J. Inst. Math. Jussieu (2020) **19**(3), 965–970
doi:10.1017/S1474748018000294

COMPARING THE BRAUER GROUP TO THE TATE–SHAFAREVICH GROUP

THOMAS H. GEISSER

Rikkyo University, Ikebukuro, Tokyo, Japan (geisser@rikkyo.ac.jp)

(Received 21 February 2018; revised 4 June 2018; accepted 18 June 2018; first published online 19 July 2018)

Abstract We give a formula relating the order of the Brauer group of a surface fibered over a curve over a finite field to the order of the Tate–Shafarevich group of the Jacobian of the generic fiber. The formula implies that the Brauer group of a smooth and proper surface over a finite field is a square if it is finite.

Keywords: Brauer group; Tate–Shafarevich group

2010 *Mathematics subject classification*: 11G40; 14G17; 14J20; 11G25; 11G35

1. Introduction

Let K be a global field, and let V be the smooth and proper model if K has characteristic p, or the spectrum of the ring of integers of K in the number field case. Let X be a regular surface and $X \to V$ a projective flat map with geometrically connected fibers such that $X_K = X \times_V K$ is smooth over K. For a point $v \in V$, let K_v be the completion of K and $X_{K_v} = X \times_V K_v$.

It is a classical result of Artin and Grothendieck [5] that the Brauer group of X is finite if and only if the Tate–Shafarevich group of the Jacobian $A = \mathrm{Pic}^0_{X_K}$ of X_K is finite. Grothendieck [5, (4.7)], Milne [10], and Gonzalez-Aviles [2] gave formulas relating the order of the Brauer group of X to the order of the Tate–Shafarevich group $Ш(A)$ under some conditions on the periods of X_{K_v}. We give a general formula without any conditions. Let δ and δ_v be the indices of X_K and X_{K_v}, respectively, and let α and α_v be the orders of the cokernel of the inclusion $\mathrm{Pic}^0(X_K) \to H^0(K, \mathrm{Pic}^0_{X_K})$ and $\mathrm{Pic}^0(X_{K_v}) \to H^0(K_v, \mathrm{Pic}^0_{X_K})$, respectively. By Lichtenbaum [6, Theorem 3 (proof)], α_v is equal to the period δ'_v of X_{K_v}.

Theorem 1.1. *If K has no real embeddings and if the Brauer group* $\mathrm{Br}(X)$ *is finite, then*

$$|\mathrm{Br}(X)|\alpha^2\delta^2 = |Ш(A)| \prod_{v \in V} \alpha_v \delta_v. \tag{1}$$

This generalizes the results of Grothendieck, Milne and Gonzalez-Aviles, and corrects the formula of Liu, Lorenzini and Raynaud [8] by the factor α^2. The problem is that [8] uses the incorrect [4, Lemma 4.2], which implies that $\alpha = 1$, see their corrigendum [9]. If K is a number field with real embeddings, then the same formula holds up to a power of 2 (due to the usual problem with duality for Galois cohomology of a number ring with real places). By [7, Remark 4.5], the right-hand side in Theorem 1.1 is a square, hence the argument of [8] gives the following.

Corollary 1.2. *Let X be a smooth and proper surface over a finite field. If the Brauer group is finite, then its order is a square.*

A key ingredient in the proof is the following local-to-global result for the Brauer group:

Theorem 1.3. *If $\mathrm{Br}(X)$ is finite and K has no real embeddings, then*

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(X_K) \to \bigoplus_{v\in V} \mathrm{Br}(X_{K_v}) \to \mathrm{Hom}(\mathrm{Pic}(X_K), \mathbb{Q}/\mathbb{Z}) \to 0$$

is exact.

2. Brauer groups and Tate–Shafarevich groups

We continue to use the notation of the introduction. For a closed point v of V, we let $\mathcal{O}_v$ be the completion of V at v, k_v the residue field at v, and $Y_v = X \times_V k_v$. Let G and G_v be the Galois groups of K and K_v, respectively.

Denoting the Pontrjagin dual of the abelian group A by $A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$, we have Lichtenbaum's duality for the curve X_{K_v} [6]

$$\mathrm{Pic}(X_{K_v})^* \cong \mathrm{Br}(X_{K_v}). \tag{2}$$

This duality has been generalized by Saito to include the finite characteristic case in [12, Theorem 9.2]. Both Lichtenbaum's and Saito's pairing are defined by pulling back elements of $\mathrm{Br}(X_{K_v})$ along divisors, and checking that the result vanishes on principal divisors. Composing with the dual of the natural map $\mathrm{Pic}(X_K) \to \prod_{v\in V} \mathrm{Pic}(X_{K_v})$, we obtain a map of discrete torsion groups

$$\bigoplus_{v\in V} \mathrm{Br}(X_{K_v}) \xrightarrow{l} \mathrm{Pic}(X_K)^*.$$

Proof of Theorem 1.3. Exactness at the left two terms can be found in [10, Lemma 2.6]. Exactness on the right follows from (2) and injectivity of $\mathrm{Pic}(X_K) \to \mathrm{Pic}(X_{K_v})$ for every v. Indeed, if X_K has a point over a finite Galois extension L and w is a place of L above w, then $\mathrm{Pic}(X_K) \to \mathrm{Pic}(X_L) \to \mathrm{Pic}(X_{L_w})$ is injective, the former by the Hochschild–Serre spectral sequence and the latter because the Picard functor is representable in the presence of a rational point. It remains to show the exactness at the sum. Consider the diagram

$$\begin{array}{ccccccc}
\mathrm{Br}(X_K) & \longrightarrow & \bigoplus_{v\in V} \mathrm{Br}(X_{K_v}) & \longrightarrow & \mathrm{Pic}(X_K)^* & \xrightarrow{inj} & \mathrm{Pic}(X)^* \\
\| & & \downarrow{\scriptstyle inj} & & & & \| \\
\mathrm{Br}(X_K) & \longrightarrow & \bigoplus_{v\in V} H^3_{Y_v}(X, \mathbb{G}_m) & \longrightarrow & H^3_{\mathrm{et}}(X, \mathbb{G}_m) & \xrightarrow{\xi} & \mathrm{Pic}(X)^*.
\end{array}$$

https://doi.org/10.1017/S1474748018000294 Published online by Cambridge University Press

The left three terms of the second row arise from the localization sequence for etale cohomology for X, and the second vertical injection is the sum of the localization sequences for the $X_{\mathcal{O}_v}$, using the vanishing of $\mathrm{Br}(X_{\mathcal{O}_v})$ ([5, Theorem 3.1], see the proof of [10, Lemma 2.6]), and the fact that $H^3_{Y_v}(X, \mathbb{G}_m) \cong H^3_{Y_v}(X_{\mathcal{O}_v}, \mathbb{G}_m)$. A diagram chase shows that the exactness at the sum follows if we can define an injective map ξ such that the right rectangle commutes.

We define the map ξ by using a divisor D on X to pull back cohomology classes in $H^3_{\mathrm{et}}(X, \mathbb{G}_m)$ to the normalization $H^3_{\mathrm{et}}(\tilde{D}, \mathbb{G}_m)$, which is isomorphic to $(\mathbb{Q}/\mathbb{Z})^c$, c the number of irreducible components of D [11, II Remark 2.2 (b)], and then summing up. Then the right rectangle commutes because both compositions are defined by pulling back cohomology classes along divisors. Saito defines a map $\phi^1 : H^3_{\mathrm{et}}(X, \mathbb{G}_m) \to \mathrm{Pic}(X)^*$ and shows in [13, Theorem 5.5(2)] that it is a surjection whose kernel vanishes if $\mathrm{Br}(X)$ is finite. It suffices to show that $\phi^1 = \xi$. In the proof of loc. cit., one chooses a divisor Y whose components generate $\mathrm{Pic}(X)$, pulls back cohomology classes $H^3_{\mathrm{et}}(X, \mathbb{G}_m)$ to $H^3_{\mathrm{et}}(Y, i^*\mathbb{G}_m)$, and uses the duality between $H^3_{\mathrm{et}}(Y, i^*\mathbb{G}_m) \cong (\mathbb{Q}/\mathbb{Z})^c$ and $H^1_Y(X, \mathbb{G}_m) \cong \mathbb{Z}^c$, where c is the number of components of Y [13, Proposition 4.6]. Now it suffices to observe that under the given hypothesis, the map $H^3_{\mathrm{et}}(X, \mathbb{G}_m)$ to $H^3_{\mathrm{et}}(Y, i^*\mathbb{G}_m)$ is injective, the map $\mathbb{Z}^c \cong H^1_Y(X, \mathbb{G}_m) \to \mathrm{Pic}(X)$ sends a generator corresponding to a component of Y to its divisor class, and

$$H^3_{\mathrm{et}}(Y, i^*\mathbb{G}_m) \cong H^3_{\mathrm{et}}(Y, \mathbb{G}_m) \cong H^3_{\mathrm{et}}(\tilde{Y}, \mathbb{G}_m) \cong (\mathbb{Q}/\mathbb{Z})^c,$$

which follows from the proof of [13, (4–11)]. □

Remark 2.1. (1) In the function field case one can show that the sequence is exact except at the sum, where its cohomology is $(T\,\mathrm{Br}(X))^*$ up to p-groups.

(2) The hypothesis on 2-torsion in case of real embeddings is used to apply Saito's result, see [13, §5].

The following generalization of the Cassels–Tate exact sequence by Gonzalez-Aviles and Tan [3] can be thought of as the analog of Theorem 1.3 for the Tate–Shafarevich group. The results on flat cohomology that are used have been corrected in [1].

Theorem 2.2. *Let A be an abelian variety over K with dual A^t, and assume that the Tate–Shafarevich group $Ш(A^t)$ is finite. Then the sequence*

$$0 \to Ш(A) \to H^1(K, A) \xrightarrow{\beta^1} \bigoplus_v H^1(K_v, A) \xrightarrow{\gamma^1} H^0(K, A^t)^* \to 0$$

is exact.

Here the map γ^1 is the dual of the injection

$$\beta^0 : H^0(K, A^t)^\wedge \to \prod_v H^0(K_v, A^t)^\wedge \cong \left(\bigoplus_v H^1(K_v, A)\right)^*,$$

where $G^\wedge = \lim_m G/m$ denotes the completion of an abelian group G.

https://doi.org/10.1017/S1474748018000294 Published online by Cambridge University Press

3. Comparison

We complete the proof of Theorem 1.1 by comparing the sequences of Theorem 1.3 and of Theorem 2.2 applied to Pic^0_X via their maps to

$$H^1(K, \mathrm{Pic}_{X_K}) \to \bigoplus_v H^1(K_v, \mathrm{Pic}_{X_K}).$$

The long exact sequence of Galois cohomology groups associated to the degree map over the separable closure K^s of K

$$0 \to \mathrm{Pic}(X_{K^s})^0 \to \mathrm{Pic}(X_{K^s}) \xrightarrow{\deg} \mathbb{Z} \to 0$$

induces the middle two exact rows of the following diagram:

$$\begin{array}{ccccccccc}
 & & & & Ш(\mathrm{Pic}^0_{X_K}) & \longrightarrow & \Phi & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/\delta' & \longrightarrow & H^1(K, \mathrm{Pic}^0_{X_K}) & \longrightarrow & H^1(K, \mathrm{Pic}_{X_K}) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle\beta^1}\downarrow & & {\scriptstyle\tau}\downarrow & & \\
0 & \longrightarrow & \oplus\mathbb{Z}/\delta'_v & \longrightarrow & \bigoplus_v H^1(K_v, \mathrm{Pic}^0_{X_K}) & \longrightarrow & \bigoplus_v H^1(K_v, \mathrm{Pic}_{X_K}) & \longrightarrow & 0 \\
 & & & & {\scriptstyle\gamma^1}\downarrow & & {\scriptstyle\rho}\downarrow & & \\
 & & & & H^0(K, \mathrm{Pic}^0_{X_K})^* & \xrightarrow{\omega} & \Psi & &
\end{array} \tag{3}$$

The upper and lower rows are the kernels and cokernels of the vertical maps. Finiteness of $Ш(\mathrm{Pic}^0_{X_K})$ and of $\oplus\mathbb{Z}/\delta'_v$ implies finiteness of Φ. Counting orders we obtain the formula

$$|\Phi| = \frac{|Ш(\mathrm{Pic}^0_{X_K})| \cdot \prod_v \delta'_v}{|\ker \omega| \cdot \delta'}.$$

Now we use the (functorial) Hochschild–Serre spectral sequence

$$0 \to \mathrm{Pic}(X_K) \to H^0(K, \mathrm{Pic}_{X_K}) \to \mathrm{Br}(K) \to \mathrm{Br}(X_K) \to H^1(K, \mathrm{Pic}_{X_K}) \to 0 \tag{4}$$

for X and X_{K_v} to obtain the middle two exact rows of the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \Phi & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & P & \longrightarrow & \mathrm{Br}(K) & \longrightarrow & \mathrm{Br}(X_K) & \longrightarrow & H^1(K, \mathrm{Pic}_{X_K}) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & {\scriptstyle\tau}\downarrow & \\
0 \to & \bigoplus \mathbb{Z}/\delta_v & \longrightarrow & \bigoplus \mathrm{Br}(K_v) & \longrightarrow & \bigoplus \mathrm{Br}(X_{K_v}) & \longrightarrow & \bigoplus H^1(K_v, \mathrm{Pic}_{X_K}) & \to 0 \\
 & & & {\scriptstyle\Sigma}\downarrow & & \downarrow & & {\scriptstyle\rho}\downarrow & \\
 & & & \mathbb{Q}/\mathbb{Z} & \xrightarrow{\deg^*} & \mathrm{Pic}(X_K)^* & \xrightarrow{\sigma} & \Psi & \to 0
\end{array} \tag{5}$$

https://doi.org/10.1017/S1474748018000294 Published online by Cambridge University Press

The upper and lower rows are the kernels and cokernels of the vertical maps. The kernel of $\mathrm{Br}(K_v) \to \mathrm{Br}(X_{K_v})$ is isomorphic to $\mathbb{Z}/\delta_v$ by the Lichtenbaum–Roquette theorem [6, Theorem 3]. Lichtenbaum's result is stated in characteristic 0, but the proof works in characteristic p as soon as duality for Galois cohomology of abelian varieties holds [12, Theorems 9.2, 9.3]. The lower middle square is commutative by the definition of the pairing (2), see [6, p. 125], and functoriality of the degree map:

$$\begin{array}{ccccc}
\mathrm{Br}(K_v) & = & \mathbb{Q}/\mathbb{Z} & = & \mathbb{Q}/\mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle \deg^*} & & \downarrow{\scriptstyle \deg^*} \\
\mathrm{Br}(X_{K_v}) & \longrightarrow & \mathrm{Pic}(X_{K_v})^* & \longrightarrow & \mathrm{Pic}(X_K)^*.
\end{array}$$

We have $|\ker \deg^*| = \delta$, and $\mathrm{Pic}^0(X_K)^* \cong \operatorname{coker} \deg^* \overset{\sigma}{\cong} \Psi$. Counting orders, we obtain the formula

$$|\Phi| = \frac{|\mathrm{Br}(X)| \cdot |P| \cdot \delta}{\prod_v \delta_v}.$$

To relate ω to the canonical injection $f : \mathrm{Pic}^0(X_K) \to H^0(K, \mathrm{Pic}^0_{X_K})$, we consider the following diagram, in which the horizontal maps are surjective and all maps except the named ones are the canonical maps:

$$\begin{array}{ccccc}
\mathrm{Br}(X_{K_v}) & \longrightarrow & H^1(K_v, \mathrm{Pic}_{X_K}) & \longleftarrow & H^1(K_v, \mathrm{Pic}^0_{X_K}) \\
\| & & \| & & \| \\
\mathrm{Pic}(X_{K_v})^* & \longrightarrow & \mathrm{Pic}^0(X_{K_v})^* & \longleftarrow & H^0(K_v, \mathrm{Pic}^0_{X_K})^* \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma^1} \\
\mathrm{Pic}(X_K)^* & \longrightarrow & \mathrm{Pic}^0(X_K)^* & \overset{f^*}{\longleftarrow} & H^0(K, \mathrm{Pic}^0_{X_K})^* \\
\| & & \downarrow{\scriptstyle \cong} & & \| \\
\mathrm{Pic}(X_K)^* & \overset{\sigma}{\longrightarrow} & \Psi & \overset{\omega}{\longleftarrow} & H^0(K, \mathrm{Pic}^0_{X_K})^*
\end{array}$$

If we replace the middle composition by ρ, then the diagram defines the maps σ of (5) in the left half and ω of (3) in the right half. The upper two squares are commutative by compatibility of Lichtenbaum's perfect pairings [6, §4], and the middle squares are obviously commutative. Then the left half of the diagram shows that the middle composition as indicated agrees with the map ρ, and the right half of the diagram shows that $\omega = f^*$ so that $|\ker \omega| = |\ker f^*|$.

Finally, the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X_K) & \longrightarrow & \mathrm{Pic}(X_K) & \longrightarrow & \delta\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(K, \mathrm{Pic}^0_{X_K}) & \longrightarrow & H^0(K, \mathrm{Pic}_{X_K}) & \longrightarrow & \delta'\mathbb{Z} & \longrightarrow & 0
\end{array} \tag{6}$$

shows that $|\ker f^*| \cdot \delta = |P| \cdot \delta'$. Since $\alpha = |\ker f^*|$, we obtain Theorem 1.1 by equating the two formulas for $|\Phi|$.

https://doi.org/10.1017/S1474748018000294 Published online by Cambridge University Press

Remark 3.1. The following example that for a curve C over a global field K, the l-rank of $\operatorname{coker} \operatorname{Pic}^0(C) \to H^0(K, \operatorname{Pic}^0_C)$ can be arbitrary large for any l was communicated to us by Starr. By the sequence (4) and the diagram (6), it suffices to find C such that the l-rank of the kernel of $\operatorname{Br}(K) \to \operatorname{Br}(C)$ is arbitrary large. Let $a_0, \ldots, a_r$ be $\mathbb{Z}/l$-linearly independent classes in $\operatorname{Br}(K)[l]$, and let $P_0, \ldots, P_r$ be the associated Severi–Brauer K-schemes. Now let C be a general complete intersection curve in the product variety $P = P_0 \times_K \cdots \times_K P_r$. Then the kernel of the pullback map $\operatorname{Br}(K) \to \operatorname{Br}(P)$ contains the classes $a_0, \ldots, a_r$, hence so does the kernel of $\operatorname{Br}(K) \to \operatorname{Br}(C)$.

Acknowledgements. We thank T. Szamuely and C. Gonzalez-Aviles for comments on an earlier version of this paper, and the referee for his careful reading.

References

1. C. Demarche and D. Harari, Artin–Mazur–Milne duality Theorem for fppf cohomology, Preprint, 2018, https://arxiv.org/abs/1804.03941.
2. C. D. González-Avilés, Brauer groups and Tate–Shafarevich groups, *J. Math. Sci. Univ. Tokyo* **10**(2) (2003), 391–419.
3. C. D. González-Avilés and K.-S. Tan, A generalization of the Cassels–Tate dual exact sequence, *Math. Res. Lett.* **14**(2) (2007), 295–302.
4. W. Gordon, Linking the conjectures of Artin–Tate and Birch–Swinnerton-Dyer, *Compos. Math.* **38** (1979), 163–199.
5. A. Grothendieck, Le groupe de Brauer. III. Exemples et compléments, in *Dix exposés sur la cohomologie des schémas*, Advanced Studies in Pure Mathematics, 3, pp. 88–188 (North-Holland, Amsterdam, 1968).
6. S. Lichtenbaum, Duality theorems for curves over p-adic fields, *Invent. Math.* **7** (1969), 120–136.
7. Q. Liu, D. Lorenzini and M. Raynaud, Néron models, Lie algebras, and reduction of curves of genus one, *Invent. Math.* **157**(3) (2004), 455–518.
8. Q. Liu, D. Lorenzini and M. Raynaud, On the Brauer group of a surface, *Invent. Math.* **159**(3) (2005), 673–676.
9. Q. Liu, D. Lorenzini and M. Raynaud, Corrigendum to [8], *Invent. Math.* to appear. https://arxiv.org/abs/1804.11158.
10. J. S. Milne, Comparison of the Brauer group with the Tate–Safarevic group, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28**(3) (1981), 735–743.
11. J. S. Milne, *Arithmetic Duality Theorems*, 2nd edn (BookSurge, LLC, Charleston, SC, 2006). viii+339 pp.
12. S. Saito, Arithmetic on two-dimensional local rings, *Invent. Math.* **85**(2) (1986), 379–414.
13. S. Saito, Arithmetic theory of arithmetic surfaces, *Ann. of Math. (2)* **129**(3) (1989), 547–589.