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On supersonic plasma flow around an obstacle

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Abstract. Supersonic plasma flow around an object large compared with the Debye distance is treated using an isothermal gas dynamics model. The case of (initially) subsonic flow has been studied previously using this model, the motivation then being the use of Langmuir probes. In supersonic plasma flow Mach cones describing weak discontinuities rather than shock waves are predicted. A comparison has been made with particle-in-cell simulations carried out by Willis et al. (Willis, C. T. N., Allen, J. E., Coppins, M. and Bacharis, M. 2011 *Phys. Rev.* E, **84**, 046410), where such Mach cones are observed. Other features cannot be explained by the isothermal gas dynamics model, these include the appearance, at high supersonic velocities, of an ion-free region downstream.

1. Introduction

A review is given of an isothermal gas dynamic theory which was previously employed to identify the plasma boundary as a Mach surface (Stangeby and Allen 1970). It was then used to study subsonic flow toward an obstacle (Stangeby and Allen 1971). In that case the flow became transonic in order that the normal component of the ion velocity reached the ion acoustic velocity at the plasma boundary (Bohm 1949). In the present paper the case of supersonic flow is considered and Mach cones are predicted which are not shock waves but weak discontinuities.

A comparison is then made with particle-in-cell (PIC) simulations carried out by Willis et al. (2011), using the code SCEPTIC developed by Hutchinson (2002, 2003). The gas dynamic model is based on a zero ion temperature and a vanishingly small Debye distance, so that a complete agreement is not expected. The object of the paper is to gain some insight into the nature of the interesting wake effects found in the simulations.

The present work is part of a programme on dust in plasmas, with particular reference to dust in tokamaks or other fusion machines. Dust particles in a fusion reactor could have serious effects on its performance, partly due to bremsstrahlung radiation following evaporation of the dust. In addition, there could be serious health and safety questions concerning the retention of tritium by metallic particles. Other applications of this work, however, will be to dust in astrophysics, and in space and atmospheric physics. The charging of spacecraft and other bodies in space represents another field of application.

2. Isothermal gas dynamic theory

It is necessary to define what is meant by a 'quasi-neutral' plasma, since this concept is still imperfectly understood. Some writers believe that one of the properties of a plasma is that it shields out DC electric fields. That is not the case. Poisson's equation for space charge is

$$\nabla^2 V = -\frac{n_i - n_e}{\varepsilon_0} e,$$

which can be normalized to read

$$\left(\frac{\lambda_D}{L}\right)^2 \nabla^2 \eta = n_{en} - n_{in}, \tag{1}$$

where λ_D is the electron Debye distance, $\eta = eV/kT_e$ and L is the characteristic length of the problem in question. The Debye distance,

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k T_e}{n_0 e^2}} \tag{2}$$

is defined in terms of n_0, the electron density in the undisturbed plasma. When $(\lambda_D/L)^2 \ll 1$, $n_i \approx n_e$ and we have a plasma. This does not mean that the electric field has been screened out; very small differences between the electron and ion densities can produce weak electric fields extending over considerable distances. A very useful model is obtained by letting $(\lambda_D/L)^2 \to 0$; we then have a two-scale model comprising the plasma and sheath regions.

Consider now a collision-free plasma with cold ions ($T_i = 0$) and Boltzmann electrons in a steady state. The momentum equation for the ions can be written,

$$n_i M(\mathbf{v} \cdot \nabla)\mathbf{v} = n_i e\mathbf{E}. \tag{3}$$

The electron thermal velocity greatly exceeds the velocity of sound, which means that the electrons closely approach the equilibrium state. As a result, the electron

