DIVIDEND OPTIMIZATION FOR A REGIME-SWITCHING DIFFUSION MODEL WITH RESTRICTED DIVIDEND RATES

BY

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ABSTRACT

We consider the optimal dividend control problem to find an optimal strategy under the constraint that dividend rates is restricted such that the expected total discounted dividends are maximized for an insurance company. The evolution of the reserve is modeled by a diffusion process with drift and volatility coefficients modulated by an observable Markov chain. We consider the regime-switching threshold strategy which pays out dividends at the maximal possible rate when the current reserve is above some critical level dependent on the regime of the Markov chain at the time, and pays nothing when the reserve is below that level. We give sufficient conditions under which such type of strategy is optimal for the regime-switching model.

KEYWORDS

Dividends, optimization, regime-switching, restricted dividend rate, threshold strategy.

1. INTRODUCTION

The optimal dividend problem has attracted extensive attention with most works focusing on models with constant parameters, for example, constant drift and volatility in the diffusion setting or constant claim arrival intensity and claim size mean in the Cramér–Lundberg setting. However, empirical studies have shown that a company's earnings and decisions are affected by the (macroeconomic) environment where the company is operating (see Sotomayor and Cadenillas, 2011 and the references therein for details). The environment itself evolves dynamically over time and hence it is more appropriate to assume that the parameters evolve according to a stochastic process in continuous time and that at each time the values of the model parameters depend on the state (regime) of the environment at the time. Econometric literature has supported the use of a finite state Markov process to model the macroeconomic environment. A model with such characteristics is called Markov regime-switching model.

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Regime-switching models have been used in the contexts of option pricing, optimal consumptions, portfolio optimization, risk theory, dividend optimization (see, for example, Meng and Siu, 2011; Wei *et al.*, 2012; Zhu and Chen, 2013; Zhu, 2013) and so on.

A special type of dividend strategy – the threshold strategy, under which the company should pay out dividends at the maximal admissible rate when the reserve exceeds a certain threshold and pay nothing otherwise, is of particular interest in the literature. Such type of strategy has been proved to be optimal for the dividend optimization problem with restricted dividend rates in the Brownian motion model (Asmussen and Taksar, 1997) and in the compound Poisson model with an exponential claim size distribution (Gerber and Shiu, 2006). When a risk model has several regimes, a regime-switching threshold strategy can be defined similarly by letting the threshold levels change with the regimes. It was shown in Sotomayor and Cadenillas (2011) that the optimal strategy for a two-regime-switching Brownian motion model with restricted dividend rates is of the regime-switching threshold form. As commented in Gerber and Shiu (2006), threshold strategies are of interest even in cases where the optimal strategy is not of threshold form. Risk models applying threshold dividend strategies have been studied extensively in the area of risk theory (see, for instance, Lin and Pavlova, 2006; Cheung *et al.*, 2008; Zhu and Yang, 2009 and the references therein).

In this paper, we will examine the optimality of the regime-switching threshold strategy when the risk process is a diffusion process with drift and volatility coefficients and some other model parameters modulated by the Markov environment process (a regime-switching diffusion process), and the dividend rates are restricted. We find sufficient conditions under which such a strategy is optimal for the regime-switching diffusion model. The optimality of such strategy has been shown in Sotomayor and Cadenillas (2011) for the regime-switching model with only two regimes. When there are only two regimes involved, the traditional approach that first finds the explicit forms of the solutions to a set of two Hamilton–Jacobi–Bellman (HJB) equations with boundary conditions and then verifies that these solutions are the desired functions associated with the optimal strategy is applicable. However, when the number of regimes increases, it is impossible to find explicit expressions for the solutions of the set of HJB equations, and therefore the traditional approach no longer works. The optimality of such strategy for a regime-switching compound Poisson model has been proven by Wei *et al.* (2011) to be true under some conditions and the assumption that there exists a set of thresholds at which the corresponding value function is smooth enough. However, to verify whether the above assumption holds or not itself poses big theoretical challenges. In this paper, we address the optimal dividend problem with restricted dividend rates for a regime-switching diffusion model with a large number of regimes. We provide sufficient conditions under which the regime-switching threshold strategy is optimal. The sufficient conditions that we present are very easy to verify, which can be done by a few simple arithmetic calculations.

The paper is organized as follows. In Section 2, we state the problem and introduce a modified optimization problem. We study the modified optimization problem and obtain optimality results for this problem in Section 3. In Section 4, we present the main results, which give sufficient conditions under which the regime-switching strategy is optimal. Concluding remarks are provided in Section 5.

2. Problem formulation

Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration. Let J_t denote the external environment state at time t. The process $J = \{J_t;\, t \geq 0\}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-Markov chain with the state space $\mathbb{E} = \{1, 2, \ldots, m\}$ and the transition intensity matrix $Q = (q_{ij})_{m\times m}$. For notational convenience, we write $q_i = -q_{ii} = \sum_{j\neq i} q_{ij}$ for $i \in \mathbb{E}$. The cash reservoir (surplus) in the absence of dividend payments evolves according to a diffusion process with the drift and volatility parameters dependent on the external environment state:

$$\mathrm{d}R_t = \mu(J_t)\mathrm{d}t + \sigma(J_t)\mathrm{d}W_t, \tag{2.1}$$

where $\{W_t;\, t \geq 0\}$ is a standard Brownian motion relative to $\{\mathcal{F}_t\}_{t\geq 0}$ and is independent of the process J, $\mu(i) \geq 0$ and $\sigma(i) > 0$. We write $\mu_i = \mu(i)$ and $\sigma_i = \sigma(i)$ for $i \in \mathbb{E}$.

The company controls dynamically the dividend rate over time and the dividend rate at time t is denoted by l_t. We call $L = \{l_t;\, t \geq 0\}$ a dividend payment strategy. Let R_t^L denote the cash reservoir at time t under the dividend strategy L. Then it follows the following dynamics:

$$\mathrm{d}R_t^L = (\mu(J_t) - l_t)\mathrm{d}t + \sigma(J_t)\mathrm{d}W_t. \tag{2.2}$$

For convenience, we use R, R^L, J, (R^L, J) to represent the stochastic processes $\{R_t;\, t \geq 0\}$, $\{R_t^L;\, t \geq 0\}$, $\{J_t;\, t \geq 0\}$ and $\{(R_t^L, J_t);\, t \geq 0\}$ respectively.

For any dividend strategy L, the time of ruin of the company is defined to be the time when the controlled surplus first reaches zero,

$$T^L = \inf\left\{t \geq 0 : R_t^L \leq 0\right\}.$$

Now assume that the dividend rate l_t cannot exceed d_{J_t} with $0 \leq d_i \leq \mu_i$ for all $i \in \mathbb{E}$. Here the constraint $d_i \leq \mu_i$ means that the dividend rate cannot exceed the premium rate. A strategy is said to be admissible if the dividend strategy process L is $\{\mathcal{F}_t\}$-adapted, the dividend rate at any time t is non-negative and bounded by d_{J_t} and no dividends will be paid out on or after ruin. We use Π to denote the set of *admissible strategies*. In other words,

$$\Pi = \{L = \{l_t;\, t \geq 0\} : L \text{ is } \{\mathcal{F}_t;\, t \geq 0\}\text{-adapted},\ l_t \in [0, d_{J_t}] \text{ for all } t \geq 0, \text{ and}$$
$$l_t = 0 \text{ for } t \geq T^L\}. \tag{2.3}$$

For any $x \in \mathbb{R}$ and $i \in \mathbb{E}$, define

$$\mathrm{P}_{(x,i)}(\cdot) = \mathrm{P}(\cdot | R_0 = x, J_0 = i), \tag{2.4}$$

$$\mathrm{E}_{(x,i)}[\cdot] = \mathrm{E}[\cdot | R_0 = x, J_0 = i]. \tag{2.5}$$

Suppose the force of discount depends on the environment state at the time. Let δ_{J_s} denote the force of discount at time s and define $\Lambda_t = \int_0^t \delta_{J_s} \mathrm{d}s$. Then $e^{-\Lambda_t}$ is the present value at time 0 of 1 unit at time t.

We measure the performance of an admissible strategy L by

$$\mathcal{P}(L)(x, i) = \mathrm{E}_{(x,i)}\left[\int_0^{T^L} l_t e^{-\Lambda_t} \mathrm{d}t\right], \tag{2.6}$$

which is the expected present value of all the dividends up to the time of ruin. The function $\mathcal{P}(L)(x, i)$ is called the objective function, and we define the maximal objective function as:

$$V(x, i) = \sup_{L \in \Pi} \mathcal{P}(L)(x, i). \tag{2.7}$$

Our objective is to find the optimal admissible strategy with the best performance. That is a strategy $L \in \Pi$ such that $V(x, i) = \mathcal{P}(L)(x, i)$.

It can be seen that the process without control (dividend payments), $\{(R_t, J_t); t \geq 0\}$, is the Markov process. By applying the standard method in the stochastic control theory for the Markov process we can obtain the following dynamic programming principle: for any stopping time τ,

$$V(x, i) = \sup_{L \in \Pi} \mathrm{E}_{(x,i)}\left[\int_0^{T^L \wedge \tau} l_t e^{-\Lambda_t} \mathrm{d}t + e^{-\Lambda_{T^L \wedge \tau}} V\big(R^L_{T^L \wedge \tau}, J_{T^L \wedge \tau}\big)\right]. \tag{2.8}$$

Inspired by an early draft of Jiang and Pistorius (2012) we study a modified optimization problem first instead of directly addressing the original optimization problem. For any $g : \mathbb{R} \times \mathbb{E} \to \mathbb{R}^+$ and any $i \in \mathbb{E}$, define a modified performance functional $\mathcal{P}_{g,i}$:

$$\mathcal{P}_{g,i}(L)(x) = \mathrm{E}_{(x,i)}\left[\int_0^{T^L \wedge \tau_1} l_t e^{-\Lambda_t} \mathrm{d}t + I\{\tau_1 < T^L\} e^{-\Lambda_{\tau_1}} g\big(R^L_{\tau_1}, J_{\tau_1}\big)\right], \tag{2.9}$$

where τ_1 denotes the time when the first transition of the Markov chain $\{J_t; t \geq 0\}$ occurs, i.e.,

$$\tau_1 = \inf\{t > 0 : J_t \neq J_0\}.$$

Define an operator $\mathcal{M}$ by

$$\mathcal{M}(g)(x, i) = \sup_{L \in \Pi} \mathcal{P}_{g,i}(L)(x). \tag{2.10}$$

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We can see that for any fixed g and i, the function $\mathcal{M}(g)(x,i)$ is the maximal objective function corresponding to the new performance functional $\mathcal{P}_{g,i}$.

In the following, we will first solve the new optimization problem to find a strategy that maximizes $\mathcal{P}_{g,i}(L)(x)$. By applying a standard method in stochastic control we know that for any g and i, the following dynamic programming principle holds:

$$\mathcal{M}(g)(x,i)$$
$$= \sup_{L\in\Pi} \mathrm{E}_{(x,i)}\left[\int_0^{T^L\wedge\tau_1\wedge\tau} l_t e^{-\Lambda_t}\mathrm{d}t + e^{-\Lambda_{T^L\wedge\tau_1\wedge\tau}}\mathcal{M}(g)\big(R^L_{T^L\wedge\tau_1\wedge\tau}, J_{T^L\wedge\tau_1\wedge\tau}\big)\right]$$
$$\text{for any stopping time } \tau,$$

and that the HJB equation associated with the modified optimization problem is

$$\frac{\sigma_i^2}{2} f''(x) + \mu_i f'(x) - (\delta_i + q_i) f(x) + \sum_{j\neq i} q_{ij} g(x,j)$$
$$+ \max_{l\in[0,d_i]} (l(1 - f'(x)) = 0. \tag{2.11}$$

3. Optimality results for the modified optimization problem

In this section we study the modified optimization problem. As this optimization problem is to maximize the total dividends up to the ruin time, or the exponential random time, τ_1, independent of the reserve process, whichever is earlier, all the dividends concerned are payable prior to the first regime switch and whatever happens afterwards will not affect the performance of a strategy. Therefore, we can consider this problem as an optimization problem with no regime switches. As it has already been shown that the optimal strategy subject to a restricted dividend rate to maximize the total dividends up to ruin for the diffusion model without regime switching is a strategy of threshold type, for our problem we also start with studying such type of strategy.

For any $b \geq 0$, let L^b denote the strategy that at any time t before τ_1 either pays no dividends when the current surplus is below b or pays dividends continuously at the maximal rate d_{J_t} when the surplus is at or above b.

Then the controlled process $\{R_t^{L^b}; t \geq 0\}$ follows the following dynamics:

$$\mathrm{d}R_t^{L^b} = \big(\mu_{J_t} - d_{J_t} I\big\{R_t^{L^b} \geq b\big\}\big)\mathrm{d}t + \sigma_{J_t}\mathrm{d}W_t. \tag{3.12}$$

Let $\mathcal{D}$ denote the class of functions $g : \mathbb{R}\times\mathbb{E} \to \mathbb{R}^+$ such that $g(0,\cdot) = 0$, g is non-decreasing and continuous with respect to the first argument, and that $g(x,j) \leq \frac{\max_{i\in\mathbb{E}} d_i}{\min_{i\in\mathbb{E}} \delta_i}$ for all $x \geq 0$ and $j \in \mathbb{E}$. Write $g(+\infty, j) = \lim_{x\to+\infty} g(x,j)$.

