Free Group Algebras in Division Rings with Valuation II

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Abstract. We apply the filtered and graded methods developed in earlier works to find (noncommutative) free group algebras in division rings.

If L is a Lie algebra, we denote by $U(L)$ its universal enveloping algebra. P. M. Cohn constructed a division ring $\mathfrak{D}_L$ that contains $U(L)$. We denote by $\mathfrak{D}(L)$ the division subring of $\mathfrak{D}_L$ generated by $U(L)$.

Let k be a field of characteristic zero, and let L be a nonabelian Lie k-algebra. If either L is residually nilpotent or $U(L)$ is an Ore domain, we show that $\mathfrak{D}(L)$ contains (noncommutative) free group algebras. In those same cases, if L is equipped with an involution, we are able to prove that the free group algebra in $\mathfrak{D}(L)$ can be chosen generated by symmetric elements in most cases.

Let G be a nonabelian residually torsion-free nilpotent group, and let $k(G)$ be the division subring of the Malcev–Neumann series ring generated by the group algebra $k[G]$. If G is equipped with an involution, we show that $k(G)$ contains a (noncommutative) free group algebra generated by symmetric elements.

1 Introduction

The search for free objects in division rings has been largely motivated by the following two conjectures that still remain open:

(G) If D is a noncommutative division ring, then the multiplicative group $D\backslash\{0\}$ contains a free group of rank two.

(A) If D is a division ring which is infinite dimensional over its center Z and is finitely generated (as a division algebra over Z), then D contains a free Z-algebra of rank two.

Conjecture (G) was stated by A. I. Lichtman in [22] and has been proved when the center of D is uncountable [5] and when D is finite dimensional over its center [16], to name two important instances where it holds true. Conjecture (A) was formulated independently by L. Makar-Limanov in [28] and T. Stafford. Evidence for conjecture (A) has been given in many papers, for example [1, 2, 25, 27, 29]. In many division rings in which conjecture (A) holds, D in fact contains a noncommutative free group Z-algebra. For example, this always happens if the center of D is uncountable [17] (or [39] for a slightly more general result). Other examples of the existence of free group

algebras in division rings can be found in [4, 25, 30, 38]. Therefore, it makes sense to consider the following unifying conjecture:

(GA) Let D be a skew field with center Z. If D is finitely generated as a division ring over Z and D is infinite dimensional over Z, then D contains a noncommutative free group Z-algebra.

For more details on these and related conjectures, the reader is referred to [18].

After the work in [10–14,18] it has become apparent that an involutional version of conjectures (G) and (A) should be investigated. Part of our work deals with an involutional version of (GA). To be more specific, if D is equipped with an involution, under the hypothesis of (GA), can we find a free group algebra whose set of free generators is formed by symmetric elements (*i.e.*, $x^* = x$)?

Let k be a field. A *k-involution* on a k-algebra R is a k-linear map $*: R \to R$, $x \mapsto x^*$, such that $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for all $a, b \in R$. There are two families of k-algebras that usually are equipped with an involution. These are group k-algebras and universal enveloping algebras of Lie k-algebras. Given an involution on a group (see p. 1499 for precise a definition), it induces a k-involution on the group k-algebra $k[G]$ (p. 1499). Furthermore, if G is an orderable group (p. 1466), there is a prescribed construction of a division k-algebra, which we call $k(G)$, that contains $k[G]$; it is generated by $k[G]$ and is such that any k-involution on $k[G]$ can be extended to $k(G)$ (see Section 7 for more details). Also, a given k-involution (see p. 1472) of a Lie k-algebra L induces a k-involution on the universal enveloping algebra $U(L)$ in the natural way ([9, Section 2.2.17]). There is also a concrete construction of a division k-algebra, which we denote by $\mathfrak{D}(L)$. It contains $U(L)$; it is generated by $U(L)$ and is such that any k-involution on L can be extended to a k-involution of $\mathfrak{D}(L)$ (see Section 5 for more details). We remark that neither $k[G]$ nor $U(L)$ need to be Ore domains, but if they are, both $k(G)$ and $\mathfrak{D}(L)$ coincide with the Ore rings of fractions of $k[G]$ and $U(L)$ respectively.

