Unitary Eigenvarieties at Isobaric Points

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Abstract. In this article we study the geometry of the eigenvarieties of unitary groups at points corresponding to tempered non-stable representations with an anti-ordinary (a.k.a evil) refinement. We prove that, except in the case where the Galois representation attached to the automorphic form is a sum of characters, the eigenvariety is non-smooth at such a point, and that (under some additional hypotheses) its tangent space is big enough to account for all the relevant Selmer group. We also study the local reducibility locus at those points, proving that in general, in contrast with the case of the eigencurve, it is a proper subscheme of the fiber of the eigenvariety over the weight space.

1 Introduction

In [BC2], G. Chenevier and the author of this note studied the geometry of the eigenvariety of a unitary group at a point corresponding to a *non-tempered* automorphic representation π with an *anti-ordinary refinement*, and used the results to prove new cases of the lower bound on the Selmer group in the Bloch–Kato conjecture for an essentially self-dual Galois representation ρ of arbitrary dimension. While the methods developed in the book were more general, the case needed for the application to the Bloch–Kato conjecture was the one of an automorphic representation π with attached Galois representation $\rho_\pi = 1 \oplus \omega \oplus \rho$, where 1 is the trivial character (of motivic weight 0), ω is the cyclotomic character (of motivic weight -2 according to the most used conventions, that we shall also use) and ρ is some irreducible, essentially self-dual Galois representation of motivic weight -1. Observe that ρ_π is not *isobaric*, that is that the motivic weights of the components of ρ_π are distinct. This corresponds, in accordance to Arthur's philosophy, to the fact that the automorphic representation π is not tempered.

Let us recall that the lower bound in the Bloch–Kato conjecture is the prediction that the Bloch–Kato Selmer group $H^1_f(G_K, \rho)$ has dimension at least the order of vanishing of the L-function $L(\rho^*(1), s)$ at $s = 0$. If the motivic weight of ρ is not -1 and ρ is automorphic, then, using the functional equation and the Hadamard–De-La-Vallée-Poussin theorem of Jacquet–Shalika, the order of vanishing $\mathrm{ord}_{s=0} L(\rho^*(1), s)$ can be expressed purely in terms of local invariants of ρ. Moreover, this lower bound on $\dim H^1_f(G_K, \rho)$ can be proved by relatively elementary means (essentially by Poitou–Tate duality; *cf.* [B2, Corollary 4.1, p. 49]). Hence, working with non-

tempered automorphic representations is essential to new applications to the Bloch–Kato conjecture. We conjecture that the geometry of the eigenvariety at π is closely related to the p-adic L-function of ρ in such a way that the methods of [BC2] or variants will imply the lower bound in the Bloch–Kato conjecture for ρ. There is an ongoing program to prove this conjecture.

In this article, we study the geometry of the eigenvariety at a *tempered* automorphic representation (still with anti-ordinary refinement) with two aims in mind: to provide support for the aforementioned conjecture, and to test tools to solve it. The temperedness of π means that the components of the Galois representation ρ_π all have the same motivic weight, hence whatever lower bound in the Bloch–Kato conjecture one might deduce from our results will concern representations of motivic weight 0, and thus be already known by elementary methods. Yet the fact that, as we shall show, the geometry of the eigenvariety is rich enough to allow us to construct the whole Selmer group of a self-dual weight 0 Galois representation ρ is a strong encouragement to try to do so for weight -1 Galois representations.

This study is a sequel to my paper [B1], which deals with the case of the unitary groups with three variables U(3), and endoscopic forms of type $(2, 1)$. In this paper, we work out the general case. While the techniques are similar in nature to the ones used in [B1] (and thus relying heavily on the work of [BC2]), the algebraic and combinatorial arguments needed to solve the general case are much harder.

1.1 Reminder on Eigenvarieties, Galois Representations, and Refinements

Notations introduced here will stay in force throughout the paper.

1.1.1 Unitary Groups

We fix a prime p.

Let K be a quadratic imaginary field[1] in which p is split. We denote by G_K the absolute Galois group of K, and for every place v of K, denote by G_v the absolute Galois group of the completion K_v of K at a place v that we see as a decomposition subgroup of G_K. We fix a prime p that is split in K. Let U(d) be the unitary group over $\mathbb{Q}$ attached to an hermitian form over K in d variables, which is definite over $\mathbb{R}$.

1.1.2 Eigenvarieties

The p-adic eigenvariety X for U(d) (and a fixed level $U = K_p U^p$, where U^p is an open compact subgroup of U(d)($\mathbb{A}_f^p$) and K_p is a maximal compact subgroup of U(d)($\mathbb{Q}_p$)) has been constructed independently by Chenevier [C] and Emerton [E]. The two constructions have been shown in [BC2] to lead to the same eigenvariety, which indeed can be characterized by some simple natural properties. The eigenvariety X is a reduced rigid analytic space over $\mathbb{Q}_p$, equidimensional of dimension $d-1$ (it would be d if we were allowing central twists), provided with a Zariski-dense set

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[1]The methods of this paper could in principle be applied to any CM field K.

of so-called *classical* points, which corresponds to automorphic eigenforms for U(d) of level U and of various weights.

1.1.3 Galois Representations Attached to Automorphic Representations

Another recent development that is of importance to us is the construction, for any automorphic forms π for U(d), of an attached Galois representation $\rho_\pi \colon G_K \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$, whose restriction to G_v for any place v above a prime l such that π_l is unramified has the expected properties (namely corresponds to the base change of π_l to K_v by Local Langlands if $l \neq p$, and is crystalline with expected Hodge-Tate weights and Frobenius eigenvalues if $l = p$). Those ρ_π are *polarized of weight $d - 1$*, that is

$$\tag{1.1} \rho^\perp \simeq \rho\omega^{d-1},$$

where for a representation $\rho \colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ the representation $\rho^\perp$ is defined by

$$\rho^\perp(g) = {}^t\rho(cgc^{-1})^{-1},$$

where c is any lift of the nontrivial elements of $\mathrm{Gal}(K/\mathbb{Q})$ in $G_\mathbb{Q}$, and $\omega \colon G_K \to \mathbb{Z}_p^* \subset (\overline{\mathbb{Q}}_p)^*$ is the cyclotomic character. This construction of ρ_π is due to S. Morel [Mor], Shin [S], and the authors of the book project [GRFAbook] led by Harris, all of whom rely on works by many authors since Langlands, including fundamental contributions by Shelstad, Kottwitz, Arthur, Clozel, Waldspurger, Laumon, and Ngo.

