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Wave propagation for a class of non-local dispersal non-cooperative systems

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This paper is concerned with the travelling waves for a class of non-local dispersal non-cooperative system, which can model the prey-predator and disease-transmission mechanism. By the Schauder's fixed-point theorem, we first establish the existence of travelling waves connecting the semi-trivial equilibrium to non-trivial leftover concentrations, whose bounds are deduced from a precise analysis. Further, we characterize the minimal wave speed of travelling waves and obtain the non-existence of travelling waves with slow speed. Finally, we apply the general results to an epidemic model with bilinear incidence for its propagation dynamics.

Keywords: nonlocal dispersal; prey-predator models; minimal wave speed; traveling waves

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1. Introduction

As we know, random dispersal operators are often used to describe the diffusion process of organisms in population dynamics, which can only influence a species' immediate neighbourhood in ecological and epidemiological models, see Cantrell and Cosner [7] and Murray [32]. However, in many biological systems, the organisms can travel for a long distance and the transition probability from one location to another usually depends on the distance the organisms travelled. Consequently, non-local dispersal problems in which the diffusion process is described by an integral operator attract much attention in recent years, and have been used to model different dispersal phenomena in population ecology, material science, neurology and so on, one can see Andreu-Vaillo *et al.* [1], Bates *et al.* [3], Fife [17], Hutson *et al.* [23], Kao *et al.* [24], Li *et al.* [27, 31] for further understanding.

Travelling wave, as a special solution maintaining its shape and moving at a constant speed, is a very important dynamical issue in the field of reaction-diffusion equations. Nowadays there are many results regarding the non-local dispersal problems, mainly focussing on the monotone scalar equations (see, e.g., Chen [8], Coville *et al.* [**13**], Shen and Zhang [**34**], Sun *et al.* [**36**, **37**], Zhang *et al.* [**43**] and Zhang *et al.* [**44**, **46**]) and competition/cooperative systems (see, e.g., Bao *et al.* [**2**], Li *et al.* [**30**], Li *et al.* [**29**], Pan *et al.* [**33**], and Wang and Lv [**39**]). In addition, we refer the readers to Li and Yang [**26**], Li *et al.* [**28**], and Yang *et al.* [**41**, **42**] concerning the wave phenomena for some special SIR epidemic models with non-local dispersal. To our knowledge, the understanding of non-local dispersal models for prey-predator type is still very limited (see, e.g., Sherratt [**35**]). This motivates us to consider the following general non-cooperative system with non-local dispersal

$$\begin{cases} u_t = d_1(J_1 * u - u) + g(u) - f(u,v)v, \\ v_t = d_2(J_2 * v - v) + \alpha f(u,v)v - \delta v - \gamma v^2, \end{cases} \tag{1.1}$$

where $u = u(t,x)$ and $v = v(t,x)$ denote the densities of two populations at time t and location x, respectively; $g(u)$ is the growth rate of the prey; $f(u,v)v$ describes predation; the constant coefficients d_1, d_2, α and δ are all positive and γ is non-negative; $J_1 * u$ and $J_2 * v$ are standard spatial convolutions with the kernels $J_i(x)$ having the properties that

(J) $J_i \in C^1(\mathbb{R}), J_i(x) = J_i(-x) \geqslant 0, \int_{\mathbb{R}} J_i(x)\mathrm{d}x = 1$ and J_i satisfy the decay bounds:

$$\int_{\mathbb{R}} J_i(x)\mathrm{e}^{\lambda x}\mathrm{d}x < \infty \quad \text{for any } \lambda > 0 \quad \text{and} \quad \int_{\mathbb{R}} |J_i'(x)|\mathrm{d}x < \infty, \quad i = 1, 2.$$

Note that (1.1) is the non-local counterpart of the following diffusion-reaction system:

$$\begin{cases} u_t = d_1 u_{xx} + g(u) - f(u,v)v, \\ v_t = d_2 v_{xx} + \alpha f(u,v)v - \delta v - \gamma v^2, \end{cases} \tag{1.2}$$

which has been studied by Zhang *et al.* [**45**]. As mentioned in [**45**], following the different forms of $g(u)$, system (1.2) can describe the diffusive interaction of prey and predators (see Fu and Tsai [**18**]), the evolution of disease transmissions (see Britton [**6**]) or autocatalytic chemical reactions (Chen and Qi [**10**]) and so on. In view of Schauder's fixed-point theorem and the persistence theory proposed by Thieme [**38**], the authors obtained the existence of weak travelling wave solutions of (1.2). Also, the non-existence of travelling wave solutions was showed by the negative one-sided Laplace transform. These results can be applied to prey-predator systems and disease-transmission models with specific interaction functions, including Beddington-DeAngelis functional response (see, e.g., Ding and Huang [**14**], Huang [**20**, **21**], Huang *et al.* [**22**] and Li and Wu [**25**]).

The current paper is devoted to the existence and non-existence of travelling waves of system (1.1), which can also describe the propagation of predator's invasion or the spread of epidemic diseases. Usually, the monotonicity theories and shooting method are very useful to show the existence of travelling waves for systems. However, since the dynamical system generated by (1.1) is non-monotone, it follows that the powerful theory of monotone dynamical systems is not suitable to construct the travelling waves. At the same time, the shooting method is also not suitable for system (1.1) due to the effect of the convolution operator. In our recent works [**26**, **41**, **42**] dealing with some specific SIR models with non-local dispersal, we applied the method of constructing an invariant cone of initial functions defined in a large but bounded domain, then using a fixed point theorem on this cone, and further extending to the unbounded spatial domain $\mathbb{R}$ through a limiting argument, to show the existence of travelling waves. This method was firstly introduced by Berestycki *et al.* [**4**], and has been widely used to study the travelling wave solutions, see, e.g., Berestycki *et al.* [**5**], Ducrot and Magal [**15**] and Ducrot *et al.* [**16**].

In this paper, we will improve a little the previous method to establish the existence of travelling waves for the general system (1.1), which is different from those in Zhang *et al.* [**45**]. Since (1.1) is a non-monotonic and non-local system, it follows that the asymptotic behaviour of travelling waves is very difficult and challenging to estimate, especially the convergence to the positive constant equilibrium. From a biological point of view, the invasion of predators is successful if the travelling waves are persistent at the end. Thus, it is enough to study the so-called weak travelling wave solutions if we only want to know whether the invasion is successful and what the invasion speed is. Compared with the method used in Zhang *et al.* [**45**], the persistence theory suggested by Thieme [**38**] cannot be applied to discuss the persistence of travelling wave solutions for our non-local dispersal system (1.1) due to the deficiency of compactness. Certain ad hoc techniques that fit this non-local problem itself are necessarily needed. Inspired by Chen *et al.* [**11**] considering a lattice dynamical system, we will study the persistence of travelling waves by some detailed analysis strongly depending on the properties of the kernels and the wave equations associated to system (1.1), which leads to an important observation (see lemma 3.14). In addition, the asymptotic behaviour of travelling waves with critical speed at $-\infty$ cannot be obtained directly as it was done by Zhang *et al.* [**45**], and hence, we have to reconsider this result. Finally, we prove the non-existence of travelling waves by contradiction thanks to the detailed analysis, which is much simpler than the method of negative one-sided Laplace transform.

The rest of this paper is organized as follows. We first give some assumptions in § 2. Then in §§ 3 and 4, we show the existence and non-existence of travelling waves, respectively. Finally, we summarize the conclusions of this paper and list some applications of our main results in § 5.

2. Assumptions

Define

$$\mathbb{R}_+^2 := \{(u,v)|\ u>0, v>0\}, \quad cl(\mathbb{R}_+^2) := \{(u,v)|\ u \geqslant 0, v \geqslant 0\},$$

Furthermore, $C^1(cl(\mathbb{R}_+^2)\backslash\{(0,0)\}$ is the continuously differentiable function space defined from $cl(\mathbb{R}_+^2)\backslash\{(0,0)\}$ to $\mathbb{R}$. We assume that $f(u,v)$ and $g(u)$ satisfy the following conditions:

(A1) $g(\cdot) \in C^1([0,+\infty))$. $g(0) \geqslant 0$ and $g'(0) > 0$ if $g(0) = 0$. There exists a constant K such that $g(u) > 0$ for $u \in (0,K)$ and $g(u) < 0$ for $u > K$.

(A2) $f(\cdot,\cdot) \in C^1(cl(\mathbb{R}_+^2))$, $f(0,v) = 0$, $f_u(u,v) \geqslant 0$ and $f_v(u,v) \leqslant 0$ for any $(u,v) \in \mathbb{R}_+^2$. $\overline{f}_v(u,v) \geqslant 0$, where $\overline{f}(u,v) = f(u,v)v$. There are at most finite many

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points in $\mathbb{R}_+^2$ such that $f_u(u,v)=0$. For any positive constant V^0, there exists a positive constant K_2 such that

$$\sup_{0<u\leqslant K, 0\leqslant v\leqslant V^0}\left\{\overline{f}_u(u,v), \overline{f}_v(u,v), \frac{\overline{f}(u,v)}{u}\right\}\leqslant K_2.$$

(A3) $\gamma>0$ or $\lim_{v\to+\infty} f(u,v)=0$ for any $u\geqslant 0$.

(A4) System (1.1) admits a unique coexistence equilibrium $\mathbf{E_1}(u^*,v^*)\in\mathbb{R}_+^2$.

In fact, many reaction fields satisfy the assumptions (A1)–(A4), for example,
I. Prey-predator system:

$$\begin{cases} u_t = d_1(J_1 * u - u) + ru(K-u) - f(u,v)v, \\ v_t = d_2(J_2 * v - v) + \alpha f(u,v)v - \delta v - \gamma v^2, \end{cases}$$

in which $f(u,v)$ is the general functional response including:

(i) Holling type I: $f(u,v)=u$ $(\gamma>0)$;

(ii) Holling type II: $f(u,v)=u/(1+u)$ $(\gamma>0)$;

(iii) Holling type III: $f(u,v)=u^2/(1+u^2)$ $(\gamma>0)$;

(iv) Beddington-DeAngelis functional response: $f(u,v)=u/(1+h_1u+h_2v)$ $(\gamma\geqslant 0)$;

(v) Ivlev type: $f(u,v)=1-\mathrm{e}^{-nu}$, $n>0$ is constant $(\gamma>0)$.

II. Disease-transmission system

$$\begin{cases} u_t = d_1(J_1 * u - u) + \Lambda - \mu u - f(u,v)v, \\ v_t = d_2(J_2 * v - v) + \alpha f(u,v)v - \delta v - \gamma v^2, \end{cases}$$

where $f(u,v)$ is the infection capacity of the disease, for instance, $f(u,v)=\beta u$ (which is considered in Yang *et al.* [**41**] for $\Lambda=\mu=\gamma=0$), $f(u,v)=\beta u/(1+hv)$ (this case is included in Li *et al.* [**28**] for $\gamma=0$), $f(u,v)=\beta u/(u+v)$ (see Li and Yang [**26**] for $\Lambda=\mu=\gamma=0$) and so on.

From (A1), we find that $u^*<K$ holds and that $\mathbf{E_0}(K,0)$ is also an equilibrium of (1.1). Moreover, one can get $\alpha f(K,0)>\delta$ under the assumption (A2). Indeed, since $\alpha f(u^*,v^*)-\delta=\gamma v^*$, we have $\alpha f(K,0)>\alpha f(u^*,v^*)\geqslant\delta$.

A positive solution $(u(t,x),v(t,x))$ of (1.1) is called a travelling wave solution if it has the form $u(t,x)=u(\xi)$ and $v(t,x)=v(\xi)$ with $\xi=x+ct$, where $c>0$ is wave speed. As described in [45], a travelling wave solution $(u(\xi), v(\xi))$ is **strong** if it satisfies

$$(u(-\infty), v(-\infty)) = \mathbf{E_0}(K, 0), \quad (u(+\infty), v(+\infty)) = \mathbf{E_1}(u^*, v^*);$$

while it is **weak** or **persistent** if there exist two positive constants M_1 and M_2 such that

$$\begin{aligned}
&(u(-\infty), v(-\infty)) = \mathbf{E_0}(K, 0),\\
&M_1 < \liminf_{\xi\to+\infty} u(\xi) \leqslant \limsup_{\xi\to+\infty} u(\xi) < M_2,\\
&M_1 < \liminf_{\xi\to+\infty} v(\xi) \leqslant \limsup_{\xi\to+\infty} v(\xi) < M_2.
\end{aligned}$$

That is, behind the front, as $\xi \to +\infty$, the leftover concentrations of prey and predator individuals are non-trivial. Note that it is difficult to show the travelling waves converge to the coexistence equilibrium $\mathbf{E_1}$ for general form of $g(u)$ and $f(u, v)$. However, investigating the persistent or extinction of travelling waves at the end is enough for us to understand whether the predator's invasion is successful or not. Thus, in this work, we mainly focus on the weak travelling waves of (1.1). It should be pointed out that system (1.1) admits a strong travelling wave solution for some certain forms of $g(u)$ and $f(u, v)$, see examples in § 5.

3. The existence of travelling waves

In this section, we mainly consider the existence of weak travelling waves of system (1.1) under the assumptions (A1)–(A4). That is, we intend to find solutions of system

$$\begin{cases} cu'(\xi) = d_1 \int_{\mathbb{R}} J_1(y)(u(\xi - y) - u(\xi))\mathrm{d}y + g(u) - f(u, v)v, \\ cv'(\xi) = d_2 \int_{\mathbb{R}} J_2(y)(v(\xi - y) - v(y))\mathrm{d}y + \alpha f(u, v)v - \delta v - \gamma v^2 \end{cases} \tag{3.1}$$

with boundary conditions $(u(-\infty), v(-\infty)) = (K, 0)$ and

$$0 < \liminf_{\xi\to+\infty} u(\xi) \leqslant \limsup_{\xi\to+\infty} u(\xi) < K, \quad 0 < \liminf_{\xi\to+\infty} v(\xi) \leqslant \limsup_{\xi\to+\infty} v(\xi) < +\infty.$$

Define a function as follows

$$\Delta(\lambda, c) = d_2 \left[\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda y}\mathrm{d}y - 1\right] - c\lambda + \alpha f(K, 0) - \delta.$$

Note that

$$\begin{aligned}
\Delta(0, c) &= \alpha f(K, 0) - \delta > 0, \quad \lim_{\lambda\to+\infty} \Delta(\lambda, c) = +\infty \text{ for each given } c,\\
\frac{\partial^2 \Delta(\lambda, c)}{\partial \lambda^2} &= d_2 \int_{\mathbb{R}} J_2(y)y^2\mathrm{e}^{-\lambda y}\mathrm{d}y > 0,\\
\frac{\partial \Delta(\lambda, c)}{\partial c} &= -\lambda < 0 \text{ and } \lim_{c\to+\infty} \Delta(\lambda, c) = -\infty \text{ for all } \lambda > 0,\\
\Delta(\lambda, 0) &= d_2 \left[\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda y}\mathrm{d}y - 1\right] + \alpha f(K, 0) - \delta > 0.
\end{aligned}$$

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Thus, we have the following result.

