mathrm{p} \mathrel{-\!\!*} \mathrm{r}$, we can similarly prove that $m \bullet c_\phi \not\models \mathrm{r}$. Therefore, $m \bullet c_\phi \not\models \mathrm{q} \vee \mathrm{r}$, and, since $c_\phi \models \phi$, we have shown $m \not\models \mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r})$, which contradicts (1). Thus, there exists no regular Kripke resource countermodel for the sequent $\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r}) \vdash (\mathrm{p} \mathrel{-\!\!*} \mathrm{q}) \vee (\mathrm{p} \mathrel{-\!\!*} \mathrm{r})$. □

From the previous result, we observe that the presence of $\vee$ leads to the incompleteness of BI with respect to regular Kripke resource models. In D'Agostino and Gabbay (1994), in which the multiplicative fragment ($\otimes$, $\multimap$, I) is considered, the regularity property arises from what is called the *canonical interpretation*. The canonical interpretation is a term model in which a world, also called an *information token*, is equivalent to the set of propositions it verifies. More precisely, information tokens are sets of propositions closed under deduction and partially ordered by set-inclusion. The forcing relation $x \models \phi$ between information tokens and propositions is given by $\phi \in x$. It follows immediately that $x \models \phi$ and $y \models \phi$ imply $x \cap y \models \phi$, in other words, satisfaction is preserved under arbitrary intersections. This, in turn, entails the existence of the *least* token that satisfies ϕ whenever there exists some token that satisfies ϕ. This least token intuitively corresponds to the computational content of ϕ, that is, the set $\{\, \psi \mid \phi \vdash \psi \,\}$.

Definition 6.2. Let $\mathscr{P}(L)$ denote the collection of BI propositions over a language L of propositional letters. The mapping $\|-\|$, from $\mathscr{P}(L)$ to sets of propositions, is defined by $\|\varphi\| = \{\, \chi \mid \varphi \vdash_{\mathsf{LBI}} \chi \,\}$ for $\varphi \equiv \phi, I, \top, \bot$.

The *canonical interpretation* for BI is then given by the carrier set $H \equiv \{\, \|\phi\| \mid \phi \in \mathscr{P}(L) \,\}$, preordered by $\sqsubseteq$ seen as set-inclusion.

Whenever no confusion may arise, we shall write $\vdash$ instead of $\vdash_{\mathsf{LBI}}$.

Lemma 6.1. $(H, \sqsubseteq, \odot, \|\mathrm{I}\|)$, where $\|\phi\| \odot \|\psi\|$ is defined by $\|\phi * \psi\|$, is a Kripke resource monoid.

Proof. We check that $\odot$ is a monoidal product on H. The identity with respect to $\|\mathrm{I}\|$, associativity and commutativity properties come directly from those of $*$. Moreover, $\odot$ is order-preserving because, in LBI, one can derive $\phi * \psi \vdash \phi' * \psi'$ from the premises $\phi \vdash \phi'$ and $\psi \vdash \psi'$. □

Lemma 6.2. The canonical interpretation has the following properties:

1. $\|\psi\| \sqsubseteq \|\phi\|$ iff $\phi \vdash \psi$.
2. $\|\top\|$ is the least element.
3. $\|\bot\|$ is the greatest element.
4. $\|\phi \vee \psi\| = \|\phi\| \cap \|\psi\|$.

Proof.

1. Suppose $\|\psi\| \sqsubseteq \|\phi\|$. Since $\psi \in \|\psi\|$, we have $\psi \in \|\phi\|$ and then, by definition, $\phi \vdash \psi$. We now suppose $\phi \vdash \psi$. Then, if $\chi \in \|\psi\|$, by definition, $\psi \vdash \chi$. Using cut with $\phi \vdash \psi$ yields $\phi \vdash \chi$, that is, $\chi \in \|\phi\|$.
2. This is immediate from (1) because, for any ϕ, $\phi \vdash \top$.
3. This is immediate from (1) because, for any ϕ, $\bot \vdash \phi$.

4. We successively prove $\|\phi \vee \psi\| \sqsubseteq \|\phi\| \cap \|\psi\|$ and $\|\phi\| \cap \|\psi\| \sqsubseteq \|\phi \vee \psi\|$.
   First, since $\phi \vdash \phi \vee \psi$ and $\psi \vdash \phi \vee \psi$ in LBI, we have $\|\phi \vee \psi\| \sqsubseteq \|\phi\|$ and $\|\phi \vee \psi\| \sqsubseteq \|\psi\|$, which implies $\|\phi \vee \psi\| \sqsubseteq \|\phi\| \cap \|\psi\|$.
   Second, suppose $\chi \in \|\phi\| \cap \|\psi\|$. Then, by the definition of $\|-\|$, we have $\phi \vdash \chi$ and $\psi \vdash \chi$, which, by $\vee_L$ of LBI, implies $\phi \vee \psi \vdash \chi$. Thus, $\chi \in \|\phi \vee \psi\|$, so $\|\phi\| \cap \|\psi\| \sqsubseteq \|\phi \vee \psi\|$. □

Corollary 6.1. $(H, \sqsubseteq, \oslash, \|\bot\|, \|\top\|)$, where $\|\phi\| \oslash \|\psi\|$ is defined by $\|\phi \vee \psi\|$, is a (complete) inf-semi-lattice with $\|\bot\|$ as the greatest element and $\|\top\|$ as the least element.

Proof. The proof is immediate since property (4) of Lemma 6.2 implies that $\|\phi \vee \psi\|$ is the greatest lower bound of ϕ and ψ. □

Although the canonical interpretation is closed under intersections, it is not closed under unions. Indeed, for any two atomic propositions p and q, we have $\mathrm{p}, \mathrm{q} \in \|\mathrm{p}\| \cup \|\mathrm{q}\|$. Since neither $\mathrm{p} \vdash \mathrm{p} \wedge \mathrm{q}$ nor $\mathrm{q} \vdash \mathrm{p} \wedge \mathrm{q}$ hold in LBI, it follows that $\mathrm{p} \wedge \mathrm{q} \notin \|\mathrm{p}\| \cup \|\mathrm{q}\|$. But suppose now that $\|\mathrm{p}\| \cup \|\mathrm{q}\| = \|\chi\|$ for some χ. Then $\chi \vdash \mathrm{p}$ and $\chi \vdash \mathrm{q}$ imply $\chi \vdash \mathrm{p} \wedge \mathrm{q}$, that is, $\mathrm{p} \wedge \mathrm{q} \in \|\mathrm{p}\| \cup \|\mathrm{q}\|$, which is a contradiction. Nevertheless, since the canonical interpretation is a complete inf-semi-lattice, it can still be embedded into a complete lattice by defining the least upper bound $\|\phi\| \oslash \|\psi\|$ of $\|\phi\|$ and $\|\psi\|$ as $\oslash\{\, \|\chi\| \mid \|\phi\| \sqsubseteq \|\chi\| \text{ and } \|\psi\| \sqsubseteq \|\chi\| \,\}$, which is easily seen to be equivalent to $\|\phi \wedge \psi\|$.