When π is *stable tempered*, that is, when the base change π_K of π to $\mathrm{GL}_n(\mathbb{A}_K)$ is cuspidal, it is expected that ρ_π is irreducible. This is known in many cases, though apparently not yet in full generality. For instance, the irreducibility of ρ_π is known if $d \leq 3$ (*cf.* [BlRo]), and if $d \leq 5$ and π_K comes by base change from an automorphic representation for $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ (*cf.* [CG]). In any dimension, it is known that ρ_π is irreducible when π is square-integrable at some place ([TY]), or for any given π for a positive density of primes p ([PT]), or for a generic π in suitable p-adic families ([BC2]).

1.1.4 Families of Galois Representations

By interpolating the Galois representations attached to classical automorphic forms, a simple argument due to Chenevier gives a pseudocharacter $T \colon G_K \to \mathcal{O}(X)$ of dimension d. If $x \in X(\overline{\mathbb{Q}}_p)$, the post-composition of T with the "evaluation at x" map $\mathcal{O}(X) \to \overline{\mathbb{Q}}_p$ gives a pseudocharacter $T_x \colon G_K \to \overline{\mathbb{Q}}_p$, which is the trace of a unique semi-simple representation $\rho_x \colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ up to isomorphism. When x is attached to a classical automorphic representation π, we have $\rho_x \simeq \rho_\pi$.

1.1.5 Tempered and Isobaric Automorphic Representations

If π is an automorphic representation for U(d), then we can write ρ as a sum of irreducible representations $\rho_\pi \simeq \rho_1 \oplus \cdots \oplus \rho_r$, for some r between 1 and d, and the ρ_i's are pairwise non isomorphic, since ρ_π has distinct Hodge–Tate weights. The *type* of ρ_π is the r-tuple $(d_1, \ldots, d_r)$, where $d_i = \dim \rho_i$. Since the map $\rho \mapsto \rho^\perp(d - 1)$

is an involution, the set $\{1, \ldots, r\}$ can be partitioned into singletons $\{i\}$ such that $\rho_i \simeq \rho_i^\perp(d-1)$ and pairs $\{i, j\}$ such that $\rho_i \simeq \rho_j^\perp(d-1)$.

Definition 1.1   If for all $i = 1, \ldots, r$, we have $\rho_i \simeq \rho_i^\perp(d-1)$, we say that ρ_π (and π) is *isobaric*.

Remark 1.2   Conjecturally, the isobaric representations π should be exactly the tempered representations, and a ρ_π should be isobaric reducible if and only if π is endoscopic tempered (see *e.g.,* the review of Arthur's conjecture given in [BC2, Appendix].)

More precisely, if $\pi_1, \ldots, \pi_r$ are stable tempered representations of unitary groups $U(d_1), \ldots, U(d_r)$ with $d = d_1 + \cdots + d_r$, there should exist a tempered representation π of $U(d)$, called the *endoscopic transfer* of $(\pi_1, \ldots, \pi_r)$ such that $\rho_\pi = \rho_{\pi_1} \oplus \cdots \oplus \rho_{\pi_r}$ provided that the Hodge–Tate weights of $\rho_{\pi_1}, \ldots, \rho_{\pi_r}$ are distinct. Conversely, every tempered representation π should be *L*-equivalent to one arising this way. Those results, which have long been conjectured, are now almost known in full generality. The analog for quasi-split unitary groups of the construction of the endoscopic transfer from stable tempered representations $\pi_1, \ldots, \pi_r$ has been recently completed by Mok [Mok] extending to unitary groups earlier results of Arthur for symplectic and orthogonal groups. The transfer back and forth between definite at infinity and quasi-split unitary groups seems a comparatively easy task, even if, to our knowledge, it has been only written down for $d \leq 3$ ([R]).

Thus a representation π obtained by endoscopic transfer from stable tempered representations $\pi_1, \ldots, \pi_r$ would be isobaric of type $(d_1, \ldots, d_r)$ provided the representations $\rho_{\pi_1}, \ldots, \rho_{\pi_r}$ are irreducible, as they are expected to be. In view of the discussion at the end of Subsection 1.1.3, this provides a large supply of examples of isobaric representations.

1.1.6   Anti-ordinary Refinements

If a classical automorphic form π of level U is given, it does not yet define a point in the eigenvariety X. For this, we need to choose some combinatorial data, called either a *p-stabilization* or *a refinement* $\mathcal{R}$ of π_p. For the general definition of a refinement, we refer the reader to [BC2, §6.4]. We will place ourselves in the case where the eigenvalues of the crystalline Frobenius on $D_{\text{crys}}(\rho_{|G_v})$ (here v is one of the two places of K above p) are distinct. In that case, a refinement is simply defined by an ordering of those eigenvalues. Thus there are at most $d!$ refinements. When ρ_π is isobaric, which we shall assume, every ordering of the eigenvalues of the crystalline Frobenius defines a refinement (*cf.* [BC2, 6.4.5]), so there are exactly $d!$ refinements

Each pair $(\pi, \mathcal{R})$ defines a classical point on the eigenvariety X. For a given π, those $d!$ different points $x_1, \ldots, x_{d!}$ have in general very different Galois-theoretic behavior: even if the representation ρ_{x_i} is the same at the different x_i, namely ρ_π, its deformations to a small neighborhood of x_i are very different.

