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# A REMARK ABOUT COMPONENTS OF RELATIVE TEICHMÜLLER SPACES

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Our aim is to compute for all $n>2$, $\psi(n, h)$, the number of components of a certain quotient of the fixed point set of an involution in the "mod-$n$" Teichmüller space. This answers part of a question raised by Earle [2] and corrects and extends the answer due to Zarrow (See Theorem 2 of [6]).

NOTATION. Let $X$ be a smooth surface of genus $g\geq 2$ and $M(X)$ the space of smooth complex structures with the $C^\infty$ topology. Let $\mathrm{Diff}(X)$ (respectively $\mathrm{Diff}^+(X)$) be the group of diffeomorphisms (respectively orientation preserving) of $X$, and $\mathrm{Diff}_n^+(X)$ those elements of $\mathrm{Diff}^+(X)$ which induce the identity on homology modulo $n$. Let $\sigma:\mathrm{Diff}(X)\to\mathrm{Diff}(X)/\mathrm{Diff}_0(X)$ (here $\mathrm{Diff}_0(X)=\{f\in\mathrm{Diff}(x)\mid f$ is homotopic to the identity$\}$), and let $\sigma_n:\mathrm{Diff}(X)\to\mathrm{Diff}(X)/\mathrm{Diff}_n^+(X)=\Gamma_n(X)$. $T_n(X)=M(X)/\mathrm{Diff}_n^+(X)$ is the mod-$n$ Teichmüller space of $X$. Let $h$ be an involution in $\mathrm{Diff}^+(X)$, $T_n(X)^{\sigma_n(h)}$ the fixed point set of $\sigma_n(h)$ acting on $T_n(X)$, and $\Gamma_n(h)$ the normalizer of $\sigma_n(h)$ in $\sigma_n(\mathrm{Diff}^+(X))$.

Finally we let $\psi(n, h)$ be the number of components of $T_n(X)^{\sigma_n(h)}/\Gamma_n(h)$. We use Earle's result (Theorem 4b of [2]) that $\psi(n, h)$ is equal to the number of $\mathrm{Diff}^+(X)$ conjugacy classes of involutions $p$ in $\mathrm{Diff}^+(X)$ with $\sigma_n(p)=\sigma_n(h)$.

PROPOSITION. *If* $n>2$, $\psi(n, h)=1$.

**Proof.** We define some matrices. $I_q$ will be the $q\times q$ identity matrix;

$$L(0, t, g')=\begin{pmatrix} S & 0\\ 0 & S\end{pmatrix} \quad\text{and}\quad L(1, 0, k)=\begin{pmatrix} T & 0\\ 0 & T\end{pmatrix}$$

where

$$S=\begin{pmatrix} 0 & I_{g'} & 0\\ I_{g'} & 0 & 0\\ 0 & 0 & -I_t\end{pmatrix} \quad\text{and}\quad T=\begin{pmatrix} 0 & I_k & 0\\ I_k & 0 & 0\\ 0 & 0 & 1\end{pmatrix}. \quad \text{Here}\quad g=2g'+t=2k+1. \quad (1)$$

Here $q$, $t$, $g$, $g'$ and $k$ are integers.

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The results of Theorem 3 for $\mu(2, g, 0, T)$, of [4], together with Proposition 1 of [3] and Theorems 1(ii), 1(iii), and 2 of [5] imply that $\sigma(p)$ is conjugate to an involution whose action on a canonical homology basis is given by $L(r, s, m)$ where either $r=0$ or $r=1$ and $s=0$. Let $h(r, s, m)$ in $\text{Diff}^+(X)$ induce the matrix $L(r, s, m)$. A conjugate of $\sigma(p)$ equals $\sigma(h(r, s, m))$ for some triple $(r, s, m)$. By the remark in Section 5G of [1], we can conclude that $p$ and $h(r, s, m)$ are already conjugate in $\text{Diff}^+(X)$. Therefore, it suffices to show that there is no symplectic matrix $K$ with $KL(r, s, m) \equiv L(u, v, w)K$ modulo $n$ unless $r=u$, $s=v$, and $m=w$.

Assume that $L(r, s, m)$ and $L(u, v, w)$ are conjugate modulo $n$. Then so are $\tau = I_{2g} + L(r, s, m)$ and $\rho = I_{2g} + L(u, v, w)$. $\tau$ and $\rho$ are endomorphisms of the abelian group $G = \mathbb{Z}/n\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n\mathbb{Z}$, $2g$ copies. If $\tau$ is any group homomorphism, let $R(\tau)$ be the minimal number of generators of $\tau(G)$. Since $\tau$ and $\rho$ are homomorphisms which are conjugate by an automorphism of $G$, $\tau(G)$ and $\rho(G)$ are isomorphic, so that $R(\tau) = R(\rho)$. In our case $\tau(G)$ is generated by the rows of $I_{2g} + L(r, s, m)$. We can compute $R(I_{2g} + L(0, t, g')) = 2g'$ and $R(I_{2g} + L(1, 0, k)) = 2k + 2$. If $r=0$ and $u=0$, this forces $m=w$ and $s=v$. If $r=0$ but $u=1$ and $v=0$, this forces $2m = 2w+2$. But by (1) $g = 2m + s = 2w+1$. This contradicts the fact that $s$ is nonnegative.

REMARK. The referee has pointed out that the result of this paper provides a new proof that $R(X, H)$, the relative Riemann space, is a complex algebraic variety when $H \subset \text{Diff}^+(X)$ is generated by a sense preserving involution [see 2].

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