Theorem 6.2. Let $\mathscr{H}$ be the structure $(H, \sqsubseteq, \odot, \|\mathrm{I}\|, \oslash, \|\bot\|, \oslash, \|\top\|)$. Then $\mathscr{H}$ is a BI-algebra, that is, a Heyting algebra equipped with an additional residuated commutative monoid structure.

Proof. From Lemma 6.2 we know that $\mathscr{H}$ is a complete lattice with least and greatest elements and, from Lemma 6.1, that it is also a Kripke resource monoid. Therefore, we only need to show that the lattice and the monoidal part of $\mathscr{H}$ are residuated.
We prove $\|\phi \multimap\!\!* \psi\| = \oslash\{\, \|\chi\| \mid \|\psi\| \sqsubseteq \|\phi * \chi\| \,\}$ first. First, for any $\|\chi\|$, $\|\psi\| \sqsubseteq \|\phi * \chi\|$ iff $\phi * \chi \vdash \psi$ iff $\chi \vdash \phi \multimap\!\!* \psi$ iff $\|\phi \multimap\!\!* \psi\| \sqsubseteq \|\chi\|$. Second, $\|\phi \multimap\!\!* \psi\|$ belongs to the set because $\phi * (\phi \multimap\!\!* \psi) \vdash \psi$ entails $\|\psi\| \sqsubseteq \|\phi * (\phi \multimap\!\!* \psi)\|$. Therefore, $\|\phi \multimap\!\!* \psi\|$ is the least upper bound.
Then, $\|\phi \to \psi\| = \oslash\{\, \|\chi\| \mid \|\psi\| \sqsubseteq \|\phi \wedge \chi\| \,\}$ can be proved similarly. □

From now on, we shall refer to the canonical interpretation as the structure $\mathscr{H}$ given in Theorem 6.2. Following the work in D'Agostino and Gabbay (1994) for the fragment $(\otimes, \multimap, \mathrm{I})$, we equip the canonical interpretation with a satisfaction relation between information tokens and propositions as follows

Definition 6.3. The canonical forcing relation $\models$ is defined for an information token $\|\chi\|$ and a BI proposition ϕ by $\|\chi\| \models \phi$ iff $\phi \in \|\chi\|$, that is, iff $\chi \vdash \phi$.

Lemma 6.3. For any proposition ϕ, $\|\phi\|$ is ϕ-characteristic for $\models$ in $\mathscr{H}$.

Proof. First, we have $\|\phi\| \models \phi$ because $\phi \vdash \phi$ obviously holds in LBI. Then, if $\|\chi\|$ is such that $\|\chi\| \models \phi$, then, by definition, $\chi \vdash \phi$, which implies $\|\phi\| \sqsubseteq \|\chi\|$. □

Corollary 6.2. The canonical forcing relation $\models$ is regular on $\mathscr{H}$.

Proof. If any formula has a characteristic world in $\mathscr{H}$, the regularity condition is satisfied. □

6.2. *A special treatment of the disjunction*

In this subsection, we observe that the canonical forcing relation does not satisfy the usual clause for disjunction, that is, $\|\chi\| \models \phi \vee \psi$ iff $\|\chi\| \models \phi$ or $\|\chi\| \models \psi$. Indeed, this would require that $\chi \vdash \phi \vee \psi$ iff $\chi \vdash \phi$ or $\chi \vdash \psi$, which does not hold in general as $\mathrm{p} \vee \mathrm{q} \vdash \mathrm{p} \vee \mathrm{q}$ is provable and neither $\mathrm{p} \vee \mathrm{q} \vdash \mathrm{p}$ nor $\mathrm{p} \vee \mathrm{q} \vdash \mathrm{q}$ is provable in LBI. Nevertheless, we remark that the if-direction still holds since it simply corresponds to the pair of $\vee_R$-rules in LBI. So, what semantic clause for $\vee$ does the canonical forcing relation satisfy?

Theorem 6.3. In the canonical interpretation $\mathscr{H}$, $\|\chi\| \models \phi \vee \psi$ if and only if there exist $\|\varphi\|$,$\|\varphi'\|$ such that $\|\varphi\| \oslash \|\varphi'\| \subseteq \|\chi\|$ and $\|\varphi\| \models \phi$ and $\|\varphi'\| \models \psi$.

Proof. For the *if* direction, if $\|\chi\| \models \phi \vee \psi$, then, by definition, $\chi \vdash \phi \vee \psi$, which implies $\|\phi \vee \psi\| \subseteq \|\chi\|$. The result follows immediately since $\|\phi \vee \psi\| = \|\phi\| \oslash \|\psi\|$, $\|\phi\| \models \phi$ and $\|\psi\| \models \psi$.

For the *only if* direction, we have, on the one hand, $\varphi \vdash \phi$ and $\varphi' \vdash \psi$ because, by hypothesis, $\|\varphi\| \models \phi$ and $\|\varphi'\| \models \psi$. On the other hand, since $\|\varphi \vee \varphi'\| = \|\varphi\| \oslash \|\varphi'\|$, we have $\|\varphi\| \oslash \|\varphi'\| \subseteq \|\chi\|$ yields $\chi \vdash \varphi \vee \varphi'$. The result is obtained from the following derivation in LBI:

$$\dfrac{\chi \vdash \varphi \vee \varphi' \qquad \dfrac{\dfrac{\varphi' \vdash \psi}{\varphi' \vdash \phi \vee \psi}\vee_R \qquad \dfrac{\varphi \vdash \phi}{\varphi \vdash \phi \vee \psi}\vee_R}{\varphi \vee \varphi' \vdash \phi \vee \psi}\vee_L}{\chi \vdash \phi \vee \psi}\,Cut.$$

□

As Theorem 6.3 shows, the semantics of disjunction in the canonical interpretation is unusual for a Kripke semantics. A special treatment of the disjunction, arising from considerations in BI-algebras, is needed to make the liberalised rules work. The topological Kripke semantics, introduced in Pym (2002), Pym *et al.* (2004) and Pym (2004), which are summarised in Section 2, allows, as for BI-algebras, $\perp$ to be taken into account together with a non-indecomposable treatment of the disjunction. Note that the clause for $\vee$ is very similar to the one given in Theorem 6.3. Indeed, they are just dual since the topological semantics considers open sets, while the canonical interpretation considers sets that are closed under deduction. Therefore, the translation from one to the other is simply obtained by taking the complement and TBI′ appears as the syntactic reflection of the forcing semantics in the category of sheaves over a topological monoid.

We shall see that the dependency graphs, which are defined in the case of liberalised rules, may be viewed as (partial) topological Kripke models.

7. Liberalised resource tableaux

In this section, we give new expansion rules for TBI. The resulting tableau system is called TBI′. In our previous discussions we defined a canonical interpretation of BI and also showed that a new semantic clause for the additive disjunction is required to achieve a suitable canonical forcing relation. However, the semantic changes made to $\vee$ have a syntactic counterpart and the corresponding initial expansion rules have to be modified accordingly. For that, we make several extensions to the initial framework.

7.1. *An extended labelling algebra*

Definition 7.1. We enrich the labelling language given in Definition 3.1 with a new unit symbol t and a new binary function symbol $\sqcap$. Therefore, compound labels become expressions of the form $x \circ y$ or $x \sqcap y$ in which x and y are labels. We say that x is a *sublabel* of y (notation, $x \preceq y$), if there exists a label z such that $y = x \circ z$ or $y = x \sqcap z$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. $\mathscr{S}(x)$ denotes the set of the sublabels of x. All other definitions remain unchanged.