Among the various refinements of π we will consider a special kind that we call the *anti-ordinary* refinements. They are the generalizations of the evil refinements of Eisenstein series in the theory of *p*-adic modular forms. They are also critical

refinements, though not all critical refinements are anti-ordinary. The definition of *anti-ordinary*, and some properties, are given in §2.

1.2 Results

1.2.1 Non-smoothness of the Eigenvariety at Isobaric Anti-ordinary Points

We are almost ready to state our non-smoothness result. As we import results from [BC2], we are also obliged to import some technical hypotheses. Those technical hypotheses, called (REG) (for *regular*), (MF′) (for *multiplicity free*) and (NGD) (for *no geometric deformation*), are as follows. Let π be an automorphic representation for U(n), unramified at p, of type $(d_1, \ldots, d_r)$ with $\rho_\pi = \rho_1 \oplus \cdots \oplus \rho_r$.

(REG) For every integer a, with $1 \le a \le d - 1$, the eigenvalues of the crystalline Frobenius on $D_{\mathrm{crys}}(\Lambda^a \rho_\pi)$ have multiplicity 1.

(MF′) For every family of integers $(a_i)_{i=1,\ldots,r}$ with $1 \le a_i \le d_i$, the representation $\rho_{(a_i)} := \bigotimes_{i=1}^r \Lambda^{a_i} \rho_i$ is absolutely irreducible. Moreover, if (a_i) and (a_i') are two distinct sequences as above with $\sum_{i=1}^r a_i = \sum_{i=1}^r a_i'$, then $\rho_{(a_i)} \not\simeq \rho_{(a_i')}$.

(NGD) For $i = 1, 2, \ldots, r$, $H^1_g(G_K, \mathrm{ad}\, \rho_i) = 0$.

Both assumptions (REG) and (MF′) are multiplicity one statements, which are only necessary due to our imperfect knowledge of the trianguline nature of the Galois representation on the eigenvarieties. It should be possible to remove them in the near future.

The hypothesis (NGD) asserts that any de Rham (at every place of K dividing p) deformation of ρ_i to $\overline{\mathbb{Q}}_p[\varepsilon]/(\varepsilon)^2$ is trivial. This is a standard infinitesimal version of the Fontaine–Mazur conjecture, which states that de Rham representations of $G_{K,S}$ are geometric, and therefore form a countable set and cannot have non-trivial families. From another point of view, the hypothesis (NGD) is a special case of the assertion that for any representation ρ' of non-negative motivic weight, $H^1_g(G_{K,S}, \rho') = 0$, namely the case $\rho' = \mathrm{ad}\, \rho_i$. This assertion is the p-adic avatar of the famous "Yoga of Weights" of Grothendieck. It is also a part of the Bloch–Kato conjecture. At any rate, the assertion (NGD) is known already in a significant number of cases, for example for one-dimensional representations and for many 2-dimensional representations ρ_i coming from $G_{\mathbb{Q}}$ (by results of Weston and Kisin, see [Kis,W]). It is also provable for all representations satisfying (1.1) whose residual representations satisfy the hypotheses of the fast-growing set of theorems on potential automorphy; see forthcoming work of Davide Reduzzi.

The following result is analogous to, but much harder than, the main theorem of [B1].

Theorem 1 *Let π be an automorphic representation of U(d), unramified at p, isobaric of type $(d_1, \ldots, d_r)$ (with $r \ge 2$, that is ρ_π reducible). Assume that ρ_π satisfies (REG), (MF′), and (NGD). Let $\mathcal{R}$ be an anti-ordinary refinement of π, and let $x \in X(\overline{\mathbb{Q}}_p)$ be the point corresponding to $(\pi, \mathcal{R})$ on the eigenvariety. Assume furthermore that $(d_1, \ldots, d_r) \ne (1, \ldots, 1)$. Then X is non-smooth at x. Furthermore, the local ring at x of every irreducible component of X through x is not factorial.*

The condition that the type of π is not $(1, \ldots, 1)$ is necessary.

1.2.2 The Case of Type $(1, n)$

We shall now explain two results that provide examples of two important phenomena: a case where the method of Ribet on eigenvarieties can produce not only one, but all the n independent extensions that are supposed to exist between two Galois representations for n arbitrary large and a classical anti-ordinary point of the eigenvariety where the local reducibility locus of the Galois representation and the schematic fiber over weight space at this point do not coincide (in contrast with the case of the eigencurve, where they always coincide). As we are interested here in constructing examples (or counter-examples), we do not mind adding restrictive hypotheses (*cf.,* (IRR) and (WEI) below) that simplify the argument, since they are still satisfied by a large set of representations.

We keep the assumptions of Theorem 1, but we assume in addition that π is of type $(1, n)$, where $n = d - 1$ is an integer ≥ 2. In this case, $\rho_\pi = \rho_1 \oplus \rho_2$, with $\dim \rho_1 = 1$. We set $\rho = \rho_1^{-1} \otimes \rho_2$, which is a representation of dimension n and motivic weight 0, satisfying $\rho = \rho^\perp$. The hypotheses on π are equivalent to the following hypotheses on ρ:

(REG) For every integer a, with $1 \leq a \leq n - 1$, the eigenvalues of the crystalline Frobenius on $D_{\mathrm{crys}}(\Lambda^a \rho)$ have multiplicity 1.

(MF$'$) For every integer a, with $1 \leq a \leq n - 1$, the representation $\Lambda^a \rho$ is absolutely irreducible.

(NGD) $H_g^1(G_K, \mathrm{ad}\, \rho) = 0$.

Let us add another hypothesis:

(IRR) The restriction $\rho_{|G_v}$ is irreducible.

Let us refer to the two places of K above p as v and $\bar{v}$. By Proposition 2.3, applicable because of (IRR), π has an anti-ordinary refinement if and only if the Hodge–Tate weights of ρ at v are all negative, or all positive.[2] Those two case are symmetric (just exchange v and $\bar{v}$), so let us assume that the Hodge–Tate weights of ρ at v are all negative, and a little more to simplify some arguments:

(WEI) The Hodge–Tate weights of $\rho_{|G_v}$ are all ≤ -2.

