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GOLDIE DIMENSION FOR C*-ALGEBRAS

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Abstract In this article, we introduce and study the notion of Goldie dimension for C*-algebras. We prove that a C*-algebra A has Goldie dimension n if and only if the dimension of the center of its local multiplier algebra is n. In this case, A has finite-dimensional center and its primitive spectrum is extremally disconnected. If moreover, A is extending, we show that it decomposes into a direct sum of n prime C*-algebras. In particular, every stably finite, exact C*-algebra with Goldie dimension, that has the projection property and a strictly full element, admits a full projection and a non-zero densely defined lower semi-continuous trace. Finally we show that certain C*-algebras with Goldie dimension (not necessarily simple, separable or nuclear) are classifiable by the Elliott invariant.

Keywords: C*-algebra; local multiplier algebra; graph C*-algebra; Artinian; compact ideal; purely infinite; Elliott invariant

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1. Introduction

Noetherian and/or Artinian C*-algebras as well as C*-algebras with Krull dimension are defined and studied in [20, 34, 35, 40, 41]. In this article, we define and study C*-algebras with Goldie dimension as a generalization of all of these classes (see Figure 1), and then extend the main results obtained in [41] and present some new results and applications.

In abstract algebra, Alfred Goldie used the notion of uniform modules to construct a measure of dimension for modules, now known as the Goldie dimension (or the uniform dimension) of the module. The basic idea is to measure the number of 'building blocks' in a direct sum of non-zero submodules of M. The Goldie dimension (unlike the Krull dimension) in some cases behaves like dimension of a vector space (e.g., $\mathbb{C}^n$ is a C*-algebra with Goldie dimension n, while it has Krull dimension 0). We refer the reader to

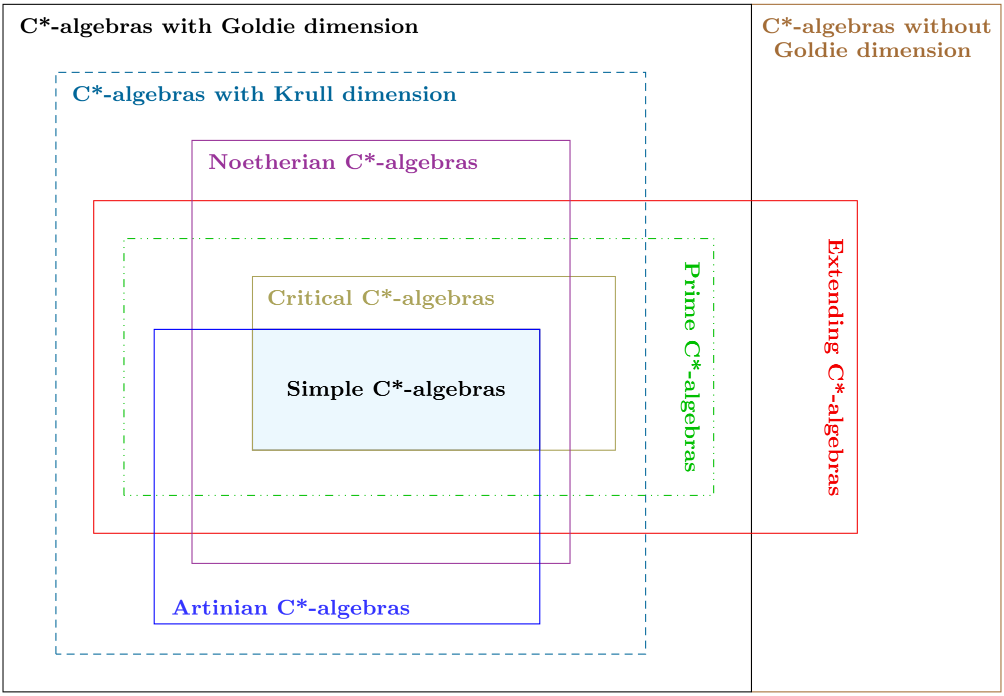

Figure 1. Classes of C*-algebras with (and without) Goldie dimension.

[8, Chapters 1 and 2], [16], [17, Chapters 1 and 3], [18, Chapter 5], [25, Section 6] and [27, Chapter 2], for results on the Goldie dimension and extending (or, CS) modules, in algebraic settings.

Our main motivation for defining the Goldie dimension for C*-algebras is to bridge between algebraic and topological properties of a large class of non-simple C*-algebras. Here, we work with (closed) two-sided ideals instead of one-sided ideals (as in the algebraic setting). We show that C*-algebras with Goldie dimension share some basic properties of C*-algebras with Krull dimension (obtained in [41]) and their local multiplier algebra and primitive ideal space can be described. We show that the Goldie dimension of a C*-algebra (if exists) is the same as the dimension of the center of its local multiplier algebra. In particular, a C*-algebra with Goldie dimension has finite-dimensional center and is boundedly centrally closed (compare with [41, Corollary 2.9 and Proposition 2.11]). We also give a classification theorem (based on the Kirchberg–Phillips classification theorem) for a subclass of C*-algebras with Goldie dimension (see Corollary 2.11 and Theorems 2.9 and 2.20).

The paper is organized as follows. In § 2, we define a notion of the Goldie dimension for C*-algebras (Definition 2.2), which is shown to be well-defined (Lemma 2.1), and conclude that every C*-algebra with Goldie dimension has a closed essential ideal which is a finite direct sum of prime C*-algebras (extending [41, Theorem 2.8]). We study the perseverance of the Goldie dimension under passing to hereditary C*-subalgebras, extensions and Morita equivalence. We show that the Goldie dimension is not preserved under

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taking quotients (unlike the Krull dimension) and crossed products in general. We provide conditions for these to happen (Theorem 2.7(iv) and Corollary 2.13). We give necessary and sufficient conditions for a C*-algebra to be have Goldie dimension (Theorems 2.7(1), 2.9 and 2.24). One of the advantages of C*-algebras with Goldie dimension is that they have extremally disconnected primitive spectrum (Theorem 2.9) and they can have compact ideals, full projections and non-zero (densely defined lower semi-continuous) traces, under certain conditions (Theorem 2.14). We introduce complete-Goldie C*-algebras and "extending" C*-algebras, and show that all complete-Goldie graph C*-algebras have real rank zero, and all extending C*-algebras with Goldie dimension are decomposable into a finite direct sum of prime C*-algebras (Corollary 2.17, and Theorem 2.24(iii)). This shows that every Hilbert C*-module, over an AW*-algebra or an extending C*-algebra with Goldie dimension n, decomposes into a direct sum of n ideal submodules (as in the case of Hilbert spaces; see Theorem 2.26).

2. Goldie dimension for C*-algebras

In this section, we define the Goldie dimension for C*-algebras and obtain the main properties of C*-algebras with Goldie dimension.

In the following results, by an ideal, we always mean a closed two-sided ideal, unless otherwise specified. For an ideal I (resp. C*-subalgebra B) in a C*-algebra A, we write $I \trianglelefteq A$ (resp. $B \leq A$). If I is an essential ideal in a C*-algebra A, then we write $I \trianglelefteq_e A$. We denote the center of a C*-algebra A by $Z(A)$. Also, for a C*-algebra A, let $Id(A)$ denote the lattice of closed ideals of A.

Let us first recall the notion of chain conditions for C*-algebras. Chain conditions on a C*-algebra are defined in its set of closed (two-sided) ideals. A C*-algebra A is called *Noetherian* if it satisfies the ascending chain condition for closed ideals, that is, for any ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq, \ldots$ of closed ideals of A, there is a positive integer n such that $I_i = I_n$, for all $i \geq n$. The dual notion to a Noetherian C*-algebra is that of an *Artinian* C*-algebra, which satisfies the descending chain condition for closed ideals. Clearly, every C*-algebra with finitely many closed ideals is Noetherian and Artinian, including simple and finite-dimensional C*-algebras. On the other hand, there are infinitely many mutually non-isomorphic Noetherian (Artinian) C*-algebras with infinitely many closed ideals [20, 40]. Recall that a Noetherian (Artinian) topological space is a space that satisfies the ascending (descending) chain condition for its open subsets. In the Noetherian case, a topological space is Noetherian if every open set is compact. Let $Prim(A)$ be the set of primitive ideals in a C*-algebra A. Then, $Prim(A)$ is a topological space with the Jacobson (or hull-kernel) topology [9, 20]. In [20], it was shown that a C*-algebra is Noetherian (Artinian) if and only if $Prim(A)$ is so as a topological space. Note that the definition of Notherian (and Artinian) C*-algebras differs from the definition of Noetherian (and Artinian) rings, which is defined on the set of one-sided ideals. Also the definition of a Noetherian C*-algebra differs from the definition of a Noetherian Banach algebra, which is defined by a chain condition on the set of closed left ideals. In the latter definition, every Noetherian Banach algebra is finite-dimensional. This is also true (under the new definition based on ideals, rather than left ideals) for commutative C*-algebras, but not in general (see [20, Theorem 3.1]). Note that being Artinian (resp. Noetherian) is stable under taking hereditary C*-subalgebras and it passes to closed ideals and quotients, and also it is preserved under extension and Morita equivalence (see [34, Lemma 2.2 and Corollary 2.5] and [35, Proposition 1.1]).

Recall that, the Krull dimension of a C*-algebra measures how close the C*-algebra is to being Artinian (see [41, Definition 2.1], for a detailed definition). The proof of the following lemma is similar to that of Theorem 2.8 in [41].