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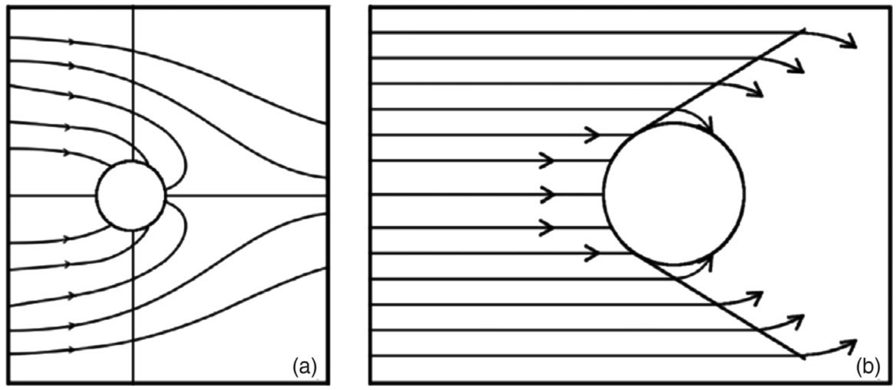

Figure 1. (a) Streamlines for transonic flow around a cylinder, the ion-acoustic Mach number is 0.3 (Stangeby and Allen 1971). (b) Schematic diagram of the predicted truncated Mach cone in supersonic flow around a sphere. A rarefaction wave, rather than a shock wave, begins at the Mach cone.

density is given by the Boltzmann relation,

$$\begin{cases} n_e = n_0 \exp(eV/kT_e) \\ \nabla p_e = -n_e e\mathbf{E} \end{cases}. \tag{4}$$

The space charge equation can be written in the form,

$$\nabla \cdot \mathbf{E} = \frac{n_i - n_e}{\varepsilon_0} e. \tag{5}$$

Using (3)–(5), the following equation can be obtained,

$$n_i M(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p_e - \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} = 0. \tag{6}$$

In the plasma region the third term is much smaller than the second term, the ratio being $(n_i - n_e)/n_e$, so that we can write

$$n_i M(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p_e = 0, \tag{7}$$

where we have put $n_i = n_e = n$ and $\nabla p_e = kT_e \nabla n$. We can note in passing that the neglected term is the divergence of the Maxwell stress tensor. The equation of continuity is

$$\nabla \cdot (n\mathbf{v}) = 0. \tag{8}$$

If we assume an axially symmetric flow, or an irrotational flow at infinity, a velocity potential ϕ can be introduced,

$$\mathbf{v} = \nabla\phi. \tag{9}$$

Comparison with gas dynamics gives (7) for momentum and (8) for continuity and (4) replaces the energy equation. Elimination, together with the introduction of the velocity potential yields the following,

$$\nabla^2\phi = \frac{1}{2c_s^2}\nabla\phi \cdot \nabla(\nabla\phi \cdot \nabla\phi), \tag{10}$$

where $c_s = (kT_e/M)^{1/2}$ and is constant.

This is the basic equation for compressible, irrotational inviscid flow, with c_s interpreted as the sound speed. The case of (initially) subsonic flows can be found in the literature (Stangeby and Allen 1970; 1971), the motivation then being the use of Langmuir probes, although reference was made to dust particles in gas discharges. Figure 1(a) is an example of transonic plasma flow around a cylinder.

3. Supersonic flow

An interesting difference between the present case and the familiar case of supersonic flow (see Fig. 2) is that there is no bow shock; this is because the plasma flow does not have to be diverted around the obstacle, part of the flow reaches it without disturbance and is thereby collected. Recombination of ions and electrons takes place on the surface, with subsequent re-emission as neutral atoms or molecules; the collision-free case is considered here so that the neutrals play no further rôle (in this model). In this way the obstacle is an effective sink for plasma reaching the plasma boundary. When the sheath is very thin, the boundary condition is essentially the result obtained by Bohm in 1949, i.e. the ion velocity must be greater than or equal to the ion acoustic velocity,

$$v \geqslant \left(\frac{kT_e}{M}\right)^{1/2}. \tag{11}$$

When the ions arrive at the boundary at an angle, the normal component of velocity is to be taken (Stangeby and Allen 1970). In the vast majority of cases the Bohm criterion with the equality sign is applicable, see the review article (Allen 2009), but the present case is an exception, another example being that of a rapidly growing sheath (Allen and Andrews 1970).

The application of the inequality form of the Bohm criterion, to the case of a spherical object, leads to the conclusion that a truncated Mach cone should appear, starting on the surface where the normal component of the ion velocity has fallen to the ion acoustic speed. This is shown in Fig. 1(b) for comparison with the (initially) subsonic case, and in Fig. 2(b) for comparison with the bow shock found in the usual supersonic gas dynamic case. The truncated Mach cone predicted in the present case is not a shock wave, but a weak discontinuity

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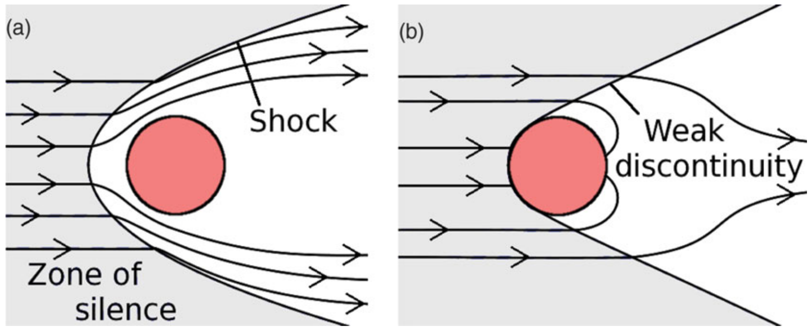

Figure 2. (Colour online) Schematic representation of the flow pattern in (a) the conventional case of an object in supersonic flow, and (b) an absorbing dust grain in supersonic flow (Willis et al. 2011).

(Landau and Lifshitz 1953) where the velocity and density do not suddenly change, but the subsequent flow is completely changed. The first spatial derivatives are discontinuous.

4. Simulation

A comparison has been made with PIC simulations carried out by Willis et al. (2011) using the code SCEPTIC developed by Hutchinson (2002, 2003). The collision-free positive ions are moved individually in the electrostatic field, the number involved being 32×10^6, and the potential distribution is given by Poisson's equation. The electrons are assumed to obey the Boltzmann relation, an assumption based on the fact that the electron thermal velocity is large compared with macroscopic plasma velocities. Another essential point is that the plasma is taken to be large compared with the radius of the sphere; as a consequence, the electron gas represents a large store of energy. Magnetic fields are not considered in the present paper, the Boltzmann relation would no longer be valid in the presence of a magnetic field. Cylindrical coordinates are employed with the z-direction being in the direction of flow. Further details of the SCEPTIC code can be found in Hutchinson (2002, 2003). The simulations discussed in the present paper are those for large values of $\rho = (a/\lambda_D)$, and for a range of values of $\beta = (T_i/T_e)$; a is the radius of the sphere. A truncated Mach cone, as predicted, is shown in Fig. 3. This cone does not represent a shock wave; plots of velocity and ion density show that there are no jumps in velocity or ion density. The cone is akin to the weak discontinuity predicted by the gas dynamic theory (Landau and Lifshitz 1953). At low ion temperatures ($\beta < 0.5$) a second (inner) cone can be seen, which had not been predicted, and which again represents a weak discontinuity rather than a shock wave. This inner cone is formed by a converging flow which has undergone acceleration and then deceleration, the cone is well defined for small values of β. The second cone is no longer to be seen at higher values of β (see Fig. 4). Experimental work by Merlino and D'Angelo (1987) has shown a rarefaction wave in the wake of a negatively charged object, but the authors assumed that the ion deflection was due to the electric field in the sheath region, rather than an electric field in the quasi-neutral plasma. Plots of the average ion velocity in the axial direction, together with the ion density, are shown in Fig. 5, these refer to a chosen plane downstream (at $z = 2a$). Figure 6 shows the position of the stagnation point, the downstream sheath boundary, and the ion density at the stagnation point. The unit of flow velocity is the Bohm velocity or ion acoustic velocity (this expression refers historically to the case of zero temperature ions). The term 'stagnation point' might be a misnomer, but it is used to indicate the point at which the average ion velocity is zero. It is noteworthy that a shock wave is not developed before the stagnation point, as would be predicted by the gas dynamic theory. It is seen in Fig. 6 that the sheath edge moves toward the sphere and then moves away from it as the flow velocity is further increased. At high velocities an 'ion-free' region develops behind the sphere where a negative space charge sheath borders part of the surface. We can note that a vacuum region cannot be developed in the case of isothermal gas dynamics (Shu 1992).

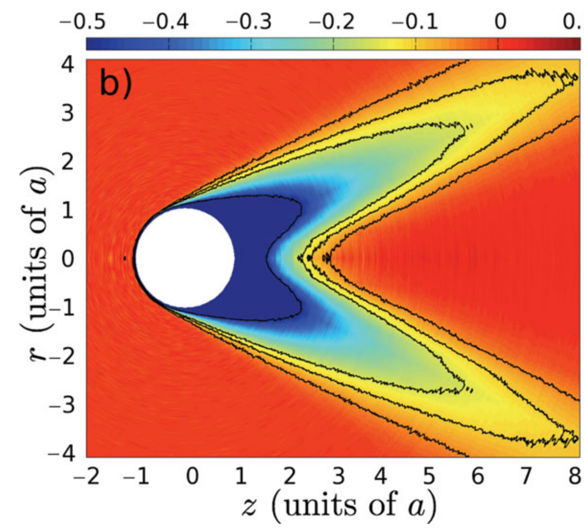

Figure 3. (Colour online) Potential distribution V (units of kT_e/e) illustrating a truncated Mach cone together with an inner cone for $\beta = 0.2, \rho = 80, u = 2.5c_s$ (Willis et al. 2011).

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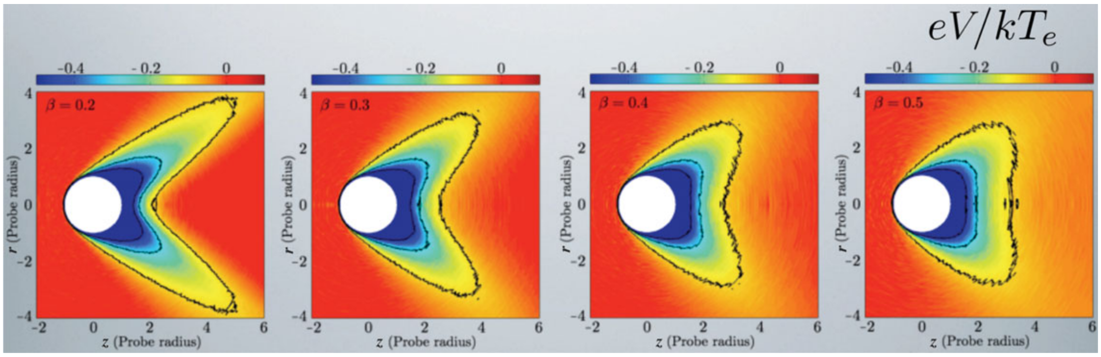

Figure 4. (Colour online) Potential plots showing the disappearance of the second (inner) Mach cone as β increases for $u = 1.75c_s$. From left to right $\beta = 0.2, 0.3, 0.4$, and 0.5.

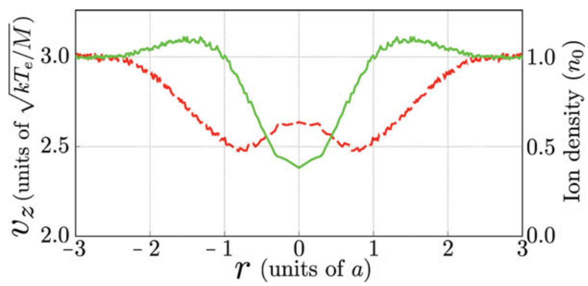

Figure 5. (Colour online) The axial component of velocity v_z (solid, green), and the ion density (dashed, red) at a distance $z = 2a$, $\beta = 0.3$, $u = 3.0c_s$, $\rho = 40$ (Willis et al. 2011).

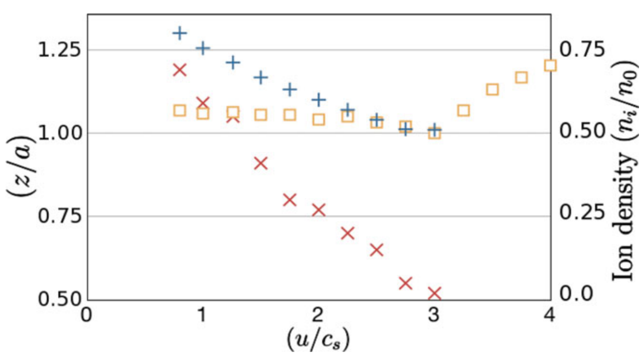

Figure 6. (Colour online) Position of the downstream sheath boundary (□, gold), position of the stagnation point (+, blue), and the ion density at the stagnation point (×, red), $\beta = 0.2$ and $\rho = 80$ (Willis et al. 2011).

Wake structures and cones have previously been observed by Hutchinson (2002), but not explored in any detail. Mach cones have also been reported by Miloch (2010), who similarly employed a PIC code, but the inherent analogy with gas dynamics was not discussed.

5. Interpretation

We can make a comparison between the 'cold ion' gas dynamic model and the results of the PIC simulation. A theory, including a finite ion temperature, has not yet been developed; it would require a solution of the Vlasov equation for collision-free ions, and the Boltzmann equation for electrons. Another point is that a quasi-neutral plasma has been considered in the gas dynamic theory, which corresponds to a vanishingly small Debye distance compared with the radius of the spherical object (probe or dust particle). Thus, we cannot expect perfect agreement between the theory and the PIC simulation, but the theory might give us some insight into the observed phenomena.

In a region of supersonic flow, a Mach surface is defined such that the fluid velocity component perpendicular to the surface is equal to the sound speed. Stangeby and Allen (1970) established that the plasma boundary was a Mach surface, but their analysis referred to (initially) subsonic flows. It corresponded to the marginal version of the Bohm criterion, i.e., when the equality sign is valid, and the plasma boundary was found to be a closed Mach surface. A primary Mach cone has been found in simulations, as predicted. It is a truncated cone starting at a circle on the spherical surface, the position depending on the flow velocity. The conical surface can be described as akin to the weak discontinuity of Fluid Mechanics (Landau and Lifshitz 1953); it is certainly not a shock wave. Near the beginning of the truncated cone the ions are deflected and accelerated in a presheath region in order to satisfy the Bohm criterion at the plasma boundary. Further along the cone the plasma is deflected and accelerated as it enters the wake region, and is then decelerated as it approaches the inner cone. The secondary (inner) cone had not been predicted when the simulations were carried out.

It is of interest to note that at high flow velocities an ion-free region appeared downstream adjacent to the spherical surface, a negative space charge sheath had replaced the usual positive space charge sheath. Clearly, the plasma boundary is not always a closed Mach surface of the kind studied in Stangeby and Allen (1970).

A major difficulty must now be discussed. In a collision-free plasma, which satisfies the Vlasov equation, it is not the case that $(p/\rho^{\gamma}) = C$, i.e., one cannot employ simple polytropic gas laws. The concept of a simple 'velocity of sound', as employed in the first part of this paper, is only valid for cold ions; in general, ion acoustic waves experience Landau damping (Clemmow

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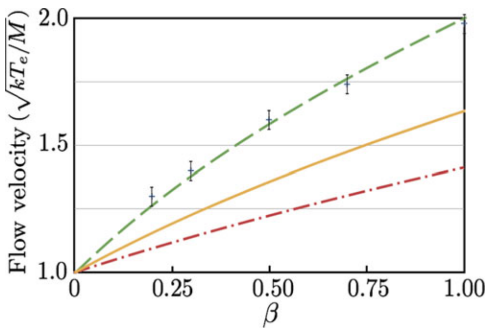

Figure 7. (Colour online) Flow velocity at which the width of the upstream presheath decreases to zero. The fits are plots of $\sqrt{1+\gamma\beta}$ with $\gamma = 1$ (dot-dashed line, red), $\gamma = 5/3$ (solid line, yellow), and $\gamma = 3$ (dashed line, green) (Willis et al. 2011).

and Dougherty 1969). The well-known Bohm criterion has been generalized by Harrison and Thompson (1959) to take account of a distribution of ion velocities at the sheath edge; it takes the form

$$\int_0^{\infty} \frac{f(v)\,\mathrm{d}v}{v^2} \leqslant \frac{M}{kT_e}. \tag{12}$$

It has been shown by Allen (1976) that this generalized form corresponds to 'sonic flow' once again at the plasma boundary, i.e., an ion acoustic wave is unable to travel back into the plasma. Figure 7 shows data from the simulations showing the flow velocities at which the presheath width goes to zero, i.e. the velocity at which ions approaching from directly upstream are unperturbed right up to the sheath edge. The Bohm velocity, or velocity of sound, is shown to be well represented in this case by the following expression,

$$c_s^* = (1+3\beta)^{1/2} c_s, \quad \text{where } \beta = \frac{T_i}{T_e} \quad \text{and } c_s = \sqrt{kT_e/M}. \tag{13}$$

It should be emphasized here that the factor of gamma ($\gamma = 3$) has not been deduced from an adiabatic gas law, but is a purely empirical quantity. An alternative method is to determine an effective sound velocity $c_s^*$to measure the half-angle of the Mach cone, and employ the relation $\sin\alpha = (c_s^*/v)$. The two methods were found to be in good agreement. It is clearly the case, however, that the generalized (kinetic) Bohm criterion is a subject requiring further investigation and relevant work is being carried out at present (Thomas et al. 2012). The disappearance of the inner cone at higher ion temperatures (Fig. 4) is to be expected since ion-acoustic waves undergo severe Landau damping unless (T_i/T_e) is small, i.e., $\beta < 0.25$ (Clemmow and Dougherty 1969). The process is one of 'phase-mixing'.

6. Conclusions

The PIC simulations reported by Willis et al. (2011) are for finite (small) values of (T_i/T_e) and for finite (large) values of (a/λ_D). As a result, the gas dynamic theory, with cold ions and a thin space charge sheath, cannot be expected to predict all the detailed results obtained from the simulations. Nevertheless the general feature of a truncated Mach cone was observed, the cone representing a weak discontinuity rather than a shock wave. We can note that the very concept of a Mach cone is inherently associated with a 'velocity of sound', whereas a collision-free plasma, which does not obey the usual gas dynamic laws, has no such simple property. Unlike ordinary sound waves in a gas, ion-acoustic waves are severely damped, unless the ion temperature is small, i.e., $(T_i/T_e) < 0.25$.

An ion-free region was observed at high flow velocities, this would not be in accord with an isothermal gas dynamics model. Such a situation is a rare case in which the Bohm criterion is irrelevant, since the positive ions do not cross the plasma boundary in that region. The effective Bohm velocity has been described by a numerical fitting procedure. It is evident, however, that the kinetic Bohm criterion, involving the distribution of ion velocities, needs further elucidation in the case of flowing plasmas.

7. Acknowledgements

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