Note that the Hodge–Tate weights of $\rho_{|G_{\bar{v}}}$ are therefore all ≥ 2. We can choose an anti-ordinary refinement of π and call it $\mathcal{R}$. Let x be the point of X corresponding to $(\pi, \mathcal{R})$.

The extension of Ribet's method, which is developed in [BC2], uses the families of Galois representations carried by the eigenvariety X around the point x (where the Galois representation is $\rho_x = \rho_1 \oplus \rho_2$) to construct extensions of ρ_1 by ρ_2. The space of extensions we can construct by this method is a subspace of $H^1(G_K, \rho) = $

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[2]Our convention regarding Hodge–Tate weight is that the cyclotomic character of G_v has Hodge–Tate weight 1.

$\mathrm{Ext}^1_{G_K}(\mathbb{Q}_p, \rho) \simeq \mathrm{Ext}^1_{G_K}(\rho_1, \rho_2)$ that we shall call $H^1_x(G_K, \rho)$. The Bloch–Kato conjecture implies [3] that $\dim H^1(G_K, \rho) = n$, and it is an easy consequence of Poitou–Tate duality that $\dim H^1(G_K, \rho) \geq n$.

The following result is analogous to, but harder than, [B1, Theorem 2].

Theorem 2   *With the above hypotheses, one has* $\dim H^1_x(G_K, \rho) \geq n$.

The significance of this result is that Ribet's method on eigenvarieties can construct all the n independent extensions in $H^1(G_K, \rho)$ that are known to exist. This is encouraging for the ability of Ribet's method on eigenvarieties to construct all extensions that are supposed (but not necessarily known) to exist in other cases (for example when ρ has motivic weight -1).

Keep the assumptions of the preceding theorem and let R_x be the local reducibility locus at x, that is the largest local closed subscheme of X containing x on which the family of Galois representations carried by X, restricted to the decomposition group D_v, is reducible. Let $\kappa \colon X \to \mathcal{W}$ be the weight map from the eigenvariety to the weight space $\mathcal{W}$ (a union of n-dimensional rigid open balls), and let F_x be the connected component at x of the schematic fiber of κ at $\kappa(x)$. Since κ is locally finite, F_x is a finite local closed subscheme of X. One of the main result of [BC2, Theorem 4.4.6] is that we have an inclusion of closed subschemes of X: $R_x \subset F_x$. In particular, R_x is a finite local scheme. In the case of the eigencurve, two proofs are given in [BC1] that $R_x = F_x$. Surprisingly (for the author), this is not the case in higher dimensions, or else the Bloch–Kato conjecture is false:

Theorem 3   *Keep the hypotheses of the above theorem, and assume that the Bloch–Kato conjecture holds for ρ and all its twists $\rho\tau$ where τ is a character of G_K satisfying $\tau = \tau^{\perp}$. (That is to say, since $\rho\tau$ has motivic weight 0, assume $H^1_f(G_K, \rho\tau) = 0$.)*

Then there are infinitely many points $x \in X$ such that the inclusion $R_x \subset F_x$ is strict, that is, such that the local reducibility locus is strictly smaller than the fiber over the weight space.

This result raises the question of what exactly is the reducibility locus R_x. This question is important if one wishes to extend the results of [BC2, Chapter 9] relating the *absolute* geometry of the eigenvariety X at non-ordinary points to the rank of suitable Selmer groups, in a way that takes into account the finer *relative* (that is, as a variety over the weight space $\mathcal{W}$) geometry of X. One reason one might wish to do that is because it is this relative geometry of X that seems more directly related to the vanishing of a suitable p-adic L-function (*cf.* [Kim],[B4, ChapterV]).

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[3]Indeed, an extension of 1 by ρ has automatically good reduction in the sense of Bloch–Kato at all places w not dividing p and also at the place $\bar{v}$ above p. This is true because ρ is of weight 0, hence $\rho_{|G_w}$ does not contain $\mathbb{Q}_p(1)$, and at $\bar{v}$ because of hypotheses (WEI). Then we can argue as in [B1, Proposition 12].

2 Anti-ordinary Refinements

We use the same notations as in the introduction: K is a quadratic imaginary field, p a prime that splits in K, v a place of K above p, and "ρ is crystalline" (or "De Rham", or "Hodge–Tate") means the same thing for the restriction ρ to the local Galois group G_v.

Let ρ be a representation of $G_{K,S}$ that is crystalline. Recall that a *refinement* $\mathcal{R}$ of ρ is an ordering of the eigenvalues of the crystalline Frobenius ϕ on $D_{\mathrm{crys}}(\rho)$ (we assume that those eigenvalues are distinct). If ρ' is a subrepresentation of ρ, a refinement of ρ induces, by restriction, a refinement of ρ'.

We assume that $\rho \simeq \rho_1 \oplus \cdots \oplus \rho_r$, where the ρ_i are irreducible of dimension d_i, and that the Hodge–Tate weights of ρ are distinct.

Recall that a refinement $\mathcal{R}$ defines a maximal flag in $D_{\mathrm{crys}}(\rho)$ (namely the flag whose space of dimension k is generated by the eigenvectors corresponding to the first k eigenvalues of the refinement) and that $\mathcal{R}$ is said to be *non-critical* if this flag is in general position with respect to the Hodge filtration in $D_{\mathrm{crys}}(\rho)$. Otherwise, it is said to be *critical*. It is clear from the definition that there always exists at least one non-critical refinement.

To a refinement $\mathcal{F}$ as above we can attach a permutation $\sigma \in S_d$ by the following recipe: let $k_1, \ldots, k_d$ be the Hodge–Tate weights of ρ in increasing order. Let $\phi_1, \ldots, \phi_d$ be the eigenvalues of ϕ on $D_{\mathrm{crys}}(\rho)$ in the order given by $\mathcal{R}$. For $i = 1, \ldots, r$, let W_i be the set of indices l such that k_l is a Hodge–Tate weight of ρ_i, and let R_i be the set of indices l such that ϕ_l is an eigenvalue of ϕ on $D_{\mathrm{crys}}(\rho_i)$. The R_i's, and the W_i's, are two partitions of $\{1, \ldots, d\}$ into r parts, with $|W_i| = |R_i| = d_i$. We define σ as the unique permutation of $\{1, \ldots, d\}$ that maps R_i to W_i and is increasing on R_i for $i = 1, \ldots, r$.

Definition 2.1   A refinement $\mathcal{R}$ of ρ is said to be *anti-ordinary* if

(INT)   The R_i are intervals of $\{1, \ldots, n\}$

(TR)   The permutation σ attached to $\mathcal{F}$ and ρ is transitive.

(NCR)   The restriction of $\mathcal{F}$ to ρ_i, $i = 1, \ldots, r$, is non-critical.

Remark 2.2   (i)   It should be noted that the anti-ordinarity of $(\rho, \mathcal{F})$ is a global property: it depends on ρ and not only on its restriction to G_v.

(ii)   If ρ is non-critical, then the permutation σ is the identity. (We leave this as an exercise for the reader.) Thus, in some sense, the anti-ordinary refinements are highly critical.

(iii)   If ρ is irreducible, then $\sigma = 1$ and no refinement is anti-ordinary. If ρ is a sum of characters, then conditions (NCR) and (INT) are automatic and is is easy to see that the number of anti-ordinary refinements is $(d-1)!$.

(iv)   For another case of existence of an anti-ordinary refinement, see [BC2, Lemma 9.3.4].

Proposition 2.3   *Assume that $d_1 = 1$ and $d_2 = d - 1$. Then ρ admits an anti-ordinary refinement if and only of the Hodge–Tate weights of ρ_2 are all bigger, or all smaller, than the Hodge–Tate weight of ρ_1.*

Proof Assume that ρ admits an anti-ordinary refinement. By hypothesis (INT), either $R_1 = \{1\}$ or $R_1 = \{d\}$. Assume first that $R_1 = \{1\}$. If $W_1 = \{i\}$ with $i < d$, then by definition σ maps the integer $j > i$ to themselves, contradicting (TR). Thus $W_1 = \{d\}$, which means that the weights of ρ_2 are all smaller than the weight of ρ_1. Similarly if $R_1 = \{d\}$, then the weight of ρ_1 is smaller than all the weights of ρ_2.

Conversely, assume that the weights of ρ_2 are all smaller than the weight of ρ_1. Then construct a refinement $\mathcal{R}$ by taking ϕ_d the eigenvalue of the crystalline Frobenius on ρ_1, and $\phi_1, \ldots, \phi_{d-1}$ the eigenvalues on ρ_2 in whatever order defines a noncritical refinement of ρ_2 (such an order always exists; see [BC2, §2.4.6]). Then (NCR) is satisfied by construction, and we have $R_1 = \{d\}$ and $R_2 = \{1, 2, \ldots, d-1\}$, so (INT) is satisfied. We also have $W_1 = \{1\}$ and $W_2 = \{2, \ldots, d\}$ by assumption, so $\sigma(d) = 1$ and $\sigma(i) = i + 1$ for $1 \le i \le d-1$, and (TR) is also satisfied. ∎

3 Eigenvarieties at Reducible, Isobaric, Anti-ordinary Points

In this section we prove Theorem 1. We use the notations introduced in the statement of that theorem: π is an automorphic forms for U(n), unramified at p, which is isobaric and such that $\rho_\pi = \rho_1 \oplus \cdots \oplus \rho_r$ with the ρ_i irreducible of dimension d_i. The representation ρ satisfies the hypotheses (REG), (NF'), (NGD), and $\mathcal{R}$ is an anti-ordinary refinement of π. The eigenvariety for U(n) (and some suitable level) is denoted by X and $x \in X(\overline{\mathbb{Q}}_p)$ is the point corresponding to $(\pi, \mathcal{R})$.

3.1 Reminder of Results of [BC2]

Let $\mathcal{O}_x$ be the local ring of X at x, and let A be the quotient of $\mathcal{O}_x$ by a minimal prime ideal (or in other words, the local ring at x of an irreducible component of X through x). The ring A is a local domain whose maximal ideal will be denoted by m and whose residue field is $\overline{\mathbb{Q}}_p$, and it carries a pseudo-character $T: G_K \to A$ of dimension d, which residually is $T \otimes A/m = \operatorname{tr} \rho_x = \operatorname{tr} \rho_\pi = \operatorname{tr} \rho_1 + \cdots + \operatorname{tr} \rho_r$. In particular, T is a residually multiplicity free pseudocharacter. Those pseudocharacters are subject to a close analysis in [BC2, chapter 1]. In particular, attached to T there exists a family $(A_{i,j})_{i,j=1,\ldots,r; i \ne j}$ of fractional ideals of A (that is, finite type A-submodules of $L = \operatorname{frac}(A)$) such that:

(a) $A_{i,j}A_{j,i} \subset m$ for every two distinct elements i, j in $\{1, \ldots, r\}$;
(b) $A_{i,j}A_{j,k} \subset A_{i,k}$ for every three distinct elements i, j in $\{1, \ldots, r\}$;
(c) $\sum_{i \ne j} A_{i,j}A_{j,i}$ is the total reducibility ideal of T, that is the smallest ideal I of A such that $T \otimes A/I$ is a sum of r pseudocharacters.

A key fact, which uses many of the main results of the book [BC2], is that the total reducibility ideal of T at x is m, which intuitively means that on any closed subscheme of $\operatorname{Spec} A$ that is strictly larger than the reduced closed point $\{x\}$ (for example, on the closed subscheme of X of ring $\overline{\mathbb{Q}}_p[\varepsilon]/(\varepsilon^2)$) that is defined by a non-zero tangent vector of $\operatorname{Spec} A$ at x), T is less reducible than it is at $\{x\}$, namely it is not the sum of as many as r pseudocharacters. This result relies on the anti-ordinarity of ρ, and the hypotheses (REG), (MF'), (NGD). For a proof, see [BC2, Proposition 9.3.7] (see also [B1, Proposition 7]).

Therefore, we can restate (c) as

(c′) $\sum_{i \neq j} A_{i,j} A_{j,i} = m$

We need one more fact. Since T is defined by interpolating representations satisfying $\rho^{\perp}(1 - d) \simeq \rho$, it satisfies $T(cgc^{-1})\omega(g)^{1-d} = T(g)$ for every $g \in G_{K,S}$, and the same relation is also satisfied by each of the tr ρ_i, since π is isobaric. Thus we have, according to [BC2, Lemma 1.8.5],

(d) for every $i \neq j$, $A_{i,j}$ and $A_{j,i}$ are isomorphic as A-modules.

3.2 An Algebraico-combinatorial Result

We now prove a purely algebraico-combinatorial result.

Proposition 3.1 *Let A be a local noetherian domain of maximal ideal m, residue field $A/m = F$, with a family of fractional ideals $(A_{i,j})_{i,j=1,\dots,r; i \neq j}$ that satisfy* (a), (b), (c′), *and* (d) *above. Then if A is factorial, we have $r > \dim A$.*

Proof If A is an UFD, there exist elements $x_1, \dots, x_r$ in L^*, such that for every $i \neq j$, $x_i x_j^{-1} A_{i,j}$ is a true ideal of A (see the proof of [BC2, Prop 1.6.1]). Changing the $A_{i,j}$ by $x_i x_j^{-1} A_{i,j}$ obviously preserves conditions (a), (b), (c′), and (d). So we can assume that the fractional ideals $A_{i,j}$ are true ideals of A.

Let us call a pair $\{i, j\} \subset \{1, \dots, r\}$ *bad* if $A_{i,j} A_{j,i} \subset m^2$, and *good* otherwise. By Nakayama's lemma, (c′) can be re-written as

(c″) $\sum_{\{i,j\} \text{ good pair} \subset \{1,\dots,r\}} A_{i,j} A_{j,i} = m$.

Let $\{i, j\}$ be a good pair. The two ideals $A_{i,j}$ and $A_{j,i}$ are not both contained in m, otherwise $A_{i,j} A_{j,i}$ would be contained in m^2. But since A is local, the only ideal not contained in m is A. Therefore, one of the ideals $A_{i,j}$ or $A_{j,i}$ is A, and in particular is free of rank one as an A-module, and by (d), so is the other, which is therefore principal. Call $X_{\{i,j\}}$ a generator of that ideal. By (a), we have $X_{\{i,j\}} \in m$, but since $\{i, j\}$ is a good pair, $X_{\{i,j\}} \notin m^2$. In particular, $X_{\{i,j\}}$ is irreducible. Note that this analysis allows to give a natural orientation to a good pair $\{i, j\}$. We orient $\{i, j\}$ as (i, j) if $A_{i,j} = AX_{\{i,j\}}$, and as (j, i) if $A_{j,i} = AX_{\{i,j\}}$.

Now reduce (c″) modulo m^2. We get

$$\sum_{\{i,j\} \text{ good pair} \subset \{1,\dots,r\}} \overline{X}_{\{i,j\}} F = m/m^2,$$

where $\overline{X}_{\{i,j\}}$ is the image of $X_{\{i,j\}}$ in m/m^2. In other words, the family of $X_{\{i,j\}}$'s for $\{i, j\}$ running among good pairs is a generating family of the cotangent space m/m^2 of A. Let t be the dimension (over the residue field F) of this space. We can choose t good pairs such that the corresponding $X_{\{i,j\}}$ are a basis over F of m/m^2. Call the chosen pairs *very good*. So we have t very good pairs, and as is well known (by the Hauptidealsatz), $t \geq \dim A$.

Consider the non-oriented graph Γ whose vertices are the elements in $\{1, \dots, r\}$, and whose edges are the very good pairs $\{i, j\}$. We claim that this graph is simply connected (that is, it is a forest). For if there is a cycle $(i_0, i_1, \dots, i_k)$, with $i_k = i_0$,

in this graph, we can assume (changing the cycle to $(i_k, i_{k-1}, \ldots, i_0)$ if necessary) that (i_1, i_0) is the natural orientation of the very good pair $(\{i_0, i_1\})$, so that $A_{i_1, i_0} = X_{\{i_0, i_1\}} A$. But by (b), we have

$$A_{i_1, i_2} A_{i_2, i_3} \cdots A_{i_{k-1}, i_0} \subset A_{i_1, i_0}.$$

The $A_{i_l, i_{l+1}}$ for $l = 1, \ldots, k-1$ are either A or of the form $X_{\{i, j\}} A$ for a very good pair $\{i, j\}$. So we obtain that, in A, $X_{\{i_0, i_1\}}$ divides a product of $X_{\{i, j\}}$'s for other very good pairs $\{i, j\}$. Since the $X_{\{i, j\}}$ are irreducible, $X_{\{i_0, i_1\}}$ has to be equal, up to a unit in A, to $X_{\{i, j\}}$ for $\{i, j\}$ another very good pair. But this contradicts the fact that the family of $X_{\{i, j\}}$'s for $\{i, j\}$ very good is a basis of m/m^2, hence a linearly independent set.