LEMMA 3.1. *There must be some* $\lambda_* > 0$ *and* $c_* > 0$ *such that* $\Delta(\lambda_*, c_*) = 0$. *For* $c > c_*$, *there are* $\lambda_1(c), \lambda_2(c) > 0$ *satisfying* $\lambda_1(c) < \lambda_2(c)$ *so that* $\Delta(\lambda_i(c), c) = 0$ $(i = 1, 2)$ *and*

$$\Delta(\lambda, c) \begin{cases} > 0, & \lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), +\infty), \\ < 0, & \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

Moreover, $\Delta(\lambda, c) > 0$ *for all* $0 < c < c_*$ *and* $\lambda > 0$.

3.1. Travelling waves for $c > c_*$

Below, we always assume $c > c_*$ and denote $\lambda_i := \lambda_i(c)$ $(i = 1, 2)$. By (A3), it follows that $\lim_{v\to+\infty} f(u, v) = 0$ if $\gamma = 0$. This implies that there exists $v_0^1 > 1$ such that $\delta > \alpha f(K, v_0^1)$. Set

$$v_0 = \begin{cases} \max\left\{1, \frac{\alpha f(K,0)-\delta}{\gamma}\right\} & \text{if } \gamma > 0, \\ v_0^1 & \text{if } \gamma = 0. \end{cases}$$

Further, define

$$\begin{aligned} &\overline{u}(\xi) = K, \quad \underline{u}(\xi) = \max\{K - \sigma \mathrm{e}^{\beta\xi}, 0\}, \\ &\overline{v}(\xi) = \min\{\mathrm{e}^{\lambda_1\xi}, v_0\}, \ \underline{v}(\xi) = \max\{\mathrm{e}^{\lambda_1\xi}(1 - M\mathrm{e}^{\varepsilon\xi}), 0\}, \end{aligned}$$

in which $\sigma, \beta, \varepsilon, M$ are all positive constants and will be determined later.

LEMMA 3.2. *The function* $\overline{v}(\xi)$ *satisfies*

$$c\overline{v}' \geqslant d_2(J_2 * \overline{v} - \overline{v}) + \alpha f(K, \overline{v})\overline{v} - \delta\overline{v} - \gamma\overline{v}^2, \quad \forall \xi \neq \frac{1}{\lambda_1}\ln v_0. \tag{3.2}$$

Proof. If $\xi < (1/\lambda_1)\ln v_0$, then $\overline{v}(\xi) = \mathrm{e}^{\lambda_1\xi}$. Moreover, (A2) implies that $f(K, \overline{v}) \leqslant f(K, 0)$. It then follows that

$$\begin{aligned} &c\overline{v}' - d_2(J_2 * \overline{v} - \overline{v}) - \alpha f(K, \overline{v})\overline{v} + \delta\overline{v} + \gamma\overline{v}^2 \\ &\geqslant c\lambda_1\mathrm{e}^{\lambda_1\xi} - d_2\mathrm{e}^{\lambda_1\xi}\left(\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda_1 y}\mathrm{d}y - 1\right) - (\alpha f(K, 0) - \delta)\mathrm{e}^{\lambda_1\xi} + \gamma e^{2\lambda_1\xi} \\ &= -\Delta(\lambda_1, c)\overline{v} + \gamma\overline{v}^2 = \gamma\overline{v}^2 > 0. \end{aligned}$$

The first inequality has used the fact that

$$J_2 * \overline{v} \leqslant \min\left\{v_0, \mathrm{e}^{\lambda_1\xi}\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda_1 y}\mathrm{d}y\right\}.$$

When $\xi > (1/\lambda_1)\ln v_0$, $\overline{v} = v_0$. By a direct computation, we have

$$\begin{aligned} &c\overline{v}' - d_2(J_2 * \overline{v} - \overline{v}) - \alpha f(K, \overline{v})\overline{v} + \delta\overline{v} + \gamma\overline{v}^2 \\ &\geqslant (-\alpha f(K, v_0) + \delta + \gamma v_0)v_0. \end{aligned}$$

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In the case where $\gamma > 0$, we have

$$(-\alpha f(K, v_0) + \delta + \gamma v_0)v_0 \geqslant (-\alpha f(K, 0) + \delta + \gamma v_0)v_0 \geqslant 0.$$

In the case where $\gamma = 0$, obviously $(-\alpha f(K, v_0) + \delta + \gamma v_0)v_0 \geqslant 0$. The proof is completed. □

Define

$$\Delta_1(\lambda, c) = d_1 \int_{\mathbb{R}} J_1(y)(\mathrm{e}^{-\lambda y} - 1)\mathrm{d}y - c\lambda.$$

Noticed that $\Delta_1(0, c) = 0$, $(\partial\Delta_1(\lambda, c))/(\partial\lambda)\big|_{\lambda=0} = -c < 0$ for $c > c_*$. Thus, for any $\lambda > 0$ small enough, it must be $\Delta_1(\lambda, c) < 0$ for $c > c_*$. Then, we can show that $\underline{u}$ is a lower solution.

Lemma 3.3. *Assume that $\beta \in (0, \lambda_1)$ is sufficiently small and $\sigma > \max\{K, (f(K, 0))/(-\Delta_1(\beta, c))\}$. Then, the function $\underline{u}(\xi)$ satisfies*

$$c\underline{u}' \leqslant d_1(J_1 * \underline{u} - \underline{u}) + g(\underline{u}) - f(\underline{u}, \overline{v})\overline{v}, \quad \forall \xi \neq \frac{1}{\beta}\ln\frac{K}{\sigma}. \tag{3.3}$$

Proof. If $\xi > (1/\beta)\ln(K/\sigma)$, $\underline{u} = 0$. Then, (3.3) is obvious. If $\xi < (1/\beta)\ln(K/\sigma)$, $\underline{u} = K - \sigma\mathrm{e}^{\beta\xi}$. Additionally, due to the definition of v_0, one can get $(1/\beta)\ln(K/\sigma) < 0 \leqslant (1/\lambda_1)\ln v_0$, and hence, $\overline{v} = \mathrm{e}^{\lambda_1\xi}$. Consequently,

$$\begin{aligned}
&c\underline{u}' - d_1(J_1 * \underline{u} - \underline{u}) - g(\underline{u}) + f(\underline{u}, \overline{v})\overline{v} \\
&\quad\leqslant -c\beta\sigma\mathrm{e}^{\beta\xi} + d_1\sigma\mathrm{e}^{\beta\xi}\left(\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\beta y}\mathrm{d}y - 1\right) + f(K, 0)\mathrm{e}^{\lambda_1\xi} - g(\underline{u}) \\
&\quad\leqslant \mathrm{e}^{\beta\xi}\left[-c\sigma\beta + \sigma d_1\left(\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\beta y}\mathrm{d}y - 1\right) + f(K, 0)\right] - g(\underline{u}) \\
&\quad= \mathrm{e}^{\beta\xi}[\sigma\Delta_1(\beta, c) + f(K, 0)] - g(\underline{u}) \leqslant 0.
\end{aligned}$$

The first inequality has used the fact that

$$J_1 * \underline{u} \geqslant \max\left\{K - \sigma\mathrm{e}^{\beta\xi}\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\beta y}\mathrm{d}y, 0\right\}.$$

This ends the proof. □

Lemma 3.4. *Let $\varepsilon < \min\{(1/2)\lambda_1, (\lambda_2 - \lambda_1)/2, \beta\}$ be small enough and $M > 0$ large enough. Then, the function $\underline{v}(\xi)$ satisfies*

$$c\underline{v}' \leqslant d_2(J_2 * \underline{v} - \underline{v}) + \alpha f(\underline{u}, \underline{v})\underline{v} - \delta\underline{v} - \gamma\underline{v}^2, \quad \forall \xi \neq \frac{1}{\varepsilon}\ln\frac{1}{M}. \tag{3.4}$$

Proof. Take $M > (\sigma/K)^{\varepsilon/\beta}$ large enough. If $\xi > (1/\varepsilon)\ln(1/M)$, then $\underline{v}(\xi) = 0$ and (3.4) holds naturally. If $\xi < \frac{1}{\varepsilon}\ln\frac{1}{M}$, we have $\underline{v}(\xi) = \mathrm{e}^{\lambda_1\xi}(1 - M\mathrm{e}^{\varepsilon\xi})$. In

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this case, since

$$\frac{1}{\varepsilon}\ln\frac{1}{M} < -\frac{1}{\varepsilon}\ln\left(\frac{\sigma}{K}\right)^{\varepsilon/\beta} = \frac{1}{\beta}\ln\frac{K}{\sigma},$$

it follows that $\underline{u}(\xi) = K - \sigma e^{\beta\xi}$. Firstly, by the mean value theorem, we have

$$\begin{aligned} f(\underline{u},\underline{v}) - f(K,0) &= f_u(p)(\underline{u} - K) + f_v(p)\underline{v} \\ &= -\sigma f_u(p)e^{\beta\xi} + f_v(p)e^{\lambda_1\xi}(1 - Me^{\varepsilon\xi}), \end{aligned}$$

where $p = (K + \theta(\underline{u} - K), \theta\underline{v})$ and $0 < \theta < 1$. Note that $\underline{u} > 0$ and $0 < \underline{v} \leqslant e^{\lambda_1\xi}$. Obviously, $\lim_{\xi\to-\infty}\underline{u}(\xi) = K$ and $\lim_{\xi\to-\infty}\underline{v}(\xi) = 0$. Thus, there exists a positive constant M_0 independent on M such that

$$(\underline{u}(\xi), \underline{v}(\xi)) \in B\left((K,0), \frac{K}{2}\right) := \left\{(u,v)\,\middle|\, (u-K)^2 + v^2 \leqslant \frac{K^2}{4}\right\}$$

for all $\xi \leqslant \xi_0 := (1/\varepsilon)\ln(1/M_0)$. In the following, we set $M > M_0$ and $\xi \leqslant \xi_0$. Let

$$K_1 = \max_{(u,v)\in B((K,0),(K/2))}\{f_u(u,v), -f_v(u,v)\}.$$

Then, we have $0 \leqslant f_u(p) \leqslant K_1$ and $0 \leqslant -f_v(p) \leqslant K_1$. In addition, since $\varepsilon < (1/2)\min\{\lambda_1, \lambda_2 - \lambda_1\}$, we have $\Delta(\lambda_1 + \varepsilon, c) < 0$ according to the discussion in lemma 3.1. Now, taking

$$M > 1 - \frac{\alpha K_1\sigma + \alpha K_1 + \gamma}{\Delta(\lambda_1 + \varepsilon, c)}$$

large enough, we have

$$\begin{aligned} & c\underline{v}' - d_2(J_2 * \underline{v} - \underline{v}) - \alpha f(\underline{u},\underline{v})\underline{v} + \delta\underline{v} + \gamma\underline{v}^2 \\ &\leqslant c[\lambda_1 e^{\lambda_1\xi} - M(\lambda_1+\varepsilon)e^{(\lambda_1+\varepsilon)\xi}] - d_2 e^{\lambda_1\xi}\int_{\mathbb{R}} J_2(y)e^{-\lambda_1 y}dy \\ &\quad + d_2 M e^{(\lambda_1+\varepsilon)\xi}\int_{\mathbb{R}} J_2(y)e^{-(\lambda_1+\varepsilon)y}dy + d_2 e^{\lambda_1\xi} - d_2 M e^{(\lambda_1+\varepsilon)\xi} \\ &\quad - \alpha[f(K,0) - \sigma f_u(p)e^{\beta\xi} + f_v(p)e^{\lambda_1\xi}(1 - Me^{\varepsilon\xi})]e^{\lambda_1\xi}(1 - Me^{\varepsilon\xi}) \\ &\quad + \delta e^{\lambda_1\xi} - \delta M e^{(\lambda_1+\varepsilon)\xi} + \gamma e^{2\lambda_1\xi}(1 - Me^{\varepsilon\xi})^2 \\ &= -\Delta(\lambda_1, c)e^{\lambda_1\xi} + \Delta(\lambda_1+\varepsilon, c)Me^{(\lambda_1+\varepsilon)\xi} + \gamma e^{2\lambda_1\xi}(1 - Me^{\varepsilon\xi})^2 \\ &\quad - \alpha[-\sigma f_u(p)e^{\beta\xi} + f_v(p)e^{\lambda_1\xi}(1 - Me^{\varepsilon\xi})]e^{\lambda_1\xi}(1 - Me^{\varepsilon\xi}) \\ &= e^{(\lambda_1+\varepsilon)\xi}[M\Delta(\lambda_1+\varepsilon, c) + \alpha e^{-\varepsilon\xi}(\sigma f_u(p)e^{\beta\xi} - f_v(p)\underline{v}(\xi)) \\ &\quad - M\alpha(\sigma f_u(p)e^{\beta\xi} - f_v(p)\underline{v}(\xi)) + \gamma e^{(\lambda_1-\varepsilon)\xi}(1 - Me^{\varepsilon\xi})^2] \\ &\leqslant e^{(\lambda_1+\varepsilon)\xi}[M\Delta(\lambda_1+\varepsilon, c) + \alpha e^{-\varepsilon\xi}(\sigma f_u(p)e^{\beta\xi} - f_v(p)\underline{v}(\xi)) + \gamma] \\ &\leqslant e^{(\lambda_1+\varepsilon)\xi}[M\Delta(\lambda_1+\varepsilon, c) + \alpha\sigma K_1 + \alpha K_1 + \gamma] \\ &\leqslant 0. \end{aligned}$$

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This ends the proof. □

Denote $\Omega_a = [-a, a]$ with $a > \max\{\frac{1}{\beta}\ln\frac{\sigma}{K}, \frac{1}{\varepsilon}\ln M\}$ and define

$$\Gamma_a = \left\{ (\varphi(\cdot),\psi(\cdot)) \in C(\Omega_a,\mathbb{R}^2) \,\middle|\, \begin{array}{l} \underline{u}(\xi) \leqslant \varphi(\xi) \leqslant \overline{u}(\xi), \\ \underline{v}(\xi) \leqslant \psi(\xi) \leqslant \overline{v}(\xi) \text{ for } \xi \in \Omega_a, \\ \varphi(-a) = \underline{u}(-a),\ \psi(-a) = \underline{v}(-a). \end{array} \right\}$$

Set $\mathcal{C}^a := C(\Omega_a) \times C(\Omega_a)$ and define $\|(u,v)\|_{\mathcal{C}^a} = \|u\|_{C(\Omega_a)} + \|v\|_{C(\Omega_a)}$. Obviously, Γ_a is a non-empty bounded closed convex set in $(\mathcal{C}^a, \|\cdot\|_{\mathcal{C}^a})$.

Next, we consider the following truncated problem

$$\begin{cases} cu' = d_1\left(\int_{\mathbb{R}} J_1(\xi-y)\hat{\varphi}(y)\mathrm{d}y - u(\xi)\right) + g(u) - f(u,\psi)\psi, & \xi \in \Omega_a\backslash\{-a\}, \\ cv' = d_2\left(\int_{\mathbb{R}} J_2(\xi-y)\hat{\psi}(y)\mathrm{d}y - v(\xi)\right) + \alpha f(\varphi,\psi)\psi - \delta v - \gamma v^2, & \xi \in \Omega_a\backslash\{-a\}, \\ u(-a) = \underline{u}(-a),\ v(-a) = \underline{v}(-a), \end{cases} \tag{3.5}$$

for any given $\varphi, \psi \in \Gamma_a$ and

$$\hat{\varphi}(\xi) = \begin{cases} \varphi(a), & \xi > a, \\ \varphi(\xi), & |\xi| \leqslant a, \\ \underline{u}(\xi), & \xi < -a, \end{cases} \qquad \hat{\psi}(\xi) = \begin{cases} \psi(a), & \xi > a, \\ \psi(\xi), & |\xi| \leqslant a, \\ \underline{v}(\xi), & \xi < -a. \end{cases}$$

Define the mapping $\mathcal{F} : \Gamma_a \to \mathcal{C}^a$ as follows

$$\mathcal{F}(\varphi,\psi) := (u,v)$$

for any $(\varphi,\psi) \in \Gamma_a$, where (u,v) is the solution of system (3.5), well-defined by the forthcoming lemma 3.5. Hence, by definition of $\mathcal{F}$, one can see that any fixed point of $\mathcal{F}$ is a solution of system

$$\begin{cases} cu' = d_1\left(\int_{\mathbb{R}} J_1(\xi-y)\hat{u}(y)\mathrm{d}y - u\right) + g(u) - f(u,v)v, & \xi \in \Omega_a\backslash\{-a\}, \\ cv' = d_2\left(\int_{\mathbb{R}} J_2(\xi-y)\hat{v}(y)\mathrm{d}y - v\right) + \alpha f(u,v)v - \delta v - \gamma v^2, & \xi \in \Omega_a\backslash\{-a\}, \\ u(-a) = \underline{u}(-a),\ v(-a) = \underline{v}(-a), \end{cases} \tag{3.6}$$

in which

$$\hat{u}(\xi) = \begin{cases} u(a), & \xi > a, \\ u(\xi), & |\xi| \leqslant a, \\ \underline{u}(-a), & \xi < -a, \end{cases} \qquad \hat{v}(\xi) = \begin{cases} v(a), & \xi > a, \\ v(\xi), & |\xi| \leqslant a, \\ \underline{v}(-a), & \xi < -a. \end{cases}$$

Thus, we only need to verify that the mapping $\mathcal{F}$ satisfies the condition of the Schauder's fixed-point theorem.

LEMMA 3.5. *The mapping $\mathcal{F}$ is well-defined. That is, for any given $(\varphi,\psi) \in \Gamma_a$, there exists a unique solution (u_a, v_a) to the Cauchy problem (3.5). Further, $\underline{u} \leqslant u_a \leqslant \overline{u}$ and $\underline{v} \leqslant v_a \leqslant \overline{v}$ on Ω_a.*

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Proof. Note that (3.5) is not a coupled system. Thus, we can deal with the existence and uniqueness of u_a and v_a separately.

First, we show the existence of u_a. Define a function as follows

$$\eta(s) = \begin{cases} 0, & s \leqslant 0, \\ s, & 0 < s < K, \\ K, & s \geqslant K \end{cases}$$

and consider the following initial value problem

$$\begin{cases} cu' = d_1 \left(\int_{\mathbb{R}} J_1(\xi - y)\hat{\varphi}(y)\mathrm{d}y - u(\xi) \right) + \tilde{g}(u) - \tilde{f}(u,\psi)\psi, & \xi \in \Omega_a \backslash \{-a\}, \\ u(-a) = \underline{u}(-a) \end{cases} \tag{3.7}$$

for any $(\varphi, \psi) \in \Gamma_a$, in which $\tilde{g}(u) = g \circ \eta(u) = g(\eta(u))$ and $\tilde{f}(u,\psi) = f \circ \eta(u,\psi) = f(\eta(u),\psi)$. According to (A1)-(A2), it easily follows that $\tilde{g}$ and $\tilde{f}$ are Lipschitz continuous on u and bounded on $\mathbb{R}$. Then, the initial value problem (3.7) has a unique solution u_a on Ω_a. By definition of $\underline{u}(\xi)$, there is some $a_0 \in (-a, a)$ such that $\underline{u}(\xi) > 0$ for all $\xi \in (-a, a_0)$ and $\underline{u}(\xi) = 0$ for all $\xi \in [a_0, a)$. Thus, the maximum principle ([**12**]) implies that $u_a(\xi) > 0$ over $(-a, a)$.

Next, we intend to show that $u_a \in \Gamma_a$. Following lemma 3.3, one can get

$$\begin{aligned} c\underline{u}' &\leqslant d_1 \left(\int_{\mathbb{R}} J_1(\xi - y)\underline{u}(y)\mathrm{d}y - \underline{u} \right) + g(\underline{u}) - f(\underline{u}, \overline{v})\overline{v} \\ &\leqslant d_1 \left(\int_{\mathbb{R}} J_1(\xi - y)\hat{\varphi}(y)\mathrm{d}y - \underline{u} \right) + \tilde{g}(\underline{u}) - \tilde{f}(\underline{u}, \psi)\psi. \end{aligned}$$

Let

$$G(u(\xi)) = \begin{cases} \frac{\tilde{g}(u) - \tilde{g}(\underline{u}) - (\tilde{f}(u,\psi) - \tilde{f}(\underline{u},\psi))\psi}{u - \underline{u}} & \text{if } u(\xi) \neq \underline{u}(\xi), \\ 0 & \text{if } u(\xi) = \underline{u}(\xi), \end{cases}$$

and set $\omega(\xi) = \mathrm{e}^{\Lambda\xi}(u_a(\xi) - \underline{u}(\xi))$ for some $\Lambda > 0$. A direct calculation can yield that

$$\omega'(\xi) \geqslant \left[\Lambda - \frac{d_1 - G(u_a(\xi))}{c} \right] \omega(\xi). \tag{3.8}$$

Set $b(\xi) := \Lambda - \frac{d_1 - G(u_a(\xi))}{c}$. In view of the boundedness of $G(u_a(\xi))$, it follows that there exists some $\Lambda > 0$ such that $\inf_{\xi \in \Omega_a} b(\xi) > 0$. Note that $\omega(-a) = 0$. Thus, we have $\omega(\xi) \geqslant 0$ on Ω_a according to (3.8), which implies that $u_a(\xi) \geqslant \underline{u}(\xi)$ on Ω_a. On the other hand, for all $\xi \in \Omega_a$, there holds

$$\begin{aligned} c\overline{u}' &\geqslant d_1 \left(\int_{\mathbb{R}} J_1(\xi - y)\overline{u}(y)\mathrm{d}y - \overline{u} \right) + g(\overline{u}) - f(\overline{u}, \underline{v})\underline{v} \\ &\geqslant d_1 \left(\int_{\mathbb{R}} J_1(\xi - y)\hat{\varphi}(y)\mathrm{d}y - \overline{u} \right) + \tilde{g}(\overline{u}) - \tilde{f}(\overline{u}, \psi)\psi. \end{aligned}$$

Since $\overline{u}(\xi) = K$, the comparison principle ([**12**]) implies $u_a(\xi) \leqslant K$ for all $\xi \in \Omega_a$. That is, $u_a \in \Gamma_a$. Hence, $\tilde{g}(u_a) = g(u_a)$ and $\tilde{f}(u_a,\psi) = f(u_a,\psi)$. Therefore, u_a is a unique solution of the first equation of (3.5).

Now, we show the existence of v_a. Define the following function:

$$h(\tau) = \begin{cases} 0, & \tau \leqslant 0, \\ \tau, & 0 < \tau < v_0, \\ v_0, & \tau \geqslant v_0, \end{cases}$$

and let $\phi(v) = v^2$, $F = \phi \circ h$. Then, consider the Cauchy problem

$$\begin{cases} cv' = d_2 \left(\int_{\mathbb{R}} J_2(\xi - y)\hat{\psi}(y)\mathrm{d}y - v \right) + \alpha f(\varphi, \psi)\psi - \delta v - \gamma F(v), \ \xi \in \Omega_a \backslash \{-a\}, \\ v(-a) = \underline{v}(-a). \end{cases} \tag{3.9}$$

Consequently, the same discussion as above, we can find a unique $v_a \in \Gamma_a$ satisfying the second equation of (3.5) and $v_a(-a) = \underline{v}(-a)$. We then complete the proof. □

LEMMA 3.6. *$\mathcal{F}$ is a continuous mapping.*

Proof. For any given $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \Gamma_a$, let

$$\mathcal{F}(\varphi_1, \psi_1) := (u_1, v_1), \quad \mathcal{F}(\varphi_2, \psi_2) := (u_2, v_2).$$

First, let $\omega_1 = u_1 - u_2$. Then, $\omega_1(-a) = 0$ and $\omega_1(\xi)$ satisfies

$$c\omega_1' + b_1(\xi)\omega_1 = \Phi_1(\xi) \quad \text{in } \Omega_a \backslash \{-a\}, \tag{3.10}$$

where

$$b_1(\xi) = d_1 - \frac{g(u_1) - g(u_2)}{u_1 - u_2} + \frac{f(u_1, \psi_1) - f(u_2, \psi_1)}{u_1 - u_2}\psi_1$$

and

$$\Phi_1(\xi) = d_1 \int_{\mathbb{R}} J_1(\xi - y)(\hat{\varphi}_1(y) - \hat{\varphi}_2(y))\mathrm{d}y + (f(u_2, \psi_2) - f(u_2, \psi_1))\psi_1 \\ + f(u_2, \psi_2)(\psi_2 - \psi_1).$$

Since $u_1, u_2 \in \Gamma_a$ and by the assumptions (A2)-(A3), we have $|b_1(\xi)| \leqslant l_0$ for some positive constant l_0. Meanwhile,

$$\left| \int_{\mathbb{R}} J_1(\xi - y)(\hat{\varphi}_1(y) - \hat{\varphi}_2(y))\mathrm{d}y \right| \\ = \left| \int_{-a}^{a} J_1(\xi - y)(\varphi_1(y) - \varphi_2(y))\mathrm{d}y + \int_{a}^{\infty} J_1(\xi - y)(\varphi_1(a) - \varphi_2(a))\mathrm{d}y \right| \\ \leqslant 2\|\varphi_1 - \varphi_2\|_{C(\Omega_a)}.$$

Thus, it is easy to verify that there are some constants $L_i > 0$ $(i = 1, 2)$ such that

$$\|\Phi_1(\cdot)\|_{C(\Omega_a)} \leqslant L_1\|\varphi_1 - \varphi_2\|_{C(\Omega_a)} + L_2\|\psi_1 - \psi_2\|_{C(\Omega_a)}.$$

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Furthermore, it follows from (3.10) that

$$\omega_1(\xi) = \frac{1}{c}\int_{-a}^{\xi} e^{\frac{1}{c}\int_{\xi}^{\eta} b_1(\tau)d\tau}\Phi_1(\eta)d\eta \quad \text{in } \Omega_a.$$

Hence, we have

$$\|\omega_1(\cdot)\|_{C(\Omega_a)} \leqslant C_1\|\varphi_1 - \varphi_2\|_{C(\Omega_a)} + C_2\|\psi_1 - \psi_2\|_{C(\Omega_a)}$$

for some constants $C_1, C_2 > 0$. Similarly, if we let $\omega_2 = v_1 - v_2$, one can get

$$\|\omega_2(\cdot)\|_{C(\Omega_a)} \leqslant \hat{C}_1\|\varphi_1 - \varphi_2\|_{C(\Omega_a)} + \hat{C}_2\|\psi_1 - \psi_2\|_{C(\Omega_a)}$$

for constants $\hat{C}_1, \hat{C}_2 > 0$. In summary, we have proved that

$$\begin{aligned}
&\|\mathcal{F}(\varphi_1,\psi_1) - \mathcal{F}(\varphi_2,\psi_2)\|_{\mathcal{C}^a}\\
&\quad = \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{C}^a} = \|u_1 - u_2\|_{C(\Omega_a)} + \|v_1 - v_2\|_{C(\Omega_a)}\\
&\quad \leqslant C_3\|\varphi_1 - \varphi_2\|_{C(\Omega_a)} + C_4\|\psi_1 - \psi_2\|_{C(\Omega_a)} \leqslant C_5\|(\varphi_1,\psi_1) - (\varphi_2,\psi_2)\|_{\mathcal{C}^a}
\end{aligned}$$

for any $(\varphi_1,\psi_1), (\varphi_2,\psi_2) \in \Gamma_a$ and some positive constants C_i $(i = 3, 4, 5)$. This implies that $\mathcal{F}$ is a continuous mapping. The proof is finished. □

Lemma 3.7. *$\mathcal{F}$ is compact.*

Proof. By the definition of the operator $\mathcal{F}$ and according to lemma 3.5, we know that the solution (u_a, v_a) of (3.5) is bounded in $\mathcal{C}^a$ for any given $(\varphi,\psi) \in \Gamma_a$. Meanwhile, it follows from (3.5) that $\|u_a\|_{C^1(\Omega_a)}$ and $\|v_a\|_{C^1(\Omega_a)}$ are both bounded. Thus, the mapping $\mathcal{F}$ is compact and this ends the proof. □

Finally, following lemmas 3.5–3.7, $\mathcal{F}$ has a fixed point by the Schauder's fixed-point theorem and this fixed point is a non-negative solution of system (3.6). That is, we have the following existence result of the truncated problem (3.6).

Lemma 3.8. *System (3.6) admits a solution (u_a, v_a) on Ω_a. Moreover,*

$$0 \leqslant \underline{u} \leqslant u_a \leqslant \overline{u} \quad \textit{and} \quad 0 \leqslant \underline{v} \leqslant v_a \leqslant \overline{v} \quad \textit{on } \Omega_a.$$

Further, we have the following result.

