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ON CERTAIN PAIRS OF AUTOMORPHISMS OF RINGS, II

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Abstract

Let R be a prime ring of characteristic not 2. Automorphisms α and β of R satisfying $\alpha \neq \beta, \alpha \neq \beta^{-1}$, and $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ are characterized. This result is an algebraic analogue of some results for operator algebras.

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The equation

$$\alpha + \alpha^{-1} = \beta + \beta^{-1}$$

where α and β are $*$-automorphisms of a von Neumann algebra, has been considered in a series of papers by Thaheem (see [6] and references given in [1, 2, 3, 6]). It has turned out that besides the obvious solution (that is, $\alpha = \beta$ or $\alpha = \beta^{-1}$, or more generally, $\alpha = \beta$ on an ideal of an algebra and $\alpha = \beta^{-1}$ on its direct summand) some unexpected situations (at least when α and β do not commute) can occur. Batty improved Thaheem's results by, on the one hand, extending the treatment of the equation to C^*-algebras, and on the other hand, by giving conditions that are both necessary and sufficient for the solution of the equation [1, Theorem 3.1].

In [2, 3] we have considered the equation from an entirely algebraic point of view. More precisely, we have dealt with pairs of automorphisms of (semi)prime rings satisfying the equation. Concerning the prime case, we can summarize the results obtained in these two papers in the following statement: Let R be a prime ring of characteristic not 2, and let α, β be automorphisms of R satisfying $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ and $\alpha \neq \beta, \alpha \neq \beta^{-1}$. Then α and β do not commute [2, Corollary 3], α and β cannot

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both be inner [3, Corollary 3], $\alpha^4 = \beta^4 = 1$ and $\alpha^2 = \beta^2$ [3, Corollary 2]. The goal of this paper is to add the conditions that are also sufficient, and not only necessary, for the solution of the equation.

We start with an observation describing a non-trivial solution of the equation. By $\mathrm{Ad}(q)$ we denote the inner automorphism given by $\mathrm{Ad}(q)(x) = qxq^{-1}$.

LEMMA. *Let R be a ring with 1 and β be an automorphism of R. Suppose that $\beta^2 = \mathrm{Ad}(q)$ for some invertible $q \in R$ such that $q \notin Z$, the center of R, $q^2 \in Z$, $1 - q^2$ is invertible and $\beta(q) = -q$. Then $\alpha = \mathrm{Ad}(1+q)\beta$ is an automorphism of R satisfying $\alpha + \alpha^{-1} = \beta + \beta^{-1}$, $\alpha \neq \beta$ and $\alpha \neq \beta^{-1}$. Moreover, $\alpha^4 = \beta^4 = 1$, $\alpha^2 = \beta^2$, and if R is 2-torsion free, then α and β are outer automorphisms.*

PROOF. Set $\lambda = q^2 \in Z$. We have $\beta^4 = (\beta^2)^2 = \mathrm{Ad}(q)^2 = \mathrm{Ad}(\lambda) = 1$, so that $\beta^{-1} = \beta^3 = \mathrm{Ad}(q)\beta$. That is, $\beta^{-1}(x) = \lambda^{-1}q\beta(x)q$. Next, noting that $(1+q)^{-1} = (1-\lambda)^{-1}(1-q)$ we have

$$\alpha(x) = (1-\lambda)^{-1}(1+q)\beta(x)(1-q).$$

It remains to compute α^{-1}. Since $\alpha = \mathrm{Ad}(1+q)\beta$, we have $\alpha^{-1} = \beta^{-1}(\mathrm{Ad}(1+q))^{-1} = \beta^{-1}\,\mathrm{Ad}((1+q)^{-1})$. Therefore, using $\beta(q) = -q$, we get

$$\begin{aligned}
\alpha^{-1}(x) &= \beta^{-1}((1+q)^{-1}x(1+q)) \\
&= \lambda^{-1}q\beta((1+q)^{-1}x(1+q))q \\
&= \lambda^{-1}q(1-q)^{-1}\beta(x)(1-q)q \\
&= \lambda^{-1}(1-\lambda)^{-1}(q+\lambda)\beta(x)(q-\lambda).
\end{aligned}$$