Lemma 2.1. *Let A be a C*-algebra and it contains an essential ideal I of the form $\bigoplus_{i=1}^{n} I_i$, where each I_i is a prime C*-algebra. Then,*

(i) Every direct sum of non-zero ideals of A has at most n summands.

(ii) Let $J = \bigoplus_{i=1}^{m} J_i$ be an ideal in A, where each J_i is a prime C-algebra. Then, $J \trianglelefteq_e A$ if and only if $m = n$.*

Definition 2.2. *A C*-algebra A is said to have Goldie dimension n ($n \in \mathbb{N}$), writing* $\mathrm{G.dim}(A) = n$, *if it contains an essential ideal which is the direct sum of n prime C*-algebras. If such an ideal exists, we call it a G-ideal in A and say that A is a C*-algebra with Goldie dimension (or A has Goldie dimension, or A is a Goldie C*-algebra). Otherwise, we say A is a C*-algebra without Goldie dimension (or A does not have Goldie dimension).*

Note that $\mathrm{G.dim}(A)$ is well-defined, by Lemma 2.1. See Examples 2.21, 2.23 and 2.25, for examples of G-ideals in certain C*-algebras with Goldie dimension. We use the convention that for a C*-algebra A, $\mathrm{G.dim}(A) = 0$ if and only if $A = \{0\}$. Also, every C*-algebra A with Krull dimension contains an essential ideal that is a finite direct sum of critical ideals [41, Theorem 2.8], here called a *K-ideal* in A.

Recall that the *local multiplier algebra* $M_{loc}(A)$ of a C*-algebra A is the C*-direct limit of the multiplier algebras $M(I)$, where I ranges over all (closed) essential ideals of A [2, Definition 2.3.1].

Remark 2.3. (a) A C*-algebra is prime if and only if $\mathrm{G.dim}(A) = 1$. Indeed, if $\mathrm{G.dim}(A) = 1$, then A contains an essential ideal I_A that is a prime C*-algebra. But in [2, Propositions 3.3.2 and 2.3.6(i)], it was shown that a C*-algebra is prime if and only if its local multiplier algebra is prime, and also the local multiplier algebra of a C*-algebra is isomorphic to the local multiplier algebra of any of its essential ideals. Thus $M_{loc}(A) \cong M_{loc}(I_A)$, and hence A is prime. The converse is clear.

(b) For an ideal I in a C*-algebra A with Goldie dimension n, $I \trianglelefteq_e A$ if and only if $\mathrm{G.dim}(A) = \mathrm{G.dim}(I)$. Because, if $I \trianglelefteq_e A$ and $\bigoplus_{i=1}^{n} I_i$ is a G-ideal in A, ($n \in \mathbb{N}$), then for each i, ($1 \le i \le n$), $I \cap I_i \neq \{0\}$. Thus, $\bigoplus_{i=1}^{n}(I \cap I_i)$ is an essential ideal of prime C*-algebras in I, and so $\mathrm{G.dim}(I)$=n. For the converse, let $\mathrm{G.dim}(A) = \mathrm{G.dim}(I)$ but I is not essential. Then, there is an ideal J in A such that $I \cap J = \{0\}$. Also I contains an essential ideal which is the direct sum of n prime C*-algebras, say K. Then, $K \oplus J$ is a direct sum of $n + 1$ summands, which is a contradiction.

(c) Let A and B be C*-algebras with Goldie dimension. Then,

$$\mathrm{G.dim}(A \oplus B) = \mathrm{G.dim}(A) + \mathrm{G.dim}(B).$$

Indeed, if $I_A = \bigoplus_{i=1}^{m} I_i^A$ and $I_B = \bigoplus_{i=1}^{n} I_i^B$ are G-ideals in A and B, respectively, then $I_A \oplus I_B \trianglelefteq_e A \oplus B$, by [41, Lemma 2.6(iii)]. Thus, $\mathrm{G.dim}(A \oplus B) = m + n$.

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Definition 2.4. *An ideal I in a C*-algebra A is said to be essentially closed, writing $I \trianglelefteq_{e.c} A$, if whenever J is an ideal in A such that $I \trianglelefteq_e J$ then $I = J$.*

Every direct summand of a C*-algebra is essentially closed. In particular, for every C*-algebra A, the trivial ideals $\{0\}$ and A are essentially closed. Also, if an ideal in a C*-algebra is injective, then it is also essentially closed. Indeed, according to [19, Proposition 4.8], a C*-algebra is injective if and only if it has no proper essential extension, (a proper essential extension of an ideal I in a C*-algebra A means the existence of a C*-subalgebra B of A that $I \trianglelefteq_e B$ but $I \neq B$. Thus unlike a proper ideal in A, a proper essential extension can be equal to the C*-algebra A). It is worth mentioning that an essentially closed ideal is not necessarily an essential ideal (see Example 2.23(b)). In a C*-algebra, the only ideal that is both essentially closed and essential is the C*-algebra itself.

Lemma 2.5. *Let A be a C*-algebra and I, J and K be ideals in A. Then, the following assertions hold: (i) (Modular Law) $I + (J \cap K) = (I + J) \cap (I + K)$. In particular, if $I \subseteq J$, then $I + (J \cap K) = J \cap (I + K)$.*
(ii) If $I \trianglelefteq J$ and $J/I \trianglelefteq_e A/I$, then $J \trianglelefteq_e A$.
(iii) If $I \trianglelefteq J \trianglelefteq_e A$ and $I \trianglelefteq_{e.c} A$, then $J/I \trianglelefteq_e A/I$.

Proof. For (i) and (ii), see [9, II.5.1.4(iv)] and [18, Proposition 5.6(c)]. For (iii), let $\bar{I} \trianglelefteq A/I$. Then, there is an ideal $K \supseteq I$ in A such that $\bar{I} = K/I$. If $(J/I) \cap (K/I) = 0$, then $(J \cap K)/I = 0$, and so $J \cap K = I$. Since $J \trianglelefteq_e A$, $I = (J \cap K) \trianglelefteq_e (A \cap K) = K$. But $I \trianglelefteq_{e.c} A$. Thus $I = K$, and hence $K/I = 0$. □

Lemma 2.6. *The relation '$\trianglelefteq_{e.c}$' is transitive.*

Proof. Let A be a C*-algebra and I, J and K be ideals in A such that $I \trianglelefteq_{e.c} J$ and $J \trianglelefteq_{e.c} K$. First, for $J \trianglelefteq K$, there is an ideal $J^{'}$ in K such that $(J \oplus J') \trianglelefteq_e K$, by [41, Lemma 2.7]. Since $J \trianglelefteq (J \oplus J') \trianglelefteq_e K$ and $J \trianglelefteq_{e.c} K$, Lemma 2.5(iii) shows that $((J \oplus J')/J) \trianglelefteq_e (K/J)$. Thus

$$(J/I) \oplus ((I \oplus J')/I) = ((J \oplus J')/I) \trianglelefteq_e (K/I),$$

by Lemma 2.5(ii). Similarly, for $I \trianglelefteq J$, there is an ideal $I^{'}$ in J such that $(I \oplus I') \trianglelefteq_e J$, and so $((I \oplus I')/I) \trianglelefteq_e (J/I)$. Therefore,

$$\begin{aligned} ((I \oplus I' \oplus J')/I) &= ((I \oplus I')/I) \oplus ((I \oplus J')/I) \\ &\trianglelefteq_e (J/I) \oplus ((I \oplus J')/I) \\ &\trianglelefteq_e (K/I). \end{aligned}$$

Now, to show that $I \trianglelefteq_{e.c} K$, let $I \trianglelefteq_e L \trianglelefteq K$. Since $I \cap (I' \oplus J') = 0$, $L \cap (I' \oplus J') = 0$. By Lemma 2.5(i), we have

$$L \cap (I \oplus I' \oplus J') = I \oplus (L \cap (I' \oplus J')) = I.$$

This means that $(L/I) \cap ((I \oplus I' \oplus J')/I) = 0$. Thus $L/I = 0$, and so $I = L$. □

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For the following theorem, we use the concepts in Section 2 of [40]. For more details on the theory of graph C*-algebras, we refer the reader to [3].

Theorem 2.7. *(i) A C*-algebra has Goldie dimension if and only if it contains no infinite direct sum of non-zero ideals.*

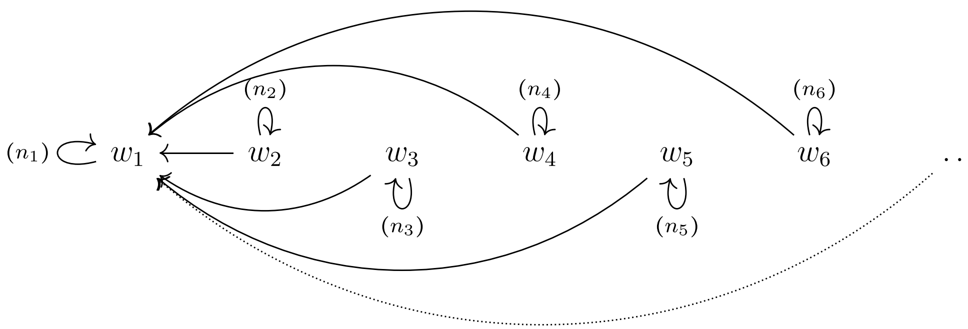
(ii) Every C-algebra with Krull dimension has Goldie dimension. The converse does not necessarily hold.*

(iii) Goldie dimension passes to hereditary C-subalgebras and is preserved under extension and Morita equivalence of C*-algebras.*

(iv) Let A be a C-algebra and $I \lhd A$. Then, $I \trianglelefteq_{e.c} A$ if and only if both I and A/I have Goldie dimension and* $\mathrm{G.dim}(A) = \mathrm{G.dim}(I) + \mathrm{G.dim}(A/I)$.