Lemma 3.1. *For any* $i \in \mathbb{E}$*,* $g \in \mathcal{D}$ *and* $b \in [0, +\infty]$*, there is a unique solution that is continuously differentiable on* $(0, +\infty)$ *to the following equations*

$$\frac{\sigma_i^2}{2} f''(x) + \mu_i f'(x) - (\delta_i + q_i) f(x) + \sum_{j \neq i} q_{ij} g(x, j) = 0, \quad 0 \leq x < b, \tag{3.13}$$

$$\frac{\sigma_i^2}{2} f''(x) + (\mu_i - d_i) f'(x) - (\delta_i + q_i) f(x) + \sum_{j \neq i} q_{ij} g(x, j) + d_i = 0, \quad x > b, \tag{3.14}$$

$$f(0) = 0, \tag{3.15}$$

$$\lim_{x \to +\infty} f(x) < +\infty. \tag{3.16}$$

Let $S_{g,i,b}(x)$ *denote the above-mentioned solution. Then, for* $0 \leq b \leq +\infty$*,*

$$S_{g,i,b}(x) = \begin{cases} C_{1i}(e^{\theta_{1i}x} - e^{-\theta_{2i}x}) - I_{1i}(x)e^{\theta_{1i}x} + I_{2i}(x)e^{-\theta_{2i}x} & 0 \leq x < b \\ \bar{I}_{3i}(x)e^{\theta_{3i}x} + C_{4i}e^{-\theta_{4i}x} + I_{4i}(x)e^{-\theta_{4i}x} & x \geq b \end{cases}, \tag{3.17}$$

where $I_{li}(x)$*,* $l = 1, 2, 3, 4$*, and* $\bar{I}_{3i}(x)$ *are defined in* (3.20)–(3.23) *and* (3.28)*, and* C_{1i} *and* C_{4i} *satisfy* (3.29) *and* (3.30)*. The function* $S_{g,i,b}(x)$ *is twice continuously differentiable for* $x \in (0, b) \cup (b, +\infty)$*.*

Furthermore, if $0 \leq b < +\infty$*,*

$$S_{g,i,b}(x) = \mathcal{P}_{g,i}(L^b)(x) \tag{3.18}$$

and

$$\lim_{x \to +\infty} \mathcal{P}_{g,i}(L^b)(x) = \frac{\sum_{j \neq i} q_{ij} g(+\infty, j) + d_i}{\delta_i + q_i}. \tag{3.19}$$

Proof. For any $i \in \mathbb{E}$, let θ_{1i} and $-\theta_{2i}$ denote respectively the positive and negative roots to the equation

$$\frac{\sigma_i^2}{2} x^2 + \mu_i x - (\delta_i + q_i) = 0,$$

and θ_{3i} and $-\theta_{4i}$ denote respectively the positive and negative roots to the equation

$$\frac{\sigma_i^2}{2} x^2 + (\mu_i - d_i) x - (\delta_i + q_i) = 0.$$

Then $e^{\theta_{1i}x}$ and $e^{-\theta_{2i}x}$ are a fundamental set of solutions to the homogeneous equation corresponding to (3.13) and the associated Wronskian is

$-(\theta_{1i}+\theta_{2i})e^{(\theta_{1i}-\theta_{2i})x}$. Similarly, $e^{\theta_{3i}x}$ and $e^{-\theta_{4i}x}$ are a fundamental set of solutions to the homogeneous equation corresponding to (3.14) and the associated Wronskian is $-(\theta_{3i}+\theta_{4i})e^{(\theta_{3i}-\theta_{4i})x}$.
Define

$$I_{1i}(x)=\frac{2}{\sigma_i^2}\int_0^x\frac{e^{-\theta_{1i}s}\sum_{j\neq i}q_{ij}g(s,j)}{\theta_{1i}+\theta_{2i}}\mathrm{d}s, \tag{3.20}$$

$$I_{2i}(x)=\frac{2}{\sigma_i^2}\int_0^x\frac{e^{\theta_{2i}s}\sum_{j\neq i}q_{ij}g(s,j)}{\theta_{1i}+\theta_{2i}}\mathrm{d}s, \tag{3.21}$$

$$I_{3i}(x)=\frac{2}{\sigma_i^2}\int_0^x\frac{e^{-\theta_{3i}s}\left(\sum_{j\neq i}q_{ij}g(s,j)+d_i\right)}{\theta_{3i}+\theta_{4i}}\mathrm{d}s, \tag{3.22}$$

and

$$I_{4i}(x)=\frac{2}{\sigma_i^2}\int_0^x\frac{e^{\theta_{4i}s}\left(\sum_{j\neq i}q_{ij}g(s,j)+d_i\right)}{\theta_{3i}+\theta_{4i}}\mathrm{d}s. \tag{3.23}$$

Let $f_{1i}(x)$ denote a solution to (3.13) and (3.15), and $f_{2i}(x)$ a solution to (3.14) and (3.16). Then by using the Variation of Parameters method we know that they have the following general forms:

$$f_{1i}(x)=C_{1i}e^{\theta_{1i}x}+C_{2i}e^{-\theta_{2i}x}-I_{1i}(x)e^{\theta_{1i}x}+I_{2i}(x)e^{-\theta_{2i}x},$$

and

$$f_{2i}(x)=C_{3i}e^{\theta_{3i}x}+C_{4i}e^{-\theta_{4i}x}-I_{3i}(x)e^{\theta_{3i}x}+I_{4i}(x)e^{-\theta_{4i}x},$$

where $C_{ji}, i\in\mathbb{E},\ j=1,2,3,4$, are constants.

Suppose $0<b<+\infty$. Note that $I_{1i}(0)=I_{2i}(0)=0$. Then $f_{1i}(0)=0$ implies

$$C_{2i}=-C_{1i}. \tag{3.24}$$

As g is positive and bounded and $\theta_{4i}>0$, by L'Hospital's rule we can see that

$$\lim_{x\to+\infty}I_{4i}(x)e^{-\theta_{4i}x}=\frac{2}{\sigma_i^2}\frac{\sum_{j\neq i}q_{ij}g(+\infty,j)+d_i}{\theta_{4i}(\theta_{3i}+\theta_{4i})}<+\infty. \tag{3.25}$$

Note that

$$\lim_{x\to+\infty}I_{3i}(x)e^{\theta_{3i}x}\geq\frac{2}{\sigma_i^2}\lim_{x\to+\infty}e^{\theta_{3i}x}\int_0^x\frac{e^{-\theta_{3i}s}d_i}{\theta_{3i}+\theta_{4i}}\mathrm{d}s=+\infty. \tag{3.26}$$

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As $\lim_{x\to+\infty} f_{2i}(x) < +\infty$, we can obtain

$$C_{3i} = I_{3i}(+\infty). \tag{3.27}$$

Define

$$\bar{I}_{3i}(x) = \frac{2}{\sigma_i^2}\int_x^{+\infty} \frac{e^{-\theta_{3i}s}\left(\sum_{j\neq i} q_{ij}g(s,j) + d_i\right)}{\theta_{3i}+\theta_{4i}}\mathrm{d}s. \tag{3.28}$$

It follows from (3.24), (3.27) and (3.28) that

$$f_{1i}(x) = C_{1i}(e^{\theta_{1i}x} - e^{-\theta_{2i}x}) - I_{1i}(x)e^{\theta_{1i}x} + I_{2i}(x)e^{-\theta_{2i}x},$$
$$f_{2i}(x) = \bar{I}_{3i}(x)e^{\theta_{3i}x} + C_{4i}e^{-\theta_{4i}x} + I_{4i}(x)e^{-\theta_{4i}x}.$$

We can choose the constants to make $f_{1i}(b) = f_{2i}(b)$ and $f'_{1i}(b) = f'_{2i}(b)$ hold, which can be done by letting C_{1i} and C_{4i} satisfy the following linear equations:

$$C_{1i}(e^{\theta_{1i}b} - e^{-\theta_{2i}b}) - I_{1i}(b)e^{\theta_{1i}b} + I_{2i}(b)e^{-\theta_{2i}b}$$
$$= \bar{I}_{3i}(b)e^{\theta_{3i}b} + C_{4i}e^{-\theta_{4i}b} + I_{4i}(b)e^{-\theta_{4i}b}, \tag{3.29}$$
$$C_{1i}(\theta_{1i}e^{\theta_{1i}b} + \theta_{2i}e^{-\theta_{2i}b}) - \theta_{1i}I_{1i}(b)e^{\theta_{1i}b} - \theta_{2i}I_{2i}(b)e^{-\theta_{2i}b}$$
$$= \theta_{3i}\bar{I}_{3i}(b)e^{\theta_{3i}b} - C_{4i}\theta_{4i}e^{-\theta_{4i}b} - \theta_{4i}I_{4i}(b)e^{-\theta_{4i}b}. \tag{3.30}$$

Define

$$h_i(x) = \begin{cases} C_{1i}(e^{\theta_{1i}x} - e^{-\theta_{2i}x}) - I_{1i}(x)e^{\theta_{1i}x} + I_{2i}(x)e^{-\theta_{2i}x} & 0 \le x < b \\ \bar{I}_{3i}(x)e^{\theta_{3i}x} + C_{4i}e^{-\theta_{4i}x} + I_{4i}(x)e^{-\theta_{4i}x} & x \ge b \end{cases}$$

with C_{1i} and C_{4i} satisfying (3.29) and (3.30). Note that such C_{1i} and C_{4i} are uniquely determined. Then $h_i(x)$ is a unique solution that is continuously differentiable on $(0, +\infty)$ to the equations (3.13), (3.14), (3.15) and (3.16).

Suppose $b = 0$. Then $f_{2i}(x)$ with C_{4i} chosen such that $f_{2i}(0) = 0$ will be the unique solution to (3.13)–(3.16). That is, $C_{4i} = -\bar{I}_{3i}(0)$, which coincides with the solution of (3.29) and (3.30) with $b = 0$.

For the case $b = +\infty$, the function $f_{1i}(x)$ with C_{1i} chosen such that

$$\lim_{x\to+\infty} f_{1i}(x) < +\infty,$$

will be the unique solution. That is, $C_{1i} = I_{1i}(+\infty)$, which is the same as the constant determined by (3.29) and (3.30) with $b = +\infty$.

In conclusion, for $0 \le b \le +\infty$,

$$S_{g,i,b}(x) = \begin{cases} C_{1i}(e^{\theta_{1i}x} - e^{-\theta_{2i}x}) - I_{1i}(x)e^{\theta_{1i}x} + I_{2i}(x)e^{-\theta_{2i}x} & 0 \le x < b \\ \bar{I}_{3i}(x)e^{\theta_{3i}x} + C_{4i}e^{-\theta_{4i}x} + I_{4i}(x)e^{-\theta_{4i}x} & x \ge b \end{cases}, \tag{3.31}$$

where C_{1i} and C_{4i} satisfy (3.29) and (3.30).

The second-order continuous differentiability of $S_{g,i,b}(x)$ on $(0, b) \cup (b, \infty)$ follows immediately.

Now we proceed to show $\mathcal{P}_{g,i}(L^b)(x) = S_{g,i,b}(x)$ for $0 \le b < +\infty$. We assume $0 \le b < +\infty$ in the rest part of the proof. Define for any $i, j \in \mathbb{E}$,

$$w_i(x, j) = \begin{cases} S_{g,i,b}(x) & j = i \\ g(x, j) & j \neq i \end{cases}. \tag{3.32}$$

We use $w_i'(x, j)$ and $w_i''(x, j)$ to denote the first and second order derivatives with respect to the first argument. By applying the Itô's lemma for semimartingales and using (3.12), we can obtain that for any $t > 0$,

$$\begin{aligned} & e^{-\Lambda_{T^{L^b}\wedge\tau_1\wedge t}} w_{J_0}\big(R^{L^b}_{T^{L^b}\wedge\tau_1\wedge t}, J_{T^{L^b}\wedge\tau_1\wedge t}\big) - w_{J_0}\big(R^{L^b}_0, J_0\big) \\ &= \int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s} G\big(R^{L^b}_{s-}, J_{s-}\big)\mathrm{d}s + \int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s}\sigma_{J_{s-}} w'_{J_0}\big(R^{L^b}_{s-}, J_{s-}\big)\mathrm{d}W_s \\ &\quad + \int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s}\left(q_{J_{s-}} w_{J_0}\big(R^{L^b}_{s-}, J_{s-}\big) - \sum_{j\neq J_{s-}} q_{J_{s-},j} w_{J_0}\big(R^{L^b}_{s-}, j\big)\right)\mathrm{d}s \\ &\quad - \int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s} d_{J_0} I\big\{R^{L^b}_{s-} \ge b\big\}\mathrm{d}s, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} G(x, i) = & \frac{\sigma_i^2}{2} w_i''(x, i) + (\mu_i - d_i I\{x \ge b\}) w_i'(x, i) - (q_i + \delta_i) w_i(x, i) \\ & + \sum_{j\neq i} q_{ij} g(x, j) + d_i I\{x \ge b\}, \end{aligned}$$

and the last equality in (3.33) follows by noting $J_{s-} = J_0$ for $s \le \tau_1$ and using (3.32). As $w_i(x, i) = S_{g,i,b}(x)$ and $S_{g,i,b}(x)$ satisfies both (3.13) and (3.14), we have

$$G(x, i) = 0 \quad \text{for } x > 0. \tag{3.34}$$

For any fixed $i \in \mathbb{E}$, it is not hard to verify that $S_{g,i,b}(x)$ and $S'_{g,i,b}(x)$ are bounded functions. By rearranging the differential equations (3.13) and (3.14) to express $S''_{g,i,b}(x)$ in terms of $S_{g,i,b}(x)$ and $S'_{g,i,b}(x)$, we can see that $S''_{g,i,b}(x)$ is also bounded. Note that $J_{s-} = J_0$ for all $s \le \tau_1$, and hence $w_{J_0}(\cdot, J_{s-}) = S_{g,J_0,b}(\cdot)$ for $s \le \tau_1$. Therefore, $w_{J_0}(\cdot, J_{s-})$ is bounded and has bounded first- and second-order derivatives. As a result, both

$$\int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s}\sigma_{J_{s-}} w''_{J_0}\big(R^{L^b}_{s-}, J_{s-}\big)\mathrm{d}W_s$$

and

$$\int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s}\left(q_{J_0}w^f_{J_0}\left(R^{L^b}_{s-}, J_s\right) - \sum_{j\neq J_0} q_{J_0 j} w^f_{J_0}\left(R^{L^b}_{s-}, J_{s-}\right)\right)\mathrm{d}s$$

are $P_{(x,i)}$-martingales, which implies

$$\mathrm{E}_{(x,i)}\left[\int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s}\sigma_{J_{s-}} w''_{J_0}\left(R^{L^b}_{s-}, J_{s-}\right)\mathrm{d}W_s\right] = 0, \tag{3.35}$$

and

$$\mathrm{E}_{(x,i)}\left[\int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s}\left(q_{J_0}w_{J_0}\left(R^{L^b}_{s-}, J_s\right) - \sum_{j\neq J_0} q_{J_0 j} w_{J_0}\left(R^{L^b}_{s-}, J_{s-}\right)\right)\mathrm{d}s\right] = 0. \tag{3.36}$$

By taking expectation $\mathrm{E}_{(x,i)}$ on (3.33) and using (3.34), (3.35) and (3.36) we can derive that

$$\begin{aligned}
&\mathrm{E}_{(x,i)}\left[e^{-\Lambda_{T^{L^b}\wedge\tau_1\wedge t}} w_{J_0}\left(R^{L^b}_{T^{L^b}\wedge\tau_1\wedge t}, J_{T^{L^b}\wedge\tau_1\wedge t}\right)\right] - w_i(x, i)\\
&\quad = -\mathrm{E}_{(x,i)}\left[\int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s} d_i I\left\{R^{L^b}_{s-} \geq b\right\}\mathrm{d}s\right].
\end{aligned} \tag{3.37}$$

It follows from (3.32) and (3.37) that

$$\begin{aligned}
S_{g,i,b}(x) = &\mathrm{E}_{(x,i)}\left[\int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s} d_i I\left\{R^{L^b}_{s-} \geq b\right\}\mathrm{d}s\right]\\
&+\mathrm{E}_{(x,i)}\left[e^{-\Lambda_{T^{L^b}\wedge\tau_1\wedge t}} w_{J_0}\left(R^{L^b}_{T^{L^b}\wedge\tau_1\wedge t}, J_{T^{L^b}\wedge\tau_1\wedge t}\right)\right].
\end{aligned} \tag{3.38}$$

For any $i, j \in \mathbb{E}$, note that $w_i(0, j) = 0$ and that $w_i(\cdot, j)$ is bounded. By the dominated convergence we obtain

$$\begin{aligned}
&\lim_{t\to+\infty} \mathrm{E}_{(x,i)}\left[e^{-\Lambda_{T^{L^b}\wedge\tau_1\wedge t}} w_{J_0}\left(R^{L^b}_{T^{L^b}\wedge\tau_1\wedge t)}, J_{T^{L^b}\wedge\tau_1\wedge t}\right)\right]\\
&= \mathrm{E}_{(x,i)}\left[e^{-\Lambda_{T^{L^b}\wedge\tau_1}} w_{J_0}\left(R^{L^b}_{T^{L^b}\wedge\tau_1}, J_{T^{L^b}\wedge\tau_1}\right)\right]\\
&= \mathrm{E}_{(x,i)}\left[e^{-\Lambda_{\tau_1}} w_{J_0}\left(R^{L^b}_{\tau_1}, J_{\tau_1}\right) I\{\tau_1 < T^{L^b}\}\right.\\
&\qquad \left. + e^{-\Lambda_{T^{L^b}}} w_{J_0}\left(R^{L^b}_{T^{L^b}}, J_{T^{L^b}}\right) I\{\tau_1 \geq T^{L^b}\}\right]\\
&= \mathrm{E}_{(x,i)}\left[e^{-\Lambda_{\tau_1}} g\left(R^{L^b}_{\tau_1}, J_{\tau_1}\right) I\{\tau_1 < T^{L^b}\}\right],
\end{aligned} \tag{3.39}$$

where the last equability follows from noticing $w_{J_0}\left(R^{L^b}_{T^{L^b}}, J_{T^{L^b}}\right) = w_{J_0}(0, J_{T^{L^b}}) = 0$ and using (3.32). It follows by the monotone convergence that

$$\lim_{t\to+\infty} \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^b}\wedge\tau_1\wedge t} e^{-\Lambda_s} d_i I\left\{R^{L^b}_{s-} \geq b\right\}\mathrm{d}s\right]$$
$$= \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^b}\wedge\tau_1} e^{-\Lambda_s} d_i I\left\{R^{L^b}_{s-} \geq b\right\}\mathrm{d}s\right]. \tag{3.40}$$

Note that

$$\mathcal{P}_{g,i}(L^b)(x)$$
$$= \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^b}\wedge\tau_1} e^{-\Lambda_t} d_i I\left\{R^{L^b}_{s-} \geq b\right\}\mathrm{d}t + I\{\tau_1 < T^{L^b}\}e^{-\Lambda_{\tau_1}} g\left(R^{L^b}_{\tau_1}, J_{\tau_1}\right)\right]. \tag{3.41}$$

By letting $t \to +\infty$ on both sides of (3.38) and then using (3.39), (3.40) and (3.41), we conclude that $S_{g,i,b}(x) = \mathcal{P}_{g,i}(L^b)(x)$.