The aim of our work is to apply the graded and filtered methods developed in [39, 40] to obtain free group algebras in division rings. Concerning conjecture (GA), we are able to prove an extension of a result by Lichtman. More precisely, [25, Theorem 4] is (ii) of the following result.

Theorem 1.1   *Let k be a field of characteristic zero and L be a nonabelian Lie k-algebra. Suppose that one of the following conditions is satisfied.*

(i)   *L is residually nilpotent.*

(ii) *The universal enveloping algebra $U(L)$ is an Ore domain.*

Then $\mathfrak{D}(L)$ contains a (noncommutative) free group k-algebra.

Notice that $\mathfrak{D}(L)$ may not contain a free k-algebra of rank two if the characteristic of k is not zero. In fact, as noted in [25, p. 147], the proof given in [20, p. 204] shows that if L is finite dimensional over k, then $\mathfrak{D}(L)$ is finite dimensional over its center. Therefore, it does not contain a noncommutative free algebra.

Concerning involutional versions of conjecture (GA), we are able to prove the following two theorems.

Theorem 1.2   *Let k be a field of characteristic zero, and let L be a nonabelian Lie k-algebra endowed with a k-involution $*\colon L \to L$. Suppose that one of the following conditions is satisfied.*

(i) *L is residually nilpotent.*

(ii) *The universal enveloping algebra $U(L)$ is an Ore domain and either*

    (a) *there exists $x \in L$ such that $[x^*, x] \neq 0$ and the Lie k-subalgebra of L generated by $\{x, x^*\}$ is of dimension at least three, or*

    (b) *$[x^*, x] = 0$ for every $x \in L$, but there exist $x, y \in L$ with $[y, x] \neq 0$ and the k-subspace of L spanned by $\{x, x^*, y, y^*\}$ is not equal to the Lie k-subalgebra of L generated by $\{x, x^*, y, y^*\}$.*

Then $\mathfrak{D}(L)$ contains a (noncommutative) free group k-algebra whose free generators are symmetric with respect to the extension of $*$ to $\mathfrak{D}(L)$.

Theorem 1.3   *Let k be a field of characteristic zero and let G be a nonabelian residually torsion-free nilpotent group endowed with an involution $*\colon G \to G$. Then $k(G)$ contains a free group k-algebra whose free generators are symmetric with respect to the extension of $*$ to $k(G)$.*

Notice that since the map $L \mapsto L$, $x \mapsto -x$, is a k-involution for any Lie k-algebra L, Theorem 1.2(i) implies Theorem 1.1(i). On the other hand, the proofs and the elements that generate the free group algebra in Theorem 1.2 are more complicated than those of Theorem 1.1.

Let k be a field of characteristic zero. The general strategy to obtain Theorems 1.1 and 1.2 goes back to Lichtman [25] and was also used in [10]. Roughly speaking, one has to obtain free (group) algebras in the division ring $\mathfrak{D}(H)$, where H is the Lie k-algebra $H = \langle x, y : [y, [y, x]] = [x, [y, x]] = 0 \rangle$. From this, one obtains free group algebras in $\mathfrak{D}(L)$, where L is a residually nilpotent Lie k-algebra. Now there is a way to obtain free (group) algebras in $\mathfrak{D}(L)$, where L is a Lie k-algebra such that $U(L)$ is an Ore domain, from the residually nilpotent case using filtered and graded methods. We have improved and somewhat clarified this strategy in order to obtain the two first theorems above. Then Theorem 1.3 is obtained from the previous results using the filtered methods from [40] and a technique from [10].

We begin Section 2.1 by introducing some basics on filtrations and valuations. In Section 2.2, we state some results on how filtrations and gradations of Lie algebras induce filtrations and gradations of their universal enveloping algebras. Section 2.3 is devoted to results about the existence of free group algebras obtained in [39, 40]. They show different ways of obtaining free group algebras in division rings generated by group graded rings and in division rings endowed with a valuation.

The results in Section 2 are stated in more generality than necessary in subsequent sections, but we believe there is some merit in the general statements and that they could be of interest to others.