Proof. (i) If A has Goldie dimension n, then by Lemma 2.1(i), every direct sum of non-zero ideals of A has at most n summands. For the converse, we first show that A contains an ideal that is a prime C*-algebra. If not, since A is not prime, there are ideals I_1 and I_1' in A such that $I_1 \cap I_1' = \{0\}$. Also I_1 is not a prime C*-algebra. Thus, there are ideals I_2 and I_2' in I_1 such that $I_2 \cap I_2' = \{0\}$. By continuing this process, we will have a direct sum as $I_1 \oplus I_2 \oplus I_3 \oplus \ldots$, that is a contradiction. Therefore, A (and also each of its ideals) has an ideal that is a prime C*-algebra. Now, similar to the first paragraph of the proof of Theorem 2.8 in [41], we can show that A contains an essential ideal which is a finite direct sum of prime C*-algebras, and so has Goldie dimension.

(ii) Let A be a C*-algebra with Krull dimension. Then, A contains an essential ideal that is a finite direct sum of critical ideals. But every critical C*-algebra is prime [41, Lemma 2.3(vi)]. Thus, A has Goldie dimension. For the next assertion, let $S = \{n_i\}_{i\in\mathbb{N}}$ be a sequence of natural numbers $n_i \geq 2$, and consider the following directed graph $\mathcal{G}$:



Here, (n_i) means that there are n_i loops at each vertex w_i. Since the graph $\mathcal{G}$ is downward directed and satisfies Condition (K) (and hence (L)), $C^*(\mathcal{G})$ is prime [1, Proposition 3.1]. Thus $\mathrm{G.dim}(C^*(\mathcal{G})) = 1$. On the other hand, $C^*(\mathcal{G})$ is without Krull dimension. Indeed, having Krull dimension passes to quotients [41, Lemma 2.3(i)], while if we consider the saturated hereditary subset $H = \{w_1\}$, then [3, Theorem 2.1.6(b)] implies that,

$$C^*(\mathcal{G})/I_H \cong C^*(\mathcal{G} \setminus H) \cong \bigoplus_{n_i \in S\setminus\{n_1\}} \mathcal{O}_{n_i},$$

where $\mathcal{O}_{n_i}$ is the Cuntz algebra, and so $C^*(\mathcal{G})/I_H$ does not have Krull dimension.

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(iii) We first show that if B is a hereditary C*-subalgebra of a C*-algebra A with Goldie dimension, then $\mathrm{G.dim}(B) \leq \mathrm{G.dim}(A)$.

Suppose that $L := \overline{ABA}$ (the closed two sided ideal generated by B) and $Id_L(A)$ is the set of closed ideals in A contained in L. Then, there is a bijection as follows:

$$\Phi : Id(B) \to Id_L(A);\ J \mapsto \overline{AJA},$$

(see [9, p. 90]), hence $Id(B) \cong Id_L(A) \cong Id(L)$. But $\mathrm{G.dim}(L) \leq \mathrm{G.dim}(A)$, by (i). Now, since L has Goldie dimension, so is B.

For the second assertion, let I be an ideal in a C*-algebra A. We show that if I and A/I have Goldie dimension, then $\mathrm{G.dim}(A) \leq \mathrm{G.dim}(I) + \mathrm{G.dim}(A/I)$.

According to [41, Lemma 2.7], there is an ideal K in A such that $I \oplus K \trianglelefteq_e A$, and

$$K \cong K/(I \cap K) \cong (I \oplus K)/I \leq A/I,$$

(see [9, p. 84]). Thus,

$$\begin{aligned}\mathrm{G.dim}(A) &= \mathrm{G.dim}(I \oplus K)\\ &= \mathrm{G.dim}(I) + \mathrm{G.dim}(K)\\ &\leq \mathrm{G.dim}(I) + \mathrm{G.dim}(A/I).\end{aligned}$$

The last assertion holds, because two Morita equivalent C*-algebras have isomorphic lattices of closed ideals [36, Theorem 3.22], and so have the same Goldie dimension.

(iv) Note first that there is an ideal Z in A such that $I \oplus Z \trianglelefteq_e A$. Furthermore, I and Z have Goldie dimension, by (iii).

Now, let $I \trianglelefteq_{e.c} A$. Then since $I \trianglelefteq I \oplus Z \trianglelefteq_e A$, by Lemma 2.5(iii), $((I \oplus Z)/I) \trianglelefteq_e (A/I)$. Thus,

$$\mathrm{G.dim}(A/I) = \mathrm{G.dim}((I \oplus Z)/I) = \mathrm{G.dim}(Z/(I \cap Z)) = \mathrm{G.dim}(Z),$$

hence,

$$\mathrm{G.dim}(A) = \mathrm{G.dim}(I \oplus Z) = \mathrm{G.dim}(I) + \mathrm{G.dim}(A/I).$$

For the converse, suppose that $I \trianglelefteq_e L \trianglelefteq A$. We show that $I = L$. Since $I \trianglelefteq A$ and $((I \oplus Z)/I) \trianglelefteq (A/I)$, both I and $(I \oplus Z)/I$ have Goldie dimension and $\mathrm{G.dim}(I \oplus Z) \leq \mathrm{G.dim}(I) + \mathrm{G.dim}((I \oplus Z)/I)$, by the proof of (ii). Thus,

$$\begin{aligned}\mathrm{G.dim}(I) + \mathrm{G.dim}(A/I) &= \mathrm{G.dim}(A)\\ &= \mathrm{G.dim}(I \oplus Z)\\ &\leq \mathrm{G.dim}(I) + \mathrm{G.dim}((I \oplus Z)/I).\end{aligned}$$

Therefore, $\mathrm{G.dim}((I \oplus Z)/I) = \mathrm{G.dim}(A/I)$, and hence $((I \oplus Z)/I) \trianglelefteq_e (A/I)$, by Remark 2.3(b). Since $L \cap Z = \{0\}$,

$$((I \oplus Z)/I) \cap (L/I) = ((I \oplus Z) \cap L)/I = (I \oplus (Z \cap L))/I = \{0\},$$

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by Lemma 2.5(i). This shows that $(L/I) = \{0\}$, and so $I = L$. $\square$

In particular, all the C*-algebras introduced in [41, Example 3.2(a–j)] have Goldie dimension.

Remark 2.8. (a) The graph C*-algebra $C^*(\mathcal{G})$, introduced in Theorem 2.7(ii), shows that Goldie dimension is not preserved under quotients.

(b) Let A be a C*-algebra with a non-trivial essentially closed ideal I. Then, $\mathrm{G.dim}(A) = 2$ if and only if both I and A/I are prime C*-algebras. In fact, if $\mathrm{G.dim}(A) = 2$, then since $I \trianglelefteq_{e.c} A$, Theorem 2.7(iv) shows that $\mathrm{G.dim}(I) + \mathrm{G.dim}(A/I) = 2$. Since I is non-trivial, we conclude that $\mathrm{G.dim}(I) = 1$ and $\mathrm{G.dim}(A/I) = 1$. The converse is clear.

(c) Let A be a C*-algebra A with Goldie dimension, B be a simple C*-algebra, with A or B nuclear. Then, $A \otimes B$ has Goldie dimension (because $Id(A) \cong Id(A \otimes B)$). In particular, $A \otimes \mathbb{K}$ and $M_n(A)$ have Goldie dimension.

Let A be a C*-algebra, $\mathcal{I}_{ce}$ be the set of all closed essential ideals in A and,

$$C(A) = \mathrm{alg}-\varinjlim_{I \in \mathcal{I}_{ce}} C(Prim(I)) \text{ and } C_b(A) = \mathrm{alg}-\varinjlim_{I \in \mathcal{I}_{ce}} C_b(Prim(I)).$$

Then, $C(A)$ (resp. $C_b(A)$) is called the (resp. *bounded*) *extended centroid* of A and we have $C(A) = Z(Q_s(A))$ (resp. $C_b(A) = Z(Q_b(A))$), where $Q_s(A)$ (resp. $Q_b(A)$) is the (resp. bounded) symmetric algebra of quotients of A (see [2, Proposition 2.2.5 and Theorem 2.2.8]).

Theorem 2.9. *Let A be a C*-algebra and $n \in \mathbb{N}$. Then,* $\mathrm{G.dim}(A) = n$ *if and only if* $Z(M_{loc}(A)) \cong \mathbb{C}^n$.

Proof. Let $\mathrm{G.dim}(A) = n$ and $I_A = \bigoplus_{i=1}^n I_i$ be a G-ideal in A. Then,

$$Z(M_{loc}(A)) \cong Z(M_{loc}(I_A)) \cong \bigoplus_{i=1}^{n} Z(M_{loc}(I_i)) \cong \mathbb{C}^n,$$

(see [2, Propositions 2.3.6 and 3.3.2]).

For the converse, let $Z(M_{loc}(A)) \cong \mathbb{C}^n$ and $C(A)$ be the extended centroid of A. According to [8, Theorem 10.3.41], $Z(M_{loc}(A)) \cong \mathbb{C}^n$ if and only if $C(A) \cong \mathbb{C}^n$, if and only if A has exactly n minimal prime ideals, say $P_1, P_2, \ldots, P_n$. Since $C(A)$ is finite dimensional,

$$C(A) = C_b(A) = Z(M(A)) = Z(M_{loc}(A)),$$

see [2, Propositions 2.2.13 and pp. 71 and 94]. Thus, A is boundedly centrally closed. But a C*-algebra A is boundedly centrally closed if and only if for every ideal J in A, $\mathrm{ann}_A(J) = eA$, for a projection $e \in Z(M(A))$ [2, Remark 3.2.7]. Now, let $e_1, e_2, \ldots, e_n$ be

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the minimal central projections in $M_{loc}(A)$. Then, we can write (by changing the index, if necessary),

$$\mathrm{ann}_A(P_1) = e_1A,\ \mathrm{ann}_A(P_2) = e_2A, \ldots,\ \mathrm{ann}_A(P_n) = e_nA.$$

Let $J_i := \mathrm{ann}_A(P_i) = e_iA$, $i \in \{1, 2, \ldots, n\}$. In this case, $J_1, J_2, \ldots, J_n$ are non-zero ideals in A. Since

$$M_{loc}(J_i) = e_iM_{loc}(A),$$

and every projection e_i is minimal, we have $Z(M_{loc}(J_i)) = \mathbb{C}$, and so every J_i is a prime C*-algebra [2, Proposition 3.3.2]. Since $\bigoplus_{i=1}^n J_i \trianglelefteq_e A$, we conclude that A has Goldie dimension n. □

A C*-algebra A is *boundedly centrally closed* if $Z(M(A)) = Z(M_{loc}(A))$; see [2, Definition 3.2.1 and Proposition 3.2.3]. Also, a topological space is *extremally disconnected* if the closure of any open subset is still an open subset, equivalently, every bounded continuous complex-valued function on a dense open subset can be (uniquely) extended to a bounded continuous function on the whole space (see, [15, Section 1H] and the proof of Proposition 3.2.4 [2]). Note that, in [2, Proposition 3.2.4], it was shown that a C*-algebra A is boundedly centrally closed if and only if $Prim(A)$ is extremally disconnected. Moreover, in the proof of above theorem, it was observed that for a C*-algebra A with Goldie dimension, we have,

$$Z(M(A)) = Z(M_{loc}(A)).$$

Therefore, the following corollary holds.

Corollary 2.10. *Let A be a C*-algebra with Goldie dimension. Then, Prim(A) is extremally disconnected.*

This result is an extension of [41, Proposition 2.11(i)] (without assumption of having a full projection).

Let us denote denote the class of all C*-algebras for which $M_{loc}(A) = A$ by $\mathcal{M}_l$. All simple and unital C*-algebras, AW*-algebras, and any finite direct sum of these, belong to $\mathcal{M}_l$.

Corollary 2.11. *Every C*-algebra with Goldie dimension has finite dimensional center. The converse also holds when the C*-algebra belongs to $\mathcal{M}_l$.*

Proof. Let A be a C*-algebra with Goldie dimension n. Then, $Z(A) \subseteq Z(M_{loc}(A)) \cong \mathbb{C}^n$, by Theorem 2.9 and [2, Lemma 3.2.2(i)]. Thus, $Z(A)$ is finite dimensional.

If $\dim(Z(A)) = n$ and $A \in \mathcal{M}_l$, then $\dim(Z(M_{loc}(A))) = n$, and hence $\mathrm{G.dim}(A) = n$, by Theorem 2.9. □

The converse direction in the above corollary does not hold in the general. For instance, if $A := \bigoplus_{i=1}^\infty A_i$, where $A_i = \mathbb{K}$, and A^1 is the unitization of A, then $Z(A) = \{0\}$ and $Z(A^1) = \mathbb{C}$, but A does not have Goldie dimension, by Theorem 2.7(i).

As an application, an AW*-algebra has Goldie dimension if and only if it has finite-dimensional center. Also, a unital C*-algebra A with $Prim(A)$ Hausdorff, has such a property. Indeed, if A has also finite-dimensional center, then $Prim(A)$ is finite, because $C(Prim(A)) \cong Z(A)$, by the Dauns-Hofmann theorem [36, Theorem A.34].

Note that for the C*-algebra $A := \bigoplus_{i=1}^{\infty} A_i$, where for all i, $A_i = \mathbb{K}$, we have

$$M_{loc}(A) = M(A) = \bigoplus_{i=1}^{\infty} \mathbb{B},$$

and hence $Z(M_{loc}(A))$ is infinite-dimensional, which of course is confirmed by Theorem 2.9.

Recall that a C*-algebra A is of type I if the von Neumann algebra generated by the range of any (non-degenerate) representation is of type I (i.e., in its lattice of projections, every non-zero central projection dominates a non-zero abelian projection).

Corollary 2.12. *(i) Let X be a compact and Hausdorff space. Then, $C(X)$ has Goldie dimension if and only if $|X| < \infty$.*

(ii) A C-algebra of type I has Goldie dimension if and only if it has Krull dimension zero.*

Proof. (i) Suppose $C(X)$ has Goldie dimension. By Corollary 2.11, every commutative C*-algebra with Goldie dimension is finite-dimensional. Thus, the cardinal of X is finite. Conversely, if $|X| < \infty$, then $Prim(C(X)) \cong X$ [9, p. 60], and hence $C(X)$ is Artinian. Now since $K.dim(C(X)) = 0$, $C(X)$ has Goldie dimension, by Theorem 2.7(ii).

(ii) Let A be a C*-algebra of type I and has Goldie dimension. A C*-algebra is of type I if and only if it is Morita equivalent to a commutative C*-algebra [7, Theorem 2.2]. Furthermore, Goldie dimension is preserved under Morita equivalence of C*-algebras. Thus, there is a finite dimensional C*-algebra B such that $A \sim_M B$, and so $Prim(A)$ is finite. This implies that $K.dim(A) = 0$. The converse is clear. □

In particular, $C(\mathbb{T})$ is a C*-algebra without Goldie dimension.

Note that Goldie dimension is not preserved under taking crossed products in general. Let α denote the action of the group $\mathbb{Z}$ on the unit circle $\mathbb{T}$, by rotations through multiples of a fixed angle $2\pi\theta$, where θ is irrational, and $(C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}, \mathbb{T}, \widehat{\alpha})$ be the dual system of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ (see [46, p. 190]). The irrational rotation C*-algebra $A = C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ is a simple C*-algebra [9, Example II.10.4.12(i)]. On the other hand,

$$\mathrm{G.dim}(A \rtimes_{\widehat{\alpha}} \mathbb{T}) = \mathrm{G.dim}(C(\mathbb{T}) \otimes K(L^2(\mathbb{Z}))) = \mathrm{G.dim}(C(\mathbb{T})),$$

by the Takai duality theorem [46, Theorem 7.1], and so $A \rtimes_{\widehat{\alpha}} \mathbb{T}$ does not have Goldie dimension.

Corollary 2.13. *Let G be a finite group acting by $\alpha : G \to Aut(A)$ on a unital C*-algebra A with Goldie dimension. If α has Rokhlin property, then the maximal crossed product $A \rtimes_\alpha G$ and the fixed point algebra A^α have Goldie dimension.*

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Proof. Since A is unital, G is finite and α is a Rokhlin action, the surjective map

$$Id(A \rtimes_\alpha G) \to Id(A)^G : J \mapsto J \cap A,$$

is a lattice isomorphism by [42, Section 1.2 and Theorem 1.30]. Thus, $A \rtimes_\alpha G$ has also Goldie dimension, by Theorem 2.7(i). Also, [9, Theorem II.10.4.18] implies that there is a projection $p \in M(A \rtimes_\alpha G)$ such that

$$A^\alpha \cong p(A \rtimes_\alpha G)p,$$

and hence A^α has Goldie dimension, by Theorem 2.7(iii). □

For a subset S of a C*-algebra A, $\overline{ASA}$ denotes the closed two sided ideal generated by S, and we simply write $\overline{AaA}$, when $S = \{a\}$. An element $a \in A$ is called *full* if $A = \overline{AaA}$ ([9], p. 91). Let A be a C*-algebra, $a \in A^+$ and $\varepsilon > 0$. Then $(a - \varepsilon)_+ := h_\varepsilon(a)$, where $h_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ is given by:

$$h_\varepsilon(t) = \max\{(t - \varepsilon), 0\}.$$

Given $a, b \in A^+$, we say that a is Cuntz subequivalent to b (and write $a \precsim b$), if there is a sequence $\{x_k\}_{k=1}^\infty \subseteq A$ such that $x_k^* b x_k \to a$ in norm. We say that a and b are Cuntz equivalent (and write $a \sim_{cu} b$), if $a \precsim b$ and $b \precsim a$ [23]. An element $a \in A$ is *strictly full* if $(a - \varepsilon)_+$ is full for some $\varepsilon > 0$, and hence for all sufficiently small $\varepsilon > 0$ [28, p. 46].

An ideal J in a C*-algebra A is called a *compact ideal* if whenever $(J_\lambda)_{\lambda \in \Lambda}$ is an increasing net of ideals in A such that $A = \overline{\bigcup_{\lambda \in \Lambda} J_\lambda}$, then $J = J_\lambda$ for some λ (see [32, Remark 2.2]). A C*-algebra A has the *projection property*, if each of its ideals has an increasing approximate unit consisting of projections (see [31, Definition 1]). Note that for the projection property, according to [11, Definition 4.8], the assumption of the approximate unit to be increasing is necessary.

Denote by $T_{lsc}(A)$ the cone of linear traces on a C*-algebra A whose domain is the Pedersen ideal $Ped(A)$ of A. We can identify $T_{lsc}(A)$ with the set of densely defined lower semi-continuous traces on A (see [39, Definition 2.12]). Let S° (resp. S^c) denote the interior (resp. the complement) of a subset S of $Prim(A)$ (in the hull-kernel topology).

Theorem 2.14. *Let A be a C*-algebra with Goldie dimension containing a strictly full element. Then, the following assertions hold:*

(i) Let $Prim(A)$ be non-Hausdorff and $J \trianglelefteq A$. Then, there is a compact ideal $\hat{J}$ in A such that:

$$\bigcap_{I \in U} I = J \subseteq \hat{J} = \bigcap_{I \in U^\circ} I; \text{ where } U := \{I \in Prim(A) \,|\, J \subseteq I\} \text{ and } U^\circ \neq \emptyset,$$

and $\hat{J}$ has a strictly full element and a maximal ideal.

(ii) Let A be exact. If A is purely infinite (resp. stably finite and has the projection property), then it has a full projection p (resp. q) and $T_{lsc}(A) = \{0\}$ (resp. $T_{lsc}(A) \neq \{0\}$ and for any $\tau \in T_{lsc}(A)$, $\tau(q) \neq 0$).

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Proof. First note that there is a lattice isomorphism ϕ between the ideal lattice $Id(A)$ of A and the lattice $\mathbb{O}(Prim(A))$ of open subsets of $Prim(A)$ such that corresponding to an ideal $J \in Id(A)$, we have the following open set,

$$\mathrm{hull}(J)^c = \{I \in Prim(A) \,|\, J \subseteq I\}^c = Prim(A/J)^c = Prim(J),$$

and, conversely, corresponding to an open set $V \subseteq Prim(A)$, we have the following ideal,

$$\ker(V^c) = \bigcap_{I \in V^c} I,$$

(see [32, Remark 2.2] and [13, Proposition 3.2.1]).

Moreover, $Prim(A)$ is compact. Indeed, let x be a strictly full element in A, $(J_\lambda)_{\lambda\in\Lambda}$ be an increasing net of ideals in A such that $A = \overline{\bigcup_{\lambda\in\Lambda} J_\lambda}$. Then, $(x-\varepsilon)_+$ is a full positive element in A for some $\varepsilon > 0$, and since

$$(x-\varepsilon)_+ \in Ped(A) \subseteq \bigcup_{\lambda\in\Lambda} J_\lambda,$$

there is λ_0 such that $(x-\varepsilon)_+ \in J_{\lambda_0}$. Hence $J_{\lambda_0} = A$. This implies that A is compact as an ideal, and so $Prim(A)$ is compact.

For (i), since $Prim(A)$ is extremally disconnected, $\overline{Prim(J)}$ is a closed and open subset. But $Prim(A)$ is compact. Thus, $\overline{Prim(J)}$ is compact-open, and hence its corresponding ideal, say $\hat{J}$, is compact and we have,

$$\hat{J} = \ker((\overline{Prim(J)})^c) = \bigcap_{I \in (\overline{Prim(J)})^c} I$$

and

$$(\overline{Prim(J)})^c = ((Prim(J))^c)^\circ = \{I \in Prim(A) \,|\, J \subseteq I\}^\circ = U^\circ.$$

Of course, corresponding to the open set $Prim(J)$ we have

$$J = \ker((Prim(J))^c) = \bigcap_{I\in U} I.$$

Note that $U^\circ \neq \emptyset$. Otherwise $\overline{Prim(J)}^c = \emptyset$, and so $\overline{Prim(J)} = Prim(A)$. On the other hand, according to [15, Section 6M(2)], a compact space X is extremally disconnected if and only if $X = \beta Y$ for every dense subspace Y, where βY is the Stone-Čech compactification of Y. Thus,

$$Prim(A) = \beta Prim(J),$$

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and hence $Prim(A)$ is Hausdorff, which is a contradiction. Now, since $U^{\circ} \neq \emptyset$, we have $J \subseteq \hat{J}$. Let $(z_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\hat{J}$, and $\hat{J}_\lambda := \overline{\hat{J}z_\lambda\hat{J}}$. Then, $\phi(\hat{J}_\lambda) = Prim(\hat{J}_\lambda)$ and

$$\hat{J} = \overline{\bigcup_{\lambda\in\Lambda} \hat{J}_\lambda} = \overline{\sum_{\lambda\in\Lambda} \hat{J}_\lambda}.$$

Since ϕ is a lattice isomorphism, $\phi(\hat{J}) = \bigcup_{\lambda\in\Lambda} Prim(\hat{J}_\lambda)$. Since $\phi(\hat{J}) = Prim(\hat{J})$ is compact, there are $\lambda_1, \lambda_2, \ldots, \lambda_n$ in Λ such that, $\phi(\hat{J}) = \bigcup_{i=1}^{n} Prim(\hat{J}_{\lambda_i})$, and,

$$\hat{J} = \overline{\sum_{i=1}^{n} \hat{J}_{\lambda_i}} = \sum_{i=1}^{n} \overline{\hat{J}z_{\lambda_i}\hat{J}}.$$

Thus, $z = \sum_{i=1}^{n} z_{\lambda_i}$ is a full positive element in $\hat{J}$.

According to [33, Proposition 4.4.4] (or [9, Proposition II.6.5.5(i)]), for $z \in \hat{J}$, there is a lower semicontinuous real function $\check{z}$ on $Prim(\hat{J})$, given by:

$$I \mapsto \|z/\ker(\pi)\|,$$

where $I \trianglelefteq \hat{J}$ and π is an irreducible representation of $\hat{J}$ such that $I = \ker(\pi)$, by [33, Proposition 4.4.2] (recall that unitarily equivalent representations have the same kernel).

Since $Prim(\hat{J})$ is compact, by [14, Theorem 7.4.14] (which states that any lower semicontinuous function on a compact space attains its minimum value), there is $K \in Prim(\hat{J})$ such that,

$$\check{z}(K) = \|z/\ker(\pi)\| = \inf_{I\in Prim(\hat{J})} \check{z}(I) > 0.$$

Note that, since z is full, it does not belong to any proper (primitive) ideal and therefore does not belong to the kernel of any non-zero irreducible representation, and hence $\|z/\ker(\pi)\| > 0$. Since

$$z = \lim_{n\to\infty}(z - 1/n)_+.$$

and $\check{z}_n \to \check{z}$ uniformly, where $z_n = (z - 1/n)_+$, for $0 < \varepsilon < \check{z}(K)$, there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and all $I \in Prim(A)$,

$$|\|z_n/\ker(\pi)\| - \|z/\ker(\pi)\|| < \varepsilon,$$

hence,

$$0 < \|z/\ker(\pi)\| - \varepsilon < \|z_n/\ker(\pi)\|.$$

This implies that z_n is full in $\hat{J}$, for all $n \geq n_0$, and hence z a strictly full element in $\hat{J}$.

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For the last assertion of (i), let Ω be the set of all proper ideals in $\hat{J}$ and Ω_0 be a chain in Ω. Since the set of all ideals in a C*-algebra is a complete lattice,

$$I_0 := \bigvee_{I\in\Omega_0} I = \overline{\bigcup_{I\in\Omega_0} I}$$

is an ideal in $\hat{J}$. Since $\hat{J}$ is a compact ideal, $I_0 \subsetneq \hat{J}$ (because if $I_0 = \hat{J}$, then for an $I \in \Omega_0$ we have $\hat{J} = I$, that is a contradiction). Hence I_0 is a proper ideal in $\hat{J}$, and in fact an upper bound for Ω_0. Thus, by Zorn's Lemma, Ω has a maximal ideal. This implies that $\hat{J}$ has a maximal ideal.

For (ii), first let A be exact and purely infinite. Since x is a strictly full element in A, for some $\varepsilon_0 > 0$, $x \precsim (x-\varepsilon_0)_+$. Now by the last two paragraphs of the proof of Proposition 2.7($i \Rightarrow ii$) in [32] (which does not indeed need the assumption of separability), there is a projection p in A such that $x \sim_{cu} p$, and so p is full in A. But $Prim(A) = Prim(A\otimes\mathbb{K})$ and $A\otimes\mathbb{K}$ is purely infinite and has also a full projection, of the form $(p\otimes e)$, where e is a rank-one projection in $\mathbb{K}$. Moreover, in [39, Theorem 2.15] it was shown that an exact C*-algebra A with $Prim(A)$ compact, admits a non-zero densely defined lower semi-continuous trace if and only if the stabilization of A does not contain a full properly infinite projection. Thus, we have $T_{lsc}(A) = \{0\}$.

Now for the second case, suppose that A is exact and stably finite and has the projection property. Then, A has an increasing approximate unit consisting of projections, say $(q_\lambda)_{\lambda\in\Lambda}$. Let $A_\lambda := \overline{Aq_\lambda A}$. Then, $(A_\lambda)_{\lambda\in\Lambda}$ is an increasing net of ideals in A. Since $A = \overline{\bigcup_{\lambda\in\Lambda} A_\lambda}$, we have $A = A_{\lambda_0}$, for some $\lambda_0 \in \Lambda$. Thus, q_{λ_0} (resp. $q_{\lambda_0}\otimes e$) is a full projection in A (resp. $A\otimes\mathbb{K}$). Note that since A is stably finite, every projection in $A\otimes\mathbb{K}$ is finite. Hence $T_{lsc}(A) \neq \{0\}$, again by [39, Theorem 2.15].

Also, suppose that $\tau \in T_{lsc}(A)$. Then, $\tau(q_{\lambda_0}) \neq 0$. Otherwise, $q_{\lambda_0} \in \ker(\tau)$. On the other hand,

$$q_{\lambda_0} \in Ped(A) = \mathrm{dom}(\tau),$$

where $\mathrm{dom}(\tau)$ is the domain of τ (see [39, Definition 2.12]). Now, since q_{λ_0} is full in A, $\ker(\tau) = \mathrm{dom}(\tau)$ and hence τ is zero, which is a contradiction. □

Definition 2.15. *A C*-algebra A is said complete-Goldie, if all its quotients have Goldie dimension.*

All C*-algebras with Krull dimension are complete-Goldie. The graph C*-algebra $C^*(\mathcal{G})$, introduced in Theorem 2.7(ii), is a Goldie C*-algebra, but it is not complete-Goldie.

Question 2.16. Is there a complete-Goldie C*-algebra that does not have Krull dimension?

Corollary 2.17. *Let E be a graph and $A = C^*(E)$ be a complete-Goldie graph C*-algebra. Then E satisfies Condition (K). In particular, $RR(A) = 0$.*

Proof. The C*-algebra $C(\mathbb{T})$ does not have Goldie dimension, by Corollary 2.11 (or Corollary 2.12). Furthermore, for a complete-Goldie C*-algebra, Goldie dimension is preserved under Morita equivalence of C*-algebras and passes to ideals and quotients, by Theorem 2.7(i) and Definition 2.22. The first assertion now follow from [40, Lemma 3.3]. The second assertion holds, because in [21, Theorem 2.5], it was shown that if E is a directed graph, then $C^*(E)$ satisfies Condition (K) if and only if the real rank of $C^*(E)$ is zero. □

In particular, every purely infinite complete-Goldie graph C*-algebra A is $\mathcal{O}_\infty$-stable and have nuclear dimension one. Indeed, since $RR(A) = 0$, A has an approximate unit consisting of projections. Furthermore, [24, Theorem 9.1] implies that a nuclear and separable C*-algebra with an approximate unit consisting of projections is $\mathcal{O}_\infty$-stable if and only if it is strongly purely infinite, and a separable C*-algebra of real rank zero is strongly purely infinite if and only if it is purely infinite. Recall that every graph C*-algebra is separable and nuclear [3, p. 65]. Thus $A \cong A \otimes \mathcal{O}_\infty$. Moreover, in [10, Theorem A], it was shown that every $\mathcal{O}_\infty$-stable, nuclear and separable C*-algebra, has nuclear dimension one. Therefore, A has nuclear dimension one. We refer the reader for the notion of nuclear dimension of a C*-algebra to [47].

Lemma 2.18. *The local multiplier algebra of an Artinian, purely infinite and separable C*-algebra is purely infinite.*

Proof. Let A be an Artinian, purely infinite and separable C*-algebra. Then, for an $n \in \mathbb{N}$, there are the 0-critical (or simple) C*-algebras A_i, $(1 \leq i \leq n)$, such that $A = \bigoplus_{i=1}^n A_i$, by [41, Lemma 2.3(v) and Theorem 2.8]. Thus $M_{loc}(A) = \bigoplus_{i=1}^n M_{loc}(A_i)$. But, for every i, $(1 \leq i \leq n)$, $M_{loc}(A_i) = M(A_i)$. Hence $M_{loc}(A) = M(A)$. By Zhang's theorem [49, Theorem 1.2(ii)], every purely infinite and simple C*-algebra has real rank zero. Furthermore, in [48, Theorem 1.3(b)], it was shown that the corona algebra of a simple and σ-unital C*-algebra of real rank zero is purely infinite. Thus, every $M(A_i)/A_i$ is purely infinite. Now being purely infinite is preserved under extensions. Therefore, every $M_{loc}(A_i) = M(A_i)$ is also purely infinite. This shows that $M_{loc}(A)$ is purely infinite. □

Recall that a Kirchberg algebra is a purely infinite, nuclear, separable and simple C*-algebra [38]. Let $\Sigma(A) = \{[p] \,|\, p \in P(A)\}$ be the dimension range $\Sigma(A)$ of a C*-algebra A [37, p. 124].

Lemma 2.19. *For every countable direct sum A of Kirchberg algebras, we have $K_0(A) = \Sigma(A)$.*

Proof. Let $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$, where Λ is a countable direct set and every A_λ is a Kirchberg algebra, and Ω be the collection of finite subsets of Λ, directed by inclusion. Then

$$A = \varinjlim_{\mathcal{F}} \bigoplus_{\lambda \in \mathcal{F}} A_\lambda,$$

where $\mathcal{F} \in \Omega$ [9, Example II.8.2.2(ii)]. According to [34, Proposition 3.4(ii)], every Noetherian, purely infinite and separable C*-algebra has the ideal property (i.e., its

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ideals are generated by thier projections). Moreover, the ideal property and purely infinite are preserved under inductive limits (see [30, Proposition 2.3] and [23, Proposition 4.18]). Also, since Λ is countable, A is separable. Thus, A is a purely infinite and separable C*-algebra with the ideal property. Now, [32, Lemma 3.8] implies that $K_0(A) = \Sigma(A)$. $\square$

Every Artinian C*-algebra A has a K-ideal J_A that is a direct sum of m simple (or 0-critical) ideals, by [41, Lemma 2.3(v) and Theorem 2.8]. On the other hand, every prime C*-algebra has only one (non-zero) simple ideal (if any). Thus, every Artinian and prime C*-algebra A has a unique simple (essential) ideal $\mathcal{J}_A$. Now, if A is a C*-algebra with Goldie dimension n and with an Artinian G-ideal $I_A = \bigoplus_{i=1}^n I_i$, then every direct summand I_i of I_A is Artinian and prime. Thus every I_i has a unique simple ideal $\mathcal{J}_i$. We call the ideal $\mathcal{J}_A := \bigoplus_{i=1}^n \mathcal{J}_i$ a G_0-*ideal* in A. Of course, $\mathcal{J}_A \trianglelefteq_e A$, by [41, Lemma 2.6].

For each class of C*-algebras that can be classified, an appropriate Elliott invariant is used, commonly shown as $Ell(.)$. In the following theorem, using Lemma 2.19, we set,

$$Ell(.) := (K_0(.), K_1(.)) = (\Sigma(.), K_1(.)),$$

for the stable case and,

$$Ell(.) := (K_0(.), [1]_0, K_1(.)) = (\Sigma(.), [1]_0, K_1(.)),$$

for the unital case.

Theorem 2.20 *Let A and B be C*-algebras with Goldie dimension n, each containing an Artinian G-ideal. Suppose that all direct summands of G_0-ideals $\mathcal{J}_A$ and $\mathcal{J}_B$ are separable, nuclear, purely infinite and in the UCT class and for $n \geq 2$, are also stable and with non-zero simple K_j-groups, $j = 0, 1$. Then, $Ell(\mathcal{J}_A) \cong Ell(\mathcal{J}_B)$ implies that $M_{loc}(A) \cong M_{loc}(B)$, and $M_{loc}(A)$ and $M_{loc}(B)$ are purely infinite. Furthermore, if $\mathcal{J}_A$ and $\mathcal{J}_B$ belong to $\mathcal{M}_l$, then*

$$Ell(A) \cong Ell(B) \textit{if and only if } A \cong B.$$

Proof. Let $\mathcal{J}_A \cong \bigoplus_{i=1}^n \mathcal{J}_i^A$ and $\mathcal{J}_B \cong \bigoplus_{i=1}^n \mathcal{J}_i^B$ be G_0-ideals in A and B. Then, all ideals $\mathcal{J}_i^A$ and $\mathcal{J}_i^B$ are Kirchberg algebras in the UCT class.

Recall that according to Zhang's Dichotomy [49, Theorem 1.2(i)], every σ-unital, purely infinite and simple C*-algebra is either unital or stable. Furthermore, according to the Kirchberg–Phillips classification theorem [38, Theorem 8.4.1(ii)], the pair $(K_0(.), K_1(.))$ (resp. the triple $(K_0(.), [1]_0, K_1(.))$) is a complete invariant for stable (resp. unital) Kirchberg algebras in the UCT class.

Let $n = 1$. Then, $\mathcal{J}_A$ and $\mathcal{J}_B$ are Kirchberg algebras in the UCT class that are either stable or unital. Therefore (in both stable and unital cases), $Ell(\mathcal{J}_A) \cong Ell(\mathcal{J}_B)$ if and only if $\mathcal{J}_A \cong \mathcal{J}_B$, and hence $Ell(\mathcal{J}_A) \cong Ell(\mathcal{J}_B)$ implies that, $M_{loc}(A) \cong M_{loc}(\mathcal{J}_A) \cong M_{loc}(\mathcal{J}_B) \cong M_{loc}(B)$. Of course, $M_{loc}(A)$ and $M_{loc}(B)$ are purely infinite, by Lemma 2.18.

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Let $n > 1$ and all $\mathcal{J}_i^A$ and $\mathcal{J}_i^B$ are also stable and with non-zero simple K_j-groups $(j = 0, 1)$. Then, $Ell(\mathcal{J}_A) \cong Ell(\mathcal{J}_B)$ implies that

$$\bigoplus_{i=1}^{n} K_j(\mathcal{J}_i^A) \cong \bigoplus_{i=1}^{n} K_j(\mathcal{J}_i^B),$$

[45, Propositions 6.2.1 and 7.1.11(4)]. Since all $\mathcal{J}_i^A$ and $\mathcal{J}_i^B$ have non-zero simple K_j-groups $(j = 0, 1)$, by changing the index, if necessary, $K_j(\mathcal{J}_i^A) \cong K_j(\mathcal{J}_i^B)$, and hence $\mathcal{J}_i^A \cong \mathcal{J}_i^B$, for any $i \in \{1, 2, \dots, n\}$. This means that,

$$M_{loc}(\mathcal{J}_A) \cong \bigoplus_{i=1}^{n} M_{loc}(\mathcal{J}_i^A) \cong \bigoplus_{i=1}^{n} M_{loc}(\mathcal{J}_i^B) \cong M_{loc}(\mathcal{J}_B),$$

and hence $M_{loc}(A) \cong M_{loc}(B)$. Also, $M_{loc}(A)$ and $M_{loc}(B)$ are purely infinite, again according to Lemma 2.18.

For the last assertion (for every $n \geq 1$), since $\mathcal{J}_A, \mathcal{J}_B \in \mathcal{M}_l$,

$$A \leq M_{loc}(A) \cong M_{loc}(\mathcal{J}_A) = \mathcal{J}_A,$$

and so $A = \mathcal{J}_A$. Similarly, $B = \mathcal{J}_B$. This implies that $Ell(A) \cong Ell(B)$ if and only if $A \cong B$, by previous assertion. □

Example 2.21. For every prime number p, let

$$\mathcal{F}_{p+1}: \quad \begin{tikzcd} w_1 \arrow[loop above, "(p+1)"] & w_2 \arrow[l] \arrow[loop above, "(p+1)"] & w_3 \arrow[l] \arrow[loop above, "(p+1)"] & \cdots \arrow[l] \end{tikzcd}$$

be the graph introduced on p. 496 of [40]. The graph C*-algebra $C^*(\mathcal{F}_{p+1})$ is Artinian, prime, purely infinite [5, Proposition 5.3], separable and nuclear [3, p. 65]. The ideal corresponding to the saturated hereditary subset $H = \{w_1\}$ is I_H [3, Theorem 2.1.6(a)] that is minimal and essential. Since

$$I_H \sim_M C^*(E_H) \cong \mathcal{O}_{p+1}$$

[3, Theorem 2.1.6(c)], the Brown–Green–Rieffel theorem [36, Theorem 5.55] implies that I_H and $C^*(E_H)$ are stably isomorphic. But the UCT class is closed under stable isomorphism [12, Remark 7.3]. Thus I_H is in the UCT class. Now, let

$$A := C^*(\mathcal{F}_{p+1}) \otimes \mathbb{B}.$$

Then, the ideal $I_A = C^*(\mathcal{F}_{p+1}) \otimes \mathbb{K}$ [29, Theorem 1.3] is an Artinian G-ideal and $\mathcal{J}_A = I_H \otimes \mathbb{K}$ is a simple G_0-ideal in A. Thus, A is a non-separable and non-nuclear

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C*-algebra with Goldie dimension one and with infinitely many ideals. The ideal $\mathcal{J}_A$ is stable, separable, nuclear, purely infinite and in the UCT class and with simple K_0-group

$$K_0(\mathcal{J}_A) \cong K_0(I_H) \cong K_0(\mathcal{O}_{p+1}) \cong \mathbb{Z}_p,$$

and K_1-group $K_1(\mathcal{J}_A) = \{0\}$ [37, Exercises 4.5 and 8.10]. Moreover, $M_{loc}(A)$ is also purely infinite.

Next, we study the decomposability of C*-algebras with Goldie dimension.

Definition 2.22. *A C*-algebra A is said extending, if every essentially closed ideal in A is a direct summand.*

Equivalently, a C*-algebra is extending if and only if for every $I \trianglelefteq A$ there exists an ideal J in A such that it is a direct summand of A and $I \trianglelefteq_e J$. Indeed, if A is an extending C*-algebra, $I \trianglelefteq A$ and $\Omega = \{K \mid I \trianglelefteq_e K \trianglelefteq A\}$, then by Zorn's Lemma, Ω has a maximal member J. Now, according to Definition 2.4 and maximality, J is essentially closed. Since A is extending, J is a direct summand. Converesly, let $I \trianglelefteq_{e.c} A$. Then for $I \trianglelefteq A$, there is a direct summand K such that $I \trianglelefteq_e K$. Thus $I = K$, and so I is a direct summand.

Example 2.23. (a) For every $n \in \mathbb{N}^{\geq 2}$, the C*-algebra $C^*(\mathcal{G})$ (resp. $B = C^*(\mathcal{G}) \oplus \mathbb{B}$) is extending and with Goldie dimension one (resp. two) that are separable and nuclear (resp. non-separable and non-nuclear) with infinitely many ideals. In general, a direct sum of simple (prime) C*-algebras is an extending C*-algebra.

(b) For $n_1, n_2 \in \mathbb{N}^{\geq 2}$, consider the following graph:

$$E_1: \qquad {\scriptstyle (n_1)} \circlearrowleft w_1 \longleftarrow v \longrightarrow w_2 \circlearrowright {\scriptstyle (n_2)}$$

The graph E_1 is row-finite and satisfies Condition (K), and so there is a lattice isomorphism from the lattice of saturated hereditary subsets of E_1 onto the lattice of ideals of $C^*(E_1)$, by [5, Theorem 4.1] and [6, Corollary 3.8]. The only non-empty saturated hereditary subsets of E_1 are $H_1 = \{w_1\}$, $H_2 = \{w_2\}$ and E_1^0. Thus, $C^*(E_1)$ has only three non-zero ideals I_{H_1}, I_{H_2} and $C^*(E_1)$ (and so it is Artinian and Noetherian). Clearly, I_{H_1} (resp. I_{H_2}) is essentially closed and isomorphic with the Cuntz algebra $\mathcal{O}_{n_1}$ (resp. $\mathcal{O}_{n_2}$). Moreover, since the sum of two closed ideals in a C*-algebra is itself a closed ideal, we must have,

$$C^*(E_1) \cong I_{H_1} \oplus I_{H_2}.$$

Of course, the only G-ideal (also K-ideal) in $C^*(E_1)$ is itself. Thus, $C^*(E_1)$ is a C*-algebra with Goldie dimension two (and with Krull dimension zero) which is extending but not prime. It is worth mentioning that the essentially closed ideals I_{H_1} and I_{H_2} are not essential ideals in $C^*(E_1)$.

Theorem 2.24. *Let A be a C*-algebra. Then, the following conditions are equivalent:*
(a) A has Goldie dimension.
(b) Every ascending (resp. descending) chain of essentially closed ideals of A stabilizes.

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If, in addition, A is extending, then conditions (a) and (b) above are equivalent to:
(c) A is a finite direct sum of prime C-algebras.*

Proof. $(a \Rightarrow b)$: Let A has Goldie dimension and $J_1 \subsetneqq J_2 \subsetneqq J_3 \subsetneqq \dots$ be a strictly ascending sequence of essentially closed ideals in A. Then for $i, j \in \mathbb{N}$, $(i \lneqq j)$, we have $J_i \subsetneqq J_j$ and, in addition, $J_i \trianglelefteq_{e.c} J_j$. Thus, J_i and J_j/J_i have Goldie dimension and,

$$\mathrm{G.dim}(J_j) = \mathrm{G.dim}(J_i) + \mathrm{G.dim}(J_j/J_i),$$

by Theorem 2.7(iv), and hence $\mathrm{G.dim}(J_i) \lneqq \mathrm{G.dim}(J_j)$. Thus, we have a strictly ascending sequence,

$$\mathrm{G.dim}(J_1) \lneqq \mathrm{G.dim}(J_2) \lneqq \mathrm{G.dim}(J_3) \lneqq \dots,$$

which is a contradiction. The descending case is proved similarly.

$(b \Rightarrow a)$: Let A does not have Goldie dimension. Thus, A contains an infinite direct sum of non-zero ideals, say $\bigoplus_{i=1}^{\infty} I_i$, (see Theorem 2.7(i)). We set $J_n := \bigoplus_{i=n}^{\infty} I_i$, $(n \in \mathbb{N})$.

First, we prove the ascending case. Since $I_1 \cap J_2 = \{0\}$, by Zorn's Lemma, the set $\Omega = \{K \,|\, K \trianglelefteq A \text{ and } K \cap J_2 = \{0\}\}$ has a maximal member L_1. Thus, $L_1 \trianglelefteq_{e.c} A$, by maximality. Since $(L_1 \oplus I_2) \cap J_3 = \{0\}$, Similarly to the previous part, there is an essentially closed ideal $L_2 \supseteq L_1 \oplus I_2$. Also, since $(L_2 \oplus I_3) \cap J_4 = \{0\}$, there is an essentially closed ideal $L_3 \supseteq L_2 \oplus I_3$. By continuing this process, we will have a strictly ascending sequence $L_1 \subsetneqq L_2 \subsetneqq L_3 \subsetneqq \dots$ of essentially closed ideals in A, which contradicts the assumption.

Now, we prove the assertion in the descending case. Let $K_1 = A$. Since $I_1 \cap J_2 = \{0\}$, there is an ideal $K_2 \supseteq J_2$ such that $K_2 \trianglelefteq_{e.c} K_1$. Also, since $I_2 \cap J_3 = \{0\}$, there is an ideal $K_3 \supseteq J_3$ such that $K_3 \trianglelefteq_{e.c} K_2$. Note that the relation $\trianglelefteq_{e.c}$ is transitive, by Lemma 2.6, and hence K_3 is also essentially closed in $K_1 = A$. By continuing this process, we get a strictly descending sequence $L_1 \supsetneqq L_2 \supsetneqq L_3 \supsetneqq \dots$ of essentially closed ideals in A, which is a contradiction.

$(c \Rightarrow a)$: Let A be an extending C*-algebra and $A = \bigoplus_{i=1}^{n} A_i$, where each A_i is a prime C*-algebra and $n \in \mathbb{N}$. Then,

$$\mathrm{G.dim}(A) = \sum_{i=1}^{n} \mathrm{G.dim}(A_i) = n.$$

$(a \Rightarrow c)$: Let A be an extending C*-algebra with Goldie dimension. Thus, A contains an ideal I_1 that is a prime C*-algebra, and there are an ideal J_1 and a C*-subalgebra B_1 of A such that $A = J_1 \oplus B_1$ and $I_1 \trianglelefteq_e J_1$. Then B_1 is also extending. Because if L_1 is an essentially closed ideal in B_1, then since B_1 is a direct summand of A (and so is essetially closed in A), L_1 is essentially closed in A, by transitivity. But A is extending, and hence L_1 is a direct summand of A (and so of B_1). Furthermore, J_1 is a prime C*-algebra ($\mathrm{G.dim}(J_1) = \mathrm{G.dim}(I_1) = 1$). Now, since $\mathrm{G.dim}(B_1) \leq \mathrm{G.dim}(A)$, B_1 has also Goldie dimension. By repeating the previous process after a finite number of times, the assertion is obtained. □

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Example 2.25. For $n, n_1, n_2 \in \mathbb{N}^{\geq 2}$, consider the following graph E_2:

$$\begin{tikzcd} w_1 \arrow[loop left, "(n_1)"] & v \arrow[l] \arrow[d] \arrow[loop right, "(n)"] \\ & w_2 \arrow[loop left, "(n_2)"] \end{tikzcd}$$

The only non-empty saturated hereditary subsets of E_2 are $H_1 = \{w_1\}$, $H_2 = \{w_2\}$, $H_3 = \{w_1, w_2\}$ and E_2^0. The graph C*-algebra $C^*(E_2)$ has Goldie dimension two, but it cannot be written as a finite direct sum of prime C*-algebras, and hence is not extending, by Theorem 2.24. In fact, its only G-ideal in $C^*(E_2)$ is $I_{H_3}(\cong I_{H_1} \oplus I_{H_2})$, which is the direct sum of two prime C*-algebras.

A Hilbert C*-module X over a C*-algebra A (or a Hilbert A-module) is a vector space over $\mathbb{C}$ and a right A-module with a compatible A-valued inner product $\langle \ldots \rangle_A$ (see [36, Definition 2.8], for a detailed definition). A Hilbert A-module X is called full if $\langle X, X \rangle_A = A$. Let $\mathcal{L}_A(X)$ be the C*-algebra of adjointable operators on X and $\mathcal{K}_A(X)$ the closed linear span of the set of all elementary operators $\theta_{x,y} : X \to X$ given by $\theta_{x,y}(z) = x\langle y, z\rangle; x, y, z \in X$ [26, 36]. For an ideal I in A, an ideal submodule for X, denoted by X_I, is defined by,

$$X_I := \overline{span}\{x.a \;:\; x \in X \;,\; a \in I\} = \{x.a \;:\; x \in X \;,\; a \in I\},$$

(see [4, Definition 1.1 and Proposition 1.2]). Let $\pi : A \to A/I$ and $q : X \to X/X_I$ be the quotient maps. Then, X/X_I with the right action $q(x)\pi(a) = q(xa)$ and the inner product $\langle q(x), q(y)\rangle = \pi(\langle x, y\rangle)$ is a Hilbert A/I-module [4].

If $A \subseteq B$ is an inclusion of C*-algebras with a common identity and a faithful canonical conditional expectation $\mathcal{E} : B \to A$, and

$$\|x\|_{\mathcal{E}}^2 = \|\mathcal{E}(x^*x)\|$$

is the norm induced by the inner product $\langle x, y\rangle_A = \mathcal{E}(x^*y)$, then, $X_{\mathcal{E}} := \overline{B}^{\|\cdot\|_{\mathcal{E}}}$ is a Hilbert A-module. A conditional expectation $\mathcal{E} : B \to A$ is said to have finite index if there exists $z_1, z_2, \ldots, z_n \in B$ such that $x = \sum_{i=1}^n z_i \mathcal{E}(z_i^* x)$, $x \in B$ [22, 44].

Theorem 2.26. *(i) Every Hilbert C*-module X over a C*-algebra A with Goldie dimension n that is extending or an AW*-algebra is decomposed into a direct sum of n ideal submodules.*

(ii) Let $A \subseteq B$ be an inclusion of C-algebras with a common identity and a conditional expectation $\mathcal{E} : B \to A$ that is faithful and of finite index. Then, A has Goldie dimension if and only if $\mathcal{K}_A(X_{\mathcal{E}})(= \mathcal{L}_A(X_{\mathcal{E}}))$ has Goldie dimension.*

Proof. (i) First we show that if X is a Hilbert C*-module over a C*-algebra A, I and J are two closed ideals in A, and X_I and X_J be the associated ideal submodules, then

$$X_I + X_J = X_{I+J} \;\text{ and }\; X_I \cap X_J = X_{I\cap J}.$$

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Since $X_I + X_J$ is closed in X [43, Lemma 2.12] and, for every ideal K in A, $X_K = \{x.k \ : \ x \in X \ , \ k \in K\}$, we have $X_I + X_J = X_{I+J}$. Note that every ideal submodule in an ideal submodule of X is also an ideal submodule in X. If $\pi_0 : I \to (I+J)/J$ is the restriction map of the quotient map

$$\pi : I + J \to (I+J)/J,$$

then $\mathrm{Ker}(\pi_0) = I \cap J$. Let $q_0 : X_I \to (X_I + X_J)/X_J$ be the restriction map of the quotient map

$$q : X_I + X_J \to (X_I + X_J)/X_J.$$

Then q_0 is a π_0-morphism (see [4, Definition 2.1]), since

$$\langle q_0(x), q_0(y)\rangle = \langle q(x), q(y)\rangle = \pi(\langle x, y\rangle) = \pi_0(\langle x, y\rangle),$$

for all $x, y \in X_I$. Thus, [4, Theorem 2.3] implies that,

$$X_I \cap X_J = \mathrm{Ker}(q_0) = X_{\mathrm{Ker}(\pi_0)} = X_{I\cap J}.$$

Now, let A be a C*-algebra with Goldie dimension n that is extending or is an AW*-algebra. If A is extending, then it is a direct sum of n prime C*-algebras, by Theorem 2.24 and Lemma 2.1. If A is an AW*-algebra and $I_A = \bigoplus_{i=1}^n I_i$ a G-ideal in A, then

$$A = M_{loc}(A) = \bigoplus_{i=1}^{n} M_{loc}(I_i),$$

where every $M_{loc}(I_i)$ is a prime C*-algebra [2, Proposition 3.3.2]. In both cases, there are prime C*-algebras $A_1, A_2, \ldots, A_n$ in A such that $A = \bigoplus_{i=1}^n A_i$. Thus,

$$X_A = X_{\bigoplus_{i=1}^n A_i} = \bigoplus_{i=1}^{n} X_{A_i}.$$

(ii) First note that since $\mathcal{E}$ has finite index, there exists $z_1, z_2, \ldots, z_n \in B$ such that,

$$x = \sum_{i=1}^{n} z_i \mathcal{E}(z_i^* x) = \sum_{i=1}^{n} \theta_{z_i, z_i}(x); \ x \in B.$$

Since B is $\|.\|_{\mathcal{E}}$-dense in $X_{\mathcal{E}}$, $\mathcal{K}(X_{\mathcal{E}})$ is unital, with $1_{\mathcal{K}(X_{\mathcal{E}})} = \sum_{i=1}^n \theta_{z_i, z_i}$, and hence $\mathcal{K}_A(X_{\mathcal{E}}) = M(\mathcal{K}_A(X_{\mathcal{E}})) = \mathcal{L}_A(X_{\mathcal{E}})$. Furthermore, observe that $X_{\mathcal{E}}$ is full. Indeed, the elements in a C*-algebra A can be expressed as linear combinations of positive elements,

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and every positive element is in the form $a^*a(=\mathcal{E}(a^*a)=\langle a,a\rangle_A)$, for an $a\in A$. Thus,

$$A\subseteq \overline{span}\{\mathcal{E}(x^*y)\,|\,x,y\in X_{\mathcal{E}}\}=\langle X_{\mathcal{E}},X_{\mathcal{E}}\rangle_A.$$

Now, since every full Hilbert A-module X is a $\mathcal{K}_A(X)$-A-imprimitivity bimodule [36, Proposition 3.8], we have $\mathcal{K}_A(X_{\mathcal{E}})\sim_M A$. Therefore, the assertion holds, by Theorem 2.7(iii). □

Theorem 2.26(ii) shows that Corollary 2.2.14 of [44] (if A is simple, then $\mathcal{K}_A(X_{\mathcal{E}})$ is simple), is also true for C*-algebras with Goldie dimension. Note that $\mathcal{K}_A(X_{\mathcal{E}})$ is exactly $C^*\langle B,e_A\rangle$, by the remark on p. 40, Definition 2.2.10 and Lemma 2.2.9 of [44].

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