Since by using L'Hospital's rule we have

$$\lim_{x\to+\infty} \bar{I}_{3i}(x)e^{\theta_{3i}x} = \frac{2}{\sigma_i^2}\frac{\sum_{j\neq i} q_{ij} g(+\infty, j) + d_i}{\theta_{3i}(\theta_{3i} + \theta_{4i})}, \tag{3.42}$$

it follows from (3.31) and (3.25) that

$$\lim_{x\to+\infty} S_{g,i,b}(x) = \frac{2}{\sigma_i^2}\frac{\sum_{j\neq i} q_{ij} g(+\infty, j) + d_i}{\theta_{3i}\theta_{4i}} = \frac{\sum_{j\neq i} q_{ij} g(+\infty, j) + d_i}{\delta_i + q_i},$$

which implies that (3.19) holds. □

Lemma 3.2. *For any $g \in \mathcal{D}$ and $i \in \mathbb{E}$, both $\mathcal{P}_{g,i}(L)(x)$ and $\mathcal{M}(g)(x, i)$ are non-negative and non-decreasing with respect to x.*

Proof. The non-negativity is obvious according to the definitions of the functions.

Let L be an admissible strategy for the process R starting with $R_0 = x$. Then the same strategy L is also admissible if $R_0 = y$ $(y > x)$. Note that τ_1 is independent of R_0 and $\{W_t; t \geq 0\}$. So the value of τ_1 will be the same for the cases with $R_0 = x$ and $R_0 = y$. As $y > x$, it is obvious that the ruin time T^L in the case $R_0 = x$ is smaller than in the case $R_0 = y$, and the value of R_t in the case $R_0 = x$ is also smaller than in the case $R_0 = y$ for any $t \leq \tau_1$. Note that g

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is non-decreasing with respect to the first argument. Hence, for $y > x \geq 0$,

$$\begin{aligned}\mathcal{P}_{g,i}(L)(x) &= \mathrm{E}_{(x,i)}\left[\int_0^{T^L\wedge\tau_1} l_t e^{-\Lambda_t}\mathrm{d}t + I\{\tau_1 < T^L\}e^{-\Lambda_{\tau_1}}g\big(R^L_{\tau_1}, J_{\tau_1}\big)\right]\\ &\leq \mathrm{E}_{(y,i)}\left[\int_0^{T^L\wedge\tau_1} l_t e^{-\Lambda_t}\mathrm{d}t + I\{\tau_1 < T^L\}e^{-\Lambda_{\tau_1}}g\big(R^L_{\tau_1}, J_{\tau_1}\big)\right]\\ &= \mathcal{P}_{g,i}(L)(y).\end{aligned}$$

Therefore, $\mathcal{P}_{g,i}(L)(x)$ is a non-decreasing function, and hence the function

$$\mathcal{M}(g)(x, i) = \sup_{x\in\Pi}\mathcal{P}_{g,i}(L)(x)$$

is also non-decreasing with respect to x. □

Lemma 3.3. *Suppose $0 \leq b < +\infty$. For any fixed $i \in \mathbb{E}$, define the stochastic process $Y^{(i)} = \{Y^{(i)}_t; t \geq 0\}$ by $Y^{(i)}_t = R_0 + (\mu_i - d_i)t + \sigma_i W_t$, and define, for any $y \geq 0$, $\tau^{Y^{(i)}}_y$ to be the first time that the process $Y^{(i)}$ hits y.*

(i) $\mathrm{E}_{(x,i)}[\int_0^{\tau_b^{Y^{(i)}}\wedge\tau_1} e^{-\delta_i s}d_i\mathrm{d}s]$ is twice continuously differentiable with respect to x for $x \in (b, +\infty)$, and

$$\limsup_{x\downarrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{E}_{(x,i)}\left[\int_0^{\tau_b^{Y^{(i)}}\wedge\tau_1} e^{-\delta_i s}d_i\mathrm{d}s\right] \leq 0. \tag{3.43}$$

(ii) $\mathrm{E}_{(x,i)}[e^{-\delta_i\tau_b^{Y^{(i)}}} I\{\tau_b^{Y^{(i)}} < \tau_1\}]$ is twice continuously differentiable with respect to x for $x \in (b, +\infty)$, and

$$\limsup_{x\downarrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{E}_{(x,i)}\left[e^{-\delta_i\tau_b^{Y^{(i)}}} I\left\{\tau_b^{Y^{(i)}} < \tau_1\right\}\right] \leq 0. \tag{3.44}$$

(iii) For any $g \in \mathcal{D}$, $\mathrm{E}_{(x,i)}[e^{-\delta_i\tau_1}g(Y^{(i)}_{\tau_1}, J_{\tau_1}); \tau_b^{Y^{(i)}} \geq \tau_1]$ is twice continuously differentiable with respect to x for $x \in (b, +\infty)$, and

$$\limsup_{x\downarrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{E}_{(x,i)}\left[e^{-\delta_i\tau_1}g(Y^{(i)}_{\tau_1}, J_{\tau_1}); \tau_b^{Y^{(i)}} \geq \tau_1\right] \leq 0. \tag{3.45}$$

Proof. Define $W_t^{\alpha,\sigma} = \alpha t + \sigma W_t$, and let $\tau_y^{W^{\alpha,\sigma}}$ denote the time that the Brownian motion $W^{\alpha,\sigma}$ hits y for the first time.

By using the reflection principle, we can derive (see, for example (Douady, 1998 (3.7))

$$
\begin{aligned}
P_{(x,i)}(\tau_b^{Y^{(i)}} \in \mathrm{d}t) &= P\big(\tau_{b-x}^{W^{(\mu_i-d_i),\sigma_i}} \in \mathrm{d}t\big) \\
&= \frac{x-b}{\sqrt{2\pi\sigma_i^2 t^3}} e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t}} \mathrm{d}t \quad \text{for } x > b. \qquad (3.46)
\end{aligned}
$$

Note that $\{W_t;\, t \geq 0\}$ and R_0 are independent of the environment process $\{J_t;\, t \geq 0\}$, and that τ_1 is the first transition time of the Markov process J. Hence, the process $Y^{(i)}$ is independent of τ_1. Note that given $J_0 = i$, τ_1 is exponentially distributed with mean $\frac{1}{q_i}$.

(i) We can have

$$
\begin{aligned}
E_{(x,i)}\left[\int_0^{\tau_b^{Y^{(i)}}\wedge\tau_1} e^{-\delta_i s} d_i \mathrm{d}s\right] &= d_i \int_0^{+\infty} q_i e^{-q_i t} \int_0^{+\infty} \\
&\times \frac{1-e^{-\delta_i(s\wedge t)}}{\delta_i} \frac{x-b}{\sqrt{2\pi\sigma_i^2 s^3}} e^{-\frac{(b-x-(\mu_i-d_i)s)^2}{2\sigma_i^2 s}} \mathrm{d}s\mathrm{d}t \quad \text{for } x > b. \qquad (3.47)
\end{aligned}
$$

It is not hard to verify that $E_{(x,i)}[\int_0^{\tau_b^{Y^{(i)}}\wedge\tau_1} e^{-\delta_i s} d_i \mathrm{d}s]$ is twice continuously differentiable with respect to x for $x > b$ and that, for $x > b$,

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2} E_{(x,i)}\left[\int_0^{\tau_b^{Y^{(i)}}\wedge\tau_1} e^{-\delta_i s} d_i \mathrm{d}s\right] \\
&= d_i \int_0^{+\infty} q_i e^{-q_i t} \int_0^{+\infty} \frac{1-e^{-\delta_i(s\wedge t)}}{\delta_i} \frac{\partial^2}{\partial x^2}\left(\frac{x-b}{\sqrt{2\pi\sigma_i^2 s^3}} e^{-\frac{(b-x-(\mu_i-d_i)s)^2}{2\sigma_i^2 s}}\right) \mathrm{d}s\mathrm{d}t.
\end{aligned}
\qquad (3.48)
$$

Note that for $x > b$,

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(\frac{x-b}{\sqrt{2\pi\sigma_i^2 s^3}} e^{-\frac{(b-x-(\mu_i-d_i)s)^2}{2\sigma_i^2 s}}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma_i^2 s^3}} \left(\frac{(b-x-(\mu_i-d_i)s)^2(x-b)}{\sigma_i^4 s^2}\right. \\
&\quad \left. - \frac{(x-b)}{\sigma_i^2 s} + \frac{2(b-x-(\mu_i-d_i)s)}{\sigma_i^2 s}\right) e^{-\frac{(b-x-(\mu_i-d_i)s)^2}{2\sigma_i^2 s}} \qquad (3.49)
\end{aligned}
$$

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$$< \frac{1}{\sqrt{2\pi\sigma_i^2 s^3}} \frac{(b-x-(\mu_i-d_i)s)^2(x-b)}{\sigma_i^4 s^2} e^{-\frac{(b-x-(\mu_i-d_i)s)^2}{2\sigma_i^2 s}}$$

$$\leq \frac{1}{\sqrt{2\pi\sigma_i^2 s^3}} \frac{(1+|\mu_i-d_i|s)^2}{\sigma_i^4 s^2} e^{-\frac{(\mu_i-d_i)^2 s - 2|\mu_i-d_i|}{2\sigma_i^2}} \quad \text{for } x \in [b, b+1]. \quad (3.50)$$

By using (3.48), (3.50) and the Fatou's lemma, we arrive at

$$\limsup_{x\downarrow b} \frac{d^2}{dx^2} \mathrm{E}_{(x,i)}\left[\int_0^{\tau_b^{Y^{(i)}}\wedge\tau_1} e^{-\delta_i s} d_i \mathrm{d}s\right] = d_i \int_0^{+\infty} q_i e^{-q_i t} \int_0^{+\infty} \frac{1-e^{-\delta_i(s\wedge t)}}{\delta_i}$$

$$\times \limsup_{x\downarrow b} \frac{\partial^2}{\partial x^2}\left(\frac{x-b}{\sqrt{2\pi\sigma_i^2 s^3}} e^{-\frac{(b-x-(\mu_i-d_i)s)^2}{2\sigma_i^2 s}}\right) \mathrm{d}s\mathrm{d}t. \quad (3.51)$$

Note from (3.49),

$$\limsup_{x\downarrow b} \frac{\partial^2}{\partial x^2}\left(\frac{x-b}{\sqrt{2\pi\sigma_i^2 s^3}} e^{-\frac{(b-x-(\mu_i-d_i)s)^2}{2\sigma_i^2 s}}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma_i^2 s^3}}\left(-\frac{2(\mu_i-d_i)}{\sigma_i^2}\right) e^{-\frac{((\mu_i-d_i)s)^2}{2\sigma_i^2 s}} \leq 0. \quad (3.52)$$

Inequality (3.43) follows immediately from (3.51) and (3.52).

(ii) Recall that given $J_0 = i$, τ_1 is independent of $Y^{(i)}$ and follows an exponential distribution with mean $\frac{1}{q_i}$. By (3.46) we have, for $x > b$,

$$\mathrm{E}_{(x,i)}\left[e^{-\delta_i \tau_b^{Y^{(i)}}} I\{\tau_b^{Y^{(i)}} < \tau_1\}\right]$$

$$= \int_0^{+\infty} q_i e^{-q_i s} \int_0^s e^{-\delta_i t} \frac{x-b}{\sqrt{2\pi\sigma_i^2 t^3}} e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t}} \mathrm{d}t\mathrm{d}s$$

$$= \int_0^{+\infty} q_i e^{-q_i s} \int_0^s e^{-\delta_i t} \frac{x-b}{\sqrt{2\pi\sigma_i^2 t^3}} e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t}} \mathrm{d}t\mathrm{d}s$$

$$= \int_0^{+\infty} \frac{x-b}{\sqrt{2\pi\sigma_i^2 t^3}} e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t} - (\delta_i+q_i)t} \mathrm{d}t. \quad (3.53)$$

It is not hard to see from (3.53) that $\mathrm{E}_x[e^{-\delta_i \tau_b^{Y(i)}} I\{\tau_b^{Y(i)} < \tau_1\}]$ is twice continuously differentiable with respect to x for $x > b$. Moreover, it follows that for $x > b$,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{E}_{(x,i)}\left[e^{-\delta_i \tau_b^{Y(i)}} I\{\tau_b^{Y(i)} < \tau_1\}\right] = \int_0^{+\infty} \frac{\partial^2}{\partial x^2}\left(\frac{x-b}{\sqrt{2\pi\sigma_i^2 t^3}} e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t}-(\delta_i+q_i)t}\right)\mathrm{d}t. \tag{3.54}$$

Note

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(\frac{x-b}{\sqrt{2\pi\sigma_i^2 t^3}} e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t}-(\delta_i+q_i)t}\right)\\
&= \frac{1}{\sqrt{2\pi\sigma_i^2 t^3}}\left(\frac{2(b-x-(\mu_i-d_i)t)}{\sigma_i^2 t}\right.\\
&\quad \left.+ (x-b)\frac{(b-x-(\mu_i-d_i)t)^2}{\sigma_i^4 t^2} - \frac{x-b}{\sigma_i^2 t}\right) e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t}-(\delta_i+q_i)t} &(3.55)\\
&\le \frac{1}{\sqrt{2\pi\sigma_i^2 t^3}}\left(\frac{-2(\mu_i-d_i)t}{\sigma_i^2 t} + \frac{(1+|\mu_i-d_i|t)^2}{\sigma_i^4 t^2}\right) e^{-\frac{(\mu_i-d_i)^2 t - 2|\mu_i-d_i|}{2\sigma_i^2}-(\delta_i+q_i)t}\\
&\qquad \text{for } x \in [b, b+1]. &(3.56)
\end{aligned}$$

Then it follows from (3.55), (3.56) and the Fatou's lemma that

$$\begin{aligned}
&\limsup_{x\downarrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{E}_{(x,i)}\left[e^{-\delta_i \tau_b^{Y(i)}} I\left\{\tau_b^{Y(i)} < \tau_1\right\}\right]\\
&\le \int_0^{+\infty}\frac{1}{\sqrt{2\pi\sigma_i^2 t^3}}\limsup_{x\downarrow b}\left[\left(\frac{2(b-x-(\mu_i-d_i)t)}{\sigma_i^2 t}\right.\right.\\
&\quad \left.\left.+ (x-b)\frac{(b-x-(\mu_i-d_i)t)^2}{\sigma_i^4 t^2} - \frac{x-b}{\sigma_i^2 t}\right)e^{-\frac{(b-x-(\mu_i-d_i)t)^2}{2\sigma_i^2 t}-(\delta_i+q_i)t}\mathrm{d}t\right]\\
&= -\int_0^{+\infty}\frac{2(\mu_i-d_i)}{\sigma_i^2\sqrt{2\pi\sigma_i^2 t^3}}e^{-\frac{((\mu_i-d_i)t)^2}{2\sigma_i^2 t}-(\delta_i+q_i)t}\mathrm{d}t < 0.
\end{aligned}$$

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(iii) By using the fact that, given $J_0 = i$, τ_1 is independent of $Y^{(i)}$ and follows an exponential distribution with mean $\frac{1}{q_i}$ again, we can obtain

$$\mathrm{E}_{(x,i)}\left[e^{-\delta_i\tau_1}g\big(Y_{\tau_1}^{(i)}, J_{\tau_1}\big);\, \tau_b^{Y^{(i)}} \geq \tau_1\right] = \int_0^{+\infty} e^{-(q_i+\delta_i)t}\sum_{j\neq i} q_{ij}\mathrm{E}_{(x,i)}\left[g\big(Y_t^{(i)}, j\big);\, \tau_b^{Y^{(i)}} \geq t\right]\mathrm{d}t. \tag{3.57}$$

Recall that $W_t^{\alpha,\sigma} = \alpha t + \sigma W_t$,. We further define $M_t^{\alpha,\sigma} = \max_{s\in[0,t]} W_s^{\alpha,\sigma}$ and $m_t^{\alpha,\sigma} = \min_{s\in[0,t]} W_s^{\alpha,\sigma}$. Using the reflection principle we can have the following joint distribution (Harrison 1990, p. 11):

$$\mathrm{P}\big(W_t^{\alpha,\sigma} \in \mathrm{d}x,\, M_t^{\alpha,\sigma} \leq y\big) = f_{\alpha,\sigma}(x, y)\mathrm{d}x,\;\; x \leq y,\;\; y \geq 0,$$

where

$$f_{\alpha,\sigma}(x, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{\alpha^2 t}{2\sigma^2}+\frac{\alpha x}{\sigma^2}}\left(e^{-\frac{x^2}{2\sigma^2 t}} - e^{-\frac{(x-2y)^2}{2\sigma^2 t}}\right). \tag{3.58}$$

Note that $W_t^{\alpha,\sigma} = -W_t^{-\alpha,-\sigma}$ and $m_t^{\alpha,\sigma} = -M_t^{-\alpha,-\sigma}$. Hence,

$$\mathrm{P}\big(W_t^{\alpha,\sigma} \leq x, m_t^{\alpha,\sigma} \geq y\big) = \mathrm{P}\big(W_t^{-\alpha,-\sigma} \geq -x,\, M_t^{-\alpha,-\sigma} \leq -y\big) = \int_{-x}^{+\infty} f_{-\alpha,-\sigma}(u, -y)\mathrm{d}u\;,\, x \geq y,\, y \geq 0.$$

As a result,

$$\mathrm{P}\big(W_t^{\alpha,\sigma} \in \mathrm{d}x, m_t^{\alpha,\sigma} \geq y\big) = f_{-\alpha,-\sigma}(-x, -y),\;\; x \geq y,\;\; y \leq 0. \tag{3.59}$$

Note that given $R_0 = x$ and $J_0 = i$, $Y_{(i)}$ has the same distribution as $x + W_t^{\mu_i-d_i,\sigma_i}$, and hence for $x \geq b$,

$$\mathrm{E}_{(x,i)}\left[g(Y_t^{(i)}, j);\;\; \tau_b^{Y^{(i)}} \geq t\right] = \mathrm{E}\left[g(x + W_t^{\mu_i-d_i,\sigma_i}, j);\;\; m_t^{\mu_i-d_i,\sigma_i} \geq b - x\right] = \int_{b-x}^{+\infty} g(x+u, j)\, f_{d_i-\mu_i,-\sigma_i}(-u, x-b)\mathrm{d}u. \tag{3.60}$$

As g is a bounded function and the function $f_{d_i-\mu_i,-\sigma_i}(x, y)$ is infinitely differentiable with respect to both arguments, it is not hard to see that $\mathrm{E}_{(x,i)}[g(Y_t^{(i)}, j);\;\; \tau_b^{Y^{(i)}} \geq t]$ is infinitely differentiable with respect to x for $x > b$. Hence, by (3.57) we can conclude that $\mathrm{E}_{(x,i)}[e^{-\delta_i\tau_1}g(Y_{\tau_1}^{(i)}, J_{\tau_1});\, \tau_b^{Y^{(i)}} \geq \tau_1]$ is also infinitely differentiable with respect to x for $x > b$. We can also see that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{E}_{(x,i)}\left[g\big(Y_t^{(i)}, j\big);\ \tau_b^{Y^{(i)}} \geq t\right] = \int_b^{+\infty} g(u, j)\frac{\partial^2}{\partial x^2} f_{d_i-\mu_i,-\sigma_i}(x-u, x-b)\mathrm{d}u. \tag{3.61}$$

Note

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2} f_{d_i-\mu_i,-\sigma_i}(x-u, x-b)\\
&= \frac{1}{\sqrt{2\pi\sigma_i^2 t}} e^{-\frac{(\mu_i-d_i)^2 t}{2\sigma_i^2}+\frac{(\mu_i-d_i)(x-u)}{\sigma_i^2}}\left(\left[\frac{1}{\sigma_i^2 t}-\left(\frac{\mu_i-d_i}{\sigma_i^2}+\frac{x+u-2b}{\sigma_i^2 t}\right)^2\right]\right.\\
&\quad\left.\times e^{-\frac{(x+u-2b)^2}{2\sigma_i^2 t}} - \left[\frac{1}{\sigma_i^2 t}-\left(\frac{\mu_i-d_i}{\sigma_i^2}+\frac{x-u}{\sigma_i^2 t}\right)^2\right]e^{-\frac{(x-u)^2}{2\sigma_i^2 t}}\right) \qquad (3.62)\\
&\leq \frac{1}{\sqrt{2\pi\sigma_i^2 t}} e^{-\frac{(\mu_i-d_i)^2 t}{2\sigma_i^2}+\frac{(\mu_i-d_i)\max\{b-u,b+1-u\}}{\sigma_i^2}}\left(\frac{1}{\sigma_i^2 t}e^{-\frac{(u-b)^2}{2\sigma_i^2 t}}\right.\\
&\quad+\max\left\{\left(\frac{\mu_i-d_i}{\sigma_i^2}+\frac{b+1-u}{\sigma_i^2 t}\right)^2, \left(\frac{\mu_i-d_i}{\sigma_i^2}+\frac{b-u}{\sigma_i^2 t}\right)^2\right\}\\
&\quad\left.\times e^{-\frac{\min\left\{(b+1-u)^2,(b-u)^2\right\}}{2\sigma_i^2 t}}\right), \qquad \text{for } x\in[b, b+1]. \qquad (3.63)
\end{aligned}$$

Letting $x \downarrow b$ in (3.62) yields

$$\lim_{x\downarrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2} f_{d_i-\mu_i,-\sigma_i}(x-u, x-b) = -\frac{4(\mu_i-d_i)(u-b)}{\sigma_i^5 t\sqrt{2\pi t}}e^{-\frac{(\mu_i-d_i)^2 t}{2\sigma_i^2}+\frac{(\mu_i-d_i)(b-u)}{\sigma_i^2}-\frac{(u-b)^2}{2\sigma_i^2 t}} \leq 0. \tag{3.64}$$

It follows from (3.61), (3.63), (3.64) and the dominated convergence that

$$\limsup_{x\downarrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{E}_{(x,i)}\left[g\big(Y_t^{(i)}, j\big);\ \tau_b^{Y^{(i)}} \geq t\right] = \int_b^{+\infty} g(u, j)\frac{\partial^2}{\partial x^2}\lim_{x\downarrow b}\frac{\mathrm{d}^2}{\mathrm{d}x^2} f_{d_i-\mu_i,-\sigma_i}(x-u, x-b)\mathrm{d}u \leq 0. \tag{3.65}$$

Combining (3.57) and (3.65) concludes the proof. □

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Corollary 3.4. *For any fixed $i \in \mathbb{E}$, any $g \in \mathcal{D}$ and any $b \in [0, +\infty)$, let $S_{g,i,b}(x)$ be the solution defined in Lemma 3.1. The function $S_{g,i,b}(x)$ is twice continuously differentiable on $(0, b) \cup (b, +\infty)$, and $\lim_{x\downarrow b} S'_{g,i,b}(x) \leq 0$ if $0 \leq b < +\infty$.*

Proof. Suppose $0 \leq b < +\infty$. Let $\tau_b^{L^b}$ denote the time that the process R^{L^b} hits b for the first time. Then given $R_0 = x > b$, $\tau_b^{L^b} > T^{L^b}$. From the structure of L^b we can see that the controlled process R^{L^b} is still the Markov process. Hence, we have for $x > b$,

$$\mathcal{P}_{g,i}(L^b)(x) = \mathrm{E}_{(x,i)}\Bigg[\int_0^{\tau_b^{L^b}\wedge\tau_1} d_{J_0} e^{-\Lambda_t}\mathrm{d}t + I\{\tau_b^{L^b} < \tau_1\} e^{-\Lambda_{\tau_b^{L^b}}}\mathcal{P}_{g,J_0}(L^b)(b) + I\{\tau_1 \leq \tau_b^{L^b}\} e^{-\Lambda_{\tau_1}} g\big(R_{\tau_1}^{L^b}, J_{\tau_1}\big)\Bigg].$$

Note that given $(R_0, J_0) = (x, i)$ with $x > b$, the path of $R^{L^b}_{\tau_b^{L^b}}$ before time $\tau_1 \wedge \tau_b^{L^b}$ is the same as the process $Y^{(i)}$ defined in Lemma 3.3. Same as in Lemma 3.3, we use $\tau_b^{Y^{(i)}}$ to denote the time that the process $Y^{(i)}$ reaches b for the first time. Then for $x > b$,

$$\mathcal{P}_{g,i}(L^b)(x) = \mathrm{E}_{(x,i)}\left[\int_0^{\tau_b^{Y^{(i)}}\wedge\tau_1} e^{-\delta_i s} d_i \mathrm{d}s\right] + \mathcal{P}_{g,i}(L^b)(b) \times \mathrm{E}_{(x,i)}\left[e^{-\delta_i \tau_b^{Y^{(i)}}} I\{\tau_b^{Y^{(i)}} < \tau_1\}\right] + \mathrm{E}_{(x,i)}\left[e^{-\delta_i \tau_1} g\big(Y^{(i)}_{\tau_1}, J_{\tau_1}\big); \tau_b^{Y^{(i)}} \geq \tau_1\right].$$

Then it follows immediately from Lemma 3.3 that $\lim_{x\downarrow b} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{P}_{g,i}(L^b)(x) \leq 0$. We conclude the proof by noticing that $S_{g,i,b}(x) = \mathcal{P}_{g,i}(L^b)(x)$ if $0 \leq b < +\infty$ (see (3.18)). □

Let $S_{g,i,b}(x)$, θ_{li}, $I_{1i}(x)$ and $\bar{I}_{li}(x)$ for $l = 1, 2, 3, 4$ and $i \in \mathbb{E}$ be defined in the same way as in Lemma 3.1. It follows from (3.17) that

$$S'_{g,i,b}(b) = C_{1i}(\theta_{1i} e^{\theta_{1i} b} + \theta_{2i} e^{-\theta_{2i} b}) - \theta_{1i} I_{1i}(b) e^{\theta_{1i} b} - \theta_{2i} I_{2i}(b) e^{-\theta_{2i} b} \quad \text{for } 0 < b < +\infty,$$

where C_{1i} satisfying (3.29) and (3.30). Solving (3.29) and (3.30) gives

$$C_{1i} = \frac{(\theta_{1i} + \theta_{4i}) I_{1i}(b) e^{\theta_{1i} b} + (\theta_{2i} - \theta_{4i}) I_{2i}(b) e^{-\theta_{2i} b} + (\theta_{3i} + \theta_{4i}) \bar{I}_{3i}(b) e^{\theta_{3i} b}}{(\theta_{1i} + \theta_{4i}) e^{\theta_{1i} b} + (\theta_{2i} - \theta_{4i}) e^{-\theta_{2i} b}}.$$

Hence, for $0 < b < +\infty$,

$$S'_{g,i,b}(b) - 1$$
$$= \frac{(\theta_{3i} + \theta_{4i})\bar{I}_{3i}(b)e^{\theta_{3i}b}}{(\theta_{1i} + \theta_{4i})e^{\theta_{1i}b} + (\theta_{2i} - \theta_{4i})e^{-\theta_{2i}b}}(\theta_{1i}e^{\theta_{1i}b} + \theta_{2i}e^{-\theta_{2i}b}) - 1$$
$$= \frac{(\theta_{3i} + \theta_{4i})\bar{I}_{3i}(b)e^{\theta_{3i}b}(\theta_{1i}e^{\theta_{1i}b} + \theta_{2i}e^{-\theta_{2i}b}) - (\theta_{1i} + \theta_{4i})e^{\theta_{1i}b} - (\theta_{2i} - \theta_{4i})e^{-\theta_{2i}b}}{(\theta_{1i} + \theta_{4i})e^{\theta_{1i}b} + (\theta_{2i} - \theta_{4i})e^{-\theta_{2i}b}}. \tag{3.66}$$

By (3.17) again we have $S_{g,i,0}(x) = \bar{I}_{3i}(x)e^{\theta_{3i}x} - \bar{I}_{3i}(0)e^{-\theta_{4i}x} + I_{4i}(x)e^{-\theta_{4i}x}$. Therefore,

$$\lim_{x\downarrow 0} S'_{g,i,0}(x) - 1 = (\theta_{3i} + \theta_{4i})\bar{I}_{3i}(0) - 1.$$

Define

$$\xi_i^g(b) = (\theta_{1i}e^{\theta_{1i}b} + \theta_{2i}e^{-\theta_{2i}b}) \int_b^{+\infty} e^{-\theta_{3i}(s-b)} \left(\sum_{j\neq i} q_{ij}g(s, j) + d_i \right) \mathrm{d}s$$
$$-(\theta_{1i} + \theta_{4i})e^{\theta_{1i}b} - (\theta_{2i} - \theta_{4i})e^{-\theta_{2i}b}. \tag{3.67}$$

Note that $\lim_{x\downarrow 0} S'_{g,i,0}(x) - 1 = \frac{\xi_i^g(0)}{\theta_{1i}+\theta_{2i}}$. Write $S'_{g,i,0}(0) = \lim_{x\downarrow 0} S'_{g,i,0}(x)$. Then for any $0 \le b < +\infty$, equations $\xi_i^g(b) > 0$, $\xi_i^g(b) = 0$ and $\xi_i^g(b) < 0$ are equivalent to $S'_{g,i,b}(b) > 1$, $S'_{g,i,b}(b) = 1$ and $S'_{g,i,b}(b) < 1$ respectively.

Define

$$b_i^g = \inf\{b \in [0, +\infty) : \xi_i^g(b) \le 0\}, \tag{3.68}$$

and $b_i^g = +\infty$ if $\xi_i^g(b) > 0$ for all $0 \le b < +\infty$.

Recall that the class of functions $\mathcal{D}$ is defined right above Lemma 3.1. Define another class of functions

$$\mathcal{C} = \{g \in \mathcal{D} : \frac{\partial^2 g(x, i)}{\partial x^2} \text{ exists and is non-positive for all } x \ge 0 \text{ and all } i \in \mathbb{E}\}. \tag{3.69}$$

We can see that $\mathcal{C}$ is a complete space.

Theorem 3.5. *For any fixed* $i \in \mathbb{E}$ *and* $g \in \mathcal{D}$,
(i) if $0 < b_i^g < +\infty$, $S'_{g,i,b_i^g}(b_i^g) = 1$;
(ii) the function $S_{g,i,b_i^g}(x)$ *is twice continuously differentiable on* $(0, +\infty)$;
(iii) for any $g \in \mathcal{C}$, *the function* $S_{g,i,b_i^g}(x)$ *is concave on* $(0, +\infty)$.

Proof. (i) This is obvious according to the definition of b_i^g.

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(ii) By Lemma 3.1 we know that $S_{g,i,b_i^g}(x)$ is continuously differentiable on $(0, +\infty)$, twice continuously differentiable on $(0, +\infty)$ if $b_i^g = +\infty$ or 0, and twice continuously differentiable on $(0, b_i^g) \cup (b_i^g, +\infty)$ if $b_i^g < +\infty$. So it is sufficient to show that, if $0 < b_i^g < +\infty$,

$$\lim_{x \uparrow b_i^g} S''_{g,i,b_i^g}(x) = \lim_{x \downarrow b_i^g} S''_{g,i,b_i^g}(x). \tag{3.70}$$

Now suppose $0 < b_i^g < +\infty$. Then

$$S'_{g,i,b_i^g}(b_i^g) = 1. \tag{3.71}$$

Then it follows from Lemma 3.1 that for $0 < x < b_i^g$,

$$S''_{g,i,b_i^g}(x) = \frac{(\delta_i + q_i) S_{g,i,b_i^g}(x) - \mu_i S'_{g,i,b_i^g}(x) - \sum_{j \neq i} q_{ij} g(x, j)}{\frac{\sigma_i^2}{2}}, \tag{3.72}$$

and that for $x > b_i^g$

$$S''_{g,i,b_i^g}(x) = \frac{(\delta_i + q_i) S_{g,i,b_i^g}(x) - (\mu_i - d_i) S'_{g,i,b_i^g}(x) - \sum_{j \neq i} q_{ij} g(x, j) - d_i}{\frac{\sigma_i^2}{2}}. \tag{3.73}$$

By letting $x \uparrow b_i^g$ and $x \downarrow b_i^g$ on (3.72) and (3.73) respectively, and then using (3.71) we can see that (3.70) holds.
(iii) Note that $S_{g,i,b_i^g}(x)$ and $g(x, j)$ are continuously differentiable with respect to x and $S_{g,i,b_i^g}(0) = g(0, j) = 0$. Letting $x \downarrow 0$ in (3.72) and then using (3.18) and Lemma 3.2 gives

$$\lim_{x \downarrow 0} S''_{g,i,b_i^g}(x) = \frac{-\mu_i \lim_{x \downarrow 0} S'_{g,i,b_i^g}(x)}{\frac{\sigma_i^2}{2}} \leq 0 \quad \text{if } b_i^g < +\infty. \tag{3.74}$$

Also note by (ii) and Corollary 3.4 that if $b_i^g < +\infty$,

$$S''_{g,i,b_i^g}(b_i^g) \leq 0. \tag{3.75}$$

Define $w_i(x) = S''_{g,i,b_i^g}(x)$ and use $g''(x, j)$ to denote the second-order derivative of $g(x, j)$ with respect to the first argument. Then it follows from Lemma 3.1

that

$$\frac{\sigma_i^2}{2} w_i''(x) + \mu_i w_i'(x) - (\delta_i + q_i) w_i(x) + \sum_{j \neq i} q_{ij} g''(x, j) = 0, \; 0 < x < b_i^g, \tag{3.76}$$

$$\frac{\sigma_i^2}{2} w_i''(x) + (\mu_i - d_i) w_i'(x) - (\delta_i + q_i) w_i(x) + \sum_{j \neq i} q_{ij} g''(x, j) = 0, \; x > b_i^g. \tag{3.77}$$

Consider the stochastic process $\{X_t^i; t \geq 0\}$ defined by

$$X_t^i = R_0 + \left(\mu_i - d_i I\left\{X_t^i \geq b_i^g\right\}\right) t + \sigma_i W_t.$$

For any $y \geq 0$, define the stopping time

$$\tau_y^i = \inf\{t \geq 0 : X_t^i = y\}.$$

For any $s > 0$, by applying Itô's formula to

$$e^{-(q_i+\delta_i)\left(\tau_0^i \wedge \tau_{b_i^g}^i \wedge s\right)} w_i(X^i_{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s}) + \sum_{j \neq i} q_{ij} \int_0^{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s} e^{-(q_i+\delta_i)} g''\left(X_t^i, j\right) \mathrm{d}t,$$

we get

$$\begin{aligned} & e^{-(q_i+\delta_i)(\tau_0^i \wedge \tau_{b_i^g}^i \wedge s)} w_i(X^i_{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s}) + \sum_{j \neq i} q_{ij} \int_0^{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s} e^{-(q_i+\delta_i)t} g''(X_t^i, j) \mathrm{d}t \\ &= w_i(R_0) + \int_0^{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s} e^{-(q_i+\delta_i)t} \sigma_i w_i'(X_t^i) \mathrm{d}W_t \\ &\quad + \int_0^{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s} e^{-(q_i+\delta_i)t} \sum_{j \neq i} q_{ij} g''(X_t^i, j) \mathrm{d}t + \int_0^{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s} e^{-(q_i+\delta_i)t} \\ &\quad \times \left(\frac{1}{2}\sigma_i^2 w_i''(X_t^i) + (\mu_i - d_i I\{X_t^i \geq b_i^g\}) w_i'(X_t^i) - (q_i + \delta_i) w_i(X_t^i)\right) \mathrm{d}t. \end{aligned}$$

Then it follows from (3.76) and (3.77) that

$$\begin{aligned} & e^{-(q_i+\delta_i)(\tau_0^i \wedge \tau_{b_i^g}^i \wedge s)} w_i\left(X^i_{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s}\right) + \sum_{j \neq i} q_{ij} \int_0^{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s} e^{-(q_i+\delta_i)t} g''\left(X_t^i, j\right) \mathrm{d}t \\ &= w_i(R_0) + \int_0^{\tau_0^i \wedge \tau_{b_i^g}^i \wedge s} e^{-(q_i+\delta_i)t} \sigma_i w_i'\left(X_t^i\right) \mathrm{d}W_t. \end{aligned} \tag{3.78}$$

Note that from Lemmas 3.1 and 3.2 we know that for any fixed $i \in \mathbb{E}$, the function $S_{g,i,b_i^g}(x)$ is positive, non-decreasing and bounded. As a result, $S'_{g,i,b_i^g}(x)$ is non-negative and bounded. As $g(\cdot, i)$ is non-decreasing, concave and bounded with respect to x, we can see that $g'(x, i)$ is bounded. Hence, it follows from (3.73) that $S''_{g,i,b_i^g}(x)$ is bounded. Note that by Lemma 3.1 we have

$$\frac{\sigma_i^2}{2} S^{(3)}_{g,i,b_i^g}(x) + (\mu_i - d_i) S''_{g,i,b_i^g}(x) - (\delta_i + q_i) S'_{g,i,b_i^g}(x) + \sum_{j \neq i} q_{ij} g'(x, j) = 0, \ x > b_i^g.$$

Then $w_i'(x) = S^{(3)}_{g,i,b_i^g}(x)$ is bounded. Therefore,

$$\mathrm{E}_{(x,i)} \left[\int_0^{\tau_0^i \wedge \tau^i_{b_i^g} \wedge s} \left(e^{-(q_i+\delta_i)t} \sigma_i w_i'\left(X_t^i\right) \right)^2 \mathrm{d}t \right] < +\infty,$$

which implies that the last term in (3.78) is a martingale. Then by taking expectations on (3.78), we can obtain

$$w_i(x) = \mathrm{E}_x \Bigg[e^{-(q_i+\delta_i)(\tau_0^i \wedge \tau^i_{b_i^g} \wedge s)} w_i(X^i_{\tau_0^i \wedge \tau^i_{b_i^g} \wedge s}) + \sum_{j \neq i} q_{ij} \int_0^{\tau_0^i \wedge \tau^i_{b_i^g} \wedge s} e^{-(q_i+\delta_i)t} g''(X_t^i, j) \mathrm{d}t \Bigg]. \tag{3.79}$$

Note that the concavity of g implies

$$g''\left(X_t^i, j\right) \leq 0. \tag{3.80}$$

Then letting $s \to +\infty$ in (3.79), it follows from the dominated convergence and the monotone convergence that

$$w_i(x) = \mathrm{E}_x \left[e^{-(q_i+\delta_i)(\tau_0^i \wedge \tau^i_{b_i^g})} w_i(X^i_{\tau_0^i \wedge \tau^i_{b_i^g}}) + \sum_{j \neq i} q_{ij} \int_0^{\tau_0^i \wedge \tau^i_{b_i^g}} e^{-(q_i+\delta_i)t} g''(X_t^i, j) \mathrm{d}t \right]. \tag{3.81}$$

If $b_i^g < +\infty$, noting that $X^i_{\tau_0^i \wedge \tau^i_{b_i^g}}$ is either 0 or b_i^g, it follows from (3.74), (3.75), (3.80) and (3.81) that $w_i(x) \leq 0$.

If $b_i^g = +\infty$, then $\tau_0^i \wedge \tau_{b_i^g}^i = \tau_0^i$ almost surely, and hence

$$w_i(x) = \mathrm{E}_x\left[\sum_{j\neq i} q_{ij}\int_0^{\tau_0^i} e^{-(q_i+\delta_i)t} g''(X_t^i, j)\mathrm{d}t\right] \leq 0.$$

Hence, we can conclude that the function $S_{g,i,b_i^g}(x)$ is concave on $(0, +\infty)$. □

Recall that, for any $b \geq 0$, the dividend strategy L^b is defined so that, at any time t before τ_1, the company either pays no dividends when the current surplus is below b or pays dividends continuously at the maximal rate d_{J_t} when the surplus is above b.

Theorem 3.6. *For any fixed $i \in \mathbb{E}$ and any $g \in \mathcal{C}$, if $b_i^g < +\infty$, then the strategy $L^{b_i^g}$ is optimal with respect to the performance functional $\mathcal{P}_{g,i}$, i.e.*

$$\mathcal{P}_{g,i}(L^{b_i^g})(x) = \mathcal{M}(g)(x, i). \tag{3.82}$$

Proof. Since $L^{b_i^g}$ is an admissible strategy, it follows immediately that

$$\mathcal{M}(g)(x, i) = \sup_{L\in\Pi} \mathcal{P}_{g,i}(L)(x) \geq \mathcal{P}_{g,i}(L^{b_i^g})(x).$$

It is sufficient to show that $\mathcal{P}_{g,i}(L^{b_i^g})(x) \geq \mathcal{M}(g)(x, i)$. For any $i \in \mathbb{E}$, define a function for $x \geq 0$,

$$h_i(x, j) = \begin{cases} \mathcal{P}_{g,i}(L^{b_i^g})(x) & j = i \\ g(x, j) & j \neq i \end{cases}. \tag{3.83}$$

Use $h_i'(x, i)$ and $h_i''(x, i)$ to denote the first- and second-order derivatives of $h(x, i)$ with respect to the first argument. Then it follows from Lemma 3.1 that for any $i \in \mathbb{E}$,

$$\frac{\sigma_i^2}{2} h_i''(x, i) + \mu_i h_i'(x, i) - (\delta_i + q_i)h_i(x, i) + \sum_{j\neq i} q_{ij} g(x, j) = 0 \quad \text{for } 0 < x < b_i^g, \tag{3.84}$$

$$\frac{\sigma_i^2}{2} h_i''(x, i) + (\mu_i - d_i)h_i'(x, i) - (\delta_i + q_i)h_i(x, i) + \sum_{j\neq i} q_{ij} g(x, j) + d_i = 0, \quad \text{for } x > b_i^g. \tag{3.85}$$

For any $s > 0$ and any admissible strategy L, by applying Itô's formula we obtain

$$
\begin{aligned}
&e^{-\delta_i(T^L\wedge\tau_1\wedge s)}h_i\big(R^L_{T^L\wedge\tau_1\wedge s}, J_{T^L\wedge\tau_1\wedge s}\big)\\
&\quad= h_i\big(R^L_0, J_0\big) + \int_0^{T^L\wedge\tau_1\wedge s} e^{-\delta_i t}\sigma_{J_{t-}} h_i'\big(R^L_{t-}, J_{t-}\big)\mathrm{d}W_t\\
&\qquad+\int_0^{T^L\wedge\tau_1\wedge s} e^{-\delta_i t}\Big(\frac{1}{2}\sigma^2_{J_{t-}} h_i''\big(R^L_{t-}, J_{t-}\big) + (\mu_{J_{t-}} - l_{t-})h_i'\big(R^L_{t-}, J_{t-}\big)\\
&\qquad-\delta_{J_{t-}} h_i\big(R^L_{t-}, J_{t-}\big)\Big)\mathrm{d}t.
\end{aligned}
\tag{3.86}
$$

Using (3.84) and (3.85) we can derive that given $J_0 = i$, for any $t \le \tau_1$,

$$
\begin{aligned}
&\frac{1}{2}\sigma^2_{J_{t-}} h_i''\big(R^L_{t-}, J_{t-}\big) + (\mu_{J_{t-}} - l_{t-})h_i'\big(R^L_{t-}, J_{t-}\big) - \delta_{J_{t-}} h_i\big(R^L_{t-}, J_{t-}\big)\\
&\quad= \frac{1}{2}\sigma_i^2 h_i''\big(R^L_{t-}, i\big) + (\mu_i - l_{t-})h_i'\big(R^L_{t-}, i\big) - \delta_i h_i\big(R^L_{t-}, i\big)\\
&\quad= (d_i I\{R^L_{t-} \ge b_i^g\} - l_{t-})h_i'\big(R^L_{t-}, i\big) + q_i h_i\big(R^L_{t-}, i\big)\\
&\qquad-\sum_{j\ne i} q_{ij} g\big(R^L_{t-}, j\big) - d_i I\{R^L_{t-} \ge b_i^g\}\\
&\quad= (d_i - l_{t-})h_i'\big(R^L_{t-}, i\big) I\{R^L_{t-} \ge b_i^g\} - l_{t-}h_i'\big(R^L_{t-}, i\big) I\{R^L_{t-} < b_i^g\}\\
&\qquad+ q_{J_{t-}} h_i\big(R^L_{t-}, J_{t-}\big) - \sum_{j\ne J_{t-}} q_{J_{t-},j}\, g\big(R^L_{t-}, j\big) - d_i I\{R^L_{t-} \ge b_i^g\}.
\end{aligned}
\tag{3.87}
$$

Write $h_i'(0, i) = \lim_{x\downarrow 0} h'(x, i)$. By (3.83), (3.18) and Theorem 3.5 we know that $h_i(x, i) = \mathcal{P}_{g,i}(L^{b_i^g})(x) = S_{g,i,b_i^g}(x)$ is concave with respect to x, and that if $0 < b_i^g < +\infty$, $h_i'(b_i^g, i) = S'_{g,i,b_i^g}(b_i^g) = 1$. Hence, if $0 < b_i^g < +\infty$,

$$
h_i'(x, i)\begin{cases}\ge 1 & 0 \le x < b_i^g\\ \le 1 & x \ge b_i^g\end{cases}.
\tag{3.88}
$$

If $b_i^g = 0$, it follows by the definition of b_i^g in (3.68) that

$$
\lim_{x\downarrow b_i^g} h_i'(x, i) = \lim_{x\downarrow 0} S'_{g,i,0}(x) \le 1.
$$

Therefore, by the concavity of h_i, we get $h_i'(x, i) \le 1$ for all $x \ge 0$. Thus, (3.88) holds in the case $b_i^g = 0$ as well.

Further note that for any admissible strategy L, $l_t \le d_{J_t}$ for all $t \ge 0$, and hence $l_t \le d_i$ for $t < \tau_1$, given $J_0 = i$. Then it follows by (3.87) and (3.88) that, given $J_0 = i$, for $t \le \tau_1$,

$$\begin{aligned}
&\frac{1}{2}\sigma_{J_{t-}}^2 h_i''\big(R_{t-}^L, J_{t-}\big) + (\mu_{J_{t-}} - l_{t-})h_i'\big(R_{t-}^L, J_{t-}\big) - \delta_{J_{t-}} h_i\big(R_{t-}^L, J_{t-}\big)\\
&\quad \le (d_i - l_{t-})I\big\{R_{t-}^L \ge b_i^g\big\} - l_{t-}I\big\{R_{t-}^L < b_i^g\big\} + q_{J_{t-}} h_i\big(R_{t-}^L, J_{t-}\big)\\
&\qquad - \sum_{j \ne J_{t-}} q_{J_{t-}j} g\big(R_{t-}^L, j\big) - d_i I\big\{R_{t-}^L \ge b_i^g\big\}\\
&\quad = -l_{t-} + q_{J_{t-}} h_i\big(R_{t-}^L, J_{t-}\big) - \sum_{j \ne J_{t-}} q_{J_{t-},j} g\big(R_{t-}^L, j\big).
\end{aligned} \tag{3.89}$$

Note that

$$\begin{aligned}
&\int_0^{T^L \wedge \tau_1 \wedge s} e^{-\delta_i t}\left(q_{J_{t-}} h_i\big(R_{t-}^L, J_{t-}\big) - \sum_{j \ne J_{t-}} q_{J_{t-},j} g\big(R_{t-}^L, j\big)\right) \mathrm{d}t\\
&\quad = \int_0^{T^L \wedge \tau_1 \wedge s} e^{-\delta_i t}\left(q_{J_{t-}} h_i\big(R_{t-}^L, J_{t-}\big) - \sum_{j \ne J_{t-}} q_{J_{t-},j} h_i\big(R_{t-}^L, j\big)\right) \mathrm{d}t,
\end{aligned}$$

which is a martingale. Then it follows by (3.89) that

$$\begin{aligned}
&\mathrm{E}_{(x,i)}\Big[\int_0^{T^L \wedge \tau_1 \wedge s} e^{-\delta_i t}\Big(\frac{1}{2}\sigma_{J_{t-}}^2 h_i''\big(R_{t-}^L, J_{t-}\big) + (\mu_{J_{t-}} - l_{t-})h_i'\big(R_{t-}^L, J_{t-}\big)\\
&\quad -\delta_{J_{t-}} h_i\big(R_{t-}^L, J_{t-}\big)\Big)\mathrm{d}t\Big] \le -\mathrm{E}_{(x,i)}\Big[\int_0^{T^L \wedge \tau_1 \wedge s} e^{-\delta_i t} l_{t-}\mathrm{d}t\Big].
\end{aligned} \tag{3.90}$$

Note from Lemmas 3.1 and 3.2 that for any fixed $i \in \mathbb{E}$, the function $h_i(x, i) = S_{g,i,b_i^g}(x)$ is positive, non-decreasing and bounded. As a result, $S'_{g,i,b_i^g}(x)$ is non-negative and bounded. Therefore,

$$\mathrm{E}_{(x,i)}\Big[\int_0^{T^L \wedge \tau_1 \wedge s} \big(e^{-\delta_i t}\sigma_{J_{t-}} h_i'\big(R_{t-}^L, i\big)\big)^2 \mathrm{d}t\Big] < +\infty,$$

which implies that $\int_0^{T^L \wedge \tau_1 \wedge s} e^{-\delta_i t}\sigma_{J_{t-}} h_i'\big(R_{t-}^L, i\big)\mathrm{d}W_t$ is a zero mean martingale. Then by taking expectations on (3.86) and using (3.90), we can obtain

$$\begin{aligned}
h_i(x, i) \ge{}& \mathrm{E}_{(x,i)}\Big[\int_0^{T^L \wedge \tau_1 \wedge s} e^{-\delta_i t} l_{t-}\mathrm{d}t\Big]\\
&+\mathrm{E}_{(x,i)}\left[e^{-\delta_i (T^L \wedge \tau_1 \wedge s)} h_i\big(R_{T^L \wedge \tau_1 \wedge s}^L, J_{T^L \wedge \tau_1 \wedge s}\big)\right].
\end{aligned}$$

As l_t is non-negative and h_i is bounded, it follows by letting $t \to +\infty$ and using the monotone convergence and dominated convergence that

$$\begin{aligned} h_i(x,i) &\geq \mathrm{E}_{(x,i)}\Big[\int_0^{T^L\wedge\tau_1} e^{-\delta_i t} l_{t-}\mathrm{d}t\Big] + \mathrm{E}_{(x,i)}\Big[e^{-\delta_i(T^L\wedge\tau_1)} h_i\big(R^L_{T^L\wedge\tau_1}, J_{T^L\wedge\tau_1}\big)\Big] \\ &= \mathrm{E}_{(x,i)}\Big[\int_0^{T^L\wedge\tau_1} e^{-\delta_i t} l_{t-}\mathrm{d}t\Big] + \mathrm{E}_{(x,i)}\Big[e^{-\delta_i \tau_1} g\big(R^L_{\tau_1}, J_{\tau_1}\big) I\{\tau_1 < T^L\}\Big], \end{aligned} \tag{3.91}$$

where the last equality follows by noting that if $\tau_1 \geq T^L$, $h_i(R^L_{T^L}, J_{T^L}) = h_i(0, J_{T^L}) = 0$, and that if $\tau_1 < T^L$, $h_i(R^L_{\tau_1}, J_{\tau_1}) = g(R^L_{\tau_1}, J_{\tau_1})$, given $J_0 = i$. Since (3.91) holds for arbitrary $L \in \Pi$, we can conclude that

$$\mathcal{P}_{g,i}(L^{b_i^g})(x) = h_i(x) \geq \mathcal{M}(g)(x,i).$$

□

Recall that the class of functions $\mathcal{C}$ is defined in (3.69).

Theorem 3.7. *For any $g \in \mathcal{C}$, if $b_i^g < +\infty$ for all $i \in \mathbb{E}$, $\mathcal{M}(g) \in \mathcal{C}$.*

Proof. Consider any fixed $i \in \mathbb{E}$. Since $b_i^g < +\infty$, by Theorem 3.6 we have $\mathcal{M}(g)(x,i) = \mathcal{P}_{g,i}(L^{b_i^g})(x)$. Therefore, by (3.18) and Theorem 3.5 we can conclude that $\mathcal{M}(g)(x,i)$ is twice continuously differentiable and concave with respect to x. It is obvious that the function $\mathcal{M}(g)(x,i)$ is non-negative.

It remains to show $\mathcal{M}(g)(x,i) \in \mathcal{D}$. From Lemma 3.2 we know that $\mathcal{M}(g)(x,i)$ is non-decreasing.

It follows by the non-decreasing property of $\mathcal{P}_{g,i}(L)(x)$ that for any $i \in \mathbb{E}$ and $x \geq 0$,

$$\begin{aligned} \mathcal{M}(g)(x,i) &= \sup_{L\in\Pi} \mathcal{P}_{g,i}(L)(x) \leq \sup_{L\in\Pi} \lim_{x\to+\infty} \mathcal{P}_{g,i}(L)(x) \\ &= \frac{\sum_{j\neq i} q_{ij} g(+\infty, j) + d_i}{\delta_i + q_i} \\ &\leq \frac{\sum_{j\neq i} q_{ij} \frac{\max_{l\in\mathbb{E}} d_l}{\min_{l\in\mathbb{E}\delta_l}} + d_i}{\delta_i + q_i} \\ &= \frac{q_i \frac{\max_{l\in\mathbb{E}} d_l}{\min_{l\in\mathbb{E}\delta_l}} + d_i}{\delta_i + q_i} \\ &\leq \frac{\max_{l\in\mathbb{E}} d_l}{\min_{l\in\mathbb{E}\delta_l}}, \end{aligned}$$

where the first inequality follows by (3.19). □

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4. MAIN RESULTS

In this section we will find sufficient conditions under which the regime-switching threshold strategy is optimal. We start with defining a condition which will be shown to be a sufficient condition for the threshold strategy to be the optimal one later on.

Condition 1: $\inf_{b\geq 0}\{(q_i\frac{\max_{l\in\mathbb{E}} d_l}{\min_{l\in\mathbb{E}}\delta_l} + d_i)\frac{\theta_{1i}e^{\theta_{1i}b}+\theta_{2i}e^{-\theta_{2i}b}}{\theta_{3i}} - (\theta_{1i}+\theta_{4i})e^{\theta_{1i}b} - (\theta_{2i}-\theta_{4i})e^{-\theta_{2i}b}\} < 0$, where θ_{1i} denote the positive root to the equation $\frac{\sigma_i^2}{2}x^2+\mu_i x-(\delta_i+q_i)=0$, and θ_{3i} and $-\theta_{4i}$ the positive and negative roots to the equation $\frac{\sigma_i^2}{2}x^2+(\mu_i-d_i)x-(\delta_i+q_i)=0$.

Remark 4.1. *For any $g\in\mathcal{D}$, it follows from* (3.67) *that*

$$\begin{aligned}
\xi_i^g(b) &\leq (\theta_{1i}e^{\theta_{1i}b}+\theta_{2i}e^{-\theta_{2i}b})\int_b^{+\infty} e^{-\theta_{3i}(s-b)}\left(\sum_{j\neq i} q_{ij}\frac{\max_{l\in\mathbb{E}} d_l}{\min_{l\in\mathbb{E}}\delta_l}+d_i\right)\mathrm{d}s \\
&\quad -(\theta_{1i}+\theta_{4i})e^{\theta_{1i}b}-(\theta_{2i}-\theta_{4i})e^{-\theta_{2i}b} \\
&= \left(q_i\frac{\max_{i\in\mathbb{E}} d_i}{\min_{i\in\mathbb{E}}\delta_i}+d_i\right)\frac{\theta_{1i}e^{\theta_{1i}b}+\theta_{2i}e^{-\theta_{2i}b}}{\theta_{3i}}-(\theta_{1i}+\theta_{4i})e^{\theta_{1i}b} \\
&\quad -(\theta_{2i}-\theta_{4i})e^{-\theta_{2i}b}.
\end{aligned}$$

If Condition 1 holds, then there exists a real number $b\in[0,+\infty)$ such that $\xi_i^g(b)<0$, and hence by (3.68), *$b_i^g\in[0,+\infty)$.*

Theorem 4.1. *If Condition 1 holds for all $i\in\mathbb{E}$, then $V\in\mathcal{C}$, and for any $i\in\mathbb{E}$, $b_i^V<+\infty$ and $\mathcal{P}_{V,i}(L^{b_i^V})(x)=V(x,i)$.*

Proof. By Remark 4.1 we know $b_i^g<+\infty$ for all $i\in\mathbb{E}$, and hence $\mathcal{M}(g)\in\mathcal{C}$ for any $g\in\mathcal{C}$ by Theorem 3.7. We first show that $\mathcal{M}$ is a contraction on $\mathcal{C}$.

Define the norm $||g||=\sup_{x\geq 0}\max_{i\in\mathbb{E}} g(x,i)$. Consider any two functions $g_1, g_2\in\mathcal{C}$.

$$\begin{aligned}
&|\mathcal{M}(g_1)(x,i)-\mathcal{M}(g_2)(x,i)| \\
&= \left|\sup_{L\in\Pi}\mathcal{P}_{g_1,i}(L)(x)-\sup_{L\in\Pi}\mathcal{P}_{g_2,i}(L)(x)\right| \\
&\leq \sup_{L\in\Pi}|\mathcal{P}_{g_1,i}(L)(x)-\mathcal{P}_{g_2,i}(L)(x)| \\
&= \left|\mathrm{E}_{(x,i)}\left[I\{\tau_1<T^L\}e^{-\Lambda_{\tau_1}}\left(g_1\left(R_{\tau_1}^L,J_{\tau_1}\right)-g_2\left(R_{\tau_1}^L,J_{\tau_1}\right)\right)\right]\right| \\
&\leq \mathrm{E}_{(x,i)}\left[e^{-\Lambda_{\tau_1}}\left|g_1\left(R_{\tau_1}^L,J_{\tau_1}\right)-g_2\left(R_{\tau_1}^L,J_{\tau_1}\right)\right|\right], \\
&\leq \mathrm{E}_{(x,i)}\left[e^{-\Lambda_{\tau_1}}\right]||g_1-g_2||.
\end{aligned}$$

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Note that $\mathrm{E}_{(x,i)}[e^{-\Lambda_{\tau_1}}] = \int_0^{+\infty} q_i e^{-q_i t} e^{-\delta_i t}\mathrm{d}t = \frac{q_i}{\delta_i + q_i}$. Hence,

$$||\mathcal{M}(g_1) - \mathcal{M}(g_2)|| \le \max_{i\in\mathbb{E}} \frac{q_i}{\delta_i + q_i}||g_1 - g_2||. \tag{4.92}$$

Since $\max_{i\in\mathbb{E}} \frac{q_i}{\delta_i+q_i} < 1$, we can conclude that $\mathcal{M}$ is a contraction with respect to the norm $||\cdot||$ on $\mathcal{C}$.

Consider two controlled stochastic processes $\{Y_t^L;\, t \ge 0\}$ and $\{Z_t^L;\, t \ge 0\}$ defined by

$$Y_t^L = Y_0 + \max_{j\in E} \mu_j t + \min_{j\in E} \sigma_j^2 W_t - \int_0^t l_s \mathrm{d}s, \tag{4.93}$$

and

$$Z_t^L = Y_0 + \min_{j\in E} \mu_j t + \max_{j\in E} \sigma_j^2 W_t - \int_0^t l_s \mathrm{d}s. \tag{4.94}$$

Define the time of ruin of these processes by $T_1^L = \inf\{t \ge 0 : Y_t^L \le 0\}$ and $T_2^L = \inf\{t \ge 0 : Z_t^L \le 0\}$ and define the sets of admissible strategies Π_1 and Π_2 by

$$\begin{aligned}\Pi_1 = \{L = \{l_t;\, t \ge 0\} :\ & L \text{ is adapted},\ 0 \le l_t \le \bar{d} \text{ for } t \ge 0,\\ & \text{and } l_t = 0 \text{ for } t \ge T_1^L\},\\ \Pi_2 = \{L = \{l_t;\, t \ge 0\} :\ & L \text{ is adapted},\ 0 \le l_t \le \underline{d} \text{ for } t \ge 0,\\ & \text{and } l_t = 0 \text{ for } t \ge T_2^L\},\end{aligned}$$

where $\bar{d} = \max_{j\in\mathbb{E}} d_j$ and $\underline{d} = \min_{j\in\mathbb{E}} d_j$.

Define

$$V_1(x) = \sup_{L\in\Pi_1} \mathrm{E}\left[\int_0^{T_1^L} e^{-\min_{j\in E}\delta_j t} l_t \mathrm{d}t \,\middle|\, Y_0 = x\right],$$

$$V_2(x) = \sup_{L\in\Pi_2} \mathrm{E}\left[\int_0^{T_2^L} e^{-\max_{j\in E}\delta_j t} l_t \mathrm{d}t \,\middle|\, Z_0 = x\right].$$

The function $V_1(x)$ can be interpreted as the value function of the dividend optimization problem of the controlled diffusion process (4.93) with restricted dividend rates and the discount rate $\min_{j\in E}\delta_j$. Similarly, $V_2(x)$ is the value function of the dividend optimization problem of the controlled diffusion process (4.94) with restricted dividend rates and the discount rate $\max_{j\in E}\delta_j$.

It has been shown in Asmussen and Taksar (1997) that the functions V_1 and V_2 are non-negative, increasing, twice continuously differentiable and concave on $[0, \infty)$, and that there exist $\bar{b}$, $\underline{b} \geq 0$ such that

$$\frac{1}{2}\min_{j\in E}\sigma_j^2 V_1''(x) + \left(\max_{j\in E}\mu_j - \bar{d}I\{x \geq \bar{b}\}\right)V_1'(x) - \min_{j\in E}\delta_j V_1(x)$$
$$+\bar{d}I\{x \geq \bar{b}\} = 0 \quad \text{for } x > 0, \tag{4.95}$$

$$V_1'(x) \geq 1\text{for } 0 < x \leq \bar{b},\ \ V_1'(x) \leq 1\text{for } x \geq \bar{b}, \tag{4.96}$$

$$V_1(0) = 0, \quad V_1(x) \leq \frac{\bar{d}}{\min_{j\in\mathbb{E}}\delta_j}, \tag{4.97}$$

$$\frac{1}{2}\max_{j\in E}\sigma_j^2 V_2''(x) + \left(\min_{j\in E}\mu_j - \underline{d}I\{x \geq \underline{b}\}\right)V_2'(x) - \max_{j\in E}\delta_j V_2(x)$$
$$+\underline{d}I\{x \geq \underline{b}\} = 0 \quad \text{for } x > 0, \tag{4.98}$$

$$V_2'(x) \geq 1 \text{ for } 0 < x \leq \underline{b},\ \ V_2'(x) \leq 1 \text{ for } x \geq \underline{b}, \tag{4.99}$$

$$V_2(0) = 0, \quad V_2(x) \leq \frac{\underline{d}}{\max_{j\in\mathbb{E}}\delta_j}. \tag{4.100}$$

We now proceed to show that

$$V_2(x) \leq V(x,i) \leq V_1(x) \quad \text{for } x \geq 0 \text{ and } i \in \mathbb{E}. \tag{4.101}$$

For any $L \in \Pi$, by applying Itô's formula to $e^{-\Lambda_{t\wedge T^L}}V_1(R^L_{t\wedge T^L})$, we obtain

$$e^{-\Lambda_{t\wedge T^L}}V_1\big(R^L_{t\wedge T^L}\big) - V_1\big(R^L_0\big)$$
$$= \int_0^{t\wedge T^L} e^{-\Lambda_s}\left[\frac{1}{2}\sigma_{J_s}^2 V_1''\big(R^L_s\big) + (\mu_{J_s} - l_s)V_1'\big(R^L_s\big) - \delta_{J_s}V_1\big(R^L_s\big)\right]\mathrm{d}s$$
$$+\int_0^{t\wedge T^L} e^{-\Lambda_s}\sigma_{J_s}V_1'\big(R^L_s\big)\mathrm{d}W_s. \tag{4.102}$$

By noticing that V_1 is increasing, non-negative and concave, we have for any $L \in \Pi$,

$$\frac{1}{2}\sigma_{J_s}^2 V_1''\big(R^L_s\big) + (\mu_{J_s} - l_s)V_1'\big(R^L_s\big) - \delta_{J_s}V_1\big(R^L_s\big)$$
$$\leq \frac{1}{2}\min_{j\in E}\sigma_j^2 V_1''\big(R^L_s\big) + \max_{j\in E}\mu_j V_1'\big(R^L_s\big) - \min_{j\in E}\delta_j V_1\big(R^L_s\big) - l_s V_1'\big(R^L_s\big)$$
$$= \bar{d}I\big\{R^L_s \geq \bar{b}\big\}\big(V_1'\big(R^L_s\big) - 1\big) - l_s V_1'\big(R^L_s\big)$$
$$= (\bar{d} - l_s)V_1'\big(R^L_s\big)I\big\{R^L_s \geq \bar{b}\big\} - \bar{d}I\big\{R^L_s \geq \bar{b}\big\} - l_s V_1'\big(R^L_s\big)I\{R^L_s < \bar{b}\}$$
$$\leq (\bar{d} - l_s)I\big\{R^L_s \geq \bar{b}\big\} - \bar{d}I\big\{R^L_s \geq \bar{b}\big\} - l_s I\{R^L_s < \bar{b}\} = -l_s, \tag{4.103}$$

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where the first equality is due to (4.95) and the last inequality follows by noticing $l_s \le \bar{d}$ and (4.96).

It is not hard to see that V_1' is bounded on $(0, +\infty)$, and hence

$$\mathrm{E}_x\left[\int_0^{t\wedge T^L} e^{-\Lambda_s}\sigma_{J_s} V_1'\big(R_s^L\big)\mathrm{d}W_s\right] = 0. \tag{4.104}$$

It follows from (4.102), (4.103) and (4.104) that

$$\mathrm{E}_{(x,i)}\left[e^{-\Lambda_{t\wedge T^L}} V_1\big(R^L_{(t\wedge T^L)}\big) - V_1\big(R_0^L\big)\right] \le -\mathrm{E}_{(x,i)}\left[\int_0^{t\wedge T^L} e^{-\Lambda_s} l_s \mathrm{d}s\right]. \tag{4.105}$$

By letting $t \to +\infty$ in (4.105) and using the dominated convergence, the monotone convergence and (4.97), we can obtain

$$V_1(x) \ge \mathrm{E}_{(x,i)}\left[\int_0^{T^L} e^{-\Lambda_s} l_s \mathrm{d}s\right].$$

Consequently,

$$V_1(x) \ge \sup_{L\in\Pi} \mathrm{E}_{(x,i)}\left[\int_0^{T^L} e^{-\Lambda_s} l_s \mathrm{d}s\right] = V(x, i).$$

Let $\hat{L}$ denote the strategy that pays dividends at rate $\underline{d}$ when the surplus is above $\underline{b}$ and nothing otherwise until the time of ruin. Note that $\underline{d} \le d_j$ for any $j \in \mathbb{E}$, and hence $\hat{L} \in \Pi$. The controlled process $R^{\hat{L}}$ has the following dynamics:

$$\mathrm{d}R_t^{\hat{L}} = \big(\mu_{J_t} - \underline{d}I\big\{R_t^{\hat{L}} \ge \underline{b}\big\}\big)\mathrm{d}t + \sigma_{J_t}\mathrm{d}W_t.$$

Applying Itô's formula to $e^{-\Lambda_{t\wedge T^{\hat{L}}}} V_2\big(R^{\hat{L}}_{t\wedge T^{\hat{L}}}\big)$ yields

$$\begin{aligned}
&e^{-\Lambda_{t\wedge T^{\hat{L}}}} V_2\big(R^{\hat{L}}_{t\wedge T^{\hat{L}}}\big) - V_2\big(R_0^{\hat{L}}\big)\\
&\quad= \int_0^{t\wedge T^{\hat{L}}} e^{-\Lambda_s}\left[\frac{1}{2}\sigma_{J_s}^2 V_2''\big(R_s^{\hat{L}}\big) + (\mu_{J_s} - \underline{d}I\big\{R_s^{\hat{L}} \ge \underline{b}\big\}) V_2'\big(R_s^{\hat{L}}\big) - \delta_{J_s} V_2\big(R_s^{\hat{L}}\big)\right]\mathrm{d}s\\
&\qquad+ \int_0^{t\wedge T^{\hat{L}}} e^{-\Lambda_s}\sigma_{J_s} V_2'\big(R_s^{\hat{L}}\big)\mathrm{d}W_s.
\end{aligned} \tag{4.106}$$

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It follows by noticing that V_2 is increasing, non-negative and concave, and that

$$\begin{aligned}
&\frac{1}{2}\sigma_{J_s}^2 V_2''\big(R_s^{\hat{L}}\big) + (\mu_{J_s} - \underline{d}I\big\{R_s^{\hat{L}} \geq \underline{b}\big\}) V_2'\big(R_s^{\hat{L}}\big) - \delta_{J_s} V_2\big(R_s^{\hat{L}}\big) \\
&\geq \frac{1}{2}\max_{j\in E}\sigma_j^2 V_2''\big(R_s^{\hat{L}}\big) + (\min_{j\in E}\mu_j - \underline{d}I\{R_s^{\hat{L}} \geq \underline{b}\}) V_2'\big(R_s^{\hat{L}}\big) - \max_{j\in E}\delta_j V_2\big(R_s^{\hat{L}}\big) \\
&= -\underline{d}I\big\{R_s^{\hat{L}} \geq \underline{b}\big\},
\end{aligned} \tag{4.107}$$

where the last equality follows from (4.98). Note that V_2' is bounded on $(0, +\infty)$, and hence

$$\mathrm{E}_{(x,i)}\left[\int_0^{t\wedge T^{\hat{L}}} e^{-\Lambda_s}\sigma_{J_s} V_2'\big(R_s^{\hat{L}}\big)\mathrm{d}W_s\right] = 0. \tag{4.108}$$

It follows from (4.106), (4.107) and (4.108) that

$$\begin{aligned}
&\mathrm{E}_{(x,i)}\left[e^{-\Lambda_{t\wedge T^{\hat{L}}}} V_2\big(R_{t\wedge T^{\hat{L}}}^{\hat{L}}\big) - V_2\big(R_0^{\hat{L}}\big)\right] \\
&\quad\geq -\mathrm{E}_{(x,i)}\left[\int_0^{t\wedge T^{\hat{L}}} e^{-\Lambda_s}\underline{d}I\big\{R_s^{\hat{L}} \geq \underline{b}\big\}\mathrm{d}s\right].
\end{aligned} \tag{4.109}$$

By letting $t \to +\infty$ in (4.109) and using the dominated convergence, the monotone convergence and (4.100), we can obtain

$$\begin{aligned}
V_2(x) &\leq \mathrm{E}_{(x,i)}\left[\int_0^{T^{\hat{L}}} e^{-\Lambda_s}\underline{d}I\big\{R_s^{\hat{L}} \geq \underline{b}\big\}\mathrm{d}s\right] \\
&\leq \sup_{L\in\Pi}\mathrm{E}_{(x,i)}\left[\int_0^{T^L} e^{-\Lambda_s} l_s\mathrm{d}s\right] = V(x, i).
\end{aligned}$$

Define $\mathcal{M}^1(g) = \mathcal{M}(g)$ and $\mathcal{M}^n(g) = \mathcal{M}^{n-1}(g)$ for $n \geq 2$. Since $V_1, V_2 \in \mathcal{C}$, it follows from Theorem 3.7 and Remark 4.1 that $\mathcal{M}^n(V_1), \mathcal{M}^n(V_2) \in \mathcal{C}$ for $n \geq 1$.

By the definition of $\mathcal{M}$ in (2.10), we can see that $\mathcal{M}$ is an increasing operator and that

$$\begin{aligned}
\mathcal{M}(V)(x, i) &= \sup_{L\in\Pi}\mathrm{E}_{(x,i)}\left[\int_0^{T^L\wedge\tau_1} l_t e^{-\Lambda_t}\mathrm{d}t + I\{\tau_1 < T^L\}e^{-\Lambda_{\tau_1}} V\big(R_{\tau_1}^L, J_{\tau_1}\big)\right] \\
&= V(x, i),
\end{aligned} \tag{4.110}$$

where the last equality follows from the dynamic programming principle (2.8). Therefore, it follows from (4.101) that $\mathcal{M}(V_2) \leq \mathcal{M}(V) \leq \mathcal{M}(V_1)$, and hence $\mathcal{M}(V_2) \leq V \leq \mathcal{M}(V_1)$. Therefore, by applying the operator $\mathcal{M}$ repeatedly we can obtain $\mathcal{M}^n(V_2) \le V \le \mathcal{M}^n(V_1)$ for $n \ge 1$. Since $\mathcal{M}$ is a contraction on $\mathcal{C}$ and $\mathcal{C}$ is a complete space, using the fixed point theory we have $\lim_{n\to+\infty} \mathcal{M}^n(V_1) = \lim_{n\to+\infty} \mathcal{M}^n(V_2)$. Consequently, $V = \lim_{n\to+\infty} \mathcal{M}^n(V_1)$, and hence V is the fixed point on $\mathcal{C}$, which implies $V \in \mathcal{C}$.

The inequality $b_i^V < +\infty$ follows immediately by Remark 4.1. It follows from Theorem 3.6 and (4.110) that $\mathcal{P}_{V,i}(L^{b_i^V})(x) = \mathcal{M}(V)(x,i) = V(x,i)$. □

Define $b_i^* = b_i^V$ and define the *regime-switching threshold strategy* $L^* = \{l_t^*; t \ge 0\}$, under which, at any time t before the time of ruin, the company pays dividends at rate d_{J_t} if the current surplus is at or above b_{J_t} and pays nothing if the current surplus is below b_{J_t}, that is, $l_t^* = d_{J_t} I\{R_t^{L^*} \ge b_{J_t}\}$.

Let $(R^{L^*}, J) : (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+} \to (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+}$ be the canonical process, and let $\mathcal{F}$ denote the right-continuous canonical filtration induced by (R^{L^*}, J). Define the shift operators $\theta_t : (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+} \to (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+}$ for $t \ge 0$ by

$$(\theta_t \omega)_s = w_{s+t}, \;\; s, t \in \mathbb{R}_+, \;\; w \in (\mathbb{R} \times \mathbb{E})^{\mathbb{R}_+}.$$

For any two random variables X and Y, we use $X \circ Y$ to denote the *composition* as long as it is well defined. It is clear that θ_t is measurable with respect to $\mathcal{F}$, and $\theta_t(R^{L^*}, J) = (R^{L^*}, J) \circ \theta_t$. Let $\tau_0 = 0$ and τ_1 be defined in the same way as before (right above (2.10)). We can further define recursively the transition times of the Markov process J,

$$\tau_{n+1} = \tau_n + \tau_1 \circ \theta_{\tau_n}, \quad n = 1, 2, \ldots. \tag{4.111}$$

The optional time τ_n is the time when the nth transition of the state of process J occurs.

Theorem 4.2. *If Condition 1 holds for all $i \in \mathbb{E}$, then the regime-switching threshold strategy L^* is an optimal strategy.*

Proof. It is not hard to verify that $L^* \in \Pi$.

Note that given the initial state $J_0 = i$, the strategy $L^{b_i^V}$ is the same as the strategy L^* before the first regime switch, which occurs at time τ_1. Hence, by Theorem 4.1 and the definition for the operator $\mathcal{P}_{V,j}$ we can see

$$V(R_{\tau_n}^{L^*}, J_{\tau_n}) = \mathcal{P}_{V,J_{\tau_n}}(L^{b_i^V})\big(R_{\tau_n}^{L^*}\big) = \mathcal{P}_{V,J_{\tau_n}}(L^*)\big(R_{\tau_n}^{L^*}\big). \tag{4.112}$$

We can see that given the history of the process (R^{L^*}, J) up to and including time t, the conditional probability distribution of L^* at any future time depends only on the current time t and the current value $(R_t^{L^*}, J_t)$. Note that the process (R, J) without dividend payments is the Markov process. Therefore, the controlled process (R^{L^*}, J) is the Markov process. By noting $V(R_{T^{L^*}}^{L^*}, J_{T^{L^*}}) = 0$

and using (2.9) we can derive that for any $x \in \mathbb{R}$ and $i \in \mathbb{E}$

$$
\begin{aligned}
&\mathcal{P}_{V,J_{\tau_n}}(L^*)\big(R^{L^*}_{\tau_n}\big)\\
&\quad = \mathrm{E}_{(R^{L^*}_{\tau_n}, J_{\tau_n})}\left[\int_{\tau_0}^{\tau_1} e^{-\Lambda_t} l^*_t \mathrm{d}t + e^{-\Lambda_{\tau_1}} V\big(R^{L^*}_{\tau_1}, J_{\tau_1}\big);\ \tau_1 < T^{L^*}\right]\\
&\qquad + \mathrm{E}_{(R^{L^*}_{\tau_n}, J_{\tau_n})}\left[\int_{\tau_0}^{T^{L^*}} e^{-\Lambda_t} l^*_t \mathrm{d}t;\ \tau_1 \geq T^{L^*}\right]\\
&\quad = \mathrm{E}\left[\int_{\tau_0\circ\theta_{\tau_n}}^{\tau_1\circ\theta_{\tau_n}} e^{-\Lambda_t} l^*_t \mathrm{d}t;\ \tau_1\circ\theta_{\tau_n} < T^{L^*}\circ\theta_{\tau_n}\,\middle|\,\mathcal{F}_{\tau_n}\right]\\
&\qquad + \mathrm{E}\left[e^{-\Lambda_{\tau_1\circ\theta_{\tau_n}}} V\big(R^{L^*}_{\tau_1}\circ\theta_{\tau_n}, J_{\tau_1}\circ\theta_{\tau_n}\big);\ \tau_1\circ\theta_{\tau_n} < T^{L^*}\circ\theta_{\tau_n}\,\middle|\,\mathcal{F}_{\tau_n}\right]\\
&\qquad + \mathrm{E}\left[\int_{\tau_0\circ\theta_{\tau_n}}^{T^{L^*}\circ\theta_{\tau_n}} e^{-\Lambda_t} l^*_t \mathrm{d}t;\ \tau_1\circ\theta_{\tau_n} \geq T^{L^*}\circ\theta_{\tau_n}\,\middle|\,\mathcal{F}_{\tau_n}\right]\ P_{(x,i)} - a.s.,
\end{aligned} \tag{4.113}
$$

where the last equality follows from the strong Markov property of L^* and (R^{L^*}, J).

Note that $\tau_n + \tau_1\circ\theta_{\tau_n} = \tau_{n+1}$ and

$$
\begin{aligned}
&\{\tau_1\circ\theta_{\tau_n} < T^{L^*}\circ\theta_{\tau_n}\} \cap \{\tau_n < T^{L^*}\}\\
&\quad = \big\{R^{L^*}_t > 0 \text{ for all } t \in [\tau_n, \tau_{n+1}]\big\} \cap \big\{R^{L^*}_t > 0 \text{ for all } t \in [0, \tau_n]\big\}\\
&\quad = \{\tau_{n+1} < T^{L^*}\}.
\end{aligned} \tag{4.114}
$$

Therefore,

$$
\begin{aligned}
&\mathrm{E}\left[\int_{\tau_0\circ\theta_{\tau_n}}^{\tau_1\circ\theta_{\tau_n}} e^{-\Lambda_t} l^*_t \mathrm{d}t; \tau_1\circ\theta_{\tau_n} < T^{L^*}\circ\theta_{\tau_n}\,\middle|\,\mathcal{F}_{\tau_n}\right] I\{\tau_n < T^{L^*}\}\\
&\quad = \mathrm{E}\left[\int_{\tau_0\circ\theta_{\tau_n}}^{\tau_1\circ\theta_{\tau_n}} e^{-\Lambda_t} l^*_t \mathrm{d}t;\ \tau_1\circ\theta_{\tau_n} < T^{L^*}\circ\theta_{\tau_n}, \tau_n < T^{L^*}\,\middle|\,\mathcal{F}_{\tau_n}\right]\\
&\quad = \mathrm{E}\left[e^{\Lambda_{\tau_n}}\int_{\tau_n}^{\tau_{n+1}} e^{-\Lambda_t} l^*_t \mathrm{d}t;\ \tau_{n+1} < T^{L^*}\,\middle|\,\mathcal{F}_{\tau_n}\right].
\end{aligned} \tag{4.115}
$$

Further, noting $\Lambda_{\tau_n} + \Lambda_{\tau_1\circ\theta_{\tau_n}} = \Lambda_{\tau_{n+1}}$, we have

$$
\begin{aligned}
&\mathrm{E}\left[e^{-\Lambda_{\tau_1\circ\theta_{\tau_n}}} V\big(R^{L^*}_{\tau_1}\circ\theta_{\tau_n}, J_{\tau_1}\circ\theta_{\tau_n}\big);\ \tau_1\circ\theta_{\tau_n} < T^{L^*}\circ\theta_{\tau_n}\,\middle|\,\mathcal{F}_{\tau_n}\right] I\{\tau_n < T^{L^*}\}\\
&\quad = \mathrm{E}\left[e^{\Lambda_{\tau_n}} e^{-\Lambda_{\tau_{n+1}}} V\big(R^{L^*}_{\tau_1}\circ\theta_{\tau_n}, J_{\tau_1}\circ\theta_{\tau_n}\big);\ \tau_1\circ\theta_{\tau_n} < T^{L^*}\circ\theta_{\tau_n}, \tau_n < T^{L^*}\,\middle|\,\mathcal{F}_{\tau_n}\right]\\
&\quad = e^{\Lambda_{\tau_n}}\mathrm{E}\left[e^{-\Lambda_{\tau_{n+1}}} V\big(R^{L^*}_{\tau_{n+1}}, J_{\tau_{n+1}}\big);\ \tau_{n+1} < T^{L^*}\,\middle|\,\mathcal{F}_{\tau_n}\right].
\end{aligned} \tag{4.116}
$$

It can also be seen that $\tau_n + T^{L^*} \circ \theta_{\tau_n} = T^{L^*}$ on $\{\tau_n < T^{L^*}\}$ and that

$$
\begin{aligned}
&\{\tau_1 \circ \theta_{\tau_n} \geq T^{L^*} \circ \theta_{\tau_n}\} \cap \{\tau_n < T^{L^*}\} \\
&\quad = \left\{ R_t^{L^*} \leq 0 \text{ for some } t \in [\tau_n, \tau_{n+1}] \right\} \cap \{\tau_n < T^{L^*}\} \\
&\quad = \{\tau_n < T^{L^*} \leq \tau_{n+1}\}.
\end{aligned}
\tag{4.117}
$$

Thus, we have

$$
\begin{aligned}
&\mathrm{E}\left[\int_{\tau_0 \circ \theta_{\tau_n}}^{T^{L^*} \circ \theta_{\tau_n}} e^{-\Lambda_t} l_t^* \mathrm{d}t;\ \tau_1 \circ \theta_{\tau_n} \geq T^{L^*} \circ \theta_{\tau_n} \middle| \mathcal{F}_{\tau_n} \right] I\{\tau_n < T^{L^*}\} \\
&\quad = \mathrm{E}\left[e^{\Lambda_{\tau_n}} \int_{\tau_n}^{T^{L^*}} e^{-\Lambda_t} l_t^* \mathrm{d}t;\ \tau_n < T^{L^*} \leq \tau_{n+1} \middle| \mathcal{F}_{\tau_n} \right].
\end{aligned}
\tag{4.118}
$$

It follows from (4.112), (4.113), (4.115) (4.116) and (4.118) that for any $x \in \mathbb{R}$ and $i \in \mathbb{E}$,

$$
\begin{aligned}
&e^{-\Lambda_{\tau_n}} V\big(R_{\tau_n}^{L^*}, J_{\tau_n}\big) I\{\tau_n < T^{L^*}\} \\
&\quad = \mathrm{E}\left[\int_{\tau_n}^{\tau_{n+1}} e^{-\Lambda_t} l_t^* \mathrm{d}t;\ \tau_{n+1} < T^{L^*} \middle| \mathcal{F}_{\tau_n} \right] \\
&\qquad + \mathrm{E}\left[e^{-\Lambda_{\tau_{n+1}}} V\big(R_{\tau_{n+1}}^{L^*}, J_{\tau_{n+1}}\big);\ \tau_{n+1} < T^{L^*} \middle| \mathcal{F}_{\tau_n} \right] \\
&\qquad + \mathrm{E}\left[\int_{\tau_n}^{T^{L^*}} e^{-\Lambda_t} l_t^* \mathrm{d}t;\ \tau_n < T^{L^*} \leq \tau_{n+1} \middle| \mathcal{F}_{\tau_n} \right] \\
&\quad = \mathrm{E}\left[I\{\tau_n < T^{L^*}\} \int_{\tau_n}^{\tau_{n+1} \wedge T^{L^*}} e^{-\Lambda_t} l_t^* \mathrm{d}t \middle| \mathcal{F}_{\tau_n} \right] \\
&\qquad + \mathrm{E}\left[e^{-\Lambda_{\tau_{n+1}}} V\big(R_{\tau_{n+1}}^{L^*}, J_{\tau_{n+1}}\big) I\{\tau_{n+1} < T^{L^*}\} \middle| \mathcal{F}_{\tau_n} \right], \quad P_{(x,i)} - a.s..
\end{aligned}
\tag{4.119}
$$

Now we proceed to show that for $i \in \mathbb{E}$,

$$
V(x, i) = \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^*} \wedge \tau_k} e^{-\Lambda_t} l_t^* \mathrm{d}t + e^{-\Lambda_{\tau_k}} V\big(R_{\tau_k}^{L^*}, J_{\tau_k}\big) I\{\tau_k < T^{L^*}\} \right] \tag{4.120}
$$

using proof by induction. Using the same argument as in the proof of (4.112), we have

$$
\begin{aligned}
V(x, i) &= \mathcal{P}_{V,i}(L^*)(x) \\
&= \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^*} \wedge \tau_1} e^{-\Lambda_t} l_t^* \mathrm{d}t + e^{-\Lambda_{\tau_1}} V\big(R_{\tau_1}^{L^*}, J_{\tau_1}\big) I\{\tau_1 < T^{L^*}\} \right],
\end{aligned}
\tag{4.121}
$$

where the last inequality follows from the definition of $\mathcal{P}$ in (2.9). Therefore, (4.120) holds for $k = 1$.

Now suppose (4.120) holds for $k = n$. Then

$$\begin{aligned} V(x, i) &= \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^*}\wedge\tau_n} e^{-\Lambda_t} l_t^* \mathrm{d}t + e^{-\Lambda_{\tau_n}} V\big(R_{\tau_n}^{L^*}, J_{\tau_n}\big) I\{\tau_n < T^{L^*}\}\right] \\ &= \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^*}\wedge\tau_n} e^{-\Lambda_t} l_t^* \mathrm{d}t\right] \\ &\quad + \mathrm{E}_{(x,i)}\left[\mathrm{E}\left[I\{\tau_n < T^{L^*}\} \int_{\tau_n}^{\tau_{n+1}\wedge T^{L^*}} e^{-\Lambda_t} l_t^* \mathrm{d}t \middle| \mathcal{F}_{\tau_n}\right]\right] \\ &\quad + \mathrm{E}_{(x,i)}\left[\mathrm{E}\left[e^{-\Lambda_{\tau_{n+1}}} V\big(R_{\tau_{n+1}}^{L^*}, J_{\tau_{n+1}}\big) I\{\tau_{n+1} < T^{L^*}\} \middle| \mathcal{F}_{\tau_n}\right]\right], \end{aligned} \quad (4.122)$$

where the last equality follows from (4.119). Consequently, by the double expectation formula it follows that

$$\begin{aligned} &V(x, i) \\ &= \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^*}\wedge\tau_{n+1}} e^{-\Lambda_t} l_t^* \mathrm{d}t + e^{-\Lambda_{\tau_{n+1}}} V\big(R_{\tau_{n+1}}^{L^*}, J_{\tau_{n+1}}\big) I\{\tau_{n+1} < T^{L^*}\}\right]. \end{aligned} \quad (4.123)$$

Note that $l_t^* \geq 0$ for $t \geq 0$, V is bounded and $\lim_{k\to+\infty} \tau_k = +\infty$ a.s.. Then it follows by letting $n \to +\infty$ on (4.123) and then using the monotone convergence and the dominated convergence that

$$V(x, i) = \mathrm{E}_{(x,i)}\left[\int_0^{T^{L^*}} e^{-\Lambda_t} l_t^* \mathrm{d}t\right],$$

which implies that L^* is an optimal strategy. □

5. Concluding remarks

We studied the optimal dividend problem for the Markovian regime-switching diffusion model with restricted dividend rates. We considered the regime-switching threshold strategy, which is a threshold dividend strategy with threshold levels modulated by the same Makov chain for the model. We found sufficient conditions under which the above-mentioned strategy is the optimal one, which maximizes the expected total discounted dividends until the time of ruin. The results we obtained will be extremely useful when there are multiple number of regimes, e.g. three or more regimes, as the traditional approach to distinguish different cases, solve the HJB equations explicitly in each case and then try to verify whether the solution in each case is or is not the optimal one, is inapplicable with multiple number of regimes. Whether the sufficient conditions presented here hold or not for a specific case can be verified simply by plugging the values of the model parameters and then performing simple arithmetic calculations.

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