Theorem 3.9. *Assume $c > c_*$. Then there is a solution $(\tilde{u}, \tilde{v})$ of system (3.1) satisfying $\tilde{u}(-\infty) = K, \tilde{v}(-\infty) = 0$ and*

$$0 < \tilde{u} < K \quad \textit{and} \quad 0 < \tilde{v} < +\infty \quad \textit{in } \mathbb{R}.$$

Proof. Let (u_a, v_a) be the solution of system (3.6). Then, following lemma 3.8, $0 \leqslant \underline{u} \leqslant u_a \leqslant K$ and $0 \leqslant \underline{v} \leqslant v_a \leqslant \overline{v}$. Now, choose some sequence $\{a_n\}$ satisfying $a_n > \max\{\frac{1}{\beta}\ln\frac{\sigma}{K}, \frac{1}{\varepsilon}\ln M\}$ and $a_n \to +\infty$ as $n \to +\infty$. For any bounded domain Ω_{a_n}, we know that (3.6) admits a solution (u_{a_n}, v_{a_n}) satisfying $\underline{u} \leqslant u_{a_n}(\xi) \leqslant K$ and $\underline{v} \leqslant v_{a_n}(\xi) \leqslant \overline{v}$. Thus, $u_{a_n}(-\infty) = K$ and $v_{a_n}(-\infty) = 0$. Meanwhile, it is easy

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to obtain that $u'_{a_n}(\xi)$ and $v'_{a_n}(\xi)$ are all uniformly bounded. Additionally, since $J_i(\cdot) \in C^1(\mathbb{R})$ $(i = 1, 2)$, we note that

$$\left|\frac{d}{d\xi}\int_{\mathbb{R}} J_1(\xi - y)\hat{u}_{a_n}(y)\mathrm{d}y\right| = \left|\int_{\mathbb{R}} \frac{d}{d\xi} J_1(\xi - y)\hat{u}_{a_n}(y)\mathrm{d}y\right| \leqslant K \int_{\mathbb{R}} |J_1'(y)|dy,$$

and

$$\left|\frac{d}{d\xi}\int_{\mathbb{R}} J_2(\xi - y)\hat{v}_{a_n}(y)\mathrm{d}y\right| = \left|\int_{\mathbb{R}} \frac{d}{d\xi} J_2(\xi - y)\hat{v}_{a_n}(y)\mathrm{d}y\right| \leqslant \overline{v} \int_{\mathbb{R}} |J_2'(y)|dy.$$

Consequently, $u''_{a_n}(\xi)$ and $v''_{a_n}(\xi)$ are all uniformly bounded on Ω_{a_n}. Hence, there exist some subsequences, denoted by $(u_{a_{n_k}}, v_{a_{n_k}})$ with $n_k \to +\infty$ as $k \to +\infty$, such that

$$u_{a_{n_k}} \to \tilde{u} \;\text{ and }\; v_{a_{n_k}} \to \tilde{v} \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}) \text{ as } k \to +\infty$$

for some continuously differentiable functions $\tilde{u}, \tilde{v}$. Moreover, the assumption (J) gives that

$$\begin{aligned}&\int_{\mathbb{R}} J_1(\xi - y)\hat{u}_{a_{n_k}}(y)\mathrm{d}y \to \int_{\mathbb{R}} J_1(\xi - y)\tilde{u}(y)\mathrm{d}y,\\ &\int_{\mathbb{R}} J_2(\xi - y)\hat{v}_{a_{n_k}}(y)\mathrm{d}y \to \int_{\mathbb{R}} J_2(\xi - y)\tilde{v}(y)\mathrm{d}y \text{ as } k \to +\infty\end{aligned}$$

for any $\xi \in \mathbb{R}$. And $(\tilde{u}, \tilde{v})$ satisfies

$$\underline{u} \leqslant \tilde{u} \leqslant K \;\text{ and }\; \underline{v} \leqslant \tilde{v} \leqslant \overline{v}. \tag{3.11}$$

Following from (3.11), it is obvious that $\tilde{u}(-\infty) = K, \tilde{v}(-\infty) = 0$. Next, we only need to prove $0 < \tilde{u} < K$ and $\tilde{v} > 0$ in $\mathbb{R}$. Assume there exists some $\xi_0 \in \mathbb{R}$ such that $\tilde{u}(\xi_0) = 0$. Then, $\tilde{u}'(\xi_0) = 0$. By the first equation of (3.1), we have $\int_{\mathbb{R}} J_1(\xi_0 - y)\tilde{u}(y)\mathrm{d}y = 0$ for all $y \in \mathbb{R}$. This implies $\tilde{u}(y) \equiv 0$ for $y \in \mathbb{R}$, which contradicts (3.11). Thus, $\tilde{u}(\xi) > 0$ in $\mathbb{R}$. Similarly, we can show $\tilde{v}(\xi) > 0$ in $\mathbb{R}$. Meanwhile, assume some $\xi_* \in \mathbb{R}$ exists so that $\tilde{u}(\xi_*) = K$. Then, $\tilde{u}'(\xi_*) = 0$. In view of the first equation of (3.1), there holds

$$\begin{aligned}0 &= d_1 \int_{\mathbb{R}} J_1(\xi_* - y)(\tilde{u}(y) - \tilde{u}(\xi_*))\mathrm{d}y + g(\tilde{u}(\xi_*)) - f(\tilde{u}(\xi_*), \tilde{v}(\xi_*))\tilde{v}(\xi_*)\\ &\leqslant -f(K, \tilde{v}(\xi_*))\tilde{v}(\xi_*).\end{aligned}$$

This contradicts the fact that $\tilde{v}(\xi_*) > 0$. Hence, $\tilde{u}(\xi) < K$, $\forall \xi \in \mathbb{R}$. This ends the proof. □

3.2. Asymptotic behaviour

Here, we mainly consider the persistence of travelling waves of system (1.1). For convenience of the description, we always assume (u, v) is the travelling wave solution of system (1.1) constructed in § 3.1. Then, theorem 3.9 implies that $u(-\infty) = K, v(-\infty) = 0$, $0 < u < K$ and $0 < v < +\infty$ in $\mathbb{R}$. To get the goal of this section, we will use repeatedly the following results obtained by Zhang *et al.* [**43**].

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Lemma 3.10 Zhang *et al.* [43]. *Assume* $c > 0$ *and* $B(\cdot)$ *is a continuous function with* $B(\pm\infty) := \lim_{\xi\to\pm\infty} B(\xi)$. *Let* $Z(\xi)$ *be a measurable function satisfying*

$$cZ(\xi) = \int_{\mathbb{R}} J_i(y)\mathrm{e}^{\int_\xi^{\xi-y} Z(s)\mathrm{d}s}\mathrm{d}y + B(\xi) \quad in\ \mathbb{R}.$$

Then, Z *is uniformly continuous and bounded. Moreover,* $\mu^\pm := \lim_{\xi\to\pm\infty} Z(\xi)$ *exist and are real roots of the characteristic equation*

$$c\mu = \int_{\mathbb{R}} J_i(y)\mathrm{e}^{-\mu y}\mathrm{d}y + B(\pm\infty) \quad (i = 1, 2).$$

Moreover, to get the persistence of travelling waves, inspired by some ideas in [9, 19], we need the following result.

Lemma 3.11. *Let* $Z \in C^1(\mathbb{R})$ *satisfy*

$$Z'(\xi) \geqslant \int_{\mathbb{R}} J_i(y)Z(\xi - y)\mathrm{d}y + b(\xi)Z(\xi) \quad in\ \mathbb{R}, \tag{3.12}$$

where $b(\xi)$ *is continuous and* $b(\xi) \geqslant -\tilde{M}$ *on* $\mathbb{R}$ *for some* $\tilde{M} > 0$. *Then there exists some constant* $C = C(\tilde{M}) > 0$ *such that*

$$C^{-1} < \int_{\mathbb{R}} J_i(y)\frac{Z(\xi - y)}{Z(\xi)}\mathrm{d}y < C \ in\ \mathbb{R}, \quad i = 1, 2.$$

Proof. Since $b(\xi) \geqslant -\tilde{M}$ on $\mathbb{R}$, it follows from (3.12) that

$$Z'(\xi) \geqslant \int_{\mathbb{R}} J_i(y)Z(\xi - y)\mathrm{d}y - \tilde{M}Z(\xi).$$

Let $\theta(\xi) = \frac{Z'(\xi)}{Z(\xi)}$ and denote $Q(\xi) = \exp\{\tilde{M}\xi + \int_0^\xi \theta(s)\mathrm{d}s\}$. Thus, a direct computation gives

$$Q'(\xi) = (\tilde{M} + \theta(\xi))Q(\xi) \geqslant \int_{\mathbb{R}} J_i(y)\mathrm{e}^{\int_\xi^{\xi-y} \theta(s)\mathrm{d}s}\mathrm{d}yQ(\xi), \tag{3.13}$$

which implies $Q(\xi)$ is non-decreasing and $Q^* := \lim_{\xi\to-\infty} Q(\xi) \geqslant 0$ exists. Let r_i be the radius of *supp*J_i, in which *supp*J_i denotes the support of functions J_i $(i = 1, 2)$. Set $r = \min\{r_1, r_2\}$. It follows from (J) that $0 < r \leqslant \infty$. Choosing some $r_0 > 0$ with

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$2r_0 < r$ and then integrating both sides of (3.13) from $-\infty$ to ξ, we get

$$\begin{aligned}Q(\xi) - Q^* &\geqslant \int_{-\infty}^{\xi} \int_{\mathbb{R}} J_i(y) \mathrm{e}^{\int_x^{x-y} \theta(s)\mathrm{d}s} \mathrm{d}y Q(x) \mathrm{d}x \\ &= \int_{\mathbb{R}} J_i(y) \mathrm{e}^{\tilde{M}y} \int_{-\infty}^{\xi} Q(x-y) \mathrm{d}x \mathrm{d}y \\ &\geqslant \int_{\mathbb{R}} J_i(y) \mathrm{e}^{\tilde{M}y} \int_{\xi - r_0}^{\xi} Q(x-y) \mathrm{d}x \mathrm{d}y \\ &\geqslant r_0 \int_{\mathbb{R}} J_i(y) \mathrm{e}^{\tilde{M}y} Q(\xi - r_0 - y) \mathrm{d}y.\end{aligned}$$

The non-negativity of Q^* implies that

$$Q(\xi) \geqslant r_0 \int_{\mathbb{R}} J_i(y) \mathrm{e}^{\tilde{M}y} Q(\xi - r_0 - y) \mathrm{d}y. \tag{3.14}$$

Moreover, integrating the two sides of (3.13) from $\xi - r_0$ to ξ yields

$$\begin{aligned}Q(\xi) - Q(\xi - r_0) &\geqslant \int_{\mathbb{R}} J_i(y) \mathrm{e}^{\tilde{M}y} \int_{\xi - r_0}^{\xi} Q(x-y) \mathrm{d}x \mathrm{d}y \\ &\geqslant r_0 \int_{-\infty}^{-2r_0} J_i(y) \mathrm{e}^{\tilde{M}y} Q(\xi - r_0 - y) \mathrm{d}y \\ &\geqslant r_0 \int_{-\infty}^{-2r_0} J_i(y) \mathrm{e}^{\tilde{M}y} \mathrm{d}y Q(\xi + r_0).\end{aligned}$$

Since $-2r_0 > -r$, it holds that $\int_{-\infty}^{-2r_0} J_i(y) \mathrm{e}^{\tilde{M}y} \mathrm{d}y > 0$. Let $\sigma_0 = \{r_0 \int_{-\infty}^{-2r_0} J_i(y) \mathrm{e}^{\tilde{M}y} \mathrm{d}y\}^{-1}$. Thus, according to the non-negativity of Q, we have

$$Q(\xi + r_0) \leqslant \sigma_0 Q(\xi), \quad \forall \xi \in \mathbb{R}. \tag{3.15}$$

Note that

$$\int_{\mathbb{R}} J_i(y) \frac{Z(\xi - y)}{Z(\xi)} \mathrm{d}y = \int_{\mathbb{R}} J_i(y) \mathrm{e}^{\tilde{M}y} \frac{Q(\xi - y)}{Q(\xi)} \mathrm{d}y.$$

Thus, it follows from (3.14) and (3.15) that

$$\begin{aligned}\int_{\mathbb{R}} J_i(y) \frac{Z(\xi - y)}{Z(\xi)} \mathrm{d}y &= \int_{-\infty}^{0} J_i(y) \mathrm{e}^{\tilde{M}y} \frac{Q(\xi - y)}{Q(\xi)} \mathrm{d}y + \int_0^{\infty} J_i(y) \mathrm{e}^{\tilde{M}y} \frac{Q(\xi - y)}{Q(\xi)} \mathrm{d}y \\ &\leqslant \int_{-\infty}^{0} J_i(y) \mathrm{e}^{\tilde{M}y} \frac{Q(\xi - y)}{Q(\xi)} \mathrm{d}y + \int_0^{\infty} J_i(y) \mathrm{e}^{\tilde{M}y} \mathrm{d}y \\ &\leqslant \sigma_0 \int_{-\infty}^{0} J_i(y) \mathrm{e}^{\tilde{M}y} \frac{Q(\xi - r_0 - y)}{Q(\xi)} \mathrm{d}y + \int_0^{\infty} J_i(y) \mathrm{e}^{\tilde{M}y} \mathrm{d}y \\ &\leqslant \frac{\sigma_0}{r_0} + \int_0^{\infty} J_i(y) \mathrm{e}^{\tilde{M}y} \mathrm{d}y.\end{aligned}$$

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Additionally,

$$\int_{\mathbb{R}} J_i(y)\frac{Z(\xi-y)}{Z(\xi)}\mathrm{d}y \geqslant \int_{-\infty}^{0} J_i(y)\mathrm{e}^{\tilde{M}y}\mathrm{d}y.$$

This completes the proof. □

The following two lemmas describe the persistence of travelling waves.

Lemma 3.12. $\inf_{\xi\in\mathbb{R}} u(\xi) > 0$.