A direct calculation now shows that $\alpha + \alpha^{-1} = \beta + \beta^{-1}$. Clearly, $1+q \notin Z$ for $q \notin Z$. This shows that $\alpha \neq \beta$. Similarly, $\alpha = \beta^{-1}$, or equivalently, $\alpha^{-1} = \beta$ is impossible for $q + \lambda \notin Z$. Next, note that $\beta\,\mathrm{Ad}(1+q) = \mathrm{Ad}(1-q)\beta$ for $\beta(q) = -q$. This yields $\alpha^2 = (\mathrm{Ad}(1+q)\beta)^2 = \mathrm{Ad}(1+q)\,\mathrm{Ad}(1-q)\beta^2 = \mathrm{Ad}(1-\lambda)\beta^2 = \beta^2$. Thus, $\alpha^2 = \beta^2$ and so $\alpha^4 = 1$.

Finally, suppose that either α or β is an inner automorphism. Then, of course, both are inner. Therefore, $\beta = \mathrm{Ad}(a)$ for some (invertible) $a \in R$. In particular, $-q = \beta(q) = aqa^{-1}$, that is, $aq + qa = 0$. But, on the other hand, $qaq^{-1} = \beta^2(a) = a^2aa^{-2} = a$ gives $aq = qa$. Therefore, if R is 2-torsion free, it follows that $aq = 0$, contradicting the invertibility of a and q.

The conditions given in the lemma are somewhat similar, but not quite the same as those obtained in [1]; of course, it is natural to expect some differences for only automorphisms preserving adjoints are considered in [1].

EXAMPLE 1. (cf. [1, Corollary 2.4] and an example in [6]) Let S be a ring with 1, containing a non-central element b such that b^2 is central and $1 - b^2$ is invertible. Let $R = S \oplus S$. Define an automorphism β of R by

$$\beta(x, y) = (byb^{-1}, x).$$

Then $\beta^2(x, y) = (bxb^{-1}, byb^{-1})$. That is, $\beta^2 = \mathrm{Ad}(q)$ where $q = (b, -b)$. It is easy to see that q satisfies all the conditions given in the lemma.

In the next example (unlike the first) the ring is prime.

EXAMPLE 2. Let R be the ring of 2 by 2 matrices over the complex numbers. We define β by

$$\beta\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \mathrm{Ad}\left(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right)\begin{pmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{pmatrix},$$

which yields

$$\beta\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \bar{x} + \bar{y} + \bar{z} + \bar{w}, & -\bar{x} + \bar{y} - \bar{z} + \bar{w} \\ -\bar{x} - \bar{y} + \bar{z} + \bar{w}, & \bar{x} - \bar{y} - \bar{z} + \bar{w} \end{pmatrix}$$

Note that $\beta^2 = \mathrm{Ad}(q)$ where

$$q = \begin{pmatrix} 0 & -it \\ it & 0 \end{pmatrix},$$

where t is any non-zero number. Note also that q is not, while q^2 is central, and that

$$1 - q^2 = \begin{pmatrix} 1 - t^2 & 0 \\ 0 & 1 - t^2 \end{pmatrix},$$

is invertible whenever $t \neq \pm 1$. Finally, $\beta(q) = -q$ for every real t.

It is our aim to show that the non-trivial solution given in the lemma is basically the only non-trivial solution in the case when the ring is prime. The only difference is that the element q does not necessarily lie in R but rather in $Q_s(R)$, the symmetric Martindale ring of quotients of R. For a definition and basic properties of $Q_s(R)$ we refer the reader to [5, Chapter 3]. Let us just recall that if R is a prime ring, then $Q_s(R)$ is a prime ring containing R, its center C (the so-called extended centroid of R) is a field, and that every automorphism of R extends uniquely to an automorphism of $Q_s(R)$. Therefore, when dealing with an automorphism α of R, there is no loss of generality in assuming that α is actually an automorphism of $Q_s(R)$. An automorphism α of R is said to be X-*inner* if $\alpha = \mathrm{Ad}(q)$ for some $q \in Q_s(R)$ (that is, α is inner as an automorphism of $Q_s(R)$). If α is not X-inner, then it is called X-*outer*.

Our main result is

THEOREM. *Let R be a prime ring of characteristic not 2, and let α and β be automorphisms of R such that $\alpha \neq \beta$ and $\alpha \neq \beta^{-1}$. Then $\alpha + \alpha^{-1} = \beta + \beta^{-1}$ if and only if there exists an invertible $q \in Q_s(R)$ such that the following conditions hold:*

(i) $q \notin C$, $q^2 \in C$, *and* $q^2 \neq 1$;

(ii) $\beta^2 = \mathrm{Ad}(q)$;

(iii) $\beta(q) = -q$;

(iv) $\alpha = \mathrm{Ad}(1+q)\beta$.

Moreover, in this case we have $\alpha^4 = \beta^4 = 1$, $\alpha^2 = \beta^2$ and both α and β are X-outer.

PROOF. We begin with some computations (cf. [2, 3]). Using $\alpha - \beta = \beta^{-1} - \alpha^{-1}$ we obtain

$$\begin{aligned}(\alpha-\beta)(x)\alpha(y) + \beta(x)(\alpha-\beta)(y) &= (\alpha-\beta)(xy)\\ &= (\beta^{-1}-\alpha^{-1})(xy)\\ &= (\beta^{-1}-\alpha^{-1})(x)\beta^{-1}(y) + \alpha^{-1}(x)(\beta^{-1}-\alpha^{-1})(y)\\ &= (\alpha-\beta)(x)\beta^{-1}(y) + \alpha^{-1}(x)(\alpha-\beta)(y).\end{aligned}$$

Thus, $(\alpha-\beta)(x)(\alpha-\beta^{-1})(y) + (\beta-\alpha^{-1})(x)(\alpha-\beta)(y) = 0$. However, as $\beta - \alpha^{-1} = \alpha - \beta^{-1}$, this can also be written in the form

$$(\alpha-\beta)(x)(\alpha-\beta^{-1})(y) + (\alpha-\beta^{-1})(x)(\alpha-\beta)(y) = 0.$$

Replacing y by $\beta(y)z$ in this relation we obtain

$$\begin{aligned}&(\alpha-\beta)(x)(\alpha-\beta^{-1})(\beta(y))\alpha(z) + (\alpha-\beta)(x)y(\alpha-\beta^{-1})(z)\\ &\quad + (\alpha-\beta^{-1})(x)(\alpha-\beta)(\beta(y))\alpha(z) + (\alpha-\beta^{-1})(x)\beta^2(y)(\alpha-\beta)(z) = 0.\end{aligned}$$

By the previous identity, the sum of the first and the third term equals zero; thus, we have derived the key relation

$$(\alpha-\beta)(x)y(\alpha-\beta^{-1})(z) + (\alpha-\beta^{-1})(x)\beta^2(y)(\alpha-\beta)(z) = 0.$$

Since $\alpha \neq \beta$ and $\alpha \neq \beta^{-1}$, it follows from [5, Lemma 12.1] that β^2 is X-inner, that is, $\beta^2 = \mathrm{Ad}(q_0)$ for some invertible $q_0 \in Q_s(R)$. Thus,

$$(\alpha-\beta)(x)y(\alpha-\beta^{-1})(z) + (\alpha-\beta^{-1})(x)q_0 y q_0^{-1}(\alpha-\beta)(z) = 0,$$

which means that

$$(\alpha-\beta)(x) \otimes_C (\alpha-\beta^{-1})(z) + (\alpha-\beta^{-1})(x)q_0 \otimes_C q_0^{-1}(\alpha-\beta)(z) = 0$$

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(see, for example, [4, Lemma 1]). Consequently, $(\alpha - \beta)(x)$ and $(\alpha - \beta^{-1})(x)q_0$ are linearly dependent for every $x \in R$. Picking x_0 such that $(\alpha - \beta^{-1})(x_0) \neq 0$ we thus have $(\alpha - \beta)(x_0) + \mu(\alpha - \beta^{-1})(x_0)q_0 = 0$ for some non-zero $\mu \in C$, whence

$$(\alpha - \beta^{-1})(x_0)q_0 \otimes_C (-\mu(\alpha - \beta^{-1})(z) + q_0^{-1}(\alpha - \beta)(z)) = 0,$$

and so

$$(\alpha - \beta^{-1})(z) = q^{-1}(\alpha - \beta)(z),$$

where $q = \mu q_0$. Of course, $\beta^2 = \mathrm{Ad}(q_0) = \mathrm{Ad}(q)$. Similarly we see that

$$(\alpha - \beta)(x) = -(\alpha - \beta^{-1})(x)q.$$

These two identities imply that

$$\begin{aligned}(\alpha - \beta)(x) &= -q^{-1}(\alpha - \beta)(x)q \\ &= -\beta^{-2}(\alpha - \beta)(x) \\ &= (-\beta^{-2}\alpha + \beta^{-1})(x).\end{aligned}$$

Thus, $-\beta^{-2}\alpha + \beta^{-1} = \alpha - \beta = \beta^{-1} - \alpha^{-1}$, and so $\beta^{-2}\alpha = \alpha^{-1}$, that is, $\alpha^2 = \beta^2$.

Similarly we see that $(\alpha - \beta^{-1})(x) = -q^{-1}(\alpha - \beta^{-1})(x)q$, which means that $\alpha - \beta^{-1} = -\beta^{-2}(\alpha - \beta^{-1})$. Noting that $-\beta^{-2}(\alpha - \beta^{-1}) = -\alpha^{-2}\alpha + \beta^{-3} = -\alpha^{-1} + \beta^{-3}$ and using $\alpha - \beta^{-1} = \beta - \alpha^{-1}$, it follows that $\beta = \beta^{-3}$, that is, $\beta^4 = 1$ (as already mentioned, the relations $\alpha^2 = \beta^2$ and $\beta^4 = 1$ have already been obtained in [3]; however, the proof given in the present paper is different).

Suppose that $\beta^2 = 1$. Then $\alpha^2 = 1$, and so $2\alpha = \alpha + \alpha^{-1} = \beta + \beta^{-1} = 2\beta$. Since the characteristic of R is not 2, this gives $\alpha = \beta$, contrary to the assumption. Thus $1 \neq \beta^2 = \mathrm{Ad}(q)$, that is, $q \notin C$. Furthermore, $\mathrm{Ad}(q^2) = (\beta^2)^2 = \beta^4 = 1$, which yields $q^2 \in C$.

The relation $(\alpha - \beta)(x) = -(\alpha - \beta^{-1})(x)q$ can be written in the form $\beta(x) = \alpha(x)(1 + q) - \beta^{-1}(x)q$. Replacing x by xy in this relation we obtain

$$\begin{aligned}\beta(x)\beta(y) &= \alpha(x)\alpha(y)(1 + q) - \beta^{-1}(x)\beta^{-1}(y)q \\ &= \alpha(x)\alpha(y)(1 + q) + \beta^{-1}(x)(\beta(y) - \alpha(y)(1 + q)).\end{aligned}$$

Thus,

$$(\beta - \beta^{-1})(x)\beta(y) = (\alpha - \beta^{-1})(x)\alpha(y)(1 + q).$$

This implies

$$\begin{aligned}(\alpha - \beta^{-1})(x)\alpha(y)(1 + q)\beta(z) &= (\beta - \beta^{-1})(x)\beta(y)\beta(z) \\ &= (\beta - \beta^{-1})(x)\beta(yz) \\ &= (\alpha - \beta^{-1})(x)\alpha(yz)(1 + q) \\ &= (\alpha - \beta^{-1})(x)\alpha(y)\alpha(z)(1 + q).\end{aligned}$$

That is,

$$(\alpha - \beta^{-1})(R)R((1+q)\beta(z) - \alpha(z)(1+q)) = 0.$$

As $\alpha \neq \beta^{-1}$, we have $(1+q)\beta(z) = \alpha(z)(1+q)$, by the primeness of R.

In order to prove that $\alpha = \mathrm{Ad}(1+q)\beta$, we now only have to show that $1+q$ is invertible. Since $(1+q)(1-q) = 1-q^2$ lies in C, we see that this is true, unless $q^2 = 1$. However, $q^2 = 1$ yields $(1+q)\beta(z)(1-q) = \alpha(z)(1+q)(1-q) = 0$, that is, $(1+q)R(1-q) = 0$, which implies $q = 1$ or $q = -1$. But this is impossible for $q \notin C$.

It remains to show that $\beta(q) = -q$. Set $p = \beta(q) \in Q_s(R)$ and $\lambda = q^2 \in C$. Of course, $q^{-1} = \lambda^{-1}q$. We have

$$\beta^{-1}(x) = \beta^3(x) = \beta^2(\beta(x)) = \lambda^{-1}q\beta(x)q.$$

Thus

$$\begin{aligned} x &= \beta(\beta^{-1}(x)) = \beta(\lambda^{-1}q\beta(x)q) \\ &= \beta(\lambda^{-1})p\beta^2(x)p \\ &= \beta(\lambda^{-1})\lambda^{-1}pqxqp, \end{aligned}$$

which means that $\beta(\lambda^{-1})\lambda^{-1}pq \otimes_C qp = 1 \otimes_C 1$. Therefore, $pq \in C$, which in turn implies that $p \in Cq$. That is, $p = \omega q$ for some $\omega \in C$. We have to show that $\omega = -1$.

We have

$$\begin{aligned} \alpha^{-1}(x) &= \alpha^3(x) = \beta^2(\alpha(x)) = \lambda^{-1}q\alpha(x)q \\ &= \lambda^{-1}q(1+q)\beta(x)(1+q)^{-1}q \\ &= \lambda^{-1}(1-\lambda)^{-1}(q+\lambda)\beta(x)(q-\lambda), \end{aligned}$$

whence

$$\begin{aligned} x &= \alpha^{-1}(\alpha(x)) = \lambda^{-1}(1-\lambda)^{-1}(q+\lambda)\beta(\alpha(x))(q-\lambda) \\ &= \lambda^{-1}(1-\lambda)^{-1}(q+\lambda)\beta((1+q)\beta(x)(1+q)^{-1})(q-\lambda) \\ &= \lambda^{-1}(1-\lambda)^{-1}(q+\lambda)(1+\omega q)\beta^2(x)(1+\omega q)^{-1}(q-\lambda) \\ &= \lambda^{-2}(1-\lambda)^{-1}(q+\lambda)(1+\omega q)qxq(1+\omega q)^{-1}(q-\lambda). \end{aligned}$$

A tensor product consideration now shows that $(q+\lambda)(1+\omega q)q \in C$, that is, $(q+\lambda)(q+\omega\lambda) = \lambda + \lambda q + \omega\lambda q + \omega\lambda^2 \in C$, so that $(1+\omega)\lambda q \in C$. Since $q \notin C$ and $\lambda \neq 0$, it follows that $\omega = -1$.

The proof of the converse, as well as the proof of X-outerness of α and β, is the same as the proof of the lemma.

Let us mention that one indeed has to exclude rings of characteristic 2. Namely, when the characteristic of a ring is 2, any two automorphisms α and β such that $\alpha^2 = \beta^2 = 1$ satisfy $\alpha + \alpha^{-1} = 0 = \beta + \beta^{-1}$. In particular, consider the ring of 2 by 2 matrices over a field of characteristic 2. Let $\alpha = 1$ and $\beta = \mathrm{Ad}(b)$ with

$$b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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