Since Γ is simply connected, a well-known and elementary result of graph theory asserts that its number of edges is strictly less that its number of vertices. So $t < r$, and therefore $\dim A < r$. ∎

3.3 End of the Proof of Theorem 1

We have now proved Theorem 1. Indeed, A is the local ring at x of an irreducible component of X through x, so what the theorem states in its precise form is that if π is not of type $(1, \ldots, 1)$, then A is not factorial. Since X is equidimensional of dimension $d-1$, $\dim A = d-1$. If A were factorial, we would have $r > \dim A = d-1$, so $r \geq d$. On the other hand, since r is the number of irreducible factors of ρ_π which has dimension d, we have $r \leq d$ with equality if and only if all factors are characters; that is, π is of type $(1, \ldots, 1)$. In all other cases, we get a contradiction proving that A cannot be factorial.

To prove the first assertion of the theorem, namely that X is nonsmooth at x if π is not of type $(1, \ldots, 1)$, we use the fact that X is smooth at x if and only if it is irreducible in a neighborhood of x, and $\mathcal{O}_x = A$ is regular, together with the Auslander–Buchsbaum theorem, which states that every regular noetherian local ring is factorial.

4 Finer Study of Eigenvarieties at Isobaric Endoscopic Anti-ordinary Points of Type $(n, 1)$

Let π be as in Theorem 2. By assumption, ρ_π satisfies all hypotheses (MF'), (NGD), and (REG) of Theorem 1. Therefore we can apply this theorem and all the tools developed for its proof. We adopt the same notations. As we recalled in Susbsection 3.1, the residually multiplicity free pseudo-character T defines fractional ideals $A_{1,2}$ and $A_{2,1}$. To simplify notations, we set $B = A_{1,2}$ and $C = A_{2,1}$. We restate the properties (a) to (d) above, and add another property (e), which is [BC2, Theorem I.5.5]:

(a) $BC \subset m$.

(b) Void here since $r = 2$.

(c),(c') The ideal BC is the total reducibility ideal of T, that is the smallest ideal I of A such that $T \otimes A/I$ is a sum of two pseudocharacters, and this ideal is m.

(d) B and C are isomorphic as A-modules.

(e)   There is a natural injective linear map

$$\iota_B \colon (B \otimes A/m)^* \to \text{Ext}^1_{G_K}(\rho_1, \rho_2) \simeq H^1(G_K, \rho).$$

There is a similar map

$$(C \otimes A/m)^* \to \text{Ext}^1_{G_K}(\rho_2, \rho_1) = H^1(G_K, \rho^*).$$

Note that the idea of (e) (in this case $r = 2$) is due to Mazur and Wiles (see [MaW] and [HP]). The subspace of $H^1(G_K, \rho)$ that we denote by $H^1_x(G_K, \rho)$ is by definition the image of ι_B.

Let us introduce the following notation: for any finite-type A-module M, the minimal number of elements of a family of generators of M is, by Nakayama's lemma, equal to the dimension of $M \otimes A/m$ over the field $A/m = \overline{\mathbb{Q}}_p$. We shall denote this number by $\text{gen}(M)$. Therefore, by (e),

$$\dim H^1_x(G_K, \rho) = \text{gen}(B).$$

Proposition 4.1   *We have* $\text{gen}(B) \geq n$ *and* $\text{gen}(C) \geq n$.

Proof   By (d), there exists an $x \in L^*$ such that $C = xB$. Let $L' = L[u]/(u^2 - x)$ if x has no square root in L. Otherwise, let $L' = L$ and denote by u a square root of x in L. Let A' be the integral closure of A in L. Since A is excellent, A' is of finite type over A, and the morphism $\text{Spec } A' \to \text{Spec } A$ is finite surjective. In particular, $\dim A' = \dim A = n$.

Now let $B' = BA'$, $C' = CA'$, and $m' = mA'$. From $BC = m$ it follows immediately that $B'C' = m'$, and from $C = xB$ that $C' = xB'$. Set $D = uB'$. Then $D^2 = u^2 B'B' = xB'B' = C'B' = m'$. The fractional ideal D is therefore a true ideal (since the square of any of its elements is in m', so in A', and A' is normal).

The closed subschemes of $\text{Spec } A'$ defined by D and by m' have the same underlying topological space, and therefore the same dimension, which is 0, since the closed subscheme attached to m' is the fiber of the closed point of $\text{Spec } A$, and therefore is finite. Since A' has dimension n, the hauptidealsatz implies that $\text{gen}(D) \geq n$. But clearly, $\text{gen}(C) = \text{gen}(B) \geq \text{gen}(B') = \text{gen}(D)$, and the proposition follows.   ∎

We deduce that $\dim H^1_x(G_K, \rho) \geq n$, which is Theorem 2.

We now turn to the proof of Theorem 3. We focus on the restriction $T_{|G_v}$ of the pseudocharacter T to the local Galois group G_v. Its residual pseudocharacter $T_{|G_v} \otimes A/m$ is $(\rho_1)_{|G_v} + \text{tr}(\rho_2)_{|G_v}$. By our hypothesis (IRR) that $\rho_{|G_v}$ is irreducible, where $\rho = \rho_1^{-1} \otimes \rho_2$, this is a residually multiplicity-free pseudocharacter with two residual factors. Therefore the same analysis as before applies: there exists two fractional ideals B_v, C_v of A such that the following hold:

(α) $B_v \subset B$ and $C_v \subset C$; in particular $B_v C_v \subset m$.

(γ) $B_v C_v$ is the reducibility ideal of $T_{|G_v}$, namely the smallest ideal I of A such that $T_{G_v} \otimes A/I$ is the sum of 2 pseudocharacters. In other words, $R_x = \text{Spec } A/BC$.

(δ)   The A-modules B_v and C_v are isomorphic as A-modules. More precisely, there is an isomorphism of B onto C that sends B_v onto C_v (*cf.* [BC2, Lemma 1.8.5]).