The first part of Section 3 is concerned with the classifications of all the k-involutions of the Heisenberg Lie k-algebra $H = \langle x, y : [y, [y, x]] = [x, [y, x]] = 0 \rangle$ over k, a field of characteristic different from two. We are able to prove that, up to equivalence, there are three involutions on H. We then use this to show that any

nilpotent Lie k-algebra endowed with an involution $*: L \to L$ contains a $*$-invariant k-subalgebra H of L whose restriction to H is one of those three involutions.

Section 4 deals with the problem of finding free (group) algebras in the Ore ring of fractions of $U(L)$, the universal enveloping algebra of a nilpotent Lie k-algebra L over a field of characteristic zero. The main result is the technical Theorem 4.5, where a lot of free (group) algebras in $\mathfrak{D}(H)$ are obtained. Each of those free (group) algebras is suitable for later applications of the results in Section 2.3. Thus, the free generators (or elements obtained from them) will be homogeneous elements of some graded rings that appear in this and subsequent sections. There could be simpler elements that do the job and avoid some technicalities, but we were not able to find them.

Let L be a nonabelian residually nilpotent Lie k-algebra over a field of characteristic zero k. The main aim of Section 5 is to obtain free (group) algebras in the division ring $\mathfrak{D}(L)$ from the ones obtained in the previous section. It is done by a method involving series that was developed in [10]. Although technical, the argument is quite natural.

Let L be a nonabelian Lie k-algebra over a field of characteristic zero such that its universal enveloping algebra $U(L)$ is an Ore domain. In Section 6, we find free group algebras in $\mathfrak{D}(L)$, the Ore ring of fractions of $U(L)$, using the results in previous sections. Roughly speaking, the idea of the proof is that for some natural filtrations of L, the associated graded Lie algebra $\mathrm{grad}(L)$ is residually nilpotent. The isomorphism of graded algebras $U(\mathrm{grad}(L)) \cong \mathrm{grad}(U(L))$ allows us to use the results in previous sections thanks to the fact that $U(L)$ is an Ore domain and the good behaviour of the Ore localization with respect to filtrations described in Section 2.

The arguments in Section 6 should clarify why some of the elements in earlier sections were chosen in that way. Here it is one of the places where Proposition 2.8 and Theorem 2.9 are strongly used.

The last section of the paper is devoted to finding free group algebras in $k(G)$ for k a field of characteristic zero and G a nonabelian residually torsion-free nilpotent group. Let $\mathbb{H} = \langle a, b : (b, (b, a)) = (a, (b, a)) = 1 \rangle$ be the Heisenberg group. There are filtrations of the group ring $k[\mathbb{H}]$ such that the induced k-algebra is isomorphic to $U(H)$ as graded k-algebras, where we consider a certain gradation in $U(H)$ induced from one of H. Again, using the crucial results of Section 2.3, one can obtain suitable free group algebras in $k(\mathbb{H})$. From this, using an argument from [14], one gets the desired free group algebras in $k(G)$.

2 Filtrations, Gradations, and Valuations

A *strict ordering* on a set S is a binary relation $<$ that is transitive and such that $s_1 < s_2$ and $s_2 < s_1$ cannot both hold for elements $s_1, s_2 \in S$. It is a *strict total ordering* if for every $s_1, s_2 \in S$, exactly one of $s_1 < s_2$, $s_2 < s_1$ or $s_1 = s_2$ holds.

A group G is called *orderable* if its elements can be given a strict total ordering $<$ that is left and right invariant. That is, $g_1 < g_2$ implies that $g_1 h < g_2 h$ and $h g_1 < h g_2$ for all $g_1, g_2, h \in G$. We call the pair $(G, <)$ an *ordered group*. Clearly, any additive subgroup of the real numbers is orderable. More generally, torsion-free abelian groups, torsion-free nilpotent groups, and residually torsion-free nilpotent groups are orderable [15].

We would like to point out that the results in this section are stated for ordered groups, but in the following ones they will be applied for the ordered group $\mathbb{Z}$ alone. We believe there is some merit in the general statements, and they could be of interest to others.