Definition 7.2. Labels and constraints are interpreted in a *labelling algebra* $\mathscr{L} = (L, \leqslant, \circ, 1, \sqcap, t)$ as follows:

1. L is a set of labels.
2. $\leqslant$ is a preordering.
3. Equality on labels is defined by $x = y$ iff $x \leqslant y$ and $y \leqslant x$.
4. $(L, \leqslant, \circ, 1)$ is an order preserving commutative monoid, that is, $\circ$ satisfies:
   - associativity: $(x \circ y) \circ z = x \circ (y \circ z)$;
   - commutativity: $x \circ y = y \circ x$;
   - identity: $x \circ 1 = 1 \circ x = x$; and
   - bifunctoriality: $x \leqslant y$ implies $x \circ z \leqslant y \circ z$.
5. $(L, \leqslant, \sqcap, t)$ is a distributive complete semi-lattice, that is, $\sqcap$ satisfies
   - associativity: $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$;
   - commutativity: $x \sqcap y = y \sqcap x$;
   - identity: $x \sqcap t = t \sqcap x = x$;
   - bifunctoriality: $x \leqslant y$ implies $x \sqcap z \leqslant y \sqcap z$;
   - contraction: $x \leqslant x \sqcap x$;
   - weakening: $x \sqcap y \leqslant x$; and
   - distributivity: $(x \circ y) \sqcap (x \circ z) \leqslant x \circ (y \sqcap z)$.

Lemma 7.1. The labelling algebra $\mathscr{L} = (L, \leqslant, \circ, 1, \sqcap, t)$ satisfies the following properties:

(1) $x \circ (y \sqcap z) \leqslant (x \circ y) \sqcap (x \circ z)$.
(2) $x \leqslant t$.
(3) $t \leqslant x \circ t$.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Proof.

(1) By the weakening axiom, we have both $y \sqcap z \leqslant y$ and $y \sqcap z \leqslant z$. Thus, by bifunctoriality of $\circ$, we can derive $x \circ (y \sqcap z) \leqslant x \circ y$ and $x \circ (y \sqcap z) \leqslant x \circ z$. The bifunctoriality of $\sqcap$ then entails $(x \circ (y \sqcap z)) \sqcap (x \circ (y \sqcap z)) \leqslant (x \circ y) \sqcap (x \circ z)$, from which the result immediately follows using the contraction axiom.

(2) The weakening axiom implies $x \sqcap t \leqslant t$, which, by identity, gives $x \leqslant t$.

(3) Property (2) yields $\sqcap\{\, y \mid t \leqslant x \circ y \,\} \leqslant t$. Therefore, $x \circ \sqcap\{\, y \mid t \leqslant x \circ y \,\} \leqslant x \circ t$ follows by bifunctoriality. Distributivity then gives $\sqcap\{\, x \circ y \mid t \leqslant x \circ y \,\} \leqslant x \circ t$. Therefore, since $\sqcap\{\, x \circ y \mid t \leqslant x \circ y \,\} = t$, we finally get $t \leqslant x \circ t$. $\square$

Notice that property (1), together with the distributivity axiom, imply that $\circ$ is 'fully' distributive over $\sqcap$. Property (2) simply means that t is the greatest element in the labelling algebra and implies, with property (3), that t absorbs any other label in a multiplication, that is, $x \circ t = t$ for any x.

Definition 7.3. The *closure* $\overline{K}$ of a set K of label constraints is extended as follows:

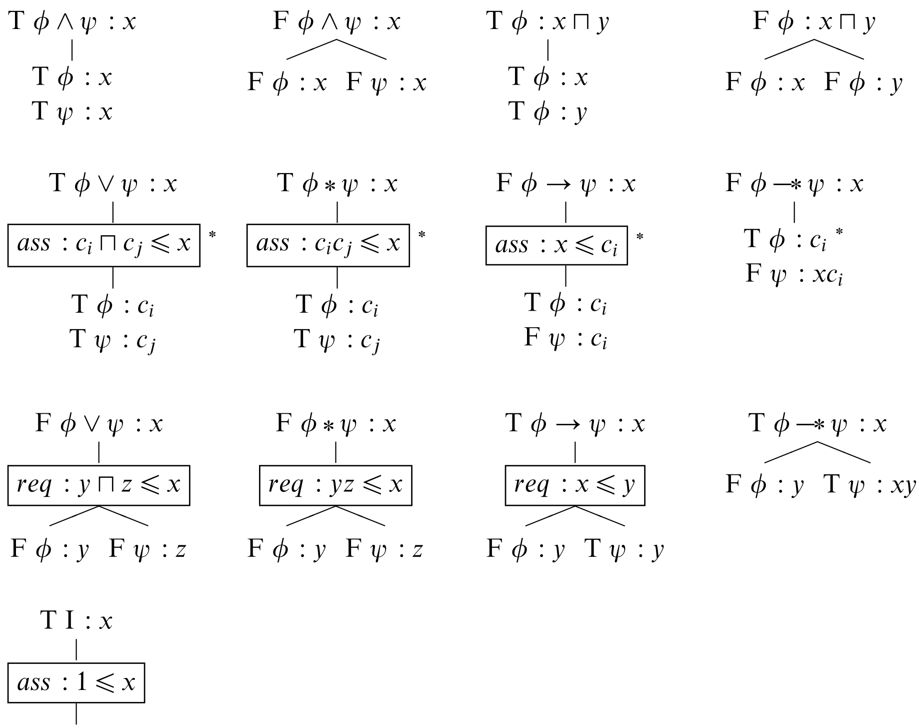
1. $K \subseteq \overline{K}$.
2. If $x \in \mathscr{D}(\overline{K})$, then $x \leqslant x \in \overline{K}$ (reflexivity).
3. If $x \leqslant y \in \overline{K}$ and $y \leqslant z \in \overline{K}$, then $x \leqslant z \in \overline{K}$ (transitivity).
4. If $y \circ z \in \mathscr{D}(\overline{K})$, then $x \leqslant y \in \overline{K}$ implies $x \circ z \leqslant y \circ z \in \overline{K}$ ($\circ$-compatibility).
5. If $y \sqcap z \in \mathscr{D}(\overline{K})$, then $x \leqslant y \in \overline{K}$ implies $x \sqcap z \leqslant y \sqcap z \in \overline{K}$ ($\sqcap$-compatibility).
6. If $x \sqcap y \in \mathscr{D}(\overline{K})$, then $x \sqcap y \leqslant x$ and $x \sqcap y \leqslant y \in \overline{K}$ (weakening).
7. If $(x \circ y) \sqcap (x \circ z)$ or $x \circ (y \sqcap z) \in \mathscr{D}(\overline{K})$, then $(x \circ y) \sqcap (x \circ z) = x \circ (y \sqcap z) \in \overline{K}$ (distributivity).

7.2. *Liberalised rules*

We now give, in Figure 8, the expansion rules of TBI', which modifies TBI in order to reflect the semantic changes previously made. Compared with the initial labelled system, we notice the presence of two new rules, namely, $T\sqcap$ and $F\sqcap$. These rules are *structural* since they only operate on the label of their signed formula without decomposing the formula itself. Their computational contents simply reflect the fact that $\sqcap$ corresponds to an intersection under the canonical interpretation. Then we note that the $T\vee$ rule has been modified and is now a $\pi\alpha$ rule, introducing two new constants, c_i, c_j, and a new assertion $c_i \sqcap c_j \leqslant x$. The $F\vee$ rule, on the other hand, becomes a $\pi\beta$ rule reusing two labels, y, z, such that the requirement $y \sqcap z \leqslant x$ is satisfied by the closure of the assertions. The new rules for $\vee$ are justified by Theorem 6.3 – they are the syntactic counterparts. The usual version of the $F\vee$ is admissible in TBI', so one could also use the following pair of rules for the disjunction:

$$
\begin{array}{c}
\mathrm{T}\ \phi \vee \psi : x \\
| \\
\boxed{ass : c_i \sqcap c_j \leqslant x} \\
| \\
\mathrm{T}\ \phi : c_i \\
\mathrm{T}\ \psi : c_j
\end{array}
\qquad\qquad
\begin{array}{c}
\mathrm{F}\ \phi \vee \psi : x \\
| \\
\mathrm{F}\ \phi : x \\
\mathrm{F}\ \psi : x
\end{array}
$$

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press



* c_i, c_j are new constants

Figure 8. TBI′ Expansion Rules.

The initial pair of rules leads to a nicely symmetric treatment of the disjunction, but has the drawback of solving requirements of the form $y \sqcap z \leqslant x$, which can be difficult. Moreover, it involves more splitting of the tableau branches, which can significantly increase the size of a tableaux proof. In the presence of explicit structural rules (Kripke monotonicity and $F\sqcap$), the two versions of the $F\vee$ are easily proved to be equivalent. Therefore, one can arbitrarily use either version in a tableaux proof.

Definition 7.4. A constant a is ϕ-*characteristic* in a tableau branch $\mathscr{B}$ if it appeared, for the first time, in a formula T $\phi : a$ that was introduced by a F $\mathrel{-\!\!*}$, T $*$ or T$\vee$ expansion.

We note that the constant c_i introduced by the F $\phi \rightarrow \psi : x$ rule is not ϕ-characteristic. This observation is semantically justified by the fact that, even if having c_i such that $x \sqsubseteq c_i$, $c_i \models \phi$ and $c_i \not\models \psi$ implies that there exists a ϕ-characteristic a, that is, an a such that $a \models \phi$ and $a \sqsubseteq c_i$, this does not necessarily imply that $x \sqsubseteq a$. However, the previous discussion shows that Definition 7.4 can still be extended to cover the case of the $F \rightarrow$ rule (and so, cover all the $\pi\alpha$ rules), by modifying it as prescribed by the fourth case of the following lemma.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Lemma 7.2. Let $\mathscr{B}$ be a tableau branch. The liberalised rules

$$\begin{array}{ccccc}
\mathrm{T}\ \phi * \psi : x^{1} & \mathrm{F}\ \phi -\!\!* \psi : x^{1} & \mathrm{T}\ \phi \vee \psi : x^{1} & \mathrm{F}\ \phi \to \psi : x^{1} & \mathrm{F}\ \phi \to \psi : x^{2} \\
| & | & | & | & | \\
\boxed{ass : ab \leqslant x} & \mathrm{T}\ \phi : a & \boxed{ass : a \sqcap b \leqslant x} & \boxed{ass : a, x \leqslant c_i} & \mathrm{T}\ \phi : x \\
| & \mathrm{F}\ \psi : xa & | & | & \mathrm{F}\ \psi : x \\
\mathrm{T}\ \phi : a & & \mathrm{T}\ \phi : a & \mathrm{T}\ \phi : a & \\
\mathrm{T}\ \psi : b & & \mathrm{T}\ \psi : b & \mathrm{F}\ \psi : c_i &
\end{array}$$

where a (b) need not be new, are admissible in TBI' provided:

1. a (b) is ϕ-characteristic (ψ-characteristic) in $\mathscr{B}$; and
2. there exists T $\phi : y$ in $\mathscr{B}$ such that $y \leqslant x \in \overline{Ass}(\mathscr{B})$.

Proof. Let $\mathscr{K}$ be a regular Kripke resource model $(\mathscr{M}, \models, [\![-]\!])$. We only prove the result for $F -\!\!*$, the other case being similar. The soundness of the liberalised $F -\!\!*$ is justified by the following semantic equivalence: $x \not\models \phi -\!\!* \psi$ iff there exists a ϕ-characteristic world a such that $x \bullet a \not\models \psi$. The *if* direction is obvious. For the *only if* direction, suppose that $x \not\models \phi -\!\!* \psi$. Then, there exists y such that $y \models \phi$ and $x \bullet y \not\models \psi$. Since $y \models \phi$, the regularity of $\models$ implies that there exists a ϕ-characteristic world a, which implies $a \sqsubseteq y$ and, by order preservation, $x \bullet a \sqsubseteq x \bullet y$. Kripke monotonicity finally entails $x \bullet a \not\models \psi$.

Now, suppose that we have a realisation $\|-\|$ of $\mathscr{B}$ in $\mathscr{K}$ with T $\phi : a$ and F $\phi -\!\!* \psi : x$ in $\mathscr{B}$. Then, $\|a\| \models \phi$ and $\|x\| \not\models \phi -\!\!* \psi$. Since $\|a\|$ is assumed to be ϕ-characteristic, it follows from $\|x\| \not\models \phi -\!\!* \psi$ and our previous discussion that $\|x\| \bullet \|a\| (= \|xa\|) \not\models \psi$, which means that the expansion of a liberalised $F -\!\!*$ preserves realisability. □

7.3. *Proof and countermodel construction*

In this section, we illustrate how TBI' works with some examples. The first example, *q.v.* Figure 9, shows two closed tableaux for $((\mathrm{p} -\!\!* \mathrm{q}) \vee (\mathrm{p} -\!\!* \mathrm{r})) -\!\!* (\mathrm{p} -\!\!* (\mathrm{q} \vee \mathrm{r}))$, which therefore holds in BI. The first tableau is obtained using the $\pi\beta$ version of the $F\vee$ rule, while the second is obtained with the usual α version. As one can see, for both tableaux, Step 2 deals with the new $T\vee$ rule and extends the branch with T $\mathrm{p} -\!\!* \mathrm{q} : c_2$ and T $\mathrm{p} -\!\!* \mathrm{r} : c_3$, introducing, by the way, two new constants c_2 and c_3 and a new assertion $c_2 \sqcap c_3 \leqslant c_1$. Step 4, which shows the correspondence between the two $F\vee$ rules, is more interesting. With the $\pi\beta$ version, we are required to find two labels x and y that verify the constraint $x \sqcap y \leqslant c_1 c_4$. Taking $x = c_2 c_4$ and $y = c_3 c_4$ is a suitable choice since, by compatibility on assertion $c_2 \sqcap c_3 \leqslant c_1$, we can deduce $(c_2 \sqcap c_3) c_4 \leqslant c_1 c_4$, which, by distributivity, gives $(c_2 c_4) \sqcap (c_3 c_4) \leqslant c_1 c_4$.

If we use the α version, Step 4 in the left-hand tableau is simulated by Steps 4, 4′ and 4″ in the right-hand one. Step 4 first expands F $\mathrm{q} \vee \mathrm{r} : c_1 c_4$ into F $\mathrm{q} \vee \mathrm{r} : (c_2 c_4) \sqcap (c_3 c_4)$ using Kripke monotonicity because, as previously explained, $(c_2 c_4) \sqcap (c_3 c_4) \leqslant c_1 c_4$ holds in the labelling algebra with respect to the assertions. Step 4′ then splits the tableau into two branches by decomposing F $\mathrm{q} \vee \mathrm{r} : (c_2 c_4) \sqcap (c_3 c_4)$ with the structural $F\sqcap$ rule. Finally, Step 4″ reapplies the $F\vee$ rule on both $\mathrm{q} \vee \mathrm{r}$ resulting from the previous step. Note that all other steps are exactly the same for both tableaux. Moreover, such a simulation of one $F\vee$ version into the other can always be performed, thus proving their equivalence.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

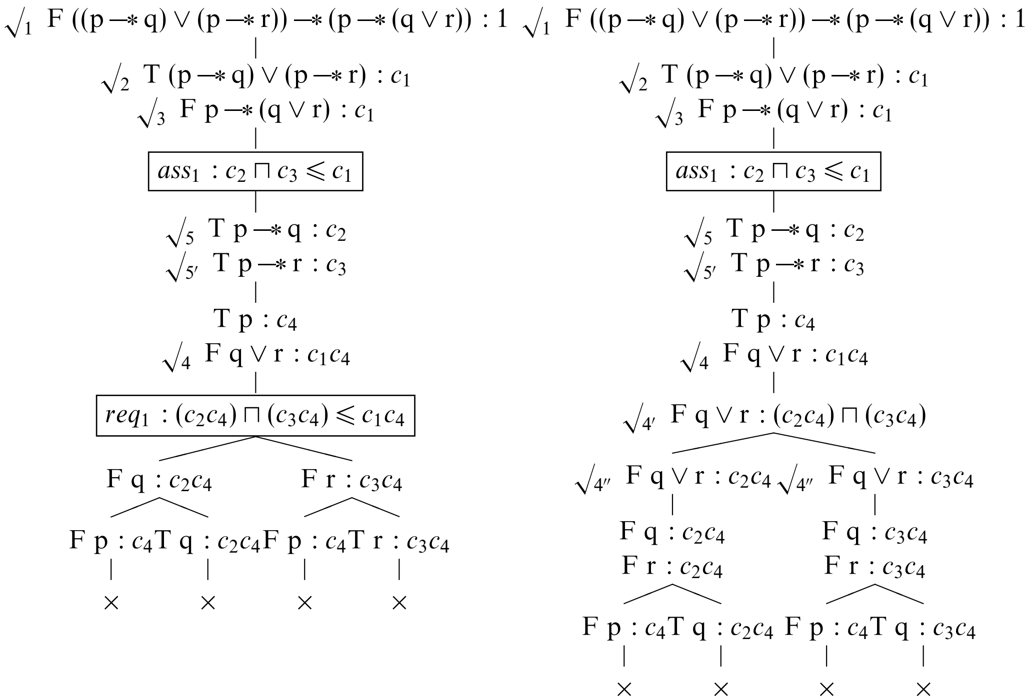


Figure 9. Two different tableaux for $((p \multimap\!\!* \, q) \vee (p \multimap\!\!* \, r)) \multimap\!\!* \, (p \multimap\!\!* \, (q \vee r))$.

For the second example, see Figure 10, the tableaux construction procedure ends up with an open branch. Therefore, the formula $(p \multimap\!\!* \, (q \vee r)) \multimap\!\!* \, ((p \multimap\!\!* \, q) \vee (p \multimap\!\!* \, r))$ does not hold in BI. Notice that, since the new version of the $T\vee$ rule introduces distinct constants for q and r when expanding $\mathrm{T}\ q \vee r : c_1c_2$ instead of propagating the c_1c_2, the fact that we reuse c_2 in Step 4 does not lead to a closed tableau as it did for the introductory example of Figure 7.

It is routine to extend the notions of fulfilled formula, completed branch, complete branch and dependency graph to cover the introduction of $\sqcap$ in the labels. The model existence theorem for TBI′ then follows immediately.

Theorem 7.1. A complete branch $\mathscr{B}$ has an (algebraic) Kripke resource model.

Proof. The problem is to embed the dependency graph of a given complete branch into a BI-algebra. The main difficulty is that $\circ$ and $\sqcap$ are only partial operations on the dependency graph. So we must complete $\circ$ and $\sqcap$ with suitable values to obtain satisfactory monoidal product and lattice meet operators. For the monoidal part, we follow the ideas of Section 4.2 and add a greatest element π, to which all undefined products are mapped. Similarly, for the lattice part, we add a least element ω to which all undefined meets are mapped (other completions, such as the Mac Neille completion, could also be used). The result then becomes a straightforward adaptation of the proof of Theorem 4.2. □

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

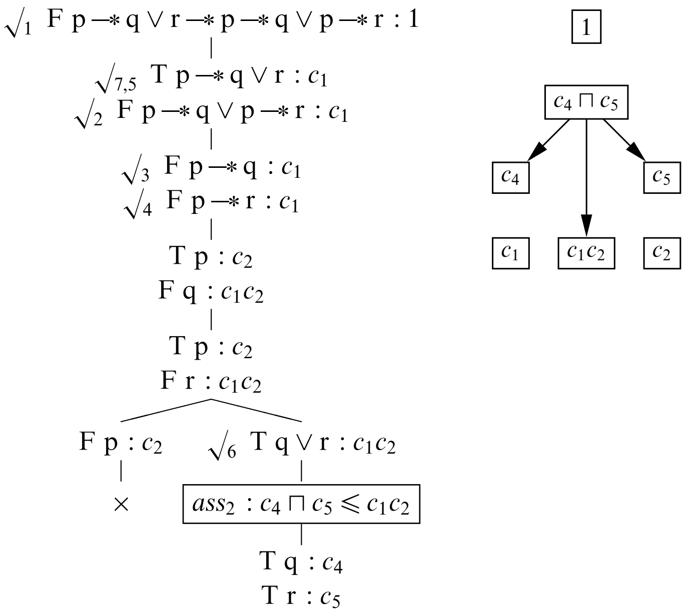


Figure 10. Tableau for $(\mathrm{p} \mathrel{-\!\!*} (\mathrm{q} \vee \mathrm{r})) \mathrel{-\!\!*} ((\mathrm{p} \mathrel{-\!\!*} \mathrm{q}) \vee (\mathrm{p} \mathrel{-\!\!*} \mathrm{r}))$.

7.4. *Properties of* TBI′

In this subsection, we prove the soundness and completeness of TBI′. For the soundness we show that each rule of TBI′ preserves realisability under the canonical interpretation.

Theorem 7.2 (Soundness of TBI′). Let ϕ be a BI proposition. If there exists a closed tableaux sequence for ϕ in TBI′, then ϕ is a theorem of LBI.

Proof. It is easy to prove that each rule of TBI′ is sound under the canonical interpretation. We do only a few cases, knowing that the others are similar.

— Case $\mathrm{T}\ \phi \vee \psi : x$.
Suppose $\|\chi\| \models \phi \vee \psi$ for some $\|\chi\|$. Then, $\|\phi \vee \psi\| \sqsubseteq \|\chi\|$. The result immediately follows since $\|\phi \vee \psi\| = \|\phi\| \oslash \|\psi\|$, $\|\phi\| \models \phi$ and $\|\psi\| \models \psi$.

— Case $\mathrm{F}\ \phi \vee \psi : x$.
Suppose $\|\chi\| \not\models \phi \vee \psi$ for some $\|\chi\|$. Then, $\|\chi\| \not\models \phi$ since, because $\phi \vdash \phi \vee \psi$, we should have $\|\chi\| \models \phi \vee \psi$. We prove $\|\chi\| \not\models \psi$ analogously.

Now, to establish the soundness of TBI′, suppose that there exists a closed tableaux sequence for ϕ. Since all rules are sound under the canonical interpretation, the initial signed formula $\mathrm{F}\ \phi : 1$ is not realisable in $\mathscr{H}$. In other words, $\|\mathrm{I}\| \not\models \phi$ is impossible. Therefore, $\|\mathrm{I}\| \models \phi$, which implies $\mathrm{I} \vdash \phi$. □

Theorem 7.3 (Completeness of TBI′). Let ϕ be a proposition. If ϕ is provable in LBI, there exists a closed tableaux sequence for ϕ in TBI′.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Proof. The completeness follows from Theorem 7.1 and the fact that the tableau construction procedure builds a tableau that is either closed or with a complete branch.

Another way to show completeness is to prove that TBI' is closed under each LBI rule, as explained in D'Agostino and Gabbay (1994). For that, we define a transformation Γ^t on a bunch Γ by replacing each proposition ϕ by a ϕ-characteristic label c_ϕ, each ',' by $\circ$ and sequences of the form $\phi_1;\phi_2;\ldots;\phi_n$ by a label $\sqcup(c_{\phi_1}, c_{\phi_2}, \ldots, c_{\phi_n})$. For example, $\phi,(\psi;\phi';\psi' \to \phi)^t = c_\phi \circ (\sqcup(c_\psi, c_{\phi'}, c_{\psi'\to\phi}))$. We write $\sqcup(c_\phi, c_\psi)$ to mean that such a label is assumed to be the least upper bound of c_ϕ and c_ψ, and, therefore, the implicit assertion $c_\phi, c_\psi \leqslant \sqcup(c_\phi, c_\psi)$ is also assumed. Then, a sequent $\Gamma \vdash \psi$, where Γ is made up of a set of propositions ϕ_i, is provable in TBI' if the following tree is closed:

$$\begin{array}{c} \mathrm{T}\ \phi_i : c_{\phi_i} \\ \mathrm{F}\ \psi : \Gamma^t \\ | \\ \mathscr{T} \\ | \\ \times \end{array}$$

Now we prove the result for the $\to_R$ rule of LBI, the others being similar. For that, we show that if $\Gamma;\phi \vdash \psi$ is provable in TBI', then, so is $\Gamma \vdash \phi \to \psi$. In other words, what we show, in the following figure, is that the tree corresponding to $\Gamma \vdash \phi \to \psi$ (on the right-hand side) can be closed if we assume that the one for $\Gamma;\phi \vdash \psi$ (on the left-hand side) is closed. We write $\mathscr{T}[x/y]$ to say that all occurrences of the label x in $\mathscr{T}$ are replaced by the label y.

$$\begin{array}{c} \mathrm{T}\ \phi_i : c_{\phi_i} \\ \mathrm{T}\ \phi : c_\phi \\ | \\ \boxed{ass : \Gamma^t, c_\phi \leqslant \sqcup\Gamma^t, c_\phi} \\ | \\ \mathrm{F}\ \psi : \sqcup\Gamma^t, c_\phi \\ | \\ \mathscr{T} \\ | \\ \times \end{array} \qquad\qquad \begin{array}{c} \mathrm{T}\ \phi_i : c_{\phi_i} \\ \surd_1\ \mathrm{F}\ \phi \to \psi : \Gamma^t \\ | \\ \boxed{ass : \Gamma^t, c_\phi \leqslant b} \\ | \\ \mathrm{T}\ \phi : c_\phi \\ \mathrm{F}\ \psi : b \\ | \\ \mathscr{T}[\sqcup\Gamma^t, c_\phi / b] \\ | \\ \times \end{array}$$

$\square$

In this section, we have presented a liberalised tableau for BI. Liberalised rules are important in practice since they give a way to control the introduction of new constants and, thus, to limit the syntactic complexity of the labels, which, in turn, leads to a more efficient proof-search (Hähnle and Schmitt 1994). However, having such liberalised rules requires a special treatment of the additive disjunction, which forces us to enrich the structure of the labels with a new symbol. For this reason, it becomes more difficult to build countermodels. Moreover, such countermodels are related to BI-algebras, which are themselves closely related to the topological Kripke semantics, of which they may be viewed as an algebraic counterpart.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

8. Decidability and finite model Property for BI

In this section, we discuss the finite model property with respect to topological resource models and, as a consequence, the decidability of BI. For that, we investigate the situations in which a tableau may have infinite branches and thus use two central notions, namely, *liberalised expansion rules* and *branch redundancy*, which have been introduced in the case of BI without $\perp$ (Galmiche and Méry 2003). The former provides a way to control the syntactic complexity of the labels by restricting the introduction of new constants during the tableau expansion. The latter characterises the potential need of an infinite expansion process to achieve a completed branch and can roughly be viewed as a kind of loop-checking.

We summarise the situation. When $\pi\alpha$ formulae are in the scope of $\pi\beta$ formulae, the fulfillment of $\pi\alpha$ formulae requires the introduction of new constants, which may destroy the fulfillment of $\pi\beta$ formulae. The first step towards termination of the complete branch construction process is to make use of the liberalised rules presented in Lemma 7.2. Doing so, only *finitely many distinct constants* can be introduced in a tableau branch $\mathcal{B}$, that is, only finitely many distinct atomic labels. But this is not yet sufficient to prevent branches from growing infinitely, because, even with a finite number of atomic labels, one can still generate an infinite number of labels through composition. Anyway, since there are only *finitely many subformulae* of the initial formula to prove, after a given finite number of expansion steps, any newly introduced signed formula must have already been introduced, up to a fixed number of occurrences of the same constant. Such a situation happens when some formulae of the form F $\phi \mathrel{-\!\!*} \psi : x$ occur in the scope of some formula T $\phi' \mathrel{-\!\!*} \psi' : y$, with y being a sublabel of x. With such expansions, we can have sequences such as F $\phi \mathrel{-\!\!*} \psi : x$, F $\phi \mathrel{-\!\!*} \psi : xc$ (c being the constant introduced by the first expansion), F $\phi \mathrel{-\!\!*} \psi : xcc, \ldots$ in a complete branch. Thus, we have repetitions of the same signed formulae differing from each other only by one occurrence of the constant c, but without any additional computational content allowing the possibility of closing the branch. In order to solve this problem, we need the following notion of *branch redundancy* (Galmiche and Méry 2003).

Definition 8.1 (Redundancy). A complete branch $\mathcal{B}$ is said to be *redundant for* the constant c if there exists $i \geqslant 1$ and $k > i$ such that for any $j \geqslant k$,

1. T $\phi : xc^j \in \mathcal{B}$ implies T $\phi : xc^{j-i} \in \mathcal{B}$;
2. F $\phi : xc^j \in \mathcal{B}$ implies F $\phi : xc^{j-i} \in \mathcal{B}$;
3. $x \leqslant yc^j \in Ass(\mathcal{B})$ implies $x \leqslant yc^{j-i} \in Ass(\mathcal{B})$; and
4. $xc^j \leqslant y \in Ass(\mathcal{B})$ implies $xc^{j-i} \leqslant y \in Ass(\mathcal{B})$.

Moreover, as Definition 8.1 suggests, considering T $\phi : xc^j$ (F $\phi : xc^j$) and T $\phi : xc^{j-i}$ (F $\phi : xc^{j-i}$) as equivalent, and since π captures the inessential parts of the model, the construction explained in Definition 4.6 always results in a finite countermodel.

Theorem 8.1. A completed branch $\mathcal{B}$ has a finite topological resource model.

https://doi.org/10.1017/S0960129505004858 Published online by Cambridge University Press

Proof. The proof is by induction on the number of constants for which $\mathscr{B}$ is redundant. *Base case.* If there is no constant for which $\mathscr{B}$ is redundant, then $\mathscr{B}$ is finite and, by Theorem 7.1, a topological resource model, which, by construction, is obviously finite. *Inductive case.* We assume that the proposition holds for any completed branch that is redundant for less than n constants. Suppose that $\mathscr{B}$ is redundant for n constants. We select one of those constants and denote it c. Then, we add the extra assertions $c^j = c^{j-i}$ for any $j \geqslant k$, where i, j and k refer to Definition 8.1. It is routine to show that the composition modulo rewriting $c^j = c^{j-i}$ preserves the properties of associativity, commutativity and identity with respect to 1. Moreover, properties (3) and (4) of Definition 8.1 ensure that the branch $\mathscr{B}$ remains non-contradictory. Since now we cannot have more than k occurrences of c in the labels, we have treated the redundancy of $\mathscr{B}$ with respect to the constant c, and we apply the induction hypothesis to obtain the result. □

Theorem 8.2 (Finite model property). If $\mathrm{I} \nVdash \phi$, there is a finite topological resource model such that $\mathrm{I} \not\models \phi$.

Proof. If $\mathrm{I} \nVdash \phi$, there is no closed tableau for ϕ. Thus, the tableau construction procedure yields a tableau with a completed branch $\mathscr{B}$. Then, by Theorem 8.1, we can build a finite topological resource model such that $\mathrm{I} \not\models \phi$. □

Hence, we have the following result.

Theorem 8.3 (Decidability). Propositional BI is decidable.

Proof. The tableau construction procedure, which is a semi-decision procedure, can be improved into a decision procedure by taking both the liberalised versions of the expansion rules and the notion of redundancy into account. Since, under the liberalised rules, an open branch cannot grow infinitely without becoming redundant, the termination of the procedure can be enforced by stopping the (potentially infinite) construction of a completed branch as soon as it has been recognised redundant. □

Note that full propositional linear logic, with exponentials, is undecidable even when restricted to the intuitionistic fragment, that the status of **MELL** is unknown, and that neither has the finite model property (Lafont 1997; Lincoln 1995).

By exploiting the capture of the semantics by labels, we have provided a decision procedure for BI that builds countermodels in Grothendieck topological semantics. Their study gives us a better understanding of the semantic information necessary to analyse provability and of the relationships between the elementary and topological settings.

9. Conclusions

Initially, resource tableaux were introduced for BI without $\perp$, with labels and constraints that directly capture the Kripke resource semantics, which is complete for this logical fragment (Galmiche and Méry 2003). This paper has presented new results for full propositional BI (with $\perp$), some of which were partially presented in Galmiche *et al.* (2002). In this context, a first non-trivial problem was: is it possible to define resource tableaux for propositional BI (with $\perp$) and then to capture the Grothendieck topological semantics that is complete for BI? In this paper we have provided a simple solution based on the existing resource tableaux, without introducing new expansion rules, but with a particular closure condition, expressed through labels, to deal with $\perp$.

We have proved the soundness and completeness of the resource tableaux method with respect to the Grothendieck topological semantics, and, as consequences, we have deduced two strong new results for BI: the decidability of propositional BI and the finite model property with respect to Grothendieck topological semantics. These results suggest that resource tableaux provide an appropriate deductive framework for logics like BI in which different kinds of connectives cohabit and interact. Through the capture of the semantics by labels, we have been able to provide a decision procedure for BI that builds countermodels in Grothendieck topological semantics. This study of such countermodels has suggested a better understanding of the semantic information necessary to analyse provability and of the relationships between the elementary and topological settings.

From a proof-search perspective, we have considered another non-trivial problem, namely, how to define a resource tableaux for BI with liberalised rules that improve efficiency in proof-search by reducing the number of new constants that must be handled. Therefore, we have defined liberalised resource tableaux that are based on a semantic analysis of the $\vee$ connective together with a specific treatment that is needed to make liberalised rules work. The related extension of the label algebra involved a less direct construction of countermodels, but, surprisingly, these countermodels are closely related to the topological Kripke semantics, which is complete for BI (Pym *et al.* 2004; Pym 2004), of which they can be viewed as an algebraic counterpart. These results emphasise the appropriateness of resource tableaux for dealing with the different semantics that are available for BI.

Another important question arises from these relationships between the semantics of BI and resource tableaux, and mainly from the extraction of countermodels from the dependency graphs: is it possible to define a new semantics of BI such that a dependency graph can be directly considered as a countermodel? We have proposed such a new Kripke semantics that can be seen as intermediate between the elementary and Grothendieck resource semantics. It emphasises the central notion of dependency graph, which captures the essential information necessary to analyse the provability in BI and leads to a simple semantics based on partially defined monoids. The definition of this semantics is important in itself, but the most interesting point is that it is strongly related to the specific models of BI known as (intuitionistic) 'pointer logic' (Ishtiaq and O'Hearn 2001) and 'separation logic' (Reynolds 2000), which were introduced in order to analyse mutable data structures.

Further work will be devoted to the study of tableaux systems for the various classical variations on BI, such as Boolean BI (that is, with classical additives). Although pointer logic (Ishtiaq and O'Hearn 2001) and separation logic (Reynolds 2000) can be formulated with intuitionistic additives, their main developments have been based on Boolean BI. Thus the development of tableaux systems for Boolean BI will facilitate the development of tableaux systems for them. In particular, an important open problem is to provide a complete semantics for Boolean BI, and so facilitate the extension of the analysis of this paper to the realm of classical pointer and separation logics. The variations of BI with classical multiplicatives are also intriguing.