(ϵ) There is a natural injective linear map

$$\iota_{B_v} \colon (B_v \otimes A/m)^* \to \mathrm{Ext}^1_{G_v}(\rho_1, \rho_2) \simeq H^1(G_v, \rho).$$

There is a similar map $\iota_{C_v} \colon (C_v \otimes A/m)^* \to \mathrm{Ext}^1_{G_v}(\rho_2, \rho_1) = H^1(G_v, \rho^*)$. The maps ι_{B_v} and ι_{C_v} are compatible with the maps ι_B and ι_C in the sense that the following diagrams commute:

$$
\begin{array}{ccc}
(B \otimes A/m)^* & \xrightarrow{\ \iota_B\ } & \mathrm{Ext}^1_{G_K}(\rho_1, \rho_2) \simeq H^1(G_K, \rho) \\
\downarrow & & \downarrow{\scriptstyle \text{restriction}} \\
(B_v \otimes A/m)^* & \xrightarrow{\ \iota_{B_v}\ } & \mathrm{Ext}^1_{G_v}(\rho_1, \rho_2) \simeq H^1(G_v, \rho)
\end{array}
$$

and

$$
\begin{array}{ccc}
(C \otimes A/m)^* & \xrightarrow{\ \iota_C\ } & \mathrm{Ext}^1_{G_K}(\rho_2, \rho_1) \simeq H^1(G_K, \rho^*) \\
\downarrow & & \downarrow{\scriptstyle \text{restriction}} \\
(C_v \otimes A/m)^* & \xrightarrow{\ \iota_{C_v}\ } & \mathrm{Ext}^1_{G_v}(\rho_2, \rho_1) \simeq H^1(G_v, \rho^*).
\end{array}
$$

We can now prove Theorem 3, by contradiction. Assume that $R_x = F_x$. Since x is not a smooth point of X, it cannot be an étale point of κ, since $\kappa(x)$ is a smooth point of $\mathcal{W}$ (which is smooth at all points, being a union of balls). Thus $R_x = F_x$ is larger than the reduced closed point x, which means that the inclusion $B_v C_v \subset m$ is strict. Since $m = BC$, then at least one of the inclusions $B_v \subset B$ and $C_v \subset C$ has to be strict. By Nakayama's lemma, this means that at least one of the linear maps $B_v \otimes A/m \to B \otimes A/m$ and $C_v \otimes A/m \to C \otimes A/m$ fails to be surjective, or equivalently, that the dual of one of these maps fails to be injective. Using (ϵ), this means that at least one of the restriction maps

(4.1) $$H^1(G_K, \rho) \to H^1(G_v, \rho)$$

and

(4.2) $$H^1(G_K, \rho^*) \to H^1(G_v, \rho^*)$$

is not injective. Assuming the Bloch–Kato conjecture, the first map (4.1) is injective. Indeed an extension of 1 by ρ has good reduction (in the sense of Bloch–Kato) everywhere but perhaps at v (*cf.* footnote 3), and if it lies in the kernel of the first map, it is trivial (and in particular has good reduction) at v; so such an extension lies in $H^1_f(G_K, \rho)$, which is zero according to the Bloch–Kato conjecture.

We thus deduce that the second restriction map, (4.2), is not injective. This does not, by itself, leads to a contradiction. However, applying this reasoning to ρ and many of its twists, we will quickly derive a contradiction. We argue as follows.

Let $\tau \colon G_K \to 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^*$ be a Hodge–Tate character of G_K satisfing $\tau^\perp = \tau$ whose Hodge–Tate weight at v is negative. Let $k \in \mathbb{Z}$. By [H], there exists an automorphic representation π_k for $U(d)$ such that $\rho_{\pi_k} = \rho_1 \oplus \rho_2 \tau^k$. When $k \geq 0$, it is obvious that π_k satisfies all conditions of Theorem 2. (When $k < 0$, the hypothesis (WEI) may and will eventually fail). Hence the same reasoning as above tells us that

for $k \geq 0$, the map (4.2) for ρ replaced by $\rho\tau^k$ is not injective. That is, for $k \geq 0$, the map

$$(4.3) \qquad H^1(G_K, \rho^*\tau^{-k}) \to H^1(G_v, \rho^*\tau^{-k})$$

is not injective.

However, the non-injectivity of a map like (4.3) for all $k \geq 0$ implies its non-injectivity for all $k \in \mathbb{Z}$ (and even, all $k \in \mathbb{Z}_p$). This is because the function

$$d(k) = \dim \mathrm{Ker}(H^1(G_K, \rho^*\tau^{-k}) \to H^1(G_v, \rho^*\tau^{-k}))$$

is lower semi-continuous (this is easy, but see *e.g.,* [B3, page 15] for a proof). Hence for all $k \in \mathbb{Z}$, (4.3) is not injective.

But if we take $k \ll 0$, the Hodge–Tate weights of $\rho_*\tau^{-k}$ at $\bar{v}$ will all be ≥ 2, and hence any extension of 1 by $\rho_*\tau^{-k}$ will have good reduction at $\bar{v}$, as well as at all places not dividing p. If such an extension lies in the kernel of (4.3), it has good reduction everywhere, and hence must be 0 by Bloch–Kato conjecture (remember that $\rho^*\tau^{-k}$ also has motivic weight 0). This contradicts the non-injectivity of (4.3) and concludes the proof of Theorem 3.

Thanks and apologies In four talks at seminars and conferences in North America during the second half of 2008, I explained Theorem 1 and Proposition 2, with ideas of their proof. I want to thank the audiences of those talks for their questions and comments. In two of those talks, I also mentioned a third result, which was that the map (4.2), page 327, was not injective. As should be clear from the reading of this paper, this result is false. The proof I had in mind used the erroneous fact that $R_x = F_x$, which I then mistakenly believed to have proved along the lines of [BC1, théorème 2]. I realized when writing the first version of this paper at the end of 2008 that that result was false (or at least, in contradiction with the Bloch–Kato conjecture), and it took me a few weeks to identify with certainty the guilty lemma (namely that $R_x = F_x$). In doing so, I was helped by an email of R. Greenberg, whom I want to thank here. I also want to offer my apologies to the audience of the two seminars where I announced results whose proofs were insufficiently checked.

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