



Matrix coefficients of cohomologically induced representations

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ABSTRACT

We define integral formulas which produce certain matrix coefficients of cohomologically induced representations of real reductive groups. They are analogous to Harish-Chandra's Eisenstein integrals for matrix coefficients of ordinary induced representations, and generalize Flensted-Jensen's fundamental functions for discrete series.

1. Introduction

Let G be a connected real Lie group such that:

- (i) the Lie algebra \mathfrak{g}_0 of G is reductive;
- (ii) the connected Lie subgroup of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ has a finite center.

Such a G is called a connected real reductive group. Fix a maximal compact subgroup K of G , which is connected and unique up to conjugation.

An elementary spherical function is a smooth function ϕ on G/K with the following three properties:

$$\left\{ \begin{array}{l} \text{(i) } \phi(1K) = 1; \\ \text{(ii) } \phi \text{ is left } K\text{-invariant;} \\ \text{(iii) } \phi \text{ is a joint eigenfunction of } D(G/K), \text{ where } D(G/K) \text{ is} \\ \quad \text{the algebra of } G\text{-invariant differential operators on } G/K. \end{array} \right. \quad (1)$$

By using the Iwasawa decomposition $G = NAK$, Harish-Chandra found in [Har58] a pretty integral formula

$$\phi_\lambda(x) = \int_K e^{\langle \lambda + \rho, H(kx) \rangle} dk, \quad x \in G, \quad (2)$$

which produces elementary spherical functions on G , where H is the map from G onto $\mathfrak{a}_0 = \text{Lie}(A)$ defined by

$$H(n \exp(a)k) = a, \quad n \in N, a \in \mathfrak{a}_0, k \in K,$$

and $\rho \in \mathfrak{a}_0^*$ is half of the sum of positive roots (the positive system is the one that corresponds to N). Moreover, he proved that a function on G is elementary spherical if and only if it is of the form ϕ_λ for some $\lambda \in \mathfrak{a}_0^* \otimes_{\mathbb{R}} \mathbb{C}$. Note that a function on G/K may be identified with a function on G which is right K -invariant.

In terms of representation theory, elementary spherical functions can be expressed as matrix coefficients of irreducible spherical representations and its trivial K -type. In general, for an

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admissible continuous representation π of G on a Hilbert space V_π , and a K -type α , define

$$\phi_{\pi,\alpha}(g) = p_\alpha \circ \pi(g) \circ j_\alpha, \quad g \in G,$$

where j_α is the embedding of the α -isotypic component $V_\pi(\alpha)$ into V_π , and p_α is the continuous linear projection from V_π onto $V_\pi(\alpha)$ defined by

$$p_\alpha(v) = \begin{cases} v, & \text{if } v \in V_\pi(\alpha), \\ 0, & \text{if } v \in V_\pi(\beta), \text{ for some } K\text{-type } \beta \neq \alpha. \end{cases}$$

It is clear that $\phi_{\pi,\alpha}$ is a real analytic function on G with values in $\text{End}_{\mathbb{C}}(V_\pi(\alpha))$. When $V_\pi(\alpha)$ has dimension 1, $\text{End}_{\mathbb{C}}(V_\pi(\alpha))$ can be canonically identified with \mathbb{C} , and therefore $\phi_{\pi,\alpha}$ is actually a complex valued function. If π is an irreducible spherical representation with the trivial K -type α_0 , then $V_\pi(\alpha_0)$ has dimension 1, and the function ϕ_{π,α_0} is an elementary spherical function. By the theory of spherical functions and spherical representations, all elementary spherical functions can be expressed as ϕ_{π,α_0} for some irreducible spherical representation π .

In the sense of Vogan, α_0 is the unique lowest K -type of every irreducible spherical representation. Therefore, it is interesting to study $\phi_{\pi,\alpha}$ for a general irreducible admissible representation π and an arbitrary lowest K -type α of π . In particular, one may consider the following.

General Problem. Find an integral formula similar to (2) for $\phi_{\pi,\alpha}$, where π is an arbitrary irreducible admissible representation, and α is an arbitrary lowest K -type of π .

Special cases of this General Problem have been extensively studied over the years. Harish-Chandra defined in [Har75] (or [Kna86, Wal92]) the Eisenstein integral which represents $\phi_{\pi,\alpha}$ when π is a parabolically induced representation. In [Fle80a], Flensted-Jensen solved the problem when π is a discrete series representation. Li and Wallach obtained a formula when π is a lowest weight unitary representation [Li90]. Harris and Li gave formulas for some very special cases of cohomologically induced representations [HL98]. Some lower rank group cases have been explicitly calculated by mathematical physicists.

It is important to point out that Flensted-Jensen also attempted to give a solution of the General Problem [Fle80b]. His formulas produce functions with good properties. However, he has to consider the convergence problem since his formulas involve integrations over non-compact groups. In order to attack the convergence problem, he has to assume that the parameters in his formulas must be sufficiently regular.

The objective of this paper is to give an integral formula for $\phi_{\pi,\alpha}$ when π is a cohomologically induced representation and α is in the bottom layer (we describe cohomological inductions and bottom layers in detail in §5). One of the good features of our formulas is that the integrations involved in our formulas are over compact groups. Hence, we do not have to deal with the convergence issue. In what follows, we explain precisely why our formulas are enough for a solution to the General Problem.

Let \mathfrak{g} be the complexification of \mathfrak{g}_0 . It is known that the function $\phi_{\pi,\alpha}$ depends only on α and the underlying (\mathfrak{g}, K) -module of π . Therefore, it makes sense to write $\phi_{M,\alpha}$ for a finitely generated admissible (\mathfrak{g}, K) -module M (every M has at least one globalization π).

Fix a Cartan involution θ on G such that the fixed point group $G^\theta = K$. Recall that a parabolic subalgebra \mathfrak{q} of \mathfrak{g} is called a θ -stable parabolic if

$$\theta(\mathfrak{q}) = \mathfrak{q},$$

and

$$\mathfrak{q} \cap \bar{\mathfrak{q}} \text{ is a Levi factor of } \mathfrak{q},$$

where ‘ $\bar{}$ ’ is the complex conjugation relative to the real form \mathfrak{g}_0 . Assume that α is a lowest K -type of an irreducible (\mathfrak{g}, K) -module M . It is automatic that α has multiplicity 1 in M . Vogan associated to α a θ -stable parabolic \mathfrak{q} (depending on some fixed positive system). Define

$$G' = \text{normalizer of } \mathfrak{q} \text{ in } G.$$

Then G' is a quasisplit connected real reductive group. Write

$$\mathfrak{g}' = \text{Lie}(G') \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad K' = K \cap G'.$$

Let α' be the K' -type which is \mathfrak{q} -associated to α (Definition 3.1). Then α' is fine with respect to G' in the sense of Vogan. By [SV98, Theorem 2.9], or [Vog81, Theorems 6.5.10 and 6.5.12], there is a (\mathfrak{g}', K') -module P' in the non-unitary principal series such that:

- (i) α' occurs in P' (automatically has multiplicity 1);
- (ii) α is in the bottom layer of X_G , occurring with multiplicity 1, where $X_G = \mathcal{R}_{\mathfrak{q}}^S(P')$ is a cohomologically induced standard (\mathfrak{g}, K) -module (this is automatic from condition (i), see the comments after Definition 5.4);
- (iii) M is isomorphic to the unique irreducible subquotient of X_G containing the K -type α .

From the third condition above we know that $\phi_{M,\alpha}$ can be canonically identified with $\phi_{X_G,\alpha}$. It is also known that $\phi_{P',\alpha'}$ can be expressed in terms of the parameters of principal series by Harish-Chandra’s Eisenstein integral. Therefore, if we can express $\phi_{X_G,\alpha}$ in terms of $\phi_{P',\alpha'}$, we give a solution to the general problem.

In this paper, we define an integral (Definition 3.2) which is analogous to Harish-Chandra’s Eisenstein integral and does express $\phi_{X_G,\alpha}$ in terms of $\phi_{P',\alpha'}$ (Theorem 6.3). It generalizes Flensted-Jensen’s fundamental functions for discrete series [Fle80a]. The crucial point is that we use the Langlands decomposition which is more general than the Iwasawa decomposition which was used by Flensted-Jensen [Fle80b].

A direct application of the formulas is to show certain positivity properties which are essential in the study of discrete spectra of theta correspondences [HL98, Li90, Li97]. For example, we find that $\phi_{\pi,\alpha}(x)$ is a positive definite operator when π is an irreducible unitary representation with non-zero cohomology, α is the unique lowest K -type, and $x \in \exp(\mathfrak{p}_0)$, where $\mathfrak{p}_0 \subset \mathfrak{g}_0$ is the -1 eigenspace of θ .

In the next section, we recall Flensted-Jensen duality which relates functions on G to functions on a Riemannian symmetric space. In §3, we introduce the integral and prove that the integral behaves well under the translations of K . In §4, we prove that the integral behaves well under the actions of $U(\mathfrak{g})^K$. In §5, we recall cohomological inductions and bottom layers, and study the actions of $U(\mathfrak{g})^K$ on bottom layers. The results of §§3, 4, and 5 imply that both our integral and $\phi_{M,\alpha}$ have three properties which are similar to those of (1), where M is a cohomologically induced module and α is in the bottom layer. In §6, we use the uniqueness of functions with these properties to get the equality between our integral and $\phi_{M,\alpha}$, and give two interesting examples.

2. Flensted-Jensen duality

2.1 Complexifications of groups

We denote by $H_{\mathbb{C}}$ the universal complexification of a Lie group H . Let $u_H : H \rightarrow H_{\mathbb{C}}$ be the canonical homomorphism, and $\bar{} : H_{\mathbb{C}} \rightarrow H_{\mathbb{C}}$ be the anti-holomorphic automorphism on $H_{\mathbb{C}}$ which is identity on $u_H(H)$. Recall that the universal complexification map u_H is defined by the following property [Hoc66]: if H' is a complex Lie group and $\phi : H \rightarrow H'$ is a Lie group homomorphism, then there is a unique holomorphic homomorphism $\phi' : H_{\mathbb{C}} \rightarrow H'$ such that

the following diagram commutes.

$$\begin{array}{ccc}
 H & \xrightarrow{u_H} & H_{\mathbb{C}} \\
 & \searrow \phi & \downarrow \phi' \\
 & & H'
 \end{array}$$

When A is a connected real analytic manifold, B is a connected complex manifold, and $c : A \rightarrow B$ is a real analytic map, we call c a complexification map if its differential is a complexification of a real vector space everywhere. The following are well known:

- (i) every Lie group has a unique universal complexification;
- (ii) the universal complexification of a connected Lie group is connected;
- (iii) if H is a connected real reductive group, then u_H is a complexification map and has a finite kernel;
- (iv) if H is compact, then u_H is injective and $H_{\mathbb{C}}$ is a reductive complex group.

2.2 Flensted-Jensen duality for groups

Recall from § 1 that θ is a Cartan involution on G such that K is its fixed point group. The involution θ induces a Lie algebra involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$. Let

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the usual decompositions corresponding to θ . Write

$$\bar{\theta} = \theta \circ \bar{} = \bar{} \circ \theta : \mathfrak{g} \rightarrow \mathfrak{g}.$$

The involution $\theta : G \rightarrow G$ also induces a holomorphic involution $\theta : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$. Still write

$$\bar{\theta} = \theta \circ \bar{} = \bar{} \circ \theta : G_{\mathbb{C}} \rightarrow G_{\mathbb{C}}.$$

Let U be the fixed points of $\bar{\theta}$ in $G_{\mathbb{C}}$. Note that U is connected, and it is a maximal compact subgroup of $G_{\mathbb{C}}$ when G has a compact center. We have a commutative diagram

$$\begin{array}{ccc}
 G/K & \xrightarrow{p_K} & G \\
 v_G \downarrow & & \downarrow u_G \\
 G_{\mathbb{C}}/U & \xrightarrow{p_U} & G_{\mathbb{C}}
 \end{array} \tag{3}$$

where v_G is the map induced by u_G , p_K is defined by

$$p_K(xK) = x\theta(x^{-1}), \quad x \in G,$$

and p_U is defined by

$$p_U(xU) = x\bar{\theta}(x^{-1}), \quad x \in G_{\mathbb{C}}.$$

Note that both u_G and p_U are complexification maps.

Fix a finite-dimensional continuous representation τ of $K \times K$ on \mathbf{V} , and still denote by τ the holomorphic extension of the representation to $K_{\mathbb{C}} \times K_{\mathbb{C}}$. We have a smooth representation T of $G \times G$ on $C^\infty(G; \mathbf{V})$ defined by

$$(T_{g,h}\phi)(x) = \phi(g^{-1}xh)$$

for all $g, h, x \in G$, $\phi \in C^\infty(G; \mathbf{V})$. By taking its differential, we get a representation T of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ on $C^\infty(G; \mathbf{V})$. We also have a smooth representation T of $G_{\mathbb{C}}$ on $C^\infty(G_{\mathbb{C}}/U; \mathbf{V})$ by

$$(T_g\psi)(x) = \psi(g^{-1}x)$$

for all $g \in G_{\mathbb{C}}, x \in G_{\mathbb{C}}/U, \psi \in C^{\infty}(G_{\mathbb{C}}/U; \mathbf{V})$. Again by taking its differential and using the complexification

$$\begin{aligned} \mathfrak{g} &\rightarrow \mathfrak{g} \times \mathfrak{g}, \\ X &\mapsto (X, \bar{\theta}(X)), \end{aligned}$$

we get a representation T of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ on $C^{\infty}(G_{\mathbb{C}}/U; \mathbf{V})$.

Define three spaces of τ -spherical functions by

$$C^{\infty}(G; \tau) = \{\phi \in C^{\infty}(G; \mathbf{V}) \mid \phi(k_1 x k_2^{-1}) = \tau(k_1, k_2)\phi(x), k_1, k_2 \in K, x \in G\}; \tag{4}$$

$$C^{\infty}(G_{\mathbb{C}}/U; \tau) = \{\phi \in C^{\infty}(G_{\mathbb{C}}/U; \mathbf{V}) \mid \phi(kx) = \tau(k, \bar{k})\phi(x), k \in K_{\mathbb{C}}, x \in G_{\mathbb{C}}/U\}; \tag{5}$$

$$C^{\infty}(G/K; \tau) = \{\phi \in C^{\infty}(G/K; \mathbf{V}) \mid \phi(kx) = \tau(k, k)\phi(x), k \in K, x \in G/K\}. \tag{6}$$

The spaces $C^{\omega}(G; \tau), C^{\omega}(G_{\mathbb{C}}/U; \tau)$ and $C^{\omega}(G/K; \tau)$ are defined similarly, here the superscript ‘ ω ’ stands for real analytic functions. Note that both $C^{\omega}(G; \tau)$ and $C^{\omega}(G_{\mathbb{C}}/U; \tau)$ are stable under $T_{U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K}$, where $U(\mathfrak{g})^K$ is as usual the K -fixed vectors in $U(\mathfrak{g})$ under the adjoint action.

The following is essentially [Fle80a, Theorem 7.1]. We omit its proof.

THEOREM 2.1. *We have the following.*

- (i) *The pull backs of p_K and v_G in the diagram (3) induce canonical isomorphisms of vector spaces:*

$$C^{\infty}(G; \tau) \cong C^{\infty}(G/K; \tau) \cong C^{\infty}(G_{\mathbb{C}}/U; \tau).$$

- (ii) *The above identifications induce linear isomorphisms*

$$C^{\omega}(G; \tau) \cong C^{\omega}(G/K; \tau) \cong C^{\omega}(G_{\mathbb{C}}/U; \tau).$$

- (iii) *The identification*

$$C^{\omega}(G; \tau) \cong C^{\omega}(G_{\mathbb{C}}/U; \tau)$$

respects the actions of $U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K$.

3. The integral

3.1 The definition

As in § 1, \mathfrak{q} is a θ -stable parabolic, G' is the normalizer of \mathfrak{q} in G ,

$$\mathfrak{g}' = \text{Lie}(G') \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad K' = K \cap G'.$$

It is known that G' is a connected real reductive group and K' is a maximal compact subgroup of it. The involution θ stabilizes G' and induces a holomorphic involution of $G'_{\mathbb{C}}$, which is also denoted by θ . Let

$$\bar{\theta} = \theta \circ - = - \circ \theta$$

as before. Denote by U' the fixed points of $\bar{\theta}$ in $G'_{\mathbb{C}}$. Let \mathfrak{n} be the nilpotent radical of $\mathfrak{q} \cap [\mathfrak{g}, \mathfrak{g}]$. Then

$$\mathfrak{q} = \mathfrak{g}' \oplus \mathfrak{n}.$$

Denote by N the connected complex subgroup of $G_{\mathbb{C}}$ with Lie algebra \mathfrak{n} . Denote by

$$j_{\mathbb{C}} : G'_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$$

the complexification of the embedding $j : G' \rightarrow G$. The Langlands decomposition

$$G_{\mathbb{C}} = N j_{\mathbb{C}}(G'_{\mathbb{C}})U$$

enables us to define a real analytic map

$$H_q : \begin{matrix} G_{\mathbb{C}}/U & \rightarrow & G'_{\mathbb{C}}/U' \\ nj_{\mathbb{C}}(g)U & \mapsto & gU' \end{matrix} \tag{7}$$

for all $n \in N$ and $g \in G'_{\mathbb{C}}$.

The following conventions will be used freely: if H is a Lie group, \mathfrak{c} is a complex subspace of a complex Lie algebra, and H acts on \mathfrak{c} by certain adjoint action, then $\delta_{\mathfrak{c}}$ is the non-unitary character on H defined by

$$\delta_{\mathfrak{c}}(x) = \text{Det}(\text{Ad}(x)|_{\mathfrak{c}}).$$

If σ is a continuous finite-dimensional representation of a Lie group H , then its differential and its holomorphic extension to $H_{\mathbb{C}}$ are still denoted by σ . Note that

$$\delta_{\mathfrak{c}}(X) = \text{Tr}(\text{ad}(X)|_{\mathfrak{c}}), \quad X \in \text{Lie}(H) \otimes_{\mathbb{R}} \mathbb{C}. \tag{8}$$

Define

$$\delta_{\mathfrak{n}}^+(xU') = \delta_{\mathfrak{n}}(x\bar{\theta}(x^{-1})), \quad xU' \in G'_{\mathbb{C}}/U'. \tag{9}$$

It is not difficult to see that $\delta_{\mathfrak{n}}^+$ has positive values everywhere. However, this fact plays no roll in this paper.

DEFINITION 3.1. (i) If σ is a finite-dimensional continuous representation of K on W , define a representation σ' of K' on

$$W' = \{v \in W \mid \sigma(\mathfrak{n} \cap \mathfrak{k})v = 0\} \tag{10}$$

by

$$\sigma'(k)v = \delta_{\mathfrak{n} \cap \mathfrak{p}}(k)\sigma(k)v, \quad k \in K', v \in W'. \tag{11}$$

Then σ' is called the representation \mathfrak{q} -associated to σ .

(ii) If τ is a finite-dimensional continuous representation of $K \times K$ on \mathbf{V} , define a representation τ' of $K' \times K'$ on

$$\mathbf{V}' = \{v \in \mathbf{V} \mid \tau(X, Y)v = 0 \text{ for all } X \in \mathfrak{n} \cap \mathfrak{k}, Y \in \bar{\mathfrak{n}} \cap \mathfrak{k}\}. \tag{12}$$

by

$$\tau'(k, l)v = \delta_{\bar{\mathfrak{n}} \cap \mathfrak{p}}(k)\delta_{\mathfrak{n} \cap \mathfrak{p}}(l)\tau(k, l)v, \quad k, l \in K', v \in \mathbf{V}'. \tag{13}$$

Then τ' is called the representation $\mathfrak{q} \times \bar{\mathfrak{q}}$ -associated to τ .

Now let τ' be the representation $\mathfrak{q} \times \bar{\mathfrak{q}}$ -associated to τ as in the second part of the definition. We define $C^\infty(G'_{\mathbb{C}}/U'; \tau')$ in the same way as that of $C^\infty(G_{\mathbb{C}}/U; \tau)$ in (5). We also use similar notation for other spaces. Now we have enough preparations to define our integral.

DEFINITION 3.2. For any

$$\phi \in C^\infty(G'_{\mathbb{C}}/U'; \tau'),$$

define the integral

$$E_q(\phi : x) = \int_K \tau(k^{-1}, k^{-1})(\delta_{\mathfrak{n}}^+ \phi)(H_q(kx)) dk, \quad x \in G_{\mathbb{C}}/U. \tag{14}$$

As usual, we use normalized Haar measures on compact groups for integration. Our integral is very similar to Harish-Chandra's Eisenstein integral for a real parabolic subgroup. The remainder of this section is used to establish one of the main results in this paper.

THEOREM 3.3. We have $E_q(\phi) \in C^\infty(G_{\mathbb{C}}/U; \tau)$ for all $\phi \in C^\infty(G'_{\mathbb{C}}/U'; \tau')$.

3.2 An integral formula

In this subsection, we establish an integral formula which is crucial for the proof of Theorem 3.3. The formula is a stronger version of [KV95, Corollary 11.40]. We formulate it in full generality, although we only need to apply it to the complex group $K_{\mathbb{C}}$, since the proof of the general case in no more difficult than that of the special case. The notation of this subsection is not used in other parts of this paper.

Let G be a real reductive Lie group (which the reader may assume to be connected), and θ be a Cartan involution on G . Let \mathfrak{q}_0 be a parabolic subalgebra of $\mathfrak{g}_0 = \text{Lie}(G)$. We introduce the following notation:

- K is the maximal compact subgroup of G corresponding to θ ;
- $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is the Cartan decomposition corresponding to θ ;
- Q is the normalizer of \mathfrak{q}_0 in G ;
- $L = Q \cap \theta(Q)$;
- $Q = LN$ is the Levi decomposition;
- $\mathfrak{q}_0 = \mathfrak{l}_0 + \mathfrak{n}_0$ is the corresponding Levi decomposition at the Lie algebra level;
- $\mathfrak{l}'_0 = \mathfrak{l}_0 \cap \mathfrak{p}_0$;
- $\delta : L \rightarrow \mathbb{R}^\times$ is the homomorphism defined by the absolute value of the determinant of the adjoint representation of L on \mathfrak{n}_0 .

By Langlands decomposition, the map

$$\begin{aligned} N \times \mathfrak{l}'_0 \times K &\rightarrow G, \\ (n, X, k) &\mapsto n \exp(X)k \end{aligned} \tag{15}$$

is a real analytic diffeomorphism. We define two maps $\kappa : G \rightarrow K$ and $I : G \rightarrow L$ by

$$\kappa(nak) = k, \quad I(nak) = a$$

for all $n \in N$, $a \in \exp(\mathfrak{l}'_0)$, $k \in K$. If $a \in G$, we define a map $\kappa^a : K \rightarrow K$ by

$$\kappa^a(k) = \kappa(ka).$$

PROPOSITION 3.4. *If f is a continuous function on K , $a \in G$, then*

$$\int_K f(\kappa^a(k)) dk = \int_K f(k)\delta(I(ka^{-1})) dk.$$

The proof of Proposition 3.4 follows from Lemma 3.5–3.8 given in the remainder of this subsection.

LEMMA 3.5. *The maps κ^a and $\kappa^{a^{-1}}$ are inverse to each other. Hence, κ^a is an analytic diffeomorphism.*

Proof. Let $k \in K$ and $ka = n_1s_1k_1$, where $n_1 \in N$, $s_1 \in \exp(\mathfrak{l}'_0)$, and $k_1 \in K$. Then we have

$$k_1a^{-1} = s_1^{-1}n_1^{-1}k = (s_1^{-1}n_1^{-1}s_1)s_1^{-1}k.$$

Hence,

$$\kappa^{a^{-1}}\kappa^a(k) = \kappa^{a^{-1}}(k_1) = \kappa(k_1a^{-1}) = k.$$

Change a to a^{-1} , we get $\kappa^a\kappa^{a^{-1}}(k) = k$. □

If $a \in G$, we denote by T_a the right translation of a on $Q \backslash G$. We give $Q \backslash G$ the unique K -invariant measure with total mass 1.

LEMMA 3.6. *If $k \in K$, then the Jacobian of T_a at Qk is*

$$J_{T_a}(Qk) = \delta(I(ka)).$$

This should be known. We give a proof for the sake of completeness.

Proof. Let $ka = ns\kappa(ka)$, where $n \in N$ and $s = I(ka)$. We have the following commutative diagram.

$$\begin{array}{ccc} Q \backslash G & \xrightarrow{T_a} & Q \backslash G \\ T_{k^{-1}} \downarrow & & \downarrow T_{\kappa(ka)^{-1}} \\ Q \backslash G & \xrightarrow{T_{ns}} & Q \backslash G \end{array}$$

Since $T_{k^{-1}}$ and $T_{\kappa(ka)^{-1}}$ preserve the measure,

$$J_{T_a}(Qk) = J_{T_{ns}}(Q1).$$

The right-hand side is the absolute value of the determinant of the tangent map $dT_{ns}|_Q$ at the point Q . We have another commutative diagram.

$$\begin{array}{ccc} G & \xrightarrow{\text{Ad}_{(ns)^{-1}}} & G \\ \downarrow & & \downarrow \\ Q \backslash G & \xrightarrow{T_{ns}} & Q \backslash G \end{array}$$

By taking the tangent map at the identity, we get the following diagram.

$$\begin{array}{ccc} \mathfrak{g}_0 & \xrightarrow{\text{Ad}_{(ns)^{-1}}} & \mathfrak{g}_0 \\ \downarrow & & \downarrow \\ \mathfrak{g}_0/\mathfrak{q}_0 & \xrightarrow{dT_{ns}|_{Q1}} & \mathfrak{g}_0/\mathfrak{q}_0 \end{array}$$

Hence,

$$\begin{aligned} J_{T_a}(Qk) &= J_{T_{ns}}(Q1) = |\text{Det}(dT_{ns}|_{Q1})| \\ &= \frac{|\text{Det}(\text{Ad}_{(ns)^{-1}})|}{|\text{Det}(\text{Ad}_{(ns)^{-1}}|_{\mathfrak{q}_0})|} = \frac{1}{\delta(s^{-1})} = \delta(s). \end{aligned} \quad \square$$

For any $a \in G$, let J_a be the Jacobian of κ^a .

LEMMA 3.7. *The Jacobian J_a is left $K \cap Q$ invariant.*

Proof. For any $k \in K$, we use L_k to denote the left translation on K by k . The lemma comes from the fact that if $k \in K \cap Q$, then the following diagram commutes.

$$\begin{array}{ccc} K & \xrightarrow{\kappa^a} & K \\ L_k \downarrow & & \downarrow L_k \\ K & \xrightarrow{\kappa^a} & K \end{array}$$

We omit the easy proof of this fact. □

LEMMA 3.8. *We have $J_a(k) = \delta(I(ka))$ for all $k \in K$.*

Proof. Let π be the map $K \rightarrow Q \backslash G, k \mapsto Qk$. By the above lemma, there is a function J'_a on $Q \backslash G$ such that $J'_a \circ \pi = J_a$. For all continuous function f on $Q \backslash G$, we have

$$\begin{aligned} \int_{Q \backslash G} f(T_a(x)) J'_a(x) dx &= \int_K f(T_a(\pi(k))) J'_a(\pi(k)) dk \\ &= \int_K f((\pi(\kappa^a(k)))) J_a(k) dk \quad (T_a \circ \pi = \pi \circ \kappa^a) \\ &= \int_K f(\pi(k)) dk \\ &= \int_{Q \backslash G} f(x) dx \\ &= \int_{Q \backslash G} f(T_a(x)) J_{T_a}(x) dx. \end{aligned}$$

Hence, $J'_a = J_{T_a}$. We conclude the proof by Lemma 3.6. □

Proof of Proposition 3.4. We have

$$\begin{aligned} \int_K f(\kappa^a(k)) dk &= \int_K f(\kappa^a(\kappa^{a^{-1}}(k))) J_{a^{-1}}(k) dk \\ &= \int_K f(k) \delta(I(ka^{-1})) dk. \end{aligned} \quad \square$$

3.3 A proof of Theorem 3.3

We now return to the notation of §3.1. We apply the results obtained in the last subsection to the group $K_{\mathbb{C}}$. Let \mathfrak{k}'_0 be the Lie algebra of K' . Let N_c be the connected subgroup of $K_{\mathbb{C}}$ with Lie algebra $\mathfrak{n} \cap \mathfrak{k}$. Now the parabolic subgroup is $N_c K'_{\mathbb{C}}$. The map (15) in this case is

$$\begin{aligned} N_c \times \sqrt{-1} \mathfrak{k}'_0 \times K &\rightarrow K_{\mathbb{C}}, \\ (n, X, k) &\mapsto n \exp(X)k. \end{aligned}$$

Define $\kappa : K_{\mathbb{C}} \rightarrow K, I : K_{\mathbb{C}} \rightarrow K'_{\mathbb{C}}, \kappa^a : K \rightarrow K, a \in K_{\mathbb{C}}$, and $\delta : K'_{\mathbb{C}} \rightarrow \mathbb{R}^{\times}$ as in the last subsection.

LEMMA 3.9. *We have*

$$H_{\mathfrak{q}}(kax) = (I(\kappa^a(k)a^{-1}))^{-1} H_{\mathfrak{q}}(\kappa^a(k)x)$$

for all $k \in K, a \in K_{\mathbb{C}}$ and $x \in G_{\mathbb{C}}/U$.

Proof. Write $ka = n'a'k'$, where $n' \in N_c, a' \in \exp(\sqrt{-1} \mathfrak{k}'_0), k' \in K'$. Then

$$\kappa^a(k)a^{-1} = k'a^{-1} = a'^{-1}n'^{-1}k = (a'^{-1}n'^{-1}a')a'^{-1}k.$$

Therefore,

$$I(\kappa^a(k)a^{-1}) = a'^{-1}.$$

As $H_{\mathfrak{q}}$ is $G'_{\mathbb{C}}$ -equivariant and N -invariant, we have

$$\begin{aligned} (I(\kappa^a(k)a^{-1}))^{-1} H_{\mathfrak{q}}(\kappa^a(k)x) &= a' H_{\mathfrak{q}}(\kappa^a(k)x) = H_{\mathfrak{q}}(a' \kappa^a(k)x) \\ &= H_{\mathfrak{q}}(a' k' x) = H_{\mathfrak{q}}(n'^{-1} kax) = H_{\mathfrak{q}}(kax). \end{aligned} \quad \square$$

Define a representation $\tau|_{K_{\mathbb{C}}}$ of $K_{\mathbb{C}}$ on \mathbf{V} by

$$\tau|_{K_{\mathbb{C}}}(k) = \tau(k, \bar{k}),$$

and a representation $\tau|_K$ of K on \mathbf{V} by

$$\tau|_K(k) = \tau(k, k).$$

LEMMA 3.10. We have

$$(\delta_n^+ \phi)(kx) = \delta_{n \cap \mathfrak{k}}(k) \delta_{\bar{n} \cap \mathfrak{k}}(\bar{k}) \tau|_{K_{\mathbb{C}}}(k) (\delta_n^+ \phi)(x)$$

for all $k \in K'_{\mathbb{C}}$, $x \in G'_{\mathbb{C}}/U'$, and $\phi \in C^\infty(G'_{\mathbb{C}}/U'; \tau')$.

Proof.

$$\begin{aligned} \delta_n^+ \phi(kx) &= \delta_n^+(kx) \phi(kx) \\ &= \delta_n(k) \delta_n(\bar{k}^{-1}) \delta_n^+(x) \tau'(k, \bar{k}) \phi(x) \quad \text{by (9)} \\ &= \delta_n(k) \delta_n(\bar{k}^{-1}) \delta_{\bar{n} \cap \mathfrak{p}}(k) \delta_{\bar{n} \cap \mathfrak{p}}(\bar{k}) \delta_n^+(x) \tau(k, \bar{k}) \phi(x) \quad \text{by (13)} \\ &= \delta_{n \cap \mathfrak{k}}(k) \delta_{\bar{n} \cap \mathfrak{k}}(\bar{k}) \tau|_{K_{\mathbb{C}}}(k) (\delta_n^+ \phi)(x). \end{aligned}$$

□

We omit the proof of the following elementary lemma.

LEMMA 3.11. We have

$$\delta(k^{-1}) \delta_{n \cap \mathfrak{k}}(k) \delta_{\bar{n} \cap \mathfrak{k}}(\bar{k}) = 1$$

for all $k \in K'_{\mathbb{C}}$.

Proof of Theorem 3.3. Let $a \in K_{\mathbb{C}}$ and $x \in G_{\mathbb{C}}/U$. Write $k' = \kappa^a(k)$. Then Lemma 3.9 implies

$$\begin{aligned} E_q(\phi : ax) &= \int_K \tau|_K(k^{-1}) (\delta_n^+ \phi)(H_q(kax)) dk \\ &= \int_K \tau|_K((\kappa^{a^{-1}}(k'))^{-1}) (\delta_n^+ \phi)((I(k'a^{-1}))^{-1}(H_q(k'x))) dk. \end{aligned}$$

Now by using Proposition 3.4, we have

$$E_q(\phi : ax) = \int_K \delta(I(ka^{-1})) \tau|_K((\kappa^{a^{-1}}(k))^{-1}) (\delta_n^+ \phi)((I(ka^{-1}))^{-1} H_q(kx)) dk.$$

Write $I' = (I(ka^{-1}))^{-1}$ and $k'' = (\kappa^{a^{-1}}(k))^{-1}$ for simplicity. By Lemma 3.10, we have

$$\begin{aligned} E_q(\phi : ax) &= \int_K \delta(I'^{-1}) \tau|_K(k'') \delta_{n \cap \mathfrak{k}}(I') \delta_{\bar{n} \cap \mathfrak{k}}(\bar{I}') \tau|_{K_{\mathbb{C}}}(I') (\delta_n^+ \phi)(H_q(kx)) dk \\ &= \int_K (\delta(I'^{-1}) \delta_{n \cap \mathfrak{k}}(I') \delta_{\bar{n} \cap \mathfrak{k}}(\bar{I}')) \tau|_{K_{\mathbb{C}}}(k'' I') (\delta_n^+ \phi)(H_q(kx)) dk. \end{aligned}$$

Note that $k'' I' = ak^{-1}n$ for some $n \in N_c$, and N_c fix the values of ϕ under the action $\tau|_{K_{\mathbb{C}}}$. Therefore, Lemma 3.11 implies

$$E_q(\phi : ax) = \int_K \tau|_{K_{\mathbb{C}}}(ak^{-1}) (\delta_n^+ \phi)(H_q(kx)) dk = \tau|_{K_{\mathbb{C}}}(a) E_q(\phi : x).$$

□

4. Differential equations satisfied by the integral

4.1 The differential equations

We recall the generalized Harish-Chandra homomorphism

$$\xi : U(\mathfrak{g})^K \rightarrow U(\mathfrak{g}')^{K'}$$

from [Vog79, § 3].

By the Poincaré–Birkhoff–Witt theorem,

$$U(\mathfrak{g}) = U(\mathfrak{g}') \oplus (\mathfrak{n} U(\mathfrak{g}) + U(\mathfrak{g})\bar{\mathfrak{n}}).$$

It is known that

$$U(\mathfrak{g})^K \subset U(\mathfrak{g}')^{K'} \oplus (\mathfrak{n} U(\mathfrak{g}) \cap U(\mathfrak{g})\bar{\mathfrak{n}}).$$

Denote the projection to the first factor by

$$\tilde{\xi} : U(\mathfrak{g})^K \rightarrow U(\mathfrak{g}')^{K'}.$$

It is also known that $\tilde{\xi}$ is an algebra homomorphism. Let

$$\eta_{\mathfrak{q}} : U(\mathfrak{g}') \rightarrow U(\mathfrak{g}')$$

be the algebra homomorphism such that

$$\eta_{\mathfrak{q}}(X) = X + \delta_{\mathfrak{n}}(X) \tag{16}$$

for all $X \in \mathfrak{g}'$. Note that

$$\delta_{\mathfrak{n}}(X) = \text{Tr}(\text{ad}(X)|_{\mathfrak{n}}),$$

which is twice the shift in the definition of Harish-Chandra homomorphism. It is easy to see that $\eta_{\mathfrak{q}}$ maps $U(\mathfrak{g}')^{K'}$ onto itself. The map ξ is now defined by

$$\xi = \eta_{\mathfrak{q}} \circ \tilde{\xi} : U(\mathfrak{g})^K \rightarrow U(\mathfrak{g}')^{K'}.$$

Recall that there is an anti-automorphism $\vee : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ which maps X to $-X$ for all $X \in \mathfrak{g}$. It preserves both $U(\mathfrak{g})^K$ and $U(\mathfrak{g}')^{K'}$. If A, B are two subalgebras of $U(\mathfrak{g})$ which are stable under \vee , and if $\eta : A \rightarrow B$ is any algebra homomorphism, then denote by $\eta^{\vee} : A \rightarrow B$ the algebra homomorphism such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & B \\ \vee \downarrow & & \downarrow \vee \\ A & \xrightarrow{\eta^{\vee}} & B \end{array}$$

commutes.

Let $\mathbf{V}, \tau, \mathbf{V}'$ and τ' be as in §3.1. Recall that we have an action T of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ on $C^{\infty}(G_{\mathbb{C}}/U; \mathbf{V})$ and $C^{\infty}(G; \mathbf{V})$. Denote by T' the analogous action of $U(\mathfrak{g}') \otimes U(\mathfrak{g}')$ on $C^{\infty}(G'_{\mathbb{C}}/U'; \mathbf{V}')$ and $C^{\infty}(G'; \mathbf{V}')$. Recall that

$$T_{U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K} \text{ stabilizes } C^{\infty}(G_{\mathbb{C}}/U; \tau),$$

and, similarly,

$$T'_{U(\mathfrak{g}')^{K'} \otimes U(\mathfrak{g}')^{K'}} \text{ stabilizes } C^{\infty}(G'_{\mathbb{C}}/U'; \tau').$$

The main result of this section is the following theorem.

THEOREM 4.1. *For all $X \otimes Y \in U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K$ and $\phi \in C^{\infty}(G'_{\mathbb{C}}/U'; \tau')$,*

$$T_{X \otimes Y} E_{\mathfrak{q}}(\phi) = E_{\mathfrak{q}}(T'_{\xi^{\vee}(X) \otimes \xi(Y)} \phi).$$

4.2 Proof of Theorem 4.1

Note that we also have

$$U(\mathfrak{g})^K \subset U(\mathfrak{g}')^{K'} \oplus (\bar{\mathfrak{n}} U(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{n}).$$

The projection to the first factor is just the algebra homomorphism

$$\tilde{\xi}^{\vee} : U(\mathfrak{g})^K \rightarrow U(\mathfrak{g}')^{K'}.$$

LEMMA 4.2. Let $X \otimes Y \in U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K$ and $f \in C^\infty(G_{\mathbb{C}}/U; \mathbf{V})$. If f is N -invariant, then

$$T_{X \otimes Y} f = T_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} f.$$

Proof. Let $X = X' + X''$, $Y = Y' + Y''$, where $X' = \tilde{\xi}^\vee(X)$, $X'' \in U(\mathfrak{g})\mathfrak{n}$, $Y' = \tilde{\xi}(Y)$, $Y'' \in U(\mathfrak{g})\bar{\mathfrak{n}}$. Note that under the complexification

$$\begin{aligned} 1 \times \bar{\theta} : \mathfrak{g} &\rightarrow \mathfrak{g} \times \mathfrak{g}, \\ v &\mapsto (v, \bar{\theta}(v)), \end{aligned}$$

the Lie algebra of N has a complexification $\mathfrak{n} \times \bar{\mathfrak{n}}$. Therefore, the N invariance of f implies $T_{\mathfrak{n} \times \bar{\mathfrak{n}}} f = 0$, i.e. $T_{\mathfrak{n} \otimes 1 + 1 \otimes \bar{\mathfrak{n}}} f = 0$. So we have

$$T_{X \otimes Y} f = T_{X' \otimes Y'} f = T_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} f. \quad \square$$

LEMMA 4.3. Let $X \otimes Y \in U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K$ and $\phi \in C^\infty(G'_{\mathbb{C}}/U'; \mathbf{V}')$. Then

$$T'_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} (\delta_{\mathfrak{n}}^+ \phi) = \delta_{\mathfrak{n}}^+ T'_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} (\phi).$$

Proof. Let

$$\eta_{\bar{\mathfrak{q}}} : U(\mathfrak{g}') \rightarrow U(\mathfrak{g}')$$

be the algebra homomorphism such that $\eta_{\bar{\mathfrak{q}}}(X) = X + \delta_{\bar{\mathfrak{n}}}(X)$ for all $X \in \mathfrak{g}'$. It is easy to see that $\eta_{\bar{\mathfrak{q}}} = \eta_{\mathfrak{q}}^\vee$ and, hence,

$$\xi^\vee = (\eta_{\bar{\mathfrak{q}}} \circ \tilde{\xi})^\vee = \eta_{\bar{\mathfrak{q}}} \circ \tilde{\xi}^\vee.$$

Define an algebra homomorphism

$$\begin{aligned} \eta_0 : U(\mathfrak{g}' \times \mathfrak{g}') = U(\mathfrak{g}') \otimes U(\mathfrak{g}') &\rightarrow U(\mathfrak{g}' \times \mathfrak{g}') = U(\mathfrak{g}') \otimes U(\mathfrak{g}'), \\ X' \otimes Y' &\mapsto \eta_{\bar{\mathfrak{q}}}(X') \otimes \eta_{\mathfrak{q}}(Y'). \end{aligned}$$

We only need to show that

$$T'_Z (\delta_{\mathfrak{n}}^+ \phi) = \delta_{\mathfrak{n}}^+ T'_{\eta_0(Z)} (\phi)$$

for all $Z \in U(\mathfrak{g}' \times \mathfrak{g}')$. As $U(\mathfrak{g}' \times \mathfrak{g}')$ is generated by $\mathfrak{g}' \times \mathfrak{g}'$, it is sufficient to show that the above equality holds for $Z \in \mathfrak{g}' \times \mathfrak{g}'$. Now assume $Z = (X', Y') \in \mathfrak{g}' \times \mathfrak{g}'$. Then T'_Z is an action defined by a vector field on $G'_{\mathbb{C}}/U'$. Hence,

$$T'_Z (\delta_{\mathfrak{n}}^+ \phi) = T'_Z (\delta_{\mathfrak{n}}^+) \phi + \delta_{\mathfrak{n}}^+ T'_Z (\phi).$$

We easily find that

$$T'_Z (\delta_{\mathfrak{n}}^+) = (\delta_{\bar{\mathfrak{n}}}(X') + \delta_{\mathfrak{n}}(Y')) \delta_{\mathfrak{n}}^+.$$

Therefore,

$$\begin{aligned} T'_Z (\delta_{\mathfrak{n}}^+ \phi) &= T'_Z (\delta_{\mathfrak{n}}^+) \phi + \delta_{\mathfrak{n}}^+ T'_Z (\phi) \\ &= \delta_{\mathfrak{n}}^+ ((\delta_{\bar{\mathfrak{n}}}(X') + \delta_{\mathfrak{n}}(Y')) \phi + T'_Z \phi) \\ &= \delta_{\mathfrak{n}}^+ T'_{\eta_0(Z)} (\phi). \end{aligned} \quad \square$$

Proof of Theorem 4.1. Denote by $T_k : G_{\mathbb{C}}/U \rightarrow G_{\mathbb{C}}/U$ the translation by k for all $k \in K$. Let $\epsilon_{\mathbf{V}}$ be the embedding of \mathbf{V}' into \mathbf{V} . Write

$$f = \epsilon_{\mathbf{V}} \circ (\delta_{\mathfrak{n}}^+ \phi) \circ H_{\mathfrak{q}} \in C^\infty(G_{\mathbb{C}}/U; \mathbf{V}).$$

Then

$$E_{\mathfrak{q}}(\phi) = \int_K \tau|_K(k^{-1}) \circ f \circ T_k dk.$$

Note that

$$T_{X \otimes Y} : C^\infty(G_{\mathbb{C}}/U; \mathbf{V}) \rightarrow C^\infty(G_{\mathbb{C}}/U; \mathbf{V})$$

is a differential operator. Therefore, we have

$$\begin{aligned} T_{X \otimes Y} E_q(\phi) &= \int_K T_{X \otimes Y}(\tau|_K(k^{-1}) \circ f \circ T_k) dk \\ &= \int_K \tau|_K(k^{-1}) \circ T_{X \otimes Y}(f \circ T_k) dk. \end{aligned}$$

Since $X \otimes Y \in U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K$, we have

$$T_{X \otimes Y}(f \circ T_k) = T_{X \otimes Y}(f) \circ T_k.$$

Hence,

$$T_{X \otimes Y} E_q(\phi) = \int_K \tau|_K(k^{-1}) \circ T_{X \otimes Y}(f) \circ T_k dk.$$

Note that the map

$$\begin{aligned} C^\infty(G'_\mathbb{C}/U'; \mathbf{V}') &\rightarrow C^\infty(G_\mathbb{C}/U; \mathbf{V}), \\ \psi &\mapsto \epsilon_{\mathbf{V}} \circ \psi \circ H_q \end{aligned}$$

is $G'_\mathbb{C}$ -equivariant. Therefore,

$$T_{X' \otimes Y'}(\epsilon_{\mathbf{V}} \circ \psi \circ H_q) = \epsilon_{\mathbf{V}} \circ T'_{X' \otimes Y'} \psi \circ H_q \tag{17}$$

for all $X' \otimes Y' \in U(\mathfrak{g}') \otimes U(\mathfrak{g}')$, $\psi \in C^\infty(G'_\mathbb{C}/U'; \mathbf{V}')$.

Now we have

$$\begin{aligned} T_{X \otimes Y}(f) &= T_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} f \quad \text{by Lemma 4.2} \\ &= T_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)}(\epsilon_{\mathbf{V}} \circ (\delta_n^+ \phi) \circ H_q) \\ &= \epsilon_{\mathbf{V}} \circ (T'_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)}(\delta_n^+ \phi)) \circ H_q \quad \text{by (17)} \\ &= \epsilon_{\mathbf{V}} \circ (\delta_n^+ T'_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} \phi) \circ H_q \quad \text{by Lemma 4.3.} \end{aligned}$$

In conclusion,

$$\begin{aligned} T_{X \otimes Y} E_q(\phi) &= \int_K \tau|_K(k^{-1}) \circ T_{X \otimes Y}(f) \circ T_k dk \\ &= \int_K \tau|_K(k^{-1}) \circ \epsilon_{\mathbf{V}} \circ (\delta_n^+ T'_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} \phi) \circ H_q \circ T_k dk \\ &= E_q(T'_{\tilde{\xi}^\vee(X) \otimes \tilde{\xi}(Y)} \phi). \end{aligned} \quad \square$$

5. Cohomological inductions and bottom layers

We call a vector space a weak $(U(\mathfrak{g})^K, K)$ -module if it is a locally finite K -module as well as a $U(\mathfrak{g})^K$ -module such that the K -action and the $U(\mathfrak{g})^K$ -action commute. Note that the α -isotypic component $M(\alpha)$ is a weak $(U(\mathfrak{g})^K, K)$ -module for every (\mathfrak{g}, K) -module M and every K -type α . In this section, we determine the weak $(U(\mathfrak{g})^K, K)$ -module structure of a bottom layer of a cohomologically induced representation.

5.1 Zuckerman functors

Let us recall cohomological inductions and bottom layer maps from [KV95]. We continue to use the notation of § 3.1.

Let Γ^0 be the Zuckerman functor from the category of (\mathfrak{g}, K') -modules to the category of (\mathfrak{g}, K) -modules. It is given by

$$\Gamma^0(M) = \text{sum of all finite-dimensional } \mathfrak{k} \text{ invariant subspaces of } M$$

for which the actions of \mathfrak{k} globalize to K .

This is a left exact covariant functor. Denote by Γ^i its i th right derived functor, $i = 0, 1, 2, \dots$. Let Γ_K^0 be the Zuckerman functor from the category of (\mathfrak{k}, K') -modules to the category of (\mathfrak{k}, K) -modules. Denote by Γ_K^i its i th right derived functor, $i = 0, 1, 2, \dots$.

Let $\text{pro}_{\mathfrak{q}, K'}^{\mathfrak{g}, K'}$ be the exact covariant functor from the category of (\mathfrak{q}, K') -modules to the category of (\mathfrak{g}, K') -modules given by

$$\text{pro}_{\mathfrak{q}, K'}^{\mathfrak{g}, K'}(M) = \text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), M)_{K'},$$

where the $U(\mathfrak{g})$ -action on $\text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), M)$ is given by

$$(X\phi)(u) = \phi(uX), \quad X, u \in U(\mathfrak{g}), \phi \in \text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), M),$$

the K' -action on $\text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), M)$ is given by

$$(k\phi)(u) = k(\phi(\text{Ad}_{k^{-1}} u)), \quad k \in K', u \in U(\mathfrak{g}), \phi \in \text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), M),$$

and

$$\text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), M)_{K'} \quad \text{is the } K' \text{-finite vectors of } \text{Hom}_{U(\mathfrak{g}')} (U(\mathfrak{g}), M).$$

Let $\text{pro}_{\mathfrak{q} \cap \mathfrak{k}, K'}^{\mathfrak{k}, K'}$ be the analogous exact covariant functor from the category of $(\mathfrak{q} \cap \mathfrak{k}, K')$ -modules to the category of (\mathfrak{k}, K') -modules. Let \mathcal{F} denote the forgetful functors in various contexts. For example, $\mathcal{F}_{\mathfrak{g}', K'}^{\mathfrak{g}', K'}$ is the forgetful functor from the category of (\mathfrak{g}', K') -modules to the category of (\mathfrak{q}, K') -modules via trivial \mathfrak{n} -action.

Let M' be a (\mathfrak{g}', K') -module and define another (\mathfrak{g}', K') -module

$$M'^{\#} = M' \otimes \wedge^{\text{top}} \mathfrak{n}.$$

Define

$$\mathcal{R}^i(M') = (\Gamma^i \circ \text{pro}_{\mathfrak{q}, K'}^{\mathfrak{g}, K'} \circ \mathcal{F}_{\mathfrak{g}', K'}^{\mathfrak{q}, K'})(M'^{\#}).$$

Then \mathcal{R}^i is the cohomological induction functor from the category of (\mathfrak{g}', K') -modules to the category of (\mathfrak{g}, K) -modules.

Define

$$\mathcal{R}_\circ^i(M') = (\Gamma_K^i \circ \mathcal{F}_{\mathfrak{g}, K'}^{\mathfrak{k}, K'} \circ \text{pro}_{\mathfrak{q}, K'}^{\mathfrak{g}, K'} \circ \mathcal{F}_{\mathfrak{g}', K'}^{\mathfrak{q}, K'})(M'^{\#}).$$

Note that

$$U(\mathfrak{g})^K \quad \text{acts on } (\mathcal{F}_{\mathfrak{g}, K'}^{\mathfrak{k}, K'} \circ \text{pro}_{\mathfrak{q}, K'}^{\mathfrak{g}, K'} \circ \mathcal{F}_{\mathfrak{g}', K'}^{\mathfrak{q}, K'})(M'^{\#})$$

by (\mathfrak{k}, K') -module endomorphisms. By using the functor Γ_K^i , we get a $U(\mathfrak{g})^K$ -action on $\mathcal{R}_\circ^i(M')$ by (\mathfrak{k}, K) -module endomorphisms.

Note that

$$\mathfrak{k}' = \text{Lie}(K') \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}' \cap \mathfrak{k} \quad \text{and} \quad \mathfrak{q} \cap \mathfrak{k} = \mathfrak{k}' \oplus (\mathfrak{n} \cap \mathfrak{k}).$$

Define

$$\mathcal{R}_K^i(M') = (\Gamma_K^i \circ \text{pro}_{\mathfrak{q} \cap \mathfrak{k}, K'}^{\mathfrak{k}, K'} \circ \mathcal{F}_{\mathfrak{g}', K'}^{\mathfrak{q} \cap \mathfrak{k}, K'})(M'^{\#}),$$

where

$$\mathcal{F}_{\mathfrak{g}', K'}^{\mathfrak{q} \cap \mathfrak{k}, K'} = \mathcal{F}_{\mathfrak{k}', K'}^{\mathfrak{q} \cap \mathfrak{k}, K'} \circ \mathcal{F}_{\mathfrak{g}', K'}^{\mathfrak{k}', K'},$$

and $\mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q} \cap \mathfrak{k},K'}$ is defined via the trivial $\mathfrak{n} \cap \mathfrak{k}$ -action. We have a $U(\mathfrak{g}')^{K'}$ -action on $\mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q} \cap \mathfrak{k},K'}(M'\#)$ by $(\mathfrak{q} \cap \mathfrak{k}, K')$ -module endomorphisms. By applying the functor $\text{pro}_{\mathfrak{q} \cap \mathfrak{k},K'}^{\mathfrak{k},K'}$, we get a $U(\mathfrak{g}')^{K'}$ -action on $(\text{pro}_{\mathfrak{q} \cap \mathfrak{k},K'}^{\mathfrak{k},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q} \cap \mathfrak{k},K'})(M'\#)$ by (\mathfrak{k}, K') -module endomorphisms. Then, by applying $\Gamma_{K'}^i$, we get a $U(\mathfrak{g}')^{K'}$ -action on $\mathcal{R}_{K'}^i(M')$ by (\mathfrak{k}, K') -module endomorphisms.

LEMMA 5.1. *For any (\mathfrak{g}', K') -module M' , $\mathcal{R}^i(M')$ is canonically isomorphic to $\mathcal{R}_\circ^i(M')$ as a weak $(U(\mathfrak{g})^K, K)$ -module.*

Proof. Write $M'' = (\text{pro}_{\mathfrak{q},K'}^{\mathfrak{g},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q},K'})(M'\#)$ and let

$$0 \rightarrow M'' \rightarrow J_0 \rightarrow J_1 \rightarrow J_2 \rightarrow \dots$$

be an injective resolution of M'' in the category of (\mathfrak{g}, K') -modules. By [KV95, Proposition 2.57], the exact functor $\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}$ sends injectives to injectives. Therefore,

$$0 \rightarrow \mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(M'') \rightarrow \mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(J_0) \rightarrow \mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(J_1) \rightarrow \mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(J_2) \rightarrow \dots$$

is an injective resolution of $\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(M'')$ in the category of (\mathfrak{k}, K') -modules.

Note that

$$0 \rightarrow \Gamma^0(J_0) \rightarrow \Gamma^0(J_1) \rightarrow \Gamma^0(J_2) \rightarrow \dots$$

and

$$0 \rightarrow \Gamma_K^0(\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(J_0)) \rightarrow \Gamma_K^0(\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(J_1)) \rightarrow \Gamma_K^0(\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'}(J_2)) \rightarrow \dots$$

are exactly the same as sequences of weak $(U(\mathfrak{g})^K, K)$ -modules. Taking the i th cohomology of both sequences, we get $\mathcal{R}^i(M') = \mathcal{R}_\circ^i(M')$ as a weak $(U(\mathfrak{g})^K, K)$ -module. \square

5.2 Bottom layer maps

Let M' be a (\mathfrak{g}', K') -module. Define a (\mathfrak{k}, K') -module homomorphism

$$\beta_{M'} : (\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'} \circ \text{pro}_{\mathfrak{q},K'}^{\mathfrak{g},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q},K'})(M'\#) \rightarrow (\text{pro}_{\mathfrak{q} \cap \mathfrak{k},K'}^{\mathfrak{k},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q} \cap \mathfrak{k},K'})(M'\#)$$

by

$$(\beta_{M'}(\phi))(r) = \phi(r), \quad \phi \in (\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'} \circ \text{pro}_{\mathfrak{q},K'}^{\mathfrak{g},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q},K'})(M'\#), \quad r \in U(\mathfrak{k}).$$

LEMMA 5.2. *For all $X \in U(\mathfrak{g})^K$, the following diagram commutes.*

$$\begin{CD} (\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'} \circ \text{pro}_{\mathfrak{q},K'}^{\mathfrak{g},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q},K'})(M'\#) @>\beta_{M'}>> (\text{pro}_{\mathfrak{q} \cap \mathfrak{k},K'}^{\mathfrak{k},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q} \cap \mathfrak{k},K'})(M'\#) \\ @VXVV @VV\tilde{\xi}(X)V \\ (\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'} \circ \text{pro}_{\mathfrak{q},K'}^{\mathfrak{g},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q},K'})(M'\#) @>\beta_{M'}>> (\text{pro}_{\mathfrak{q} \cap \mathfrak{k},K'}^{\mathfrak{k},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q} \cap \mathfrak{k},K'})(M'\#) \end{CD}$$

Proof. Let

$$X = aX_1 + X_2,$$

where $a \in \mathfrak{n}$, $X_1 \in U(\mathfrak{g})$, and

$$X_2 = \tilde{\xi}(X) \in U(\mathfrak{g}')^{K'}.$$

Let

$$\phi \in (\mathcal{F}_{\mathfrak{g},K'}^{\mathfrak{k},K'} \circ \text{pro}_{\mathfrak{q},K'}^{\mathfrak{g},K'} \circ \mathcal{F}_{\mathfrak{g}',K'}^{\mathfrak{q},K'})(M'\#) \subset \text{Hom}_{U(\mathfrak{q})}(U(\mathfrak{g}), M'\#),$$

and $r \in U(\mathfrak{k})$. We have

$$\begin{aligned} (\beta_{M'}(X\phi))(r) &= (X\phi)(r) = \phi(rX) = \phi(Xr) \\ &= \phi(aX_1r) + \phi(X_2r) \\ &= a(\phi(X_1)) + X_2(\phi(r)) \\ &= X_2(\beta_{M'}(\phi)(r)) \\ &= \tilde{\xi}(X)(\beta_{M'}(\phi))(r). \end{aligned}$$

Therefore,

$$\beta_{M'}(X\phi) = \tilde{\xi}(X)(\beta_{M'}(\phi)). \quad \square$$

By applying the functor Γ_K^i to $\beta_{M'}$, we define the bottom layer map

$$\mathcal{B}_{M'} = \Gamma_K^i(\beta_{M'}) : \mathcal{R}_\circ^i(M') \rightarrow \mathcal{R}_K^i(M')$$

which is a (\mathfrak{k}, K) -module homomorphism. Applying Γ_K^i to the commutative diagram of the above lemma, we obtain the following.

PROPOSITION 5.3. *For all $X \in U(\mathfrak{g})^K$, the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{R}_\circ^i(M') & \xrightarrow{\mathcal{B}_{M'}} & \mathcal{R}_K^i(M') \\ X \downarrow & & \tilde{\xi}(X) \downarrow \\ \mathcal{R}_\circ^i(M') & \xrightarrow{\mathcal{B}_{M'}} & \mathcal{R}_K^i(M') \end{array}$$

5.3 Bottom layers

From now on we fix a K -type α . We also fix an irreducible representation σ_0 of K of type α on a vector space W_α . Let σ'_0 be the representation of K' on $W_{\alpha'}$ which is \mathfrak{q} -associated to σ_0 (Definition 3.1). The representation σ'_0 is also irreducible by the highest weight theory. We call the K' -type α' of σ'_0 the K' -type \mathfrak{q} -associated to α .

Set $S = \dim(\mathfrak{n} \cap \mathfrak{k})$ as usual. The most interesting case of cohomological induction is that when $i = S$. Write

$$M = \mathcal{R}^S(M') \quad \text{and} \quad L = \text{Hom}_{K'}(W_{\alpha'}, M').$$

Then L is a $U(\mathfrak{g}')^{K'}$ -module by the action on M' . We make L a $U(\mathfrak{g})^K$ -module by the formula

$$Xv = \xi(X)v, \quad X \in U(\mathfrak{g})^K, v \in L. \quad (18)$$

Now $L \otimes W_\alpha$ is a weak $(U(\mathfrak{g})^K, K)$ -module by the action of $U(\mathfrak{g})^K$ on the first factor, and the action of K on the second factor.

DEFINITION 5.4. *Let M' be a (\mathfrak{g}', K') -module, $M = \mathcal{R}^S(M')$. The K -type α is said to be in the bottom layer of M if the map induced by $\mathcal{B}_{M'}$,*

$$\mathcal{B}_{M'}(\alpha) : M(\alpha) = \mathcal{R}_\circ^S(M')(\alpha) \rightarrow \mathcal{R}_K^S(M')(\alpha)$$

is bijective and nonzero.

We know that under a weak condition on M' , α is in the bottom layer of M if and only if $M'(\alpha') \neq 0$. The condition holds when M' has an infinitesimal character [KV95, Theorem 5.80, Corollary 5.72]. The main result of this section is the following theorem.

THEOREM 5.5. *With the notation as above, if α is in the bottom layer of M , then $M(\alpha)$ is isomorphic to $L \otimes W_\alpha$ as a weak $(U(\mathfrak{g})^K, K)$ -module.*

Proof. We easily check that the diagram

$$\begin{array}{ccc} M'^{\#} & \xlongequal{\quad} & M' \otimes \wedge^{\text{top}} \mathfrak{n} \\ X \downarrow & & \eta_{\mathfrak{q}}(X) \otimes 1 \downarrow \\ M'^{\#} & \xlongequal{\quad} & M' \otimes \wedge^{\text{top}} \mathfrak{n} \end{array}$$

commutes for every $X \in U(\mathfrak{g}')$, where $\eta_{\mathfrak{q}}$ is defined in (16). This implies the diagram

$$\begin{array}{ccc} \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M'^{\#}) & \xlongequal{\quad} & \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M' \otimes \wedge^{\text{top}} \mathfrak{n}) \\ X \downarrow & & \eta_{\mathfrak{q}}(X) \otimes 1 \downarrow \\ \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M'^{\#}) & \xlongequal{\quad} & \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M' \otimes \wedge^{\text{top}} \mathfrak{n}) \end{array}$$

commutes for every $X \in U(\mathfrak{g}')^{K'}$. It is obvious that the diagram

$$\begin{array}{ccc} \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M' \otimes \wedge^{\text{top}} \mathfrak{n}) & \xlongequal{\quad} & \text{Hom}_{K'}(W_{\alpha'}, M') \\ \eta_{\mathfrak{q}}(X) \otimes 1 \downarrow & & \eta_{\mathfrak{q}}(X) \downarrow \\ \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M' \otimes \wedge^{\text{top}} \mathfrak{n}) & \xlongequal{\quad} & \text{Hom}_{K'}(W_{\alpha'}, M') \end{array}$$

commutes for every $X \in U(\mathfrak{g}')^{K'}$. By combining these two diagrams, and replacing X by $\tilde{\xi}(X)$, we get a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M'^{\#}) & \xlongequal{\quad} & \text{Hom}_{K'}(W_{\alpha'}, M') \\ \tilde{\xi}(X) \downarrow & & \xi(X) \downarrow \\ \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M'^{\#}) & \xlongequal{\quad} & \text{Hom}_{K'}(W_{\alpha'}, M') \end{array}$$

for every $X \in U(\mathfrak{g})^K$.

Now (11) implies

$$H^0(\mathfrak{n} \cap \mathfrak{k}, W_{\alpha}) \otimes \wedge^{\text{top}}(\mathfrak{n} \cap \mathfrak{k}) \cong W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n} \tag{19}$$

as a K' -module. Theorem 4.155 of [KV95] states that there is a natural isomorphism

$$\text{Hom}_K(W_{\alpha}, \mathcal{R}_K^S(M')) \cong \text{Hom}_{K'}(H^0(\mathfrak{n} \cap \mathfrak{k}, W_{\alpha}) \otimes \wedge^{\text{top}}(\mathfrak{n} \cap \mathfrak{k}), M'^{\#}). \tag{20}$$

The isomorphism (19) and the naturalness of (20) imply that we actually have a $U(\mathfrak{g}')^{K'}$ -module isomorphism

$$\text{Hom}_K(W_{\alpha}, \mathcal{R}_K^S(M')) \cong \text{Hom}_{K'}(W_{\alpha'} \otimes \wedge^{\text{top}} \mathfrak{n}, M'^{\#}). \tag{21}$$

Lemma 5.1 and Proposition 5.3 give a commutative diagram

$$\begin{array}{ccc} \text{Hom}_K(W_{\alpha}, \mathcal{R}^S(M')) & \longrightarrow & \text{Hom}_K(W_{\alpha}, \mathcal{R}_K^S(M')) \\ X \downarrow & & \tilde{\xi}(X) \downarrow \\ \text{Hom}_K(W_{\alpha}, \mathcal{R}^S(M')) & \longrightarrow & \text{Hom}_K(W_{\alpha}, \mathcal{R}_K^S(M')) \end{array}$$

for every $X \in U(\mathfrak{g})^K$, where each row is the map induced by $\mathcal{B}_{M'}$, which is an isomorphism as α is in the bottom layer. We put the last two commutative diagrams and the isomorphism (21) together, and obtain a commutative diagram

$$\begin{array}{ccc} \text{Hom}_K(W_{\alpha}, \mathcal{R}^S(M')) & \longrightarrow & L = \text{Hom}_{K'}(W_{\alpha'}, M') \\ X \downarrow & & \xi(X) \downarrow \\ \text{Hom}_K(W_{\alpha}, \mathcal{R}^S(M')) & \longrightarrow & L = \text{Hom}_{K'}(W_{\alpha'}, M') \end{array}$$

for every $X \in U(\mathfrak{g})^K$, with bijective horizontal maps. This finishes the proof. \square

6. Matrix coefficients of cohomologically induced representations

In this section, we use the results of the previous sections to find integral representations of matrix coefficients of bottom layers of cohomologically induced representations.

6.1 Matrix coefficient of a K -type

Let α be a K -type as before. Assume that M is a general finitely generated admissible (\mathfrak{g}, K) -module, with the actions π of $U(\mathfrak{g})$ and K on it. Write $W = M(\alpha)$. Then W is a finite-dimensional weak $(U(\mathfrak{g})^K, K)$ -module. We easily check that the function $\phi_{M,\alpha} \in C^\omega(G; \text{End}_{\mathbb{C}}(W))$ defined in the introduction satisfies

$$\begin{cases} \text{(i)} \ \phi_{M,\alpha}(1) = \text{identity map of } W; \\ \text{(ii)} \ \phi_{M,\alpha}(kxl) = \pi(k) \circ \phi_{M,\alpha}(x) \circ \pi(l), \quad x \in G, k, l \in K; \\ \text{(iii)} \ T_{X \otimes Y} \phi_{M,\alpha}(x) = \pi(X^\vee) \circ \phi_{M,\alpha}(x) \circ \pi(Y), \quad x \in G, X \otimes Y \in U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K. \end{cases} \tag{22}$$

These conditions are generalizations of those of (1) for elementary spherical functions.

PROPOSITION 6.1. *Let W be a weak $(U(\mathfrak{g})^K, K)$ -module which is isomorphic to $M(\alpha)$ for some finitely generated admissible (\mathfrak{g}, K) -module M . Then there is a unique function in $C^\omega(G; \text{End}_{\mathbb{C}}(W))$ which satisfies the three conditions of (22).*

The existence is clear. We omit the standard proof of uniqueness (as indicated in the proof of [Fle80a, Lemma 7.6]). Write ϕ_W for this unique function. Of course,

$$\phi_{M(\alpha)} = \phi_{M,\alpha},$$

and the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi_W} & \text{End}_{\mathbb{C}}(W) \\ & \searrow \phi_{M(\alpha)} & \downarrow \\ & & \text{End}_{\mathbb{C}}(M(\alpha)) \end{array}$$

commutes, where the vertical arrow is the linear isomorphism which is induced by an arbitrary $(U(\mathfrak{g})^K, K)$ -isomorphism of W and $M(\alpha)$. Note that if M is irreducible, then $M(\alpha)$ is irreducible as a weak $(U(\mathfrak{g})^K, K)$ -module, and the vertical arrow is uniquely determined.

Write $\psi_W, \psi_{M,\alpha} \in C^\omega(G_{\mathbb{C}}/U; \text{End}_{\mathbb{C}}(W))$ for the functions corresponding to ϕ_W and $\phi_{M,\alpha}$, respectively, in Theorem 2.1. The function ψ_W is characterized by

$$\begin{cases} \text{(i)} \ \psi_W(1U) = \text{identity map of } W; \\ \text{(ii)} \ \psi_W(kx) = \sigma(k) \circ \phi_{M,\alpha}(x) \circ \sigma(\bar{k}^{-1}), \quad x \in G_{\mathbb{C}}/U, k \in K_{\mathbb{C}}; \\ \text{(iii)} \ T_{X \otimes Y} \psi_W(x) = \sigma(X^\vee) \circ \psi_W(x) \circ \sigma(Y), \quad x \in G_{\mathbb{C}}/U, X \otimes Y \in U(\mathfrak{g})^K \otimes U(\mathfrak{g})^K; \end{cases} \tag{23}$$

where we write σ for the actions of $U(\mathfrak{g})^K$ and $K_{\mathbb{C}}$ on W . Similarly, we have

$$\psi_{M(\alpha)} = \psi_{M,\alpha},$$

and the diagram

$$\begin{array}{ccc} G_{\mathbb{C}}/U & \xrightarrow{\psi_W} & \text{End}_{\mathbb{C}}(W) \\ & \searrow \psi_{M(\alpha)} & \downarrow \\ & & \text{End}_{\mathbb{C}}(M(\alpha)) \end{array}$$

commutes.

6.2 Matrix coefficient of a bottom layer

Let us return to the situation of § 5.3. Assume that M' is admissible, then $M = \mathcal{R}^S(M')$ is also admissible. Assume that α is in the bottom layer of M . Write

$$W' = M'(\alpha') \cong L \otimes W_{\alpha'}$$

as weak $(U(\mathfrak{g}')^{K'}, K')$ -modules. Set

$$W = L \otimes W_{\alpha}$$

and view it as a weak $(U(\mathfrak{g})^K, K)$ -module. Theorem 5.5 states that $W \cong M(\alpha)$ as weak $(U(\mathfrak{g})^K, K)$ -modules. Denote by σ the actions of $U(\mathfrak{g})^K$ and K on W and by σ' the actions of $U(\mathfrak{g}')^{K'}$ and K' on W' .

Define a representation τ of $K \times K$ on $\text{End}_{\mathbb{C}}(W)$ by

$$\tau(k, l)(f) = \sigma(k) \circ f \circ \sigma(l^{-1}), \quad k, l \in K, f \in \text{End}_{\mathbb{C}}(W),$$

and a representation τ' of $K' \times K'$ on $\text{End}_{\mathbb{C}}(W')$ by

$$\tau'(k, l)(f) = \sigma'(k) \circ f \circ \sigma'(l^{-1}), \quad k, l \in K', f \in \text{End}_{\mathbb{C}}(W').$$

We have a decomposition

$$W = W' \oplus \sigma(\bar{\mathfrak{n}} \cap \mathfrak{k})W.$$

Therefore, we may view an element of $\text{End}_{\mathbb{C}}(W')$ as an element of $\text{End}_{\mathbb{C}}(W)$ which vanishes on $\sigma(\bar{\mathfrak{n}} \cap \mathfrak{k})W$. Denote by

$$j_{\mathfrak{q}} : \text{End}_{\mathbb{C}}(W') \rightarrow \text{End}_{\mathbb{C}}(W)$$

the corresponding embedding. A routine verification of Definition 3.1 gives the following lemma.

LEMMA 6.2. *An isomorphism is induced by $j_{\mathfrak{q}}$ from τ' to the representation which is $\mathfrak{q} \times \bar{\mathfrak{q}}$ -associated to τ .*

The second condition of (23) for W' states that

$$\psi_{W'} \in C^{\omega}(G'_{\mathbb{C}}/U'; \tau').$$

Therefore, the above lemma enables us to define the integral $E_{\mathfrak{q}}(j_{\mathfrak{q}} \circ \psi_{W'})$. Write

$$E = \frac{\text{deg}(\alpha)}{\text{deg}(\alpha')} E_{\mathfrak{q}}(j_{\mathfrak{q}} \circ \psi_{W'}).$$

An easy calculation shows that the function E satisfies the first condition of (23). Theorem 3.3 states that E satisfies the second condition of (23). It is a consequence of Theorem 4.1 and the $U(\mathfrak{g})^K$ -action on W of (18) that E satisfies the third condition of (23). We leave the routine details to the reader. We finally get the main theorem of this paper by Proposition 6.1, as follows.

THEOREM 6.3. *We have*

$$\psi_W = \frac{\text{deg}(\alpha)}{\text{deg}(\alpha')} E_{\mathfrak{q}}(j_{\mathfrak{q}} \circ \psi_{W'}). \tag{24}$$

6.3 Two examples

The first example we discuss is the case when $\dim(M') = 1$, i.e. $A_{\mathfrak{q}}(\lambda)$ modules. Let

$$\lambda : G' \rightarrow \mathbb{C}^{\times}$$

be a continuous character, unitary or not. Still denote by λ its holomorphic extension to $G'_{\mathbb{C}}$ and its differential. Recall that $A_{\mathfrak{q}}(\lambda)$ is just the cohomologically induced module $\mathcal{R}^S(\lambda)$.

Fix a maximal torus T_{\max} of K' with complexified Lie algebra \mathfrak{t} . Then T_{\max} is also a maximal torus of K . Let

$$\Lambda = \lambda|_{\mathfrak{t}} + \delta_{\mathfrak{n} \cap \mathfrak{p}}|_{\mathfrak{t}}.$$

Assume that Λ is dominant with respect to $\mathfrak{n} \cap \mathfrak{k}$, and α is the K -type with an extremal weight Λ . Then $\lambda|_{K'}$ is the K' -type \mathfrak{q} -associated to α , α is the unique K -type which is in the bottom layer of $A_{\mathfrak{q}}(\lambda)$, and α has multiplicity 1 in $A_{\mathfrak{q}}(\lambda)$.

Let σ_0 be an irreducible unitary representation of K on W_{α} , of type α , as before. Fix a unit vector v_0 in W_{α} , of weight Λ . Let $P_{\Lambda} \in \text{End}_{\mathbb{C}}(W_{\alpha})$ be the orthogonal projection onto $\mathbb{C}v_0$. Define a map

$$\begin{aligned} \tilde{H}_{\mathfrak{q}} : G &\rightarrow G'_{\mathbb{C}}, \\ x &\mapsto y\overline{\theta(y^{-1})}, \end{aligned} \tag{25}$$

where $y \in G'_{\mathbb{C}}$ is an element so that $H_{\mathfrak{q}}(xU) = yU'$. Fix an identification of $A_{\mathfrak{q}}(\lambda)(\alpha)$ and W_{α} as K -modules. Now Theorem 6.3 easily implies

$$\phi_{A_{\mathfrak{q}}(\lambda), \alpha}(kx^2) = \text{deg}(\alpha) \int_K (\delta_{\mathfrak{n}}\lambda)(\tilde{H}_{\mathfrak{q}}(lx))\sigma_0(kl^{-1}) \circ P_{\Lambda} \circ \sigma_0(l) dl \tag{26}$$

for all $k \in K$, $x \in \exp(\mathfrak{p}_0)$. Write

$$\tilde{\phi}_{\mathfrak{q}, \lambda}(x) = \frac{1}{\text{deg}(\alpha)} \text{Tr}(\phi_{A_{\mathfrak{q}}(\lambda), \alpha}(x)), \quad x \in G.$$

It is a direct consequence of (26) that

$$\tilde{\phi}_{\mathfrak{q}, \lambda}(kx^2) = \int_K \langle \sigma_0(lkl^{-1})v_0, v_0 \rangle (\delta_{\mathfrak{n}}\lambda)(\tilde{H}_{\mathfrak{q}}(lx)) dl, \quad k \in K, x \in \exp(\mathfrak{p}_0).$$

An example of the above formula confirms Li's assumption in [Li97, Theorem 4.3], which enables us to get more information on Gross' conjecture on the discrete spectrum of $(G_2, PGS p_6)$.

The second example we will discuss is the lowest weight modules. Assume that \mathfrak{g} is simple, G is of Hermitian type and

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

where both \mathfrak{p}^+ and \mathfrak{p}^- are K invariant abelian subspaces of \mathfrak{p} , and

$$\mathfrak{p}^+ = \overline{\mathfrak{p}^-}.$$

Assume

$$\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+.$$

Fix an irreducible continuous unitary representation σ_0 of K on W_{α} , of type α , as before. Define a (\mathfrak{g}, K) -module

$$M = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{q}})} W_{\alpha},$$

where we view W_{α} as a $\bar{\mathfrak{q}}$ -module via the trivial \mathfrak{p}^- -action. The $U(\mathfrak{g})$ -action on M is defined via left multiplication, and the K -action on M is defined via the tensor product. It is clear that

$$M(\alpha) = W_{\alpha}.$$

Note that M is contragredient to a certain cohomologically induced module, with α the unique K -type in the bottom layer. By Theorem 6.3, or just check the three conditions of (23), we have

$$\phi_{M, \alpha}(kx^2) = \int_K \sigma_0(kl^{-1}\tilde{H}_{\mathfrak{q}}(lx)l) dl, \quad k \in K, x \in \exp(\mathfrak{p}_0),$$

where

$$\tilde{H}_{\mathfrak{q}} : G \rightarrow K_{\mathbb{C}} = G'_{\mathbb{C}}$$

is defined in (25). This formula looks prettier than that given by Li and Wallach in [Li90].

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