We will also study the relationships with other recent work on proof-search in BI based on free variable tableaux (Galmiche and Méry 2003) and on connection methods (Galmiche and Méry 2002). Moreover, as BI is conservative over intuitionistic logic (IL) and multiplicative intuitionistic linear logic (MILL) (O'Hearn and Pym 1999), these results on resource tableaux can be restricted to both logics in order to propose new proof-search methods based on labels and constraints. For instance, we will compare such a method for IL with existing methods from the efficiency and countermodels construction perspectives. In the case of MILL, we will compare it with methods based on connections and proof nets construction (Galmiche 2000; Galmiche and Méry 2002). Moreover, from the semantic perspective, the impact of the results on MILL will be analysed and compared with previous proposals about resource models (Galmiche and Larchey-Wendling 2000) and the analysis of countermodels.

Acknowledgements

We are grateful to Hongseok Yang, Peter O'Hearn and the anonymous referees for their help in finding errors in, and with the presentation of, this long and complex paper. Pym is partially supported by a Royal Society Industry Fellowship at Hewlett-Packard Laboratories, Bristol.

References

Balat, V. and Galmiche, D. (2000) Labelled Proof Systems for Intuitionistic Provability. In: Labelled Deduction. *Applied Logic Series* **17**, Kluwer Academic Publishers.

D'Agostino, M. and Gabbay, D. M. (1994) A Generalization of Analytic Deduction via Labelled Deductive Systems. Part I: Basic substructural logics. *Journal of Automated Reasoning* **13** 243–281.

Day, B. (1970) On closed categories of functors. *Springer-Verlag Lecture Notes in Computer Science* **137** 1–38.

Dunn, M. (1986) Relevance Logic and Entailment. In: *Handbook of Philosophical Logic* **3** 117–224.

Fitting, M. (1990) *First-Order Logic and Automated Theorem Proving*, Texts and Monographs in Computer Science, Springer-Verlag.

Gabbay, D. M. (1996) *Labelled Deductive Systems, Volume I – Foundations*, Oxford University Press.

Galmiche, D. (2000) Connection Methods in Linear Logic and Proof nets Construction. *Theoretical Computer Science* **232** (1–2) 231–272.

Galmiche, D. and Larchey-Wendling, D. (2000) Quantales as completions of ordered monoids: revised semantics for Intuitionistic Linear Logic. *Electronic Notes in Theoretical Computer Science* **35**.

Galmiche, D. and Méry, D. (2001) Proof-search and countermodel generation in propositional BI logic (extended abstract). In: 4th Int. Symposium on Theoretical Aspects of Computer Software, Sendai, Japan. *Springer-Verlag Lecture Notes in Computer Science* **2215** 263–282

Galmiche, D. and Méry, D. (2002) Connection-based proof-search in propositional BI logic. In: *18th Int. Conference on Automated Deduction, CADE-18, LNAI 2392*, Copenhagen, Denmark 111–128.

Galmiche, D. and Méry, D. (2003) Semantic labelled tableaux for propositional BI (without bottom). *Journal of Logic and Computation* **13** (5) 708–753.

Galmiche, D., Méry, D. and Pym, D. J. (2002) Resource Tableaux (extended abstract). In: 16th Int. Workshop on Computer Science Logic, Edinburgh, Scotland. *Springer-Verlag Lecture Notes in Computer Science* **2471** 183–199.

Girard, J.-Y. (1987) Linear logic. *Theoretical Computer Science* **50** (1) 1–102.

Hähnle, R. and Schmitt, P. (1994) The liberalized δ-rule in free variable semantic tableaux. *Journal of Automated Reasoning* **13** 211–221.

Harland, J. and Pym, D. J. (2003) Resource-distribution via Boolean constraints. *ACM Transactions on Computational Logic* **4** (1) 56–90.

Ishtiaq, S. and O'Hearn, P. W. (2001) BI as an assertion language for mutable data structures. In: *28th ACM Symposium on Principles of Programming Languages*, London, U.K. 14–26.

Kripke, S. A. (1965) Semantical Analysis of Intuitionistic Logic I. In: Crossley, J. N. and Dummett, M. A. E. (eds.) *Formal Systems and Recursive Functions*, North-Holland 92–130.

Lafont, Y. (1997) The finite model property for various fragments of linear logic. *Journal of Symbolic Logic* **62** (4) 1202–1208.

Lambek, J. and Scott, P. (1986) *Introduction to higher-order categorical logic*, Cambridge University Press.

Lincoln, P. D. (1995) Deciding provability in linear logic formulas. In: Girard, J.-Y., Lafont, Y. and Regnier, L. (eds.) *Advances in Linear Logic*, Cambridge University Press 109–122.

Mac Lane, S. and Moerdijk, I. (1992) *Sheaves in Geometry and Logic*, Springer-Verlag.

O'Hearn, P. W. (1999) Resource interpretations, bunched implications and the $\alpha\lambda$-calculus. In: 4th Int. Conference on Typed Lambda Calculi and Applications, L'Aquila, Italy. *Springer-Verlag Lecture Notes in Computer Science* **1581** 258-279.

O'Hearn, P. W. and Pym, D. J. (1999) The Logic of Bunched Implications. *Bulletin of Symbolic Logic* 5 (2) 215–244.

O'Hearn, P. W., Reynolds, J. and Yang, H. (2001) Local reasoning about programs that alter data structures. In: 15th Int. Workshop on Computer Science Logic, Paris, France. *Springer-Verlag Lecture Notes in Computer Science* **2142** 1–19.

Prawitz, D. (1965) *Natural Deduction: A Proof-Theoretical Study*, Almquist and Wiksell, Stockholm.

Pym, D. J. (1999) On Bunched Predicate Logic. In: *14h Symposium on Logic in Computer Science*, Trento, Italy, IEEE Computer Society Press 183–192.

Pym, D. J. (2002) The Semantics and Proof Theory of the Logic of Bunched Implications. *Applied Logic Series* **26**, Kluwer Academic Publishers. (See Pym (2004) for errata and remarks.)

Pym, D. J., O'Hearn, P. W. and Yang, H. (2004) Possible worlds and resources: The semantics of BI. *Theoretical Computer Science* **315** (1) 257–305.
(Erratum: page 285, line -12: ', for some $P', Q \equiv P; P'$ ' should be '$P \vdash Q$'.

Pym, D. J. (2004) Errata and Remarks for *The Semantics and Proof Theory of the Logic of Bunched Implications* (Pym 2002).
Available at: `http://www.cs.bath.ac.uk/~pym/BI-monograph-errata.pdf`.

Reynolds, J. C. (2000) Lectures on reasoning about shared mutable data structure, Tandil, Argentina.

Reynolds, J. C. (2002) Separation logic: A logic for shared mutable data structures. In: *IEEE Symposium on Logic in Computer Science*, Copenhagen, Denmark, IEEE Computer Society Press 55–74.

Routley, R. and Meyer, R. (1972) The Semantics of Entailment, II-III. *Journal of Philosophical Logic* **1** 53–73, 192–208.

Urquhart, A. (1972) Semantics for relevant logics. *Journal of Symbolic Logic* **49** 1059–1073.