2.1 On Filtrations and Valuations

Let R be a ring and $(G, <)$ an ordered group. A family $F_G R = \{F_g R\}_{g \in G}$ of additive subgroups of R is a (descending) *G-filtration* if it satisfies the following four conditions:

(F1) $F_g R \supseteq F_h R$ for all $g, h \in G$ with $g \leq h$;
(F2) $F_g R \cdot F_h R \subseteq F_{gh} R$ for all $g, h \in G$;
(F3) $1 \in F_1 R$;
(F4) $\bigcup_{g \in G} F_g R = R$.

We say that the *G*-filtration is *separating* if it also satisfies the following:

(F5) For every $x \in R$, there exists $g \in G$ such that $x \in F_g R$ and $x \notin F_h R$ for all $h \in G$ with $g < h$.

Let R be a ring, $(G, <)$ an ordered group, and $F_G R = \{F_g R\}_{g \in G}$ a *G*-filtration of R. For each $g \in G$, define

$$F_{>g} R = \sum_{h>g} F_h R \quad \text{and} \quad R_g = F_g R / F_{>g} R.$$

The fact that G is an ordered group and the definition of *G*-filtration imply that

$$F_{>g} R \cdot F_{>h} R \subseteq F_{>gh} R, \quad F_{>h} R \cdot F_g R \subseteq F_{>gh} R, \quad F_g R \cdot F_{>h} R \subseteq F_{>gh} R$$

for any $g, h \in G$. Thus, a multiplication can be defined by

$$(2.1) \qquad R_g \times R_h \longrightarrow R_{gh}, \quad (x + F_{>g} R)(y + F_{>h} R) = xy + F_{>gh} R.$$

The *associated graded ring* of $F_G R$ is defined to be

$$\operatorname{grad}_{F_G}(R) = \bigoplus_{g \in G} R_g.$$

The addition on $\operatorname{grad}_{F_G}(R)$ arises from the addition on each component R_g. The multiplication is defined by extending the multiplication (2.1) on the components bilinearly to all $\operatorname{grad}_{F_G}(R)$. Notice that $\operatorname{grad}_{F_G}(R)$ may not have an identity element. If $F_G R$ is separating, then $\operatorname{grad}_{F_G}(R)$ is a ring with identity element $1 + F_{>1} R$.

The *Rees ring* of the filtration is

$$\operatorname{Rees}_{F_G}(R) = \bigoplus_{g \in G} (F_g R) g,$$

which is a subring of the group ring $R[G]$. Thus, an element of $\operatorname{Rees}_{F_G}(R)$ is a finite sum $\sum_{g \in G} a_g g$ where $a_g \in F_g R$. Notice that $\operatorname{Rees}_{F_G}(R)$ is a *G*-graded ring with identity element $1_{\operatorname{Rees}_{F_G}(R)} = 1_R 1_G$.

The next lemma is well known. It can be proved as in [32, Section 1.8], where the filtrations are ascending.

Lemma 2.1 *Let R be a ring, let $(G, <)$ be an ordered group, and let $F_G R = \{F_g R\}_{g \in G}$ be a G-filtration of R. The following hold true.*

(i)   *The subset $G_1 = \{g \in G : g \leq 1\}$ is an Ore subset of $\mathrm{Rees}_v(R)$, and the Ore localization $G_1^{-1}\mathrm{Rees}_{F_G}(R) = R[G]$ is the group ring.*

(ii) *Let J be the ideal of $\mathrm{Rees}_{F_G}(R)$ generated by $G^- = \{g \in G : g < 1\}$. Then $J = \bigoplus_{g\in G}(F_{>g}R)g$ and $\mathrm{Rees}_{F_G}(R)/J \cong \mathrm{grad}_{F_G}(R)$ as graded rings.*

(iii) *Let I be the ideal of $\mathrm{Rees}_{F_G}(R)$ generated by the elements $\{1 - g : g \in G^-\}$. Then $\mathrm{Rees}_{F_G}(R)/I \cong R$.*

Let R be a ring and $(G, <)$ be an ordered group. A map $v\colon R \to G \cup \{\infty\}$ is a *valuation* if it satisfies

(V1) $v(x) = \infty$ if, and only if, $x = 0$;

(V2) $v(x + y) \geq \min\{v(x), v(y)\}$;

(V3) $v(xy) = v(x)v(y)$.

Notice that $v(1) = 1_G$ and $v(-x) = v(x)$ for all $x \in R$. For each $g \in G$, we set $R_{\geq g} = \{f \in R : v(f) \geq g\}$ and $R_{>g} = \{f \in R : v(f) > g\}$. Defining $F_g R = R_{\geq g}$ for each $g \in G$, we obtain a separating filtration $F_G R = \{F_g R\}_{g\in G}$. We will denote the graded ring and the Rees ring associated with this filtration as $\mathrm{grad}_v(R)$ and $\mathrm{Rees}_v(R)$, respectively. Furthermore, observe that $\mathrm{grad}_v(R)$ is a domain because of (V3). It is well known that the converse is also true [32, p. 91]. That is, given a separating filtration $F_G R = \{F_g R\}_{g\in G}$ of R such that the associated graded ring $\mathrm{grad}_{F_G}(R)$ is a domain, one can define a valuation $v\colon R \to G \cup \{\infty\}$ by $v(x) = \max\{g \in G : x \in F_g R\}$ for each $x \in R\backslash\{0\}$.

If X is an Ore domain, we denote by $Q_{\mathrm{cl}}(X)$ the Ore ring of fractions of X, that is, the Ore localization of X at the multiplicative set $X\backslash\{0\}$.

The following lemma is a generalization of [25, Propositions 16, 17, 18], with a somewhat different proof.

Lemma 2.2   *Let R be an Ore domain, $(G, <)$ an ordered group, and $v\colon R \to G \cup \{\infty\}$ a valuation. Let D be the Ore ring of fractions of R. The following hold true.*

(i)   *The valuation v can be extended to a valuation $v\colon D \to G \cup \{\infty\}$.*

(ii) *The set $\mathcal{H}$ of nonzero homogeneous elements of $\mathrm{grad}_v(R)$ is an Ore subset of $\mathrm{grad}_v(R)$.*

(iii) *There exists an isomorphism of G-graded rings $\lambda\colon \mathcal{H}^{-1}\mathrm{grad}_v(R) \to \mathrm{grad}_v(D)$ given by $f + R_{>v(f)} \mapsto f + D_{>v(f)}$ for all $f \in R$.*

(iv) *If G is poly-(torsion-free abelian), then $\mathrm{grad}_v(R)$ is an Ore domain.*

(v)   *If G is poly-(torsion-free abelian), then $\mathrm{Rees}_v(R)$ is an Ore domain.*

(vi) *If G is torsion-free abelian, then $\mathcal{J} = \mathrm{Rees}_v(R)\backslash J$ is an Ore subset of $\mathrm{Rees}_v(R)$, and $\mathcal{J}^{-1}\mathrm{Rees}_v(R)$ is a local ring with residue division ring $Q_{\mathrm{cl}}(\mathrm{grad}_v(R))$.*

Proof   The proof of (i) can be found in [7, Proposition 9.1.1], for example.

(ii) Let $f_1, f_2 \in R\backslash\{0\}$. Consider the nonzero homogeneous elements $f_1 + R_{>v(f_1)}$, $f_2 + R_{>v(f_2)} \in \mathrm{grad}_v(R)$. Since R is an Ore domain, there exist $q_1, q_2 \in R$ such that $q_1 f_1 = q_2 f_2 \neq 0$. Consider the nonzero homogeneous elements $q_1 + R_{>v(q_1)}, q_2 + R_{>v(q_2)} \in \mathrm{grad}_v(R)$. Then

$$(q_1 + R_{>v(q_1)})(f_1 + R_{>v(f_1)}) = (q_2 + R_{>v(q_2)})(f_2 + R_{>v(f_2)}).$$

Now [33, Lemma 8.1.1] implies the result.

(iii) First note that $\mathrm{grad}_v(D)$ is a G-graded skew field, and the natural maps $\iota\colon \mathrm{grad}_v(R) \hookrightarrow \mathrm{grad}_v(D)$, $\kappa\colon \mathrm{grad}_v(R) \hookrightarrow \mathcal{H}^{-1}\mathrm{grad}_v(R)$ are embeddings of G-graded rings. Thus, for each element in $\mathcal{H}$, the image by ι is an homogeneous invertible element in $\mathrm{grad}_v(D)$. By the universal property of the Ore localization, there exists a homomorphism $\lambda\colon \mathcal{H}^{-1}\mathrm{grad}_v(R) \to \mathrm{grad}_v(D)$ such that $\iota = \lambda\kappa$. The homomorphism λ is injective, since it is so when restricted to homogeneous elements. Now let $f, q \in R\backslash\{0\}$. Consider $q^{-1}f + D_{>v(q^{-1}f)}$. This element is the image by λ of $(q + R_{>v(q)})^{-1}(f + R_{>v(f)})$. Thus, λ is surjective.

(iv) The graded division ring $\mathrm{grad}_v(D)$ is a crossed product of the division ring D_0 over the subgroup $\{g \in G : D_g \neq 0\}$, which is again poly-(torsion-free abelian). Thus, $\mathrm{grad}_v(D)$ is an Ore domain by, for example, [36, Corollary 37.11]. We show that the Ore ring of fractions $Q_{\mathrm{cl}}(\mathrm{grad}_v(D))$ of $\mathrm{grad}_v(D)$ is also the Ore ring of fractions of $\mathrm{grad}_v(R)$. For that, it is enough to show that every element of $Q_{\mathrm{cl}}(\mathrm{grad}_v(D))$ is of the form $b^{-1}a$ with $a, b \in Q_{\mathrm{cl}}(\mathrm{grad}_v(R))$, $b \neq 0$. An element of $f \in Q_{\mathrm{cl}}(\mathrm{grad}_v(D))$ is of the form $(d_{g_i} + \cdots + d_{g_r})^{-1}(e_{h_1} + \cdots + e_{h_s})$, where $d_{g_i} \in D_{g_i}$, $e_{h_j} \in D_{h_j}$. By (ii), (iii), and after bringing it to a common denominator, we can suppose that there exist $t, a_i, b_j \in \mathcal{H}$ such that

$$f = (t^{-1}a_1 + \cdots + t^{-1}a_r)^{-1}(t^{-1}b_1 + \cdots + t^{-1}b_s) = (a_1 + \cdots + a_r)^{-1}(b_1 + \cdots + b_s).$$

(v) In the same way as (iv), one can show that the group ring $D[G]$ and $R[G]$ are Ore domains with the same Ore ring of fractions $Q_{\mathrm{cl}}(R[G])$. By Lemma 2.1(i), $R[G]$ is the localization of $\mathrm{Rees}_v(R)$ at G_1. Hence, one can proceed as in (iv) to show that $\mathrm{Rees}_v(R)$ is an Ore domain with Ore ring of fractions $Q_{\mathrm{cl}}(R[G])$.

(vi) Let $x = \sum_{i=1}^{n} a_i g_i \in \mathrm{Rees}_v(R)$ where we suppose that $a_i \neq 0$, $i = 1, \ldots, n$. Hence, $v(a_i) \geq g_i$ for all i. We suppose that if $i < j$ either $v(a_i)^{-1}g_i < v(a_j)^{-1}g_j$ or $v(a_i)^{-1}g_i = v(a_j)^{-1}g_j$ and $g_i < g_j$. We define $\omega(x) = v(a_n)^{-1}g_n \leq 1_G$. Observe that $x = x'\omega(x)$, where

$$x' = \sum_{i=1}^{n} a_i g_i g_n^{-1} v(a_n).$$

Since $v(a_n)^{-1}g_n \geq v(a_i)^{-1}g_i$, $g_n^{-1}v(a_i) \geq g_n^{-1}v(a_n)$. This implies that $v(a_i) \geq g_i g_n^{-1}v(a_n)$. Hence, $x' \in \mathcal{J}$. Note that $x \in \mathcal{J}$ if and only if $\omega(x) = 1$, since $\mathcal{J} = \bigoplus_{g \in G} F_{>g} R \cdot g$.

If $a, b \in R$, $g \in G$ such that $ag, bg \in \mathrm{Rees}_v(R)$, then

$$\omega((a + b)g) \leq \max\{\omega(ag), \omega(bg)\}.$$

Now let $y = \sum_{j=1}^{p} b_j h_j \in \mathrm{Rees}_v(R)$ where we suppose that $b_j \neq 0$, $j = 1, \ldots, p$, and if $j < l$ either $v(b_j)^{-1}h_j < v(b_l)^{-1}h_l$ or $v(b_j)^{-1}h_j = v(b_l)^{-1}h_l$ and $h_j < h_l$.

Now note that

$$xy = a_n b_p g_n h_p + \sum_{\{(i,j):(i,j)\neq(n,p)\}} a_i b_j g_i h_j,$$

and if $(i, j) \neq (n, p)$, either $v(a_i b_j)^{-1}g_i h_j < v(a_n b_p)^{-1}g_n h_p$ or $g_i h_j < g_n h_p$. Therefore $\omega(xy) = \omega(x)\omega(y)$.

Let $u \in \mathcal{J}$ and $v \in \mathrm{Rees}_v(R)$. Since $\mathrm{Rees}_v(R)$ is an Ore domain, there exist $x, y \in \mathrm{Rees}_v(R)$ such that $xu = yv$. We have to prove that y can be chosen such that $y \in \mathcal{J}$. From $xu = x'u\omega(x) = y'v'\omega(y)\omega(v) = yv$, we get $x\omega(y)^{-1}u = y'v$, where $y' \in \mathcal{J}$.

Since $\omega(x) = \omega(y)\omega(v)$ with $\omega(v) \leq 1$, $\omega(x) \leq \omega(y) \leq 1$. It implies that $x\omega(y)^{-1} \in$ $\mathrm{Rees}_v(R)$ and $\omega(y') = 1$. ∎

2.2 On Gradations and Filtrations of Universal Enveloping Algebras

If L is a Lie algebra, we denote its *universal enveloping algebra* by $U(L)$.

Let k be a field, L a Lie k-algebra, and G a commutative group. We say that L is a *G-graded Lie k-algebra* if there exists a decomposition of L as $L = \bigoplus_{g \in G} L_g$ satisfying the following:

(a) L_g is a k-subspace of L for each $g \in G$,
(b) $[L_g, L_h] \subseteq L_{g+h}$ for all $g, h \in G$.

The elements of $\bigcup_{g \in G} L_g$ are the *homogeneous elements* of L. If $x \in L_g$, we say that x is *homogeneous of degree g*.

The main examples we will deal with are the following. Examples (i) and (ii) are important in Section 4, while examples (iii) and (iv) are useful in Section 7

Example 2.3 Let k be a field. We can endow the Heisenberg Lie k-algebra H with different $\mathbb{Z}$-gradings. We will use the following ones.

(i) $H = \bigoplus_{n \in \mathbb{Z}} H_n$, where $H_{-1} = kx + ky$, $H_{-2} = kz$, and $H_n = 0$ for all $n \neq -1, -2$.
(ii) $H = \bigoplus_{n \in \mathbb{Z}} H_n$, where $H_{-1} = kx$, $H_{-2} = ky$, $H_{-3} = kz$, and $H_n = 0$ for all $n \neq -1, -2, -3$.
(iii) $H = \bigoplus_{n \in \mathbb{Z}} H_n$, where $H_1 = kx + ky$, $H_2 = kz$, and $H_n = 0$ for all $n \neq 1, 2$.
(iv) $H = \bigoplus_{n \in \mathbb{Z}} H_n$, where $H_1 = kx$, $H_2 = ky$, $H_3 = kz$, and $H_n = 0$ for all $n \neq 1, 2, 3$.

For each $g \in G$, let $\mathcal{B}_g = \{e_i^g\}_{i \in I_g}$ be a k-basis of L_g. Then $\mathcal{B} = \bigcup_{g \in G} