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ON OPTIMAL PERIODIC DIVIDEND AND CAPITAL INJECTION STRATEGIES FOR SPECTRALLY NEGATIVE LÉVY MODELS

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Abstract

De Finetti's optimal dividend problem has recently been extended to the case when dividend payments can be made only at Poisson arrival times. In this paper we consider the version with bail-outs where the surplus must be nonnegative uniformly in time. For a general spectrally negative Lévy model, we show the optimality of a Parisian-classical reflection strategy that pays the excess above a given barrier at each Poisson arrival time and also reflects from below at 0 in the classical sense.

Keywords: Dividends; capital injection; spectrally negative Lévy process; scale function; periodic barrier strategy

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1. Introduction

In the bail-out model of de Finetti's dividend problem, a joint optimal dividend and capital injection strategy is pursued so as to maximize the total expected dividend payments minus the cost of capital injections. In the past decade, the classical Cramér–Lundberg model has been generalized to a spectrally negative Lévy model. In particular, Avram *et al.* [4] showed the optimality of a double barrier strategy that reflects from below at 0 and also from above at a certain barrier.

In this paper we consider its extension with a periodic dividend constraint. Periodic observation models have recently been studied widely in the insurance literature; see, e.g. [1], [2]. For the case without capital injections in which dividends are paid until the time of ruin, a periodic barrier strategy that pays any excess above a certain barrier at each payment opportunity is expected to be optimal. Its optimality has been confirmed for the spectrally positive Lévy (dual) models by Avanzi *et al.* [3] and Pérez and Yamazaki [15], and for the spectrally negative Lévy models with a completely monotone Lévy density by Noba *et al.* [14]. On the other hand,

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regarding the bail-out case, the optimality results are available only for the dual model given in the second problem considered in [15], to the best of the authors' knowledge.

The objective of this paper is to show the optimality of a *periodic-classical barrier strategy* under a general spectrally negative Lévy model. This can be seen as the bail-out extension of [14] and also as the spectrally negative version of the bail-out model in [15].

We follow the *guess-and-verify* procedure to tackle the problem. Under a periodic-classical barrier strategy, the expected net present values (NPVs) of dividends and capital injections can be written in terms of the scale function using the results in [16]. The candidate optimal barrier is first chosen using the conjecture that the slope of the value function at the barrier becomes 1. The optimality of the selected strategy is then confirmed by showing that the candidate value function solves the proper variational inequalities. This is indeed satisfied by the convexity of the candidate value function, that is shown by our observation that its slope becomes proportional to a certain ruin identity, which is monotone in the starting value of the process.

Regarding the comparison with the dual model [15], there are both similarities and differences. In this paper we focus on the differences and omit similar results, such as the verification lemma, that can be attained similarly to [15]. In the dual model, only minimal modifications are necessary to solve the bail-out case from the case with ruin. As shown in [15], the value functions for both cases admit exactly the same expressions except that the optimal barriers are different. On the other hand, this is not expected in the spectrally negative Lévy model. The expressions of the optimal solutions are different, and we use different approaches to show the variational inequalities. It is noted that in this paper we *do not assume* the completely monotone density assumption which was needed in [14].

The rest of the paper is organized as follows. The considered problem is formulated and a review of the spectrally negative Lévy process is given in Section 2. In Section 3 we define the periodic-classical barrier strategies and construct the corresponding surplus process. We also provide a review of the scale function and obtain the expected NPVs corresponding to these strategies. In Section 4 we obtain the optimal barrier for the periodic-classical strategy, and in Section 5 we prove that the expected NPVs associated with this strategy solves the proper variational inequalities. Finally, in Section 6, we provide some numerical results.

2. Preliminaries

2.1. Spectrally negative Lévy processes

Let $X = (X(t);\ t \geq 0)$ be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, modeling the surplus of an insurance company in the absence of control. For $x \in \mathbb{R}$, we denote by $\mathbb{P}_x$ the law of X when it starts at x and write for convenience $\mathbb{P}$ in place of $\mathbb{P}_0$. Accordingly, we shall write $\mathbb{E}_x$ and $\mathbb{E}$ for the associated expectation operators.

In this paper we assume that X is a spectrally negative Lévy process that is not the negative of a subordinator, and its Laplace exponent $\psi(\theta)\colon [0, \infty) \to \mathbb{R}$ is such that

$$\mathbb{E}[\mathrm{e}^{\theta X(t)}] =: \mathrm{e}^{\psi(\theta)t}, \qquad t, \theta \geq 0,$$

given by the *Lévy–Khintchine formula*

$$\psi(\theta) := \gamma\theta + \frac{1}{2}\eta^2\theta^2 + \int_{(-\infty,0)} (\mathrm{e}^{\theta z} - 1 - \theta z \mathbf{1}_{\{z>-1\}})\Pi(\mathrm{d}z), \qquad \theta \geq 0. \tag{2.1}$$

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Here, $\gamma \in \mathbb{R}$, $\eta \geq 0$, and Π is a Lévy measure on $(-\infty, 0)$ such that

$$\int_{(-\infty,0)} (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

The process X has paths of bounded variation if and only if $\eta = 0$ and $\int_{(-1,0)} |z|\Pi(\mathrm{d}z)$ is finite. In this case, X can be written as

$$X(t) = ct - S(t), \qquad t \geq 0,$$

where

$$c := \gamma - \int_{(-1,0)} z\Pi(\mathrm{d}z)$$

and $(S(t);\ t \geq 0)$ is a driftless subordinator. By the assumption that it does not have monotone paths, we must have $c > 0$ and we can write

$$\psi(\theta) = c\theta + \int_{(-\infty,0)} (\mathrm{e}^{\theta z} - 1)\Pi(\mathrm{d}z), \qquad \theta \geq 0.$$

2.2. The optimal Poissonian dividend problem with classical capital injection

A (dividend/capital injection) strategy is a pair of processes $\pi := (L^\pi(t), R^\pi(t);\ t \geq 0)$ consisting of the cumulative amount of dividends L^π and those of capital injection R^π.

Regarding the dividend strategy, we assume that the dividend payments can be made only at the arrival times $\mathcal{T}_r := (T(i);\ i \geq 1)$ of a Poisson process $N^r = (N^r(t);\ t \geq 0)$ with intensity $r > 0$, which is independent of the Lévy process X. In other words, $T(i) - T(i-1)$, $i \geq 1$, (with $T(0) := 0$) are independent and exponentially distributed with mean $1/r$. More precisely, L^π admits the form

$$L^\pi(t) = \int_{[0,t]} \nu^\pi(s)\,\mathrm{d}N^r(s), \qquad t \geq 0, \tag{2.2}$$

for some càglàd process ν^π adapted to the filtration $\mathbb{F} := (\mathcal{F}(t);\ t \geq 0)$ generated by the processes (X, N^r).

Regarding the capital injection, we assume that R^π is a nondecreasing, right-continuous, and $\mathbb{F}$-adapted process with $R^\pi(0^-) = 0$. Contrary to the dividend payments, capital injection can be made continuously.

The corresponding risk process is given by $U^\pi(0^-) = X(0)$ and

$$U^\pi(t) := X(t) - L^\pi(t) + R^\pi(t), \qquad t \geq 0,$$

and (L^π, R^π) must be chosen so that $U^\pi(t) \geq 0$ for all $t \geq 0$ almost surely.

Assuming that $\beta > 1$ is the cost per unit injected capital and $q > 0$ is the discount factor, the objective is to maximize

$$v_\pi(x) := \mathbb{E}_x\bigg(\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}L^\pi(t) - \beta\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}R^\pi(t)\bigg), \qquad x \geq 0,$$

over the set of all admissible strategies $\mathcal{A}$ that satisfy all the constraints described above and

$$\mathbb{E}_x\bigg(\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}R^\pi(t)\bigg) < \infty. \tag{2.3}$$

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Hence, the problem is to compute the value function

$$v(x) := \sup_{\pi \in \mathcal{A}} v_\pi(x), \qquad x \geq 0, \tag{2.4}$$

and obtain an optimal strategy π^* that attains it, if such a strategy exists. Throughout the paper, for the solution to be nontrivial, we assume that

$$\mathbb{E}[X(1)] = \psi'(0^+) > -\infty. \tag{2.5}$$

3. Periodic-classical barrier strategies

As in the spectrally positive case [15], the objective of this paper is to show the optimality of the periodic-classical barrier strategy

$$\bar{\pi}^{0,b} := \{(L_r^{0,b}(t), R_r^{0,b}(t));\ t \geq 0\}.$$

The controlled process $U_r^{0,b}$ becomes the *Lévy process with Parisian reflection above and classical reflection below* considered in [15], which can be constructed as follows.

Let $R(t) := (-\inf_{0\leq s\leq t} X(s)) \vee 0$ for $t \geq 0$, and then we have

$$U_r^{0,b}(t) = X(t) + R(t), \qquad 0 \leq t < \widehat{T}_b^+(1),$$

where $\widehat{T}_b^+(1) := \inf\{T(i)\colon X(T(i)) + R(T(i)) > b\}$. The process then jumps down by $X(\widehat{T}_b^+(1)) + R(\widehat{T}_b^+(1)) - b$ so that $U_r^{0,b}(\widehat{T}_b^+(1)) = b$. For $\widehat{T}_b^+(1) \leq t < \widehat{T}_b^+(2) := \inf\{T(i) > \widehat{T}_b^+(1)\colon U_r^{0,b}(T(i)^-) > b\}$, $U_r^{0,b}(t)$ is the process reflected at 0 of the process $(X(t) - X(\widehat{T}_b^+(1)) + b;\ t \geq \widehat{T}_b^+(1))$. The process $U_r^{0,b}$ can be constructed by repeating this procedure. It is clear that it admits a decomposition

$$U_r^{0,b}(t) = X(t) - L_r^{0,b}(t) + R_r^{0,b}(t), \qquad t \geq 0,$$

where $L_r^{0,b}(t)$ and $R_r^{0,b}(t)$ are, respectively, the cumulative amounts of Parisian and classical reflection until time t.

We shall see that the strategy $\bar{\pi}^{0,b} := \{(L_r^{0,b}(t), R_r^{0,b}(t));\ t \geq 0\}$ for $b \geq 0$ is admissible for the problem described in Section 2.2 (since (2.3) holds by Lemma 3.1 and our assumption (2.5)). Its expected net present value of dividends minus the cost of capital injection is

$$v_b(x) := \mathbb{E}_x\bigg(\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}L_r^{0,b}(t) - \beta \int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}R_r^{0,b}(t)\bigg), \qquad x \geq 0. \tag{3.1}$$

3.1. Scale functions

For fixed $q \geq 0$, let $W^{(q)}\colon \mathbb{R} \to [0,\infty)$ be the scale function of the spectrally negative Lévy process X. This takes the value 0 on the negative half-line, and on the positive half-line it is a continuous and strictly increasing function defined by its Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \qquad \theta > \Phi(q), \tag{3.2}$$

where ψ is as defined in (2.1) and

$$\Phi(q) := \sup\{\lambda \geq 0\colon \psi(\lambda) = q\}. \tag{3.3}$$

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We also define, for all $x \in \mathbb{R}$,

$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(y)\,\mathrm{d}y, \qquad \overline{\overline{W}}^{(q)}(x) := \int_0^x \int_0^z W^{(q)}(w)\,\mathrm{d}w\,\mathrm{d}z,$$

$$Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x), \qquad \overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(z)\,\mathrm{d}z = x + q\overline{\overline{W}}^{(q)}(x).$$

Since $W^{(q)}(x) = 0$ for $-\infty < x < 0$, we have

$$\overline{W}^{(q)}(x) = 0, \qquad \overline{\overline{W}}^{(q)}(x) = 0, \qquad Z^{(q)}(x) = 1, \qquad \overline{Z}^{(q)}(x) = x, \quad x \le 0.$$

Remark 3.1. (i) $W^{(q)}$ is differentiable almost everywhere. In particular, if X is of unbounded variation or the Lévy measure does not have an atom, it is known that $W^{(q)}$ is $C^1(\mathbb{R} \setminus \{0\})$; see, e.g. [6, Theorem 3].

(ii) As in [9, Lemma 3.1],

$$W^{(q)}(0) = \begin{cases} 0 & \text{if } X \text{ is of unbounded variation,} \\ c^{-1} & \text{if } X \text{ is of bounded variation.} \end{cases}$$

We also use $W^{(q+r)}$ and $\Phi(q+r)$, which are defined by (3.2) and (3.3) with q replaced by $q + r$. By the convexity of ψ on $(0, \infty)$, we have $\Phi(q+r) > \Phi(q)$ for $r > 0$, and, from [12, Equation (6)],

$$W^{(q+r)}(x) - W^{(q)}(x) = r \int_0^x W^{(q+r)}(u) W^{(q)}(x-u)\,\mathrm{d}u, \qquad x \in \mathbb{R}.$$

We let, for $q, r > 0$ and $x \in \mathbb{R}$,

$$\begin{aligned} Z^{(q)}(x, \Phi(q+r)) &:= \mathrm{e}^{\Phi(q+r)x}\left(1 - r\int_0^x \mathrm{e}^{-\Phi(q+r)z} W^{(q)}(z)\,\mathrm{d}z\right) \\ &= r\int_0^\infty \mathrm{e}^{-\Phi(q+r)z} W^{(q)}(z+x)\,\mathrm{d}z \\ &> 0. \end{aligned}$$

Here, the second equality holds since (3.2) yields $\int_0^\infty \mathrm{e}^{-\Phi(q+r)x} W^{(q)}(x)\,\mathrm{d}x = r^{-1}$. Differentiating this with respect to the first argument, we have

$$\begin{aligned} Z^{(q)\prime}(x, \Phi(q+r)) &:= \frac{\partial}{\partial x} Z^{(q)}(x, \Phi(q+r)) \\ &= \Phi(q+r) Z^{(q)}(x, \Phi(q+r)) - rW^{(q)}(x), \qquad x > 0. \end{aligned}$$

Finally, for $b \ge 0$ and $x \in \mathbb{R}$, we define

$$\begin{aligned} W_{-b}^{(q,r)}(x) &:= W^{(q)}(x+b) + r\int_0^x W^{(q+r)}(y) W^{(q)}(x-y+b)\,\mathrm{d}y, \\ Z_{-b}^{(q,r)}(x) &:= Z^{(q)}(x+b) + r\int_0^x W^{(q+r)}(y) Z^{(q)}(x-y+b)\,\mathrm{d}y, \qquad (3.4) \\ \overline{Z}_{-b}^{(q,r)}(x) &:= \overline{Z}^{(q)}(x+b) + r\int_0^x W^{(q+r)}(y) \overline{Z}^{(q)}(x-y+b)\,\mathrm{d}y. \end{aligned}$$

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Remark 3.2. Using the identities from [12, Equation (5)], we have

$$W_0^{(q,r)}(x) = W^{(q+r)}(x) \quad \text{and} \quad Z_0^{(q,r)}(x) = Z^{(q+r)}(x), \quad x \in \mathbb{R}.$$

Remark 3.3. Fix $b \geq 0$. Let X_r be the Parisian reflected process of X from above at level 0 (without classical reflection) as studied in [16], and

$$\tau_{-b}^{-}(r) := \inf\{t > 0 \colon X_r(t) < -b\}.$$

Then, using [16, Corollary 3], for any $x \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}_{x-b}[\exp(-\tau_{-b}^{-}(r))] \\ &\quad = Z_{-b}^{(q,r)}(x-b) - rZ^{(q)}(b)\overline{W}^{(q+r)}(x-b) \\ &\quad\quad - q\frac{Z^{(q)}(b,\Phi(q+r))}{Z^{(q)\prime}(b,\Phi(q+r))}(W_{-b}^{(q,r)}(x-b) - rW^{(q)}(b)\overline{W}^{(q+r)}(x-b)), \end{aligned} \tag{3.5}$$

where, in particular,

$$\mathbb{E}_0[\exp(-\tau_{-b}^{-}(r))] = Z^{(q)}(b) - q\frac{Z^{(q)}(b,\Phi(q+r))}{Z^{(q)\prime}(b,\Phi(q+r))}W^{(q)}(b). \tag{3.6}$$

These identities are used later in Remark 4.1 and the proof of Lemma 5.2.

For a detailed study on the scale function and its applications, see [9] and [10].

3.2. Expression of v_b via the scale function

Using the functions given in (3.4), we can compute (3.1) immediately using [16, Corollaries 10 and 11] via the scale function. Recall in our assumption (2.5) that $\psi'(0^+)$ is finite. Below, we extend the domain of v_b to $\mathbb{R}$ by setting $v_b(x) = \beta x + v_b(0)$ for $x < 0$ so as to include the case when the process is started at a negative value and is pushed up to 0 immediately.

Lemma 3.1. *For $b \geq 0$ and $x \in \mathbb{R}$,*

$$\begin{aligned} v_b(x) = {} & -C_b(Z_{-b}^{(q,r)}(x-b) - rZ^{(q)}(b)\overline{W}^{(q+r)}(x-b)) - r\overline{\overline{W}}^{(q+r)}(x-b) \\ & + \beta\bigg(\overline{Z}_{-b}^{(q,r)}(x-b) + \frac{\psi'(0^+)}{q} - r\overline{Z}^{(q)}(b)\overline{W}^{(q+r)}(x-b)\bigg), \end{aligned} \tag{3.7}$$

where

$$C_b := \frac{r(\beta Z^{(q)}(b) - 1)}{q\Phi(q+r)Z^{(q)}(b,\Phi(q+r))} + \frac{\beta}{\Phi(q+r)}. \tag{3.8}$$

In particular, for $x \leq b$, we obtain

$$v_b(x) = -C_bZ^{(q)}(x) + \beta\bigg(\overline{Z}^{(q)}(x) + \frac{\psi'(0^+)}{q}\bigg). \tag{3.9}$$

Proof. Using [16, Corollaries 10 and 11] for all $b \geq 0$ and $x \in \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{E}_x\bigg(\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}L_r^{0,b}(t)\bigg) \\ &\quad = r\bigg(\frac{Z_{-b}^{(q,r)}(x-b) - rZ^{(q)}(b)\overline{W}^{(q+r)}(x-b)}{q\Phi(q+r)Z^{(q)}(b,\Phi(q+r))} - \overline{\overline{W}}^{(q+r)}(x-b)\bigg), \end{aligned}$$

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and

$$\mathbb{E}_x\bigg(\int_{[0,\infty)} \mathrm{e}^{-qt}\,\mathrm{d}R_r^{0,b}(t)\bigg)$$
$$= \bigg(\frac{rZ^{(q)}(b)}{q\Phi(q+r)Z^{(q)}(b,\Phi(q+r))} + \frac{1}{\Phi(q+r)}\bigg)$$
$$\times\,(Z_{-b}^{(q,r)}(x-b) - rZ^{(q)}(b)\overline{W}^{(q+r)}(x-b))$$
$$- \bigg(\overline{Z}_{-b}^{(q,r)}(x-b) + \frac{\psi'(0^+)}{q} - r\overline{Z}^{(q)}(b)\overline{W}^{(q+r)}(x-b)\bigg).$$

Combining these, we have the claim. □

3.3. Smoothness of v_b

Here we analyze the smoothness of the function v_b. The proof of the following lemma is straightforward and is hence omitted.

Lemma 3.2. *For all* $b \geq 0$,

$$v_b'(x) = -qC_bW_{-b}^{(q,r)}(x-b) - r\overline{W}^{(q+r)}(x-b) + \beta Z_{-b}^{(q,r)}(x-b), \qquad x \in \mathbb{R}\setminus\{0\}, \quad (3.10)$$

and

$$v_b''(x^+) = -qC_b\bigg(W^{(q)\prime}(x^+) + rW^{(q+r)}(x-b)W^{(q)}(b) + r\int_0^{x-b} W^{(q+r)}(y)W^{(q)\prime}(x-y)\,\mathrm{d}y\bigg) - rW^{(q+r)}(x-b)$$
$$+ \beta(qW_{-b}^{(q,r)}(x-b) + rW^{(q+r)}(x-b)Z^{(q)}(b)), \qquad x \in \mathbb{R}\setminus\{0,b\}. \quad (3.11)$$

By the smoothness of the scale function on $\mathbb{R}\setminus\{0\}$ as in Remark 3.1(i), the derivative (3.10) is continuous on $\mathbb{R}\setminus\{0\}$. In particular, in the case that X is of unbounded variation, by Remarks 3.1(i) and 3.1(ii), the second derivative, given by (3.11), is continuous on $\mathbb{R}\setminus\{0\}$. Hence, we have the following results.

Lemma 3.3. *For all* $b \geq 0$, *we have the following:*

(i) *when* X *is of bounded variation,* v_b *is continuously differentiable on* $\mathbb{R}\setminus\{0\}$*;*

(ii) *when* X *is of unbounded variation,* v_b *is twice continuously differentiable on* $\mathbb{R}\setminus\{0\}$.

Remark 3.4. (*Continuity/smoothness at* 0.) For $b \geq 0$, we have the following:

(i) by Lemma 3.1, it follows that v_b is continuous at 0.

(ii) for the case that X is of unbounded variation, v_b is continuously differentiable at 0 since, using Lemma 3.2 and Remark 3.1(ii), $v_b'(0^+) = -qC_bW^{(q)}(0) + \beta = \beta = v_b'(0^-)$.

4. Selection of a candidate optimal barrier b^*

In this section we focus on the periodic barrier strategy defined in the previous section and choose the candidate barrier b^*, which satisfies $v_{b^*}'(b^*) = 1$ if such $b^* > 0$ exists, and set it to be 0 otherwise.

Recall, as in Lemma 3.3, that v_b is continuously differentiable except at 0. If $b > 0$, using (3.8) and (3.10), we have

$$v_b'(b) = -qC_b W^{(q)}(b) + \beta Z^{(q)}(b) = g(b) + 1, \tag{4.1}$$

where we define, for $b \geq 0$,

$$\begin{aligned} g(b) &:= \bigg(1 - \frac{rW^{(q)}(b)}{\Phi(q+r)Z^{(q)}(b, \Phi(q+r))}\bigg)(\beta Z^{(q)}(b) - 1) - \frac{\beta q}{\Phi(q+r)} W^{(q)}(b) \\ &= \frac{Z^{(q)\prime}(b, \Phi(q+r))}{\Phi(q+r)Z^{(q)}(b, \Phi(q+r))}(\beta Z^{(q)}(b) - 1) - \frac{\beta q}{\Phi(q+r)} W^{(q)}(b). \end{aligned} \tag{4.2}$$

In other words, for $b > 0$, $v_b'(b) = 1$ if and only if $g(b) = 0$.

Remark 4.1. (*Probabilistic representation of* g.) Using (3.6) and (4.2),

$$g(b) = \frac{q}{\Phi(q+r)} \frac{\beta \mathbb{E}_0[\exp(-q\tau_{-b}^-(r))] - 1}{Z^{(q)}(b) - \mathbb{E}_0[\exp(-q\tau_{-b}^-(r))]} W^{(q)}(b). \tag{4.3}$$

(i) Since $Z^{(q)}(b) - \mathbb{E}_0[\exp(-q\tau_{-b}^-(r))] > 0$ for $b > 0$ and $b \mapsto \beta\mathbb{E}_0[\exp(-q\tau_{-b}^-(r))] - 1$ is strictly decreasing, there exists at most one root of $g(b) = 0$.

(ii) Using, in (4.3), the fact that $\lim_{b\uparrow\infty} \mathbb{E}_0[\exp(-q\tau_{-b}^-(r))] = 0$, and $W^{(q)}(x)/Z^{(q)}(x) \to \Phi(q)/q$ as $x \uparrow \infty$, as in [10, Exercise 8.5(i)], it follows that

$$\lim_{b\uparrow\infty} g(b) = -\frac{\Phi(q)}{\Phi(q+r)} < 0.$$

Therefore, $g(b)$ must be negative for sufficiently large b.

In order to also handle the case where such a b does not exist, we define

$$b^* := \inf\{b \geq 0 \colon g(b) \leq 0\}, \tag{4.4}$$

which is well defined since, using Remark 4.1(ii), the set $\{b \geq 0 \colon g(b) \leq 0\} \neq \varnothing$.

Below, we provide a necessary and sufficient condition for the optimal barrier b^* to be 0.

Lemma 4.1. *We have* $b^* = 0$ *if and only if* X *is of bounded variation and*

$$\beta - 1 - \frac{r(\beta - 1) + q\beta}{c\Phi(q+r)} \leq 0. \tag{4.5}$$

Proof. By the definition of b^* as in (4.4), we have $b^* = 0$ if and only if $g(0) \leq 0$ where, by (4.2),

$$g(0) = \beta - 1 - (r(\beta - 1) + q\beta)\frac{W^{(q)}(0)}{\Phi(q+r)}.$$

For the case of unbounded variation (where $W^{(q)}(0) = 0$ by Remark 3.1(ii)), we have $g(0) = \beta - 1 > 0$ and, hence, $b^* > 0$. On the other hand, for the case of bounded variation, by Remark 3.1(ii), $b^* = 0$ if and only if (4.5) holds. □

Remark 4.2. (*Slope at* b^*.) (i) If $b^* > 0$ (i.e. $g(b^*) = 0$), (4.1) implies that $v_{b^*}'(b^*) = 1$.

(ii) If $b^* = 0$ (i.e. $g(0) \leq 0$), (4.1) yields $v_{b^*}'(0+) \leq 1$.

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Remark 4.3. Suppose that $b^* > 0$ (i.e. $g(b^*) = 0$). Then, by (4.1), we have

$$C_{b^*} = \frac{\beta Z^{(q)}(b^*) - 1}{q W^{(q)}(b^*)}.$$

Remark 4.4. From (4.3) and (4.4), we have

$$b^* = \inf\left\{b > 0\colon \mathbb{E}_0[\exp(-q\tau_{-b}^-(r))] \leq \frac{1}{\beta}\right\}. \tag{4.6}$$

Identity (4.6) implies that b^* decreases as the discount rate q increases.

5. Verification of optimality

In this section we shall show the optimality of the strategy $\overline{\pi}^{0,b^*}$ for the value of b^* selected in the previous section.

Theorem 5.1. *The strategy $\overline{\pi}^{0,b^*}$ is optimal and the value function of problem (2.4) is given by $v = v_{b^*}$.*

In order to prove Theorem 5.1, it suffices to prove the variational inequalities. We omit the proof of the following proposition since it is essentially the same as the spectrally positive case [15, Lemma 5.3]. Here we slightly relax the assumption on the smoothness at 0, which can be done by applying the Meyer–Itô formula as in [17, Theorem IV.71].

Let $\mathcal{L}$ be the infinitesimal generator associated with the process X applied to a measurable function f on $\mathbb{R}$, i.e. $C^1(0,\infty)$ (respectively, $C^2(0,\infty)$) for the case in which X is of bounded (respectively, unbounded) variation with

$$\mathcal{L}f(x) := \gamma f'(x) + \frac{1}{2}\eta^2 f''(x) + \int_{(-\infty,0)} [f(x+z) - f(x) - f'(x)z\mathbf{1}_{\{-1<z<0\}}]\Pi(\mathrm{d}z).$$

Below, as in [4], we extend the domain of v_π of (3.1) to $\mathbb{R}$ by setting $v_\pi(x) = \beta x + v_\pi(0)$ for $x < 0$.

Proposition 5.1. *Suppose that $\widehat{\pi} \in \mathcal{A}$ is such that $v_{\widehat{\pi}}$ is $C^1(0,\infty)$ (respectively, $C^2(0,\infty)$) for the case that X is of bounded (respectively, unbounded) variation, continuous on $\mathbb{R}$, and, for the case of unbounded variation, continuously differentiable at* 0*. In addition, suppose that*

$$\begin{gathered}(\mathcal{L} - q)v_{\widehat{\pi}}(x) + r\max_{0\leq l\leq x}\{l + v_{\widehat{\pi}}(x-l) - v_{\widehat{\pi}}(x)\} \leq 0, \qquad v'_{\widehat{\pi}}(x) \leq \beta, \quad x > 0, \\ \inf_{x\geq 0} v_{\widehat{\pi}}(x) > -m \quad \text{for some } m > 0.\end{gathered} \tag{5.1}$$

Then $\widehat{\pi}$ is an optimal strategy and $v_{\widehat{\pi}}(x) = v(x)$ for all $x \geq 0$.

We shall provide some preliminary results in order to show the variational inequalities (5.1).

Lemma 5.1. *For $b \geq 0$, we have*

$$(\mathcal{L} - q)v_b(x) = \begin{cases} 0 & \text{if } x \in (0,b), \\ -r\{(x-b) + v_b(b) - v_b(x)\} & \text{if } x \in [b,\infty). \end{cases} \tag{5.2}$$

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Proof. (i) Suppose that $0 < x < b$. By the proof of [5, Theorem 2.1], we have

$$(\mathcal{L} - q)Z^{(q)}(y) = (\mathcal{L} - q)\bigg(\overline{Z}^{(q)}(y) + \frac{\psi'(0^+)}{q}\bigg) = 0, \qquad y > 0. \tag{5.3}$$

Applying these in (3.9), we obtain (5.2).

(ii) Suppose that $x > b$. From the proof of [14, Lemma 5.2], we have

$$\begin{aligned}(\mathcal{L} - q)\overline{W}^{(q+r)}(x-b) &= 1 + r\overline{W}^{(q+r)}(x-b),\\ (\mathcal{L} - q)\overline{\overline{W}}^{(q+r)}(x-b) &= (x-b) + r\overline{\overline{W}}^{(q+r)}(x-b).\end{aligned} \tag{5.4}$$

On the other hand, from the proof of [8, Lemma 4.5], we have

$$\begin{aligned}&(\mathcal{L} - (q+r))\bigg(\int_0^{x-b} W^{(q+r)}(y)Z^{(q)}(x-y)\,\mathrm{d}y\bigg)\\ &\quad= (\mathcal{L} - (q+r))\bigg(\int_0^{x-b} W^{(q+r)}(x-b-y)Z^{(q)}(b+y)\,\mathrm{d}y\bigg)\\ &\quad= Z^{(q)}(x),\end{aligned}$$

and, hence,

$$(\mathcal{L} - q)\bigg(\int_0^{x-b} W^{(q+r)}(y)Z^{(q)}(x-y)\,\mathrm{d}y\bigg) = Z_{-b}^{(q,r)}(x-b). \tag{5.5}$$

Combining (5.3) and (5.5), we obtain

$$(\mathcal{L} - q)Z_{-b}^{(q,r)}(x-b) = rZ_{-b}^{(q,r)}(x-b). \tag{5.6}$$

In a similar way, we see that

$$(\mathcal{L} - q)\bigg(\int_0^{x-b} W^{(q+r)}(y)\overline{Z}^{(q)}(x-y)\,\mathrm{d}y\bigg) = \overline{Z}_{-b}^{(q,r)}(x-b), \tag{5.7}$$

and, using (5.3) and (5.7), we obtain

$$(\mathcal{L} - q)\bigg(\overline{Z}_{-b}^{(q,r)}(x-b) + \frac{\psi'(0^+)}{q}\bigg) = r\overline{Z}_{-b}^{(q,r)}(x-b). \tag{5.8}$$

Therefore, applying (5.4), (5.6), and (5.8) in (3.7),

$$\begin{aligned}(\mathcal{L} - q)v_b(x) &= -C_b(rZ_{-b}^{(q,r)}(x-b) - rZ^{(q)}(b)(1 + r\overline{W}^{(q+r)}(x-b)))\\ &\quad - r((x-b) + r\overline{\overline{W}}^{(q+r)}(x-b))\\ &\quad - r\beta\overline{Z}^{(q)}(b)(1 + r\overline{W}^{(q+r)}(x-b)) + r\beta\overline{Z}_{-b}^{(q,r)}(x-b)\\ &= -r((x-b) + v_b(b) - v_b(x)),\end{aligned}$$

where in the last equality we use the fact that $v_b(b) = -C_bZ^{(q)}(b) + \beta(\overline{Z}^{(q)}(b) + \psi'(0^+)/q)$. This completes the proof. □

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Lemma 5.2. *We have* $1 \le v'_{b^*}(x) \le \beta$ *for* $x \in (0, b^*)$ *and* $0 \le v'_{b^*}(x) \le 1$ *for* $x \in (b^*, \infty)$.

Proof. We prove separately for the cases (i) $b^* > 0$ and (ii) $b^* = 0$.

(i) Suppose that $b^* > 0$. Then, using (3.10) and Remark 4.3, we obtain

$$v'_{b^*}(x) = \beta Z^{(q,r)}_{-b^*}(x - b^*) - r\overline{W}^{(q+r)}(x - b^*) - \frac{\beta Z^{(q)}(b^*) - 1}{W^{(q)}(b^*)} W^{(q,r)}_{-b^*}(x - b^*). \tag{5.9}$$

From the second equality of (4.2) and the fact that $g(b^*) = 0$, we obtain

$$q \frac{Z^{(q)}(b^*, \Phi(q + r))}{Z^{(q)\prime}(b^*, \Phi(q + r))} W^{(q)}(b^*) = \frac{\beta Z^{(q)}(b^*) - 1}{\beta}.$$

Hence, using the above expression and (5.9) in (3.5), we obtain, for $x > 0$,

$$\begin{aligned} &\beta \mathbb{E}_{x-b^*}[\exp(-q\tau^-_{-b^*}(r))] \\ &\quad = \beta Z^{(q,r)}_{-b^*}(x - b^*) - r\beta Z^{(q)}(b^*)\overline{W}^{(q+r)}(x - b^*) \\ &\qquad - \frac{\beta Z^{(q)}(b^*) - 1}{W^{(q)}(b^*)} (W^{(q,r)}_{-b^*}(x - b^*) - rW^{(q)}(b^*)\overline{W}^{(q+r)}(x - b^*)) \\ &\quad = v'_{b^*}(x), \end{aligned} \tag{5.10}$$

where the last inequality follows from (5.9).

From (5.10), we then deduce that $0 \le v'_{b^*}(x) \le \beta$ and that v'_{b^*} is decreasing on $(0, \infty)$. This and the fact that $v'_{b^*}(b^*) = 1$ as in Remark 4.2 complete the proof.

(ii) Suppose that $b^* = 0$ (then necessarily X is of bounded variation by Lemma 4.1). Due to

$$C_0 = \frac{r(\beta - 1) + q\beta}{q\Phi(q + r)},$$

we have, by (3.10) and Remark 3.2,

$$\begin{aligned} v'_0(x) &= -\frac{r(\beta - 1) + q\beta}{\Phi(q + r)} W^{(q,r)}_0(x) - r\overline{W}^{(q+r)}(x) + \beta Z^{(q,r)}_0(x) \\ &= \frac{r(\beta - 1) + q\beta}{r + q} \bigg(Z^{(q+r)}(x) - \frac{r + q}{\Phi(q + r)} W^{(q+r)}(x) \bigg) + \frac{r}{r + q}. \end{aligned} \tag{5.11}$$

Differentiating (5.11) further, we obtain

$$v''_0(x^+) = (r(\beta - 1) + q\beta) \bigg(1 - \frac{1}{\Phi(q + r)} \frac{W^{(q+r)\prime}(x^+)}{W^{(q+r)}(x)} \bigg) W^{(q+r)}(x). \tag{5.12}$$

Since $\beta > 1$, we have $r(\beta - 1) + q\beta > 0$. In addition, $x \mapsto W^{(q+r)\prime}(x^+)/W^{(q+r)}(x)$ is monotonically decreasing in x as in [10, Equation (8.18) and Lemma 8.2] and converges to $\Phi(r + q)$ as $x \to \infty$. By these facts and (5.12), we have $v''_0(x^+) < 0$, meaning v_0 is concave.

Recall, as in Remark 4.2, that $v'_0(0^+) \le 1$. Hence, we have $v'_0(x) \le 1$ for all $x \in (0, \infty)$. Finally, we have $v'_0(x) \to r/(r+q) > 0$ as $x \uparrow \infty$, since $Z^{(q+r)}(x) - (r + q)W^{(q+r)}(x)/\Phi(r + q)$ vanishes in the limit by [10, Theorem 8.1(ii)], and, hence, $v'_0(x) > 0$. □

Next, by applying Lemma 5.2 for $b^* > 0$ and $b^* = 0$, the following results are immediate.

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Lemma 5.3. *For* $b^* \geq 0$*, we have*

$$\max_{0 \leq l \leq x} \{l + v_{b^*}(x - l) - v_{b^*}(x)\} = \begin{cases} 0 & \text{if } x \in [0, b^*], \\ x - b^* + v_{b^*}(b^*) - v_{b^*}(x) & \text{if } x \in (b^*, \infty). \end{cases}$$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. We shall show that v_{b^*} satisfies all the conditions given in Proposition 5.1. The desired smoothness/continuity of v_{b^*} holds by Lemma 3.3 and Remark 3.4. Hence, we need to prove only the variational inequalities given in (5.1).

Lemmas 5.1 and 5.3 yield the validity of the first item of (5.1) with equality. The second item holds by Lemma 5.2. Finally, the third item follows since, by the monotonicity of v_{b^*} in view of Lemma 5.2 and (2.5), we have $\inf_{x \geq 0} v_{b^*}(x) \geq v_{b^*}(0) > -\infty$. □

Remark 5.1. Let X be the spectrally negative Lévy process used in this paper with $\eta > 0$. We consider another spectrally negative Lévy process $\widetilde{X}$ with a lower value of the diffusion coefficient $\widetilde{\eta} \in (0, \eta)$ (with the same values of (γ, Π)). By $\widetilde{v}$ and $\widetilde{\mathcal{L}}$ we denote the corresponding value function and infinitesimal generator, respectively.

With the assumption $\widetilde{\eta} > 0$, $\widetilde{v}$ is sufficiently smooth for the original process X and, hence, $\mathcal{L}\widetilde{v}$ is well defined. Since $\widetilde{v}$ solves the variational inequality for the generator $\widetilde{\mathcal{L}}$ and by the concavity of $\widetilde{v}$,

$$\begin{aligned} (\mathcal{L} - q)\widetilde{v}(x) &+ r \max_{0 \leq l \leq x} \{l + \widetilde{v}(x - l) - \widetilde{v}(x)\} \\ &= (\widetilde{\mathcal{L}} - q)\widetilde{v}(x) + r \max_{0 \leq l \leq x} \{l + \widetilde{v}(x - l) - \widetilde{v}(x)\} + \tfrac{1}{2}(\eta^2 - \widetilde{\eta}^2)\widetilde{v}''(x) \\ &\leq \tfrac{1}{2}(\eta^2 - \widetilde{\eta}^2)\widetilde{v}''(x) \\ &< 0. \end{aligned}$$

In addition, since $\widetilde{v}$ is the optimal value function under $\widetilde{X}$, which solves (5.1) as in Proposition 5.1, we have $\widetilde{v}'(x) \leq \beta$ for $x > 0$ and $\inf_{x \geq 0} \widetilde{v}(x) > -m$ for some $m > 0$.

In summary, $\widetilde{v}$ satisfies the requirements given in (5.1) under the generator $\mathcal{L}$ and, hence, proceeding as the proof of [15, Lemma 5.3], we obtain, for any $\pi \in \mathcal{A}$, $\widetilde{v}(x) \geq v_\pi(x)$, implying the inequality $\widetilde{v}(x) \geq v(x)$ for all $x \geq 0$.

6. Numerical results

In this section we give numerical results using the spectrally negative Lévy process with phase-type jumps of the form

$$X(t) - X(0) = ct + \eta B(t) - \sum_{n=1}^{N(t)} Z_n, \qquad 0 \leq t < \infty,$$

where $B = (B(t);\ t \geq 0)$ is a standard Brownian motion, $N = (N(t);\ t \geq 0)$ is a Poisson process with arrival rate 1, and $Z = (Z_n;\ n = 1, 2, \ldots)$ is an independent and identically distributed sequence of phase-type random variables that approximate the (folded) normal distribution with mean 0 and variance 1 (the phase-type parameters can be found in [11]); see, e.g. [13] for a review of the phase-type distribution. The processes B, N, and Z are assumed to be mutually independent. We refer the reader to [7] and [9] for the forms of the corresponding scale functions. Throughout we set $q = 0.05$.

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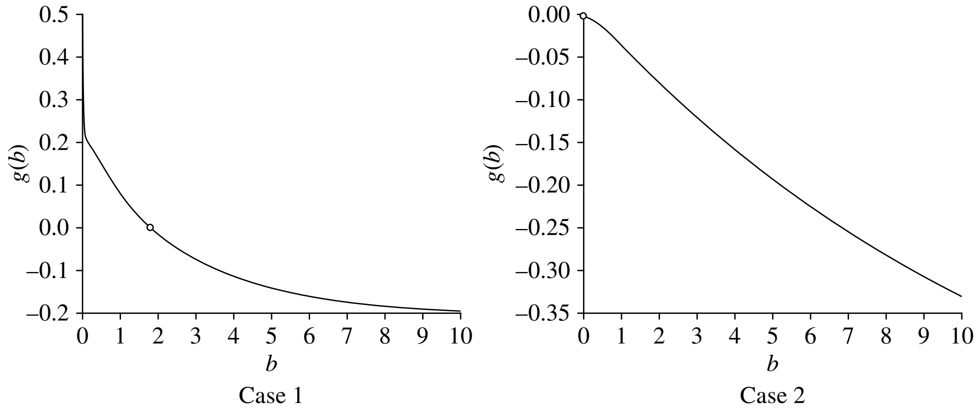

FIGURE 1: Plots of $b \mapsto g(b)$ for cases 1 and 2. The values of b^* are indicated by the circles.

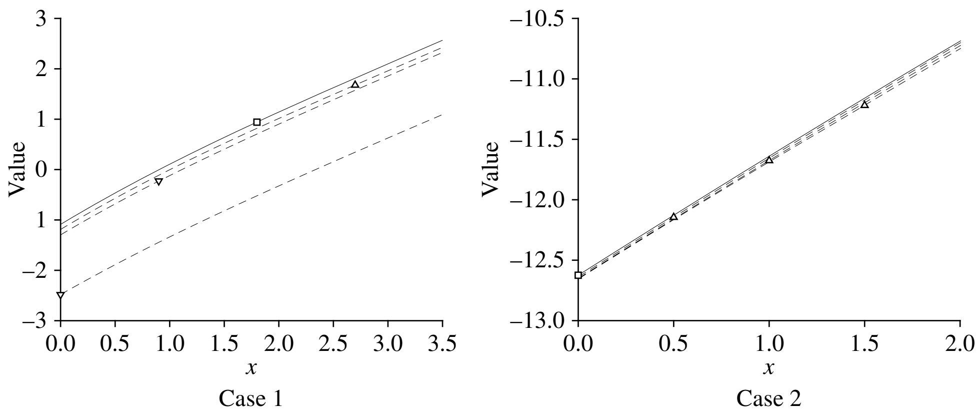

FIGURE 2: Plots of v_{b^*} (*solid*) for cases 1 and 2 in comparison to v_b (*dashed*) with $b = 0, \frac{1}{2}b^*, \frac{3}{2}b^*$ for case 1 and $b = 0.5, 1.0, 1.5$ for case 2. The points $(b^*, v_{b^*}(b^*))$ are indicated by the squares and the points $(b, v_b(b))$ are indicated by the down-pointing (respectively, up-pointing) triangles if $b < b^*$ (respectively, $b > b^*$).

We first illustrate the implementation procedure using *case 1* (unbounded variation) with $\eta = 0.2$, $c = 1$, and $\beta = 1.5$ and *case 2* (bounded variation) with $\eta = 0$, $c = 0.3$, and $\beta = 1.05$, with the common value of $r = 0.5$.

Recall the definition of b^* as in (4.4). In Figure 1 we present the plot of the function $g(b)$ as in (4.2) for cases 1 and 2. As studied in Remark 4.1 and Lemma 4.1, if $g(0) > 0$ as in case 1, there exists a unique value b^* such that $g(b^*) = 0$, and, hence, this can be computed using the bisection method. For the case of $g(0) \leq 0$ as in case 2, we set $b^* = 0$. Using the selected b^*, the optimal value functions v_{b^*} are computed, and we present the plots in Figure 2 for both cases 1 and 2. In the same graphs, in order to confirm the optimality, we plot the function v_b for a different selection of b. It is confirmed that v_{b^*} dominates v_b for $b \neq b^*$, uniformly in x.

In Figure 3 we present the behaviors of the optimal solutions with respect to the parameters β and r using the same parameters as case 1 (unless stated otherwise for the values of β and r). In the left panel we present the plot of v_{b^*} for β ranging from 1.01 to 20. As expected, v_{b^*}

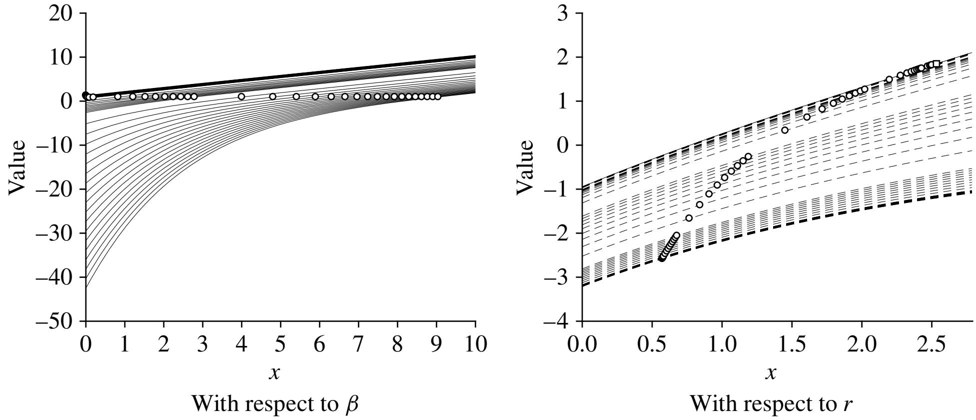

FIGURE 3: *Left*: plots of v_{b^*} for $\beta = 1.01, 1.02, \ldots, 1.09, 1.1, 1.2, \ldots, 1.9, 2, 3, \ldots, 19, 20$ with the points $(b^*, v_{b^*}(b^*))$ indicated by the circles. *Right*: plots of v_{b^*} (*dashed*) for $r = 0.0001, 0.0002, \ldots, 0.0009, 0.001, 0.002, \ldots, 0.009, 0.01, 0.02, \ldots, 0.1, 0.2, \ldots, 0.9, 1, 2, \ldots, 9, 10, 20, 30, 40, 50$ with the points $(b^*, v_{b^*}(b^*))$ indicated by circles, along with the classical case $\overline{v}_{b^\dagger}$ (*solid line*) as in [4] with the point $(b^\dagger, \overline{v}_{b^\dagger}(b^\dagger))$ indicated by the square.

is decreasing in β uniformly in x. In addition, we observe that b^* increases as β increases. In the right panel we present the plot of v_{b^*} for various values of r ranging from 0.0001 to 50 along with the results in the classical bail-out case (without the restriction (2.2)), say $\overline{v}_{b^\dagger}$ with the optimal classical barrier $b^\dagger$, as in [4]. It is observed that the value function converges increasingly to that in [4]. It is also confirmed that b^* increases in r and converges to $b^\dagger$ of [4] as $r \to \infty$.

7. Concluding remarks

In this paper we obtained an optimal dividend-capital injection strategy under the assumption that dividend payment opportunities arrive at Poisson arrival times while capital injection can be made continuously. Under this setting, we showed that the optimal strategy is of periodic-barrier type, and the optimal barrier as well as the value function can be written in terms of the scale function.

It is a natural and interesting question for future research to consider an extension where capital injection opportunities are also restricted to Poisson arrival times. Contrary to the problem considered in this paper, ruin may not be avoidable and, hence, the positivity constraint on the surplus process must be removed.

In light of the conclusions given in this paper, a double periodic barrier strategy—that pushes the process into a certain interval whenever dividend/capital injection opportunities arrive—is a reasonable candidate for the optimal strategy. On the other hand, as in [14], certain assumptions (such as complete monotonicity) on the Lévy measure might be needed.

As the arrival rate of the capital injection opportunities increases to ∞, the problem considered in this paper can be seen as the limiting case. In reality, companies are expected to increase the rate of observations when the surplus is near the ruin boundary and, hence, we conjecture that the results of this paper can potentially be a good approximation in these scenarios. It is worthy of investigation to confirm this approximation both analytically and numerically so as to see the link between these models.

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