Proof. Assume, on the contrary, that $\inf_{\xi\in\mathbb{R}} u(\xi) = 0$. It follows that there exists some real number sequence $\{\xi_n\}$ such that $u(\xi_n) \to 0$ as $n \to +\infty$. Let $u_n(\xi) := u(\xi+\xi_n)$ and $v_n(\xi) := v(\xi+\xi_n)$ in $\mathbb{R}$. Then, $0 < u_n(\xi) < K$ in $\mathbb{R}$, $\lim_{\xi\to-\infty} u_n(\xi) = K$ for each given n and $0 < v_n(\xi) < +\infty$ in $\mathbb{R}$. Since $u_n(\xi)$ and $v_n(\xi)$ satisfy

$$cu_n'(\xi) = d_1\left(\int_{\mathbb{R}} J_1(y)u_n(\xi-y)\mathrm{d}y - u_n\right) + g(u_n) - f(u_n,v_n)v_n,$$

there exists some subsequence, still denoted by $\{u_n\}$ and $\{v_n\}$, such that $u_n \to u_\infty$ and $v_n \to v_\infty$ locally uniformly in $C^1(\mathbb{R})$ for some functions u_∞, v_∞ as $n \to +\infty$. Note that $u_\infty(0) = 0$ and u_∞ satisfies

$$cu_\infty'(\xi) = d_1\left(\int_{\mathbb{R}} J_1(y)u_\infty(\xi-y)\mathrm{d}y - u_\infty\right) + g(u_\infty) - f(u_\infty,v_\infty)v_\infty. \tag{3.16}$$

In view of $0 < u_n < K$ in $\mathbb{R}$, we have $u_\infty'(0) = 0$. Thus, (3.16) gives that

$$d_1\int_{\mathbb{R}} J_1(y)u_\infty(-y)\mathrm{d}y + g(u_\infty(0)) = 0. \tag{3.17}$$

Note that (3.17) is impossible if $g(0) > 0$.

If $g(0) = 0$, the equation (3.17) gives that $u_\infty(y) \equiv 0$ in $\mathbb{R}$. Then, applying the second equation of (3.1) yields that

$$cv_\infty'(\xi) = d_2\left(\int_{\mathbb{R}} J_2(y)v_\infty(\xi-y)\mathrm{d}y - v_\infty\right) - \delta v_\infty - \gamma v_\infty^2 \quad \text{in } \mathbb{R}. \tag{3.18}$$

For any $x, z \in \mathbb{R}$, integrating both sides of the above equation from x to z, we have

$$\begin{aligned} c[v_\infty(z) - v_\infty(x)] = d_2\int_x^z\int_{\mathbb{R}} J_2(y)(v_\infty(\xi-y) - v_\infty(\xi))\mathrm{d}y\mathrm{d}\xi \\ - \delta\int_x^z v_\infty\mathrm{d}\xi - \gamma\int_x^z v_\infty^2\mathrm{d}\xi. \end{aligned} \tag{3.19}$$

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Note that

$$\begin{aligned}
&\int_x^z \int_{\mathbb{R}} J_2(y)(v_\infty(\xi-y)-v_\infty(\xi))\mathrm{d}y\mathrm{d}\xi \\
&\quad = \int_{\mathbb{R}} J_2(y) \int_x^z (v_\infty(\xi-y)-v_\infty(\xi))\mathrm{d}\xi\mathrm{d}y \\
&\quad = \int_{\mathbb{R}} J_2(y)(-y) \int_x^z \int_0^1 v_\infty'(\xi-\theta y)\mathrm{d}\theta\mathrm{d}\xi\mathrm{d}y \\
&\quad = \int_{\mathbb{R}} J_2(y)(-y) \int_0^1 [v_\infty(z-\theta y)-v_\infty(x-\theta y)]\mathrm{d}\theta\mathrm{d}y.
\end{aligned}$$

Since v_∞ is bounded in $\mathbb{R}$, some constant $M>0$ exists such that $\sup_{\mathbb{R}} v_\infty(\xi) < M$. Thus, applying (3.19), we have

$$\delta \int_x^z v_\infty \mathrm{d}\xi + \gamma \int_x^z v_\infty^2 \mathrm{d}\xi \leqslant 2d_2 M \int_{\mathbb{R}} J_2(y)|y|\mathrm{d}y + 2cM.$$

This gives that $v_\infty \in L^1(\mathbb{R})$. Further, in view of (3.18), $v'(\xi)$ is bounded in $\mathbb{R}$. Therefore, $v_\infty(\pm\infty)=0$. Now, integrating (3.18) over $\mathbb{R}$, we obtain

$$\delta \int_x^z v_\infty \mathrm{d}\xi + \gamma \int_x^z v_\infty^2 \mathrm{d}\xi = 0,$$

which gives that $v_\infty(\xi)=0$ in $\mathbb{R}$. Now, define

$$\Xi_n(\xi) := \frac{u_n(\xi)}{u(\xi_n)} = \exp\left\{ \int_{\xi_n}^{\xi+\xi_n} \frac{u'(s)}{u(s)}\mathrm{d}s \right\}.$$

Since $u(\xi)$ satisfies

$$c\frac{u'(\xi)}{u(\xi)} = d_1\left(\int_{\mathbb{R}} J_1(y)\frac{u(\xi-y)}{u(\xi)}\mathrm{d}y - 1 \right) + \frac{g(u)}{u} - \frac{f(u,v)v}{u},$$

it follows from (A1), (A2) and lemma 3.11 that $|(u'(\xi))/(u(\xi))|$ is bounded in $\mathbb{R}$. Since $\Xi_n(\xi)$ satisfy

$$c\Xi_n'(\xi) = d_1\left(\int_{\mathbb{R}} J_1(y)\Xi_n(\xi-y)\mathrm{d}y - \Xi_n(\xi) \right) + \frac{g(u_n(\xi))}{u(\xi_n)} - \frac{f(u_n,v_n)v_n(\xi)}{u(\xi_n)},$$

one can get that $\Xi_n(\xi)$ is locally uniformly bounded in $C^1(\mathbb{R})$. Up to extraction of a subsequence, there is some function $\Xi_\infty(\xi)$ such that $\Xi_n(\xi) \to \Xi_\infty(\xi)$ in $C_{\mathrm{loc}}(\mathbb{R})$ as $n \to +\infty$. Thus, letting $n \to +\infty$, we have

$$c\Xi_\infty'(\xi) = d_1\left(\int_{\mathbb{R}} J_1(y)\Xi_\infty(\xi-y)\mathrm{d}y - \Xi_\infty(\xi) \right) + g'(0)\Xi_\infty(\xi). \tag{3.20}$$

We then claim that $\Xi_\infty(\xi) > 0$ in $\mathbb{R}$. In fact, if some ξ_* exists so that $\Xi_\infty(\xi_*) = 0$, it follows from (3.20) that $\int_{\mathbb{R}} J_1(y)\Xi_\infty(\xi_*-y)\mathrm{d}y = 0$. This implies that $\Xi_\infty(y) = 0$

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in $\mathbb{R}$ and contradicts the fact that $\Xi_\infty(0) = 1$. Let $\Pi(\xi) = (\Xi'_\infty(\xi))/(\Xi_\infty(\xi))$. Then $\Pi(\xi)$ satisfies

$$c\Pi(\xi) = d_1 \int_{\mathbb{R}} J_1(y) \mathrm{e}^{\int_\xi^{\xi-y} \Pi(s)\mathrm{d}s} \mathrm{d}y - d_1 + g'(0).$$

According to lemma 3.10, $\Pi(\pm\infty)$ exist and satisfy

$$c\Pi(\pm\infty) = d_1 \int_{\mathbb{R}} J_1(y) \mathrm{e}^{-\Pi(\pm\infty)y} \mathrm{d}y - d_1 + g'(0).$$

Since $g'(0) > 0$, we have $\Pi(\pm\infty) > 0$. By the definition of $\Pi(\xi)$, some $\xi_0 < 0$ exists such that $\Xi'_\infty(\xi) > 0$ for any $\xi \leqslant \xi_0$. However, since $0 < u_n < K$ in $\mathbb{R}$ and $u_n(-\infty) = K$, for each given n, some $\xi_1 < 0$ exists so that $u'_n(\xi) \leqslant 0$ for any $\xi \leqslant \xi_1$. Thus, we can obtain a contradiction by taking $\xi \leqslant \min\{\xi_0, \xi_1\}$. The proof is completed. □

Lemma 3.13. $\liminf\limits_{\xi \to +\infty} v(\xi) > 0$.

The above lemma is a straightforward consequence of the following important observation.

Lemma 3.14. *Let $0 < c_1 \leqslant c_2$ be given and (u, v) be a solution of system (3.1) with speed $c \in [c_1, c_2]$ satisfying $0 < u < K$ and $v > 0$. Then there exists some $\tau > 0$ such that $v'(\xi) > 0$ provided that $v(\xi) \leqslant \tau$ for $\xi \in \mathbb{R}$.*

Proof. On the contrary, assume that there is no such τ. Choose a sequence $\{c_k\}$ of real numbers such that $c_k \in [c_1, c_2]$ for each $k \in \mathbb{N}$ and let $\{(u_k, v_k)\}$ be the associated solutions of system (3.1) with $0 < u_k < K$ and $v_k > 0$. Thus, there is a sequence $\{\xi_k\}$ so that $v_k(\xi_k) \to 0$ as $k \to +\infty$ and $v'_k(\xi_k) \leqslant 0$ for all $k \in \mathbb{N}$. Up to extraction of a subsequence, one can assume without loss of generality that $\xi_k = 0$ for all $k \in \mathbb{N}$ and $c_k \to c_\infty \in [c_1, c_2]$ as $k \to +\infty$.

Since $v_k(0) \to 0^+$ as $k \to +\infty$, it follows that

$$v_k \to 0 \text{ locally uniformly in } \mathbb{R} \text{ as } k \to +\infty.$$

By the second equation of (3.1), we have

$$|v'_k(\xi)| \leqslant \frac{d_2}{c_k} \int_{\mathbb{R}} J_2(y) v_k(\xi - y)\mathrm{d}y + \frac{1}{c_k}(d_2 + \alpha f(K, 0) + \delta + \gamma\overline{v}) v_k(\xi),$$

and then $v'_k(\xi) \to 0$ locally uniformly in $\mathbb{R}$ as $k \to +\infty$. Additionally, there exists a function u_∞ such that $u_k \to u_\infty$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ as $k \to +\infty$ and $0 \leqslant u_\infty \leqslant K$ solving

$$c_\infty u'_\infty = d_1 \left(\int_{\mathbb{R}} J_1(\xi - y) u_\infty(y)\mathrm{d}y - u_\infty \right) + g(u_\infty) \quad \text{in } \mathbb{R}.$$

Let $\alpha_0 = \inf\limits_{\mathbb{R}} u_\infty$ and $\{\zeta_m\}$ be sequence of real numbers so that $u_\infty(\zeta_m) \to \alpha_0$ as $m \to +\infty$. Up to extraction of a subsequence, the functions $\xi \mapsto u_\infty(\xi + \zeta_m)$

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converge as $m \to +\infty$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ to a function Φ_∞ solving

$$c_\infty \Phi'_\infty = d_1 \left(\int_{\mathbb{R}} J_1(\xi - y)\Phi_\infty(y)\mathrm{d}y - \Phi_\infty \right) + g(\Phi_\infty) \quad \text{in } \mathbb{R}.$$

Notice that $\alpha_0 \leqslant \Phi_\infty \leqslant K$ in $\mathbb{R}$ and $\Phi_\infty(0) = \alpha_0$. Consequently, $\Phi'_\infty(0) = 0$ and

$$\int_{\mathbb{R}} J_1(-y)\Phi_\infty(y)\mathrm{d}y - \Phi_\infty(0) \geqslant 0,$$

hence $g(\Phi_\infty(0)) \leqslant 0$ and then $\alpha_0 \geqslant K$. Since $\alpha_0 = \inf_{\mathbb{R}} u_\infty$ and $u_\infty \leqslant K$, we conclude that $u_\infty = K$ in $\mathbb{R}$.

Now, set $\Psi_k(\xi) = (v_k(\xi))/(v_k(0))$ for $k \in \mathbb{N}$ and $\xi \in \mathbb{R}$. Since $v_k(\xi)$ is bounded and positive in $\mathbb{R}$, by the similar analysis as in lemma 3.11, we can get that $(v'_k(\xi))/(v_k(\xi))$ is uniformly bounded in $\mathbb{R}$. Noting that

$$\Psi_k(\xi) = \frac{v_k(\xi)}{v_k(0)} = \exp\left\{ \int_0^\xi \frac{v'_k(s)}{v_k(s)}\mathrm{d}s \right\},$$

we know $\Psi_k(\xi)$ is locally uniformly bounded in $\mathbb{R}$. Therefore, the functions

$$\Psi'_k(\xi) = \frac{v'_k(\xi)}{v_k(\xi)}\Psi_k(\xi)$$

are also locally unifomly bounded in $\mathbb{R}$. Moreover, since $\Psi_k(\xi)$ satisfies

$$\begin{aligned} c_k \Psi'_k(\xi) = d_2 \left(\int_{\mathbb{R}} J_2(\xi - y)\Psi_k(y)\mathrm{d}y - \Psi_k(\xi) \right) + \alpha f(u_k, v_k)\Psi_k(\xi) \\ - \delta\Psi_k(\xi) - \gamma v_k(\xi)\Psi_k(\xi), \end{aligned}$$

we know $\Psi_k(\xi)$ is bounded in $C^2_{\mathrm{loc}}(\mathbb{R})$. Then, the Arzela-Ascoli theorem gives that, up to extraction of a subsequence, the positive functions Ψ_k converge in $C^1_{\mathrm{loc}}(\mathbb{R})$ to a non-negative solution Ψ_∞ of equation

$$c_\infty \Psi'_\infty(\xi) = d_2 \left(\int_{\mathbb{R}} J_2(\xi - y)\Psi_\infty(y)\mathrm{d}y - \Psi_\infty(\xi) \right) + (\alpha f(K, 0) - \delta)\Psi_\infty(\xi) \quad (3.21)$$

in $\mathbb{R}$. Thus, $\Psi_\infty(\xi) > 0$ in $\mathbb{R}$. In fact, if there is some $\xi_0 \in \mathbb{R}$ so that $\Psi_\infty(\xi_0) = 0$, then $\Psi'_\infty(\xi_0) = 0$. It follows from (3.21) that

$$\int_{\mathbb{R}} J_2(\xi_0 - y)\Psi_\infty(y)\mathrm{d}y = 0,$$

and this implies $\Psi_\infty(y) \equiv 0$ in $\mathbb{R}$, which contradicts the fact that $\Psi_\infty(0) = 1$.

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Below, define $z := (\Psi'_\infty/\Psi_\infty)$. Thus, $z(\xi)$ satisfies

$$c_\infty z(\xi) = d_2 \int_{\mathbb{R}} J_2(y) \mathrm{e}^{\int_\xi^{\xi-y} z(s)\mathrm{d}s} \mathrm{d}y + \alpha f(K,0) - \delta - d_2. \tag{3.22}$$

According to lemma 3.10, $z(\xi)$ has finite limits $z(\pm\infty)$ as $\xi \to \pm\infty$ satisfying

$$c_\infty z(\pm\infty) = d_2 \int_{\mathbb{R}} J_2(y) \mathrm{e}^{-z(\pm\infty)y} \mathrm{d}y + \alpha f(K,0) - \delta - d_2.$$

From lemma 3.1, $z(\pm\infty)$ are necessarily positive. Indeed, by definition of z, Ψ'_∞ is also positive at $\pm\infty$. Further, differentiating both sides of (3.22) on ξ, one gives

$$\begin{aligned} c_\infty z'(\xi) &= d_2 \int_{\mathbb{R}} J_2(y)(z(\xi-y) - z(\xi)) \mathrm{e}^{\int_\xi^{\xi-y} z(s)\mathrm{d}s} \mathrm{d}y \\ &= d_2 \int_{\mathbb{R}} J_2(y)(z(\xi-y) - z(\xi)) \frac{\Psi_\infty(\xi-y)}{\Psi_\infty(\xi)} \mathrm{d}y \quad \text{in } \mathbb{R}. \end{aligned}$$

Therefore, if z has a minimum or maximum point ξ_* in $\mathbb{R}$, then $z'(\xi_*) = 0$. It follows that $z(\xi_* - y) = z(\xi_*)$ for all $y \in \mathbb{R}$. That is, $z(\xi) = z(\xi_*)$ for all $\xi \in \mathbb{R}$. Hence, following (3.22) and lemma 3.1, there are two different positive roots of (3.22) if $z(\xi)$ is a constant. Consequently, we have

$$\inf_{\mathbb{R}} z \geqslant \min\{z(-\infty), z(+\infty)\} > 0.$$

This gives that $\Psi'_\infty > 0$ in $\mathbb{R}$. Thus,

$$0 < \Psi'_\infty(0) = \lim_{k\to+\infty} \Psi'_k(0) = \lim_{k\to+\infty} \frac{v'_k(0)}{v_k(0)}$$

and $v'_k(0) > 0$ for k large enough. This contradicts the fact that $v'_k(0) \leqslant 0$ for all $k \in \mathbb{N}$ and then the proof is completed. □

Remark 3.15. Note that lemma 3.14 applied with $c_1 = c_*$ and $c_2 = c_* + 1$ yields the existence of $\tau > 0$ such that $v'(\xi) > 0$ provided that $v(\xi) \leqslant \tau$ for $\xi \in \mathbb{R}$. This will be used to show the persistence of travelling waves with $c = c_*$ below.

Lemma 3.16. $\limsup_{\xi\to+\infty} u(\xi) < K$.

Proof. Assume there exists a sequence $\{\xi_n\}$ converging to $+\infty$ as $n \to +\infty$ such that $u(\xi_n) \to K$ as $n \to +\infty$. Denote $u_n(\xi) := u(\xi + \xi_n)$ and $v_n(\xi) := v_n(\xi + \xi_n)$. Thus, up to extraction of a subsequence, $u_n(\xi) \to u_\infty(\xi)$ and $v_n(\xi) \to v_\infty(\xi)$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ for some non-negative functions u_∞ and v_∞. Furthermore, $0 < u_\infty \leqslant K$ and $v_\infty > 0$ according to theorem 3.9 and lemmas 3.12, 3.13. Since $u_\infty(0) = K$, we have $u'_\infty(0) = 0$. Thus, following the first equation of (3.1), there holds

$$0 = d_1 \left(\int_{\mathbb{R}} J_1(y) u_\infty(-y) \mathrm{d}y - K \right) - f(K, v_\infty) v_\infty.$$

This is impossible because

$$\int_{\mathbb{R}} J_1(y) u_\infty(-y) \mathrm{d}y - K \leqslant 0 \quad \text{and} \quad f(K, v_\infty) v_\infty > 0.$$

The proof is finished. □

Denote $u(+\infty) := \lim_{\xi \to +\infty} u(\xi)$ and $v(+\infty) := \lim_{\xi \to +\infty} v(\xi)$. Then, the following result states the convergence to the positive equilibrium under certain conditions.

LEMMA 3.17. *If $u(+\infty)$ (or $v(+\infty)$) exists, then both $v(+\infty)$ and $u(+\infty)$ exist, and $u(+\infty) = u^*$, $v(+\infty) = v^*$.*

Proof. First, suppose $\lim\limits_{\xi \to +\infty} u(\xi) = u_0$. Obviously, $u_0 > 0$ according to lemma 3.12. Assume on the contrary that

$$\underline{\tilde{v}} := \liminf_{\xi \to +\infty} v(\xi) < \limsup_{\xi \to +\infty} v(\xi) := \overline{\tilde{v}}.$$

Then, some sequences $\{\xi_n^1\}$ and $\{\xi_n^2\}$ exist satisfying $\xi_n^1 \to +\infty$ and $\xi_n^2 \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n \to +\infty} v(\xi_n^1) = \underline{\tilde{v}}, \;\; v'(\xi_n^1) = 0 \quad \text{and} \quad \lim_{n \to +\infty} v(\xi_n^2) = \overline{\tilde{v}}, \;\; v'(\xi_n^2) = 0.$$

Applying the first equation of (3.1), we have

$$g(u_0) - f(u_0, \underline{\tilde{v}})\underline{\tilde{v}} = 0 \quad \text{and} \quad g(u_0) - f(u_0, \overline{\tilde{v}})\overline{\tilde{v}} = 0.$$

On the other hand, since

$$\begin{cases} 0 = cv'(\xi_n^1) = d_1 \int_{\mathbb{R}} J_1(y)(v(\xi_n^1 - y) - v(\xi_n^1))\mathrm{d}y + \alpha f(u,v) v(\xi_n^1) - \delta v(\xi_n^1) - \gamma v^2(\xi_n^1), \\ 0 = cv'(\xi_n^2) = d_2 \int_{\mathbb{R}} J_2(y)(v(\xi_n^2 - y) - v(\xi_n^2))\mathrm{d}y + \alpha f(u,v) v(\xi_n^2) - \delta v(\xi_n^2) - \gamma v^2(\xi_n^2), \end{cases}$$

it follows that

$$\alpha f(u_0, \overline{\tilde{v}})\overline{\tilde{v}} - \delta \overline{\tilde{v}} - \gamma \overline{\tilde{v}}^2 \geqslant 0 \geqslant \alpha f(u_0, \underline{\tilde{v}})\underline{\tilde{v}} - \delta \underline{\tilde{v}} - \gamma \underline{\tilde{v}}^2$$

as $n \to +\infty$. In view of the fact that $\alpha f(u_0, \underline{\tilde{v}})\underline{\tilde{v}} = \alpha f(u_0, \overline{\tilde{v}})\overline{\tilde{v}}$, we have

$$0 \geqslant \delta \overline{\tilde{v}} + \gamma \overline{\tilde{v}}^2 - \delta \underline{\tilde{v}} - \gamma \underline{\tilde{v}}^2 = (\overline{\tilde{v}} - \underline{\tilde{v}})[\delta + \gamma(\overline{\tilde{v}} + \underline{\tilde{v}})] > 0,$$

which is a contradiction. Consequently, we have $\overline{\tilde{v}} = \underline{\tilde{v}}$. That is, $v(+\infty)$ exists. Following the assumption (A4) and lemma 3.13, there hold $\lim\limits_{\xi \to +\infty} u(\xi) = u^*$ and $\lim\limits_{\xi \to +\infty} v(\xi) = v^*$.

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On the other hand, assume $\lim_{\xi\to+\infty} v(\xi) = v_0$. According to lemma 3.13, we know $v_0 > 0$. For any sequence $\{\xi_n\}_{n=1}^{\infty}$ satisfying $\xi_n \to +\infty$ as $n \to +\infty$, let $u_n(\xi) := u(\xi_n + \xi)$ and $v_n(\xi) := v(\xi_n + \xi)$. Thus, some function $u_\infty(\xi)$ exists such that

$$u_n(\xi) \to u_\infty(\xi) \text{ and } v_n(\xi) \to v_0 \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}) \text{ as } n \to +\infty.$$

Since (u_n, v_n) satisfy

$$cv_n'(\xi) = d_2\left(\int_{\mathbb{R}} J_2(y)v_n(\xi - y)\mathrm{d}y - v_n\right) + \alpha f(u_n, v_n)v_n - \delta v_n - \gamma v_n^2,$$

letting $n \to +\infty$, we have

$$0 = \alpha f(u_\infty, v_0)v_0 - \delta v_0 - \gamma v_0^2.$$

That is

$$f(u_\infty(\xi), v_0) = \frac{1}{\alpha}(\delta + \gamma v_0) \text{ in } \mathbb{R}.$$

Now, following the assumption (A2) and lemma 3.13, there is some constant $\tilde{u}_0 > 0$ so that $u_\infty(\xi) \equiv \tilde{u}_0$ in $\mathbb{R}$. Thus, by arbitrariness of the sequence $\{\xi_n\}$, we have $\lim_{\xi\to+\infty} u(\xi) = \tilde{u}_0$. Consequently, the same arguments as above can get $\tilde{u}_0 = u^*$ and $v_0 = v^*$. This ends the proof. □

In summary, the main result of this section is the following.

THEOREM 3.18. *For any $c > c_*$, there exist travelling waves of system (1.1) satisfying*

$$0 < u < K,\ v > 0 \quad \text{in } \mathbb{R}$$

and

$$\lim_{\xi\to-\infty} u(\xi) = K, \qquad \lim_{\xi\to-\infty} v(\xi) = 0 \tag{3.23}$$

together with

$$\begin{aligned} 0 < \liminf_{\xi\to+\infty} u(\xi) \leqslant \limsup_{\xi\to+\infty} u(\xi) < K, \\ 0 < \liminf_{\xi\to+\infty} v(\xi) \leqslant \limsup_{\xi\to+\infty} v(\xi) < +\infty. \end{aligned} \tag{3.24}$$

3.3. Travelling waves for $c = c_*$

This subsection is devoted to the existence of travelling wave solution (u, v) for system (1.1) satisfying the asymptotic behaviour (3.23) and (3.24) with the critical wave speed c_*. This proof depends on a limiting argument. That is, choose a sequence $\{c_n\}$ firstly, and then pass to the limit $c_n \to c_*^+$. Noting that the lower solutions $\underline{u}$ and $\underline{v}$ depend on c_n, they will be degenerate as $c_n \to c_*^+$. Thus, we need to find a new method to prove the asymptotic behaviour (3.23).

THEOREM 3.19. *When $c = c_*$, system (1.1) admits a positive bounded travelling wave solution $(u(\xi), v(\xi))$ satisfying the asymptotic behaviour (3.23) and (3.24).*

Proof. Choose some sequence $\{c_n\}$ of real numbers such that $c_n \in (c_*, c_* + 1]$ with $c_n > c_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} c_n = c_*$. Let (u_n, v_n) be the solutions of (3.1) associated with c_n. It then follows from theorem 3.18 that $0 < u_n < K, 0 < v_n < +\infty$ and (u_n, v_n) satisfy (3.23)-(3.24).

To complete the proof, we first claim that $\liminf_{n\to+\infty} \|v_n\|_{L^\infty(\mathbb{R})} > 0$. Otherwise, up to extraction of a subsequence, assume $\lim_{n\to+\infty} \|v_n\|_{L^\infty(\mathbb{R})} = 0$ without loss of generality. Denote $\nu_n := \|v_n\|_{L^\infty(\mathbb{R})}$. Hence, there exist some $\tau_0 > 0$ small enough and $n_0 > 0$ large enough such that $\nu_n < \tau_0$ for all $n \geqslant n_0$. Now, the same arguments as in lemma 3.14 give that $v_n'(\xi) > 0$ for $n \geqslant n_0$ (see also remark 3.15). This implies that $\lim_{\xi\to+\infty} v_n(\xi)$ exists for $n \geqslant n_0$. Thus, following lemma 3.17, we know

$$\lim_{\xi\to+\infty} v_n(\xi) = v^* > 0,$$

for all $n \geqslant n_0$, which contradicts the fact that $\|v_n\|_{L^\infty(\mathbb{R})} \to 0$ as $n \to +\infty$. Therefore, due to the fact that $v_n(-\infty) = 0$ and $v_n > 0$ for each $n \in \mathbb{N}$, there is some $\xi_n \in \mathbb{R}$ so that $v_n(\xi_n) = \tau_*$ for some sufficiently small $0 < \tau_* \leqslant v^*$. Let

$$\tilde{u}_n(\xi) := u_n(\xi + \xi_n) \quad \text{and} \quad \tilde{v}_n(\xi) := v_n(\xi + \xi_n)$$

for all $\xi \in \mathbb{R}$. Without loss of generality, one can assume $\tilde{v}_n(\xi) \leqslant \tau_*$ for all $\xi < 0$. Note that both $\tilde{u}_n(\cdot)$ and $\tilde{v}_n(\cdot)$ are all uniformly bounded in $C^2(\mathbb{R})$. Thus, up to extracting a subsequence for necessary, $(\tilde{u}_n(\cdot), \tilde{v}_n(\cdot))$ converge to $(u(\cdot), v(\cdot))$ in $C^1_{\text{loc}}(\mathbb{R}) \times C^1_{\text{loc}}(\mathbb{R})$ as $n \to +\infty$. Here, $(u(\xi), v(\xi))$ satisfies (3.1) with wave speed c_*. Meanwhile, $0 \leqslant u(\xi) \leqslant K$, $v(\xi) \geqslant 0$ and $v(0) = \tau_*$.

Below, we prove that $0 < u(\xi) < K$ and $v(\xi) > 0$ in $\mathbb{R}$. Assume there is some $\xi_0 \in \mathbb{R}$ so that $v(\xi_0) = 0$. Then $v'(\xi_0) = 0$. Applying the second equation of system (3.1), we have

$$d_2 \int_{\mathbb{R}} J_2(\xi_0 - y)v(y)\mathrm{d}y = 0.$$

Thus, $v(y) \equiv 0$ in $\mathbb{R}$. This contradicts the fact $v(0) = \tau_* > 0$. Additionally, if some $\xi_* \in \mathbb{R}$ exists so that $u(\xi_*) = 0$, then $u'(\xi_*) = 0$. The first equation of system (3.1) gives that

$$0 = cu'(\xi_*) = d_1 \int_{\mathbb{R}} J_1(y)u(\xi_* - y)\mathrm{d}y + g(0). \tag{3.25}$$

Seen from (3.25), a direct contradiction happens if $g(0) > 0$. On the other hand, if $g(0) = 0$, then (3.25) implies that $u(y) \equiv 0$ in $\mathbb{R}$. In this case, it follows from the second equation of system (3.1) that

$$c_* v'(\xi) = d_2 \int_{\mathbb{R}} J_2(y)(v(\xi - y) - v(\xi))\mathrm{d}y - \delta v - \gamma v^2. \tag{3.26}$$

Analogous arguments to those presented in lemma 3.12, we know $v \in L^1(\mathbb{R})$. Then, integrating (3.26) on $\mathbb{R}$, one can get $v \equiv 0$ in $\mathbb{R}$. This contradicts the fact that $v > 0$ in $\mathbb{R}$ and it then follows that $u > 0$ in $\mathbb{R}$. Moreover, we also can get $u < K$ in $\mathbb{R}$.

Finally, we show the asymptotic behaviour. The similar discussion as lemmas 3.12 and 3.13 can show that

$$\liminf_{\xi\to+\infty} u(\xi) > 0 \quad \text{and} \quad \liminf_{\xi\to+\infty} v(\xi) > 0.$$

On the other hand, by the choice of τ_* and $v(0) = \tau_*$, we have $v'(\xi) > 0$ for $\xi < 0$. Thus, $\lim\limits_{\xi\to-\infty} v(\xi)$ exists, denoted by $v(-\infty) := \lim\limits_{\xi\to-\infty} v(\xi)$. If $v(-\infty) > 0$, then lemma 3.17 gives that $v(-\infty) = v^* \geqslant \tau_*$ and $u(-\infty) = u^*$. A contradiction happens because $v(-\infty) < \tau_*$. This implies that $v(-\infty) = 0$. Further, assume

$$u_{\inf} := \liminf_{\xi\to-\infty} u(\xi) < \limsup_{\xi\to-\infty} u(\xi) =: u_{\sup} \leqslant K$$

for contradiction. Thus, there is some real number sequence $\{\tilde{\xi}_n\}$ satisfying $\lim\limits_{n\to+\infty} \tilde{\xi}_n = -\infty$ such that $\lim\limits_{n\to+\infty} u(\tilde{\xi}_n) = u_{\inf}$ and $u'(\tilde{\xi}_n) = 0$. We then have

$$0 = c_* u'(\tilde{\xi}_n) = d_1 \int_{\mathbb{R}} J_1(y) u(\tilde{\xi}_n - y)\mathrm{d}y - d_1 u(\tilde{\xi}_n) + g(u(\tilde{\xi}_n)) - f(u(\tilde{\xi}_n), v(\tilde{\xi}_n))v(\tilde{\xi}_n).$$

Since $\lim\limits_{n\to+\infty} v(\tilde{\xi}_n) = 0$, letting $n \to +\infty$ on both sides of the above equation, it follows that

$$d_1 u_{\inf} - g(u_{\inf}) = d_1 \lim_{n\to\infty} \int_{\mathbb{R}} J_1(y) u(\tilde{\xi}_n - y)\mathrm{d}y \geqslant d_1 u_{\inf}.$$

Hence, $g(u_{\inf}) \leqslant 0$. In view of $u_{\inf} \geqslant 0$, (A1) implies that $u_{\inf} \geqslant K$. Therefore, we get $\lim\limits_{\xi\to-\infty} u(\xi) = K$. The proof is completed. □

4. The non-existence of travelling waves

Now, we are concerned with the non-existence of travelling waves of system (1.1) when the wave speed is below its critical value.

THEOREM 4.1. *For any $0 < c < c_*$, there exist no bounded non-negative travelling waves of system* (1.1) *with $\lim\limits_{\xi\to-\infty} u(\xi) = K$ and $\lim\limits_{\xi\to-\infty} v(\xi) = 0$.*

Proof. From the second equation of system (3.1), we have

$$\begin{aligned} c\frac{v'(\xi)}{v(\xi)} &= d_2 \int_{\mathbb{R}} J_2(y) \mathrm{e}^{\int_{\xi}^{\xi-y} \frac{v'(s)}{v(s)}\mathrm{d}s}\mathrm{d}y - d_2 + \alpha f(u,v) - \delta - \gamma v \\ &\geqslant d_2 \int_{\mathbb{R}} J_2(y) \mathrm{e}^{\int_{\xi}^{\xi-y} \frac{v'(s)}{v(s)}\mathrm{d}s}\mathrm{d}y - d_2 - (\delta + \gamma \|v\|_{L^\infty(\mathbb{R})}). \end{aligned}$$

Then, it follows from lemma 3.11 that $\frac{v'(\xi)}{v(\xi)}$ is bounded in $\mathbb{R}$. Take some point sequence $\{\xi_n\}_{n=1}^{\infty}$ with $\xi_n \to -\infty$ as $n \to +\infty$ and define

$$u_n(\xi) := u(\xi_n + \xi),\ v_n(\xi) := \frac{v(\xi_n + \xi)}{v(\xi_n)}.$$

Note that

$$v_n(\xi) := \frac{v(\xi_n + \xi)}{v(\xi_n)} = \exp\left\{\int_{\xi_n}^{\xi_n+\xi} \frac{v'(s)}{v(s)}\mathrm{d}s\right\}.$$

Then $v_n(\xi)$ is locally uniformly bounded in $\mathbb{R}$. Moreover, $(u_n(\xi), v_n(\xi))$ satisfy

$$\begin{aligned} cv_n'(\xi) &= d_2 \int_{\mathbb{R}} J_2(y)(v_n(\xi - y) - v_n(\xi))\mathrm{d}y + \alpha f(u_n(\xi), v(\xi_n + \xi))v_n(\xi) \\ &\quad - \delta v_n(\xi) - \gamma v(\xi_n + \xi)v_n(\xi). \end{aligned} \tag{4.1}$$

It is noticed that $v_n'(\xi)$ and $v_n''(\xi)$ are also locally uniformly bounded in $\mathbb{R}$. Thus, some $v_\infty(\xi)$ exists such that

$$v_n(\xi) \to v_\infty(\xi) \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}) \text{ as } n \to +\infty.$$

Since $\lim_{\xi\to-\infty} u(\xi) = K$ and $\lim_{\xi\to-\infty} v(\xi) = 0$, it follows that $\lim_{n\to+\infty} u_n(\xi) = K$ and $\lim_{n\to+\infty} v_n(\xi_n + \xi) = 0$ locally uniformly in $\mathbb{R}$. Now, letting $n \to +\infty$ on both sides of (4.1), we have

$$cv_\infty'(\xi) = d_2 \int_{\mathbb{R}} J_2(y)(v_\infty(\xi - y) - v_\infty(\xi))\mathrm{d}y + (\alpha f(K, 0) - \delta)v_\infty(\xi). \tag{4.2}$$

Since $v_\infty(0) = 1$ by the definition of $v_n(\xi)$, it follows from (4.2) that $v_\infty(\xi) > 0$ in $\mathbb{R}$. Set $z(\xi) = (v_\infty'(\xi))/(v_\infty(\xi))$. Then, applying (4.2) yields that $z(\xi)$ satisfies

$$cz(\xi) = d_2 \int_{\mathbb{R}} J_2(y)\mathrm{e}^{\int_\xi^{\xi-y} z(s)\mathrm{d}s}\mathrm{d}y - d_2 + \alpha f(K, 0) - \delta.$$

According to lemma 3.10, we know $\lim_{\xi\to\pm\infty} z(\xi)$ exist and satisfy the equation

$$c\lambda = d_2 \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda y}\mathrm{d}y - d_2 + \alpha f(K, 0) - \delta.$$

This gives $c > c_*$ and one contradiction happens. The proof is completed. □

5. Conclusions and applications

In this paper, we mainly consider the wave propagation for a class of non-cooperative system with non-local dispersal, which can be applied to some prey-predator models as well as disease-transmission models. The existence of travelling waves is obtained by using the upper-lower solutions and combining with the Schauder's fixed-point theorem, and the non-existence of travelling waves can also be shown by skilled analysis. theorems 3.18, 3.19 and 4.1 combined provide a threshold condition for the existence and non-existence of travelling wave solutions in terms of the minimal wave speed c_*.

Compared with the work in Zhang *et al.* [**45**], we must overcome the difficulties brought both by the non-compactness of solution maps due to the appearance of the convolution operator and by the non-preservation of system (1.1). In this work,

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we exploit a different method to show the existence of travelling waves, which is also effective for system (1.2). Moreover, one of the important issues is that of the estimation of the eventual states of travelling waves. That is the problem of the persistence of travelling waves. The method applied in Zhang *et al.* [**45**] is not suitable for our non-local system (3.1) and we overcome this difficulty by some skilled analysis. It should be pointed out that we can further confirm that the travelling waves should connect two equilibria (strong travelling waves) in some certain cases. As an application, we consider the following disease-transmission model:

$$\begin{cases} S_t = d_1(J_1 * S - S) + b(K - S) - \beta SI, \\ I_t = d_2(J_2 * I - I) + \beta SI - \delta I - \gamma I^2, \end{cases} \tag{5.1}$$

in which $d_1, d_2, b, K, \beta, \delta, \gamma$ are all positive constants. By the simple calculations, we can obtain that (5.1) admits a unique positive constant equilibrium denoted by (S^*, I^*) if $\beta K > \delta$. Let $\xi = x + ct$ and $(S(\xi), I(\xi))$ satisfies

$$\begin{cases} cS'(\xi) = d_1(\int_{\mathbb{R}} J_1(y)S(\xi - y)\mathrm{d}y - S) + b(K - S) - \beta SI, \\ cI'(\xi) = d_2(\int_{\mathbb{R}} J_2(y)I(\xi - y)\mathrm{d}y - I) + \beta SI - \delta I - \gamma I^2. \end{cases} \tag{5.2}$$

Thus, we have the following result.

THEOREM 5.1. *Assume $\beta K > \delta$ and $\beta\delta < b\gamma$. Suppose (J) holds. Then, there is some $c_* > 0$ such that system (5.1) admits a travelling wave solution $(S(\xi), I(\xi))$ with*

$$\begin{aligned} &\lim_{\xi \to -\infty} S(\xi) = K, \qquad \lim_{\xi \to -\infty} I(\xi) = 0, \\ &\lim_{\xi \to +\infty} S(\xi) = S^*, \qquad \lim_{\xi \to +\infty} I(\xi) = I^* \end{aligned} \tag{5.3}$$

for any $c \geqslant c_*$, and admits no travelling waves satisfying (5.3) when $0 < c < c_*$.

Proof. The existence and non-existence of travelling waves are straightforward consequences of theorems 3.18, 3.19 and 4.1. It only needs us to verify the asymptotic behaviour at $+\infty$ if the travelling waves exist.

Consider the following problem

$$\begin{cases} u_t = d_1(J_1 * u - u)(x, t) + b(K - u) - \beta uv & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ v_t = d_2(J_2 * v - v)(x, t) + \beta uv - \delta v - \gamma v^2 & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ u(x, 0) = S(x),\ v(x, 0) = I(x) & \text{in } \mathbb{R}. \end{cases} \tag{5.4}$$

Thus, $(u(x, t), v(x, t)) = (S(x + ct), I(x + ct))$ is the unique solution of problem (5.4). Note that $u(x, t) = S(x + ct) \leqslant K$ for all $x \in \mathbb{R}$ and $t \geqslant 0$. So, we know

$v(x,t) = I(x+ct)$ satisfies

$$\begin{cases} v_t \leqslant d_2(J_2 * v - v)(x,t) + (\beta K - \delta - \gamma v)v & \text{in } \mathbb{R}\times\mathbb{R}^+, \\ v(x,0) = I(x) & \text{in } \mathbb{R}. \end{cases}$$

The comparison principle [36, theorem 2.2] gives that $v(x,t) \leqslant (\beta K - \delta)/\gamma$ for all $x \in \mathbb{R}$ and $t \geqslant 0$. Consequently, $u(x,t) = S(x+ct)$ satisfies

$$\begin{cases} u_t \geqslant d_1(J_1 * u - u)(x,t) + b(K-u) - \frac{\beta(\beta K-\delta)}{\gamma}u & \text{in } \mathbb{R}\times\mathbb{R}^+, \\ u(x,0) = S(x) & \text{in } \mathbb{R}. \end{cases}$$

Therefore, we have

$$u(x,t) \geqslant \frac{bK\gamma}{b\gamma + \beta(\beta K - \delta)} > 0 \text{ in } \mathbb{R}\times\mathbb{R}^+.$$

Now, define

$$S_\Delta := \liminf_{\xi\to+\infty} S(\xi), \quad S^\Delta := \limsup_{\xi\to+\infty} S(\xi),$$
$$I_\Delta := \liminf_{\xi\to+\infty} I(\xi), \quad I^\Delta := \limsup_{\xi\to+\infty} I(\xi).$$

Obviously, it follows from the above discussion that

$$0 < \frac{bK\gamma}{b\gamma + \beta(\beta K - \delta)} \leqslant S_\Delta \leqslant S^\Delta \leqslant K.$$

Assume that $S_\Delta < S^\Delta$ and $I_\Delta < I^\Delta$ without loss of the generality, and the other cases can be obtained similarly. Then, there exist some sequences $\{\xi_n^1\}$ and $\{\xi_n^2\}$, satisfying $\xi_n^1 \to +\infty$ and $\xi_n^2 \to +\infty$ as $n \to +\infty$, such that

$$S'(\xi_n^1) = 0,\ S(\xi_n^1) \to S_\Delta \text{ as } n\to\infty,$$
$$S'(\xi_n^2) = 0,\ S(\xi_n^2) \to S^\Delta \text{ as } n\to\infty.$$

Hence, following the first equation of (5.2), we have

$$\begin{cases} 0 = d_1 \int_{\mathbb{R}} J_1(y)(S(\xi_n^1 - y) - S(\xi_n^1))\mathrm{d}y + b(K - S(\xi_n^1)) - \beta S(\xi_n^1)I(\xi_n^1), \\ 0 = d_1 \int_{\mathbb{R}} J_1(y)(S(\xi_n^2 - y) - S(\xi_n^2))\mathrm{d}y + b(K - S(\xi_n^2)) - \beta S(\xi_n^2)I(\xi_n^2). \end{cases}$$

Letting $n \to \infty$ on both sides of the above equations, we have

$$\begin{cases} 0 \geqslant b(K - S_\Delta) - \beta S_\Delta I^\Delta, \\ 0 \leqslant b(K - S^\Delta) - \beta S^\Delta I_\Delta. \end{cases} \tag{5.5}$$

Similarly, some sequences $\{\eta_n^1\}$ and $\{\eta_n^2\}$ exist, satisfying $\eta_n^1 \to \infty$ and $\eta_n^2 \to \infty$ as $n \to \infty$, such that

$$\begin{cases} 0 = d_2 \int_{\mathbb{R}} J_2(y)(I(\eta_n^1 - y) - I(\eta_n^1))\mathrm{d}y + \beta S(\eta_n^1)I(\eta_n^1) - \delta I(\eta_n^1) - \gamma I^2(\eta_n^1), \\ 0 = d_2 \int_{\mathbb{R}} J_2(y)(I(\eta_n^2 - y) - I(\eta_n^2))\mathrm{d}y + \beta S(\eta_n^2)I(\eta_n^2) - \delta I(\eta_n^2) - \gamma I^2(\eta_n^2). \end{cases}$$

Therefore,

$$\begin{cases} 0 \geqslant (\beta S_\Delta - \delta - \gamma I_\Delta) I_\Delta, \\ 0 \leqslant (\beta S^\Delta - \delta - \gamma I^\Delta) I^\Delta, \end{cases} \tag{5.6}$$

which gives that

$$\begin{cases} \beta S_\Delta \leqslant \delta + \gamma I_\Delta, \\ \beta S^\Delta \geqslant \delta + \gamma I^\Delta. \end{cases} \tag{5.7}$$

This shows that $\beta(S^\Delta - S_\Delta) \geqslant \gamma(I^\Delta - I_\Delta)$. At the same time, by (5.7), we have

$$\begin{cases} \beta S_\Delta I^\Delta \leqslant \delta I^\Delta + \gamma I_\Delta I^\Delta, \\ \beta S^\Delta I_\Delta \geqslant \delta I_\Delta + \gamma I^\Delta I_\Delta. \end{cases}$$

It then follows that $\beta(S_\Delta I^\Delta - S^\Delta I_\Delta) \leqslant \delta(I^\Delta - I_\Delta)$. On the other hand, we can conclude from (5.5) that $\beta(S_\Delta I^\Delta - S^\Delta I_\Delta) \geqslant b(S^\Delta - S_\Delta)$. Hence, we have

$$\frac{\gamma}{\beta}(I^\Delta - I_\Delta) \leqslant \frac{\delta}{b}(I^\Delta - I_\Delta).$$

That is, $(\beta\delta - b\gamma)(I^\Delta - I_\Delta) \geqslant 0$. Since $\beta\delta < b\gamma$, we get $I_\Delta = I^\Delta$ and it is easy to obtain that $S_\Delta = S^\Delta$ according to (5.5). Now, due to the fact that (5.2) admits a unique positive constant equilibrium (S^*, I^*), there are

$$S_\Delta = S^\Delta = S^* \text{ and } I_\Delta = I^\Delta = I^*.$$

This ends the proof. □

Note that the results and the methods in § 3 strongly depend on the assumption (A3). That is, when $\gamma = 0$, it must be required that $\lim_{v \to +\infty} f(u, v) = 0$ for any $u \geqslant 0$. However, for system (5.1), (A3) does not hold when $\gamma = 0$, and this leads to the fact that system (5.1) has no bounded super solution. Therefore, we must discuss the boundedness of $I(\xi)$. Below, we always assume $\gamma = 0$ and $\beta K > \delta$. Additionally, assume that J_2 is **compactly supported** on $\mathbb{R}$. Define

$$f(\lambda, c) = d_2 \int_{\mathbb{R}} J_2(y) \mathrm{e}^{-\lambda y} \mathrm{d}y - d_2 - c\lambda + \beta K - \delta.$$

Similarly, we can show lemma 3.1 holds for $f(\lambda, c)$. Meanwhile, the functions

$$\overline{S}(\xi) = K, \quad \underline{S}(\xi) = \max\{K - \sigma \mathrm{e}^{\beta\xi}, 0\},$$
$$\overline{I}(\xi) = \mathrm{e}^{\lambda_1 \xi}, \ \underline{I}(\xi) = \max\{\mathrm{e}^{\lambda_1 \xi}(1 - M\mathrm{e}^{\varepsilon\xi}), 0\}$$

are super and lower solutions of system (5.2), in which $\sigma, \beta, \varepsilon, M$ are all positive constants. Hence, we have the following results by the analogous arguments as in § 3.

THEOREM 5.2. *Assume $\beta K > \delta$ and $c > c_*$. Then (5.2) admits a travelling wave solution $(S(\xi), I(\xi))$ satisfying*

$$S(-\infty) = K, \quad I(-\infty) = 0, \quad 0 < S(\xi) < K \quad \textit{and} \quad I(\xi) > 0 \quad \textit{in } \mathbb{R}.$$

In the rest of this section, we always assume $(S(\xi), I(\xi))$ is the solution of (5.2) satisfying theorem 5.2 without other description.

THEOREM 5.3. *$I(\xi)$ is bounded in $\mathbb{R}$. Moreover, $(S(\xi), I(\xi))$ satisfies*

$$\liminf_{\xi \to +\infty} S(\xi) > 0 \quad \textit{and} \quad \liminf_{\xi \to +\infty} I(\xi) > 0. \tag{5.8}$$

Before showing this theorem, we prove some technical results, see also [**11**, **40**] for the discrete endemic model.

LEMMA 5.4. *If some sequence $\{\xi_n\}$ exists so that $I(\xi_n) \to \infty$ as $n \to +\infty$, then $S(\xi_n) \to 0$ as $n \to +\infty$.*

Proof. On the contrary, assume there is some subsequence of $\{\xi_n\}_{n\in\mathbb{N}}$, still denoted by $\{\xi_n\}$, such that $S(\xi_n) \geqslant \varepsilon_0$ for some positive constant ε_0 and all $n \in \mathbb{N}$. For the first equation of (5.2), there is $cS'(\xi) \leqslant (2d_1 + b)K$ in $\mathbb{R}$. Hence, some positive constant δ_0 exists so that $S(\xi) \geqslant \varepsilon_0/2$ for $\xi \in [\xi_n - \delta_0, \xi_n]$ and all $n \in \mathbb{N}$. Following lemma 3.11 and the second equation of (5.2), there is some constant $C_0 > 0$ such that

$$\left| \frac{I'(\xi)}{I(\xi)} \right| \leqslant C_0 \quad \text{in } \mathbb{R}.$$

Thus, we have

$$\frac{I(\xi_n)}{I(\xi)} = \exp\left\{ \int_{\xi}^{\xi_n} \frac{I'(s)}{I(s)} \mathrm{d}s \right\} \leqslant \mathrm{e}^{C_0\delta_0} \quad \text{for all } \xi \in [\xi_n - \delta_0, \xi_n].$$

This gives that

$$\min_{\xi \in [\xi_n - \delta_0, \xi_n]} I(\xi) \geqslant I(\xi_n)\mathrm{e}^{-C_0\delta_0} \to \infty \quad \text{as } n \to +\infty.$$

Additionally, applying the first equation of (5.2), one can get

$$\max_{\xi \in [\xi_n - \delta_0, \xi_n]} cS'(\xi) \leqslant (2d_1 + b)K - \frac{\beta\varepsilon_0}{2} \min_{\xi \in [\xi_n - \delta_0, \xi_n]} I(\xi) \to -\infty \quad \text{as } n \to +\infty.$$

Thus, taking $\tilde{M} = 2K/\delta_0$, there is some $n_0 > 0$ such that

$$S'(\xi) < -\tilde{M} \quad \text{for all } \xi \in [\xi_n - \delta_0, \xi_n],$$

provided $n \geqslant n_0$. Integrating this inequality from $\xi_n - \delta_0$ to ξ_n, one can get

$$S(\xi_n) < S(\xi_n - \delta_0) - \delta_0\tilde{M} \leqslant K - \delta_0\tilde{M} = -K,$$

which contradicts the fact that $S(\xi) > 0$ for all $\xi \in \mathbb{R}$. The proof is completed. □

LEMMA 5.5. *If* $\limsup_{\xi\to\infty} I(\xi) = \infty$, *then* $\lim_{\xi\to\infty} I(\xi) = \infty$.

Proof. Assume $I_{\inf} := \liminf_{\xi\to\infty} I(\xi) < \infty$ for the contrary. Then, there exists some sequence $\{\xi_k\}$ such that $\lim_{k\to\infty} I(\xi_k) = I_{\inf}$ with $\xi_k \to \infty$ as $k \to \infty$. Without loss of generality, we can assume $I(\xi_k) \leqslant I_{\inf} + 1$ for all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, choose a point $\eta_k \in [\xi_k, \xi_{k+1}]$ such that $I(\eta_k) = \max_{[\xi_k,\xi_{k+1}]} I(\xi)$. Since $\limsup_{\xi\to\infty} I(\xi) = \infty$, we have $\lim_{k\to\infty} I(\eta_k) = \infty$. Thus, following lemma 5.4, $S(\eta_k) \to 0$ as $k \to \infty$. Denote $m_0 = \sup_{\xi\in\mathbb{R}} |(I'(\xi))/(I(\xi))|$ and assume $I(\eta_k) > (I_{\inf} + 1)\mathrm{e}^{m_0 r}$ without loss of generality, in which r is the radius of *supp*J_2. Note that

$$\frac{I(\eta_k)}{I(\xi)} = \mathrm{e}^{\int_\xi^{\eta_k} \frac{I'(s)}{I(s)}\mathrm{d}s} \leqslant \mathrm{e}^{m_0|\xi-\eta_k|} \leqslant \mathrm{e}^{m_0 r} \quad \text{if} \quad |\xi - \eta_k| \leqslant r.$$

This gives $I(\xi) > I_{\inf} + 1$ for all $\xi \in [\eta_k - r, \eta_k + r]$. Hence, $[\eta_k - r, \eta_k + r] \subset (\xi_k, \xi_{k+1})$. Now, in view of the second equation of (5.2), it follows that

$$\begin{aligned} 0 = cI'(\eta_k) &= d_2 \int_{\mathbb{R}} J_2(y)(I(\eta_k - y) - I(\eta_k))\mathrm{d}y + \beta S(\eta_k)I(\eta_k) - \delta I(\eta_k) \\ &\leqslant (\beta S(\eta_k) - \delta)I(\eta_k), \end{aligned}$$

which contradicts the fact that $-\delta I(\eta_k) < 0$ for all $k \in \mathbb{N}$. The proof is finished. □

The proof of theorem 5.3. Since $I(-\infty) = 0$ and $I(\xi)$ is continuous in $\mathbb{R}$, we only need to prove the case that $\limsup_{\xi\to+\infty} I(\xi) < +\infty$. On the contrary, assume that this is not true. Thus, lemma 5.5 implies that $\lim_{\xi\to+\infty} I(\xi) = +\infty$. Meanwhile, lemma 5.4 gives $\lim_{\xi\to+\infty} S(\xi) = 0$. Let $w(\xi) = (I'(\xi))/(I(\xi))$. It follows from the second equation of (5.2) that $w(\xi)$ satisfies

$$cw'(\xi) = d_2 \int_{\mathbb{R}} J_2(y)\mathrm{e}^{\int_\xi^{\xi-y} w(s)\mathrm{d}s}\mathrm{d}y - d_2 + \beta S(\xi) - \delta \quad \text{in } \mathbb{R}.$$

Since $\lim_{\xi\to-\infty} S(\xi) = K$ and $\lim_{\xi\to+\infty} S(\xi) = 0$, applying lemma 3.10, we know some $\nu_0 \in \mathbb{R}$ exists satisfying $\lim_{\xi\to+\infty} w(\xi) = \nu_0$ and

$$c\nu_0 = d_2 \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\nu_0 y}\mathrm{d}y - d_2 - \delta.$$

Since $I(\xi) \to +\infty$ as $\xi \to +\infty$ and $I(\xi) > 0$ in $\mathbb{R}$, there is $\nu_0 > 0$. Additionally, due to the fact that

$$f(\nu_0, c) = \beta K > 0,$$

we have $\nu_0 > \max\{\lambda_1, \lambda_2\}$. Thus, some $\xi_0 > 0$ exists such that for any $\xi \geqslant \xi_0$

$$I(\xi) \geqslant C\mathrm{e}^{\frac{\lambda_1+\lambda_2}{2}\xi}$$

with some constant $C > 0$ large enough. However, this contradicts the fact that $I(\xi) \leqslant e^{\lambda_1 \xi}$ in $\mathbb{R}$ and then $I(\xi)$ is bounded in $\mathbb{R}$. Once this fact holds, one can get the later results by the arguments analogous to those of § 3. The proof is completed. □

THEOREM 5.6. *Suppose $\beta K > \delta$. If $c = c_*$, then system* (5.2) *admits travelling waves with* (5.8).

Proof. Choose some sequence $c_n \in (c_*, c_* + 1]$ and c_n satisfies $c_n > c_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} c_n = c_*$. Let $(S_n(\xi), I_n(\xi))$ be the solutions of system (5.2) associated with c_n.

Here, seen from the proof of theorem 3.19, we only need to prove that $I_n(\xi)$ is uniformly bounded in $\mathbb{R}$. That is, there is some constant $\tilde{C} > 0$ independent on n so that $\|I_n(\cdot)\|_{L^\infty(\mathbb{R})} < \tilde{C}$. On the contrary, assume some sequence $\{\xi_n\}_{n\in\mathbb{N}}$ exist such that $I_n(\xi_n) \to +\infty$ as $n \to +\infty$, and it then follows from lemma 5.4 that $S_n(\xi_n) \to 0$ as $n \to +\infty$. Without loss of generality, we may assume that $I_n(\xi_n) = \max_{\mathbb{R}} I_n(\xi)$ for each n. Hence, we have $I_n'(\xi_n) = 0$ and

$$\begin{aligned} 0 = c_n I_n'(\xi_n) &= d_2 \int_{\mathbb{R}} J_2(y)(I_n(\xi_n - y) - I_n(\xi_n))\mathrm{d}y + \beta S_n(\xi_n) I_n(\xi_n) - \delta I_n(\xi_n) \\ &\leqslant (\beta S_n(\xi_n) - \delta) I_n(\xi_n). \end{aligned} \tag{5.9}$$

However, since $S_n(\xi_n) \to 0$ as $n \to +\infty$, there is some $n_0 > 0$ large enough so that $\beta S_n(\xi_n) < \delta$ and $I_n(\xi_n) > 0$ for all $n \geqslant n_0$. Thus, applying (5.9) yields a contradiction and the proof is completed. □

As a direct application of theorem 4.1, we have the following non-existence result.

THEOREM 5.7. *Suppose $\beta K > \delta$ and $0 < c < c_*$. Then there are no bounded travelling waves of system* (5.2) *with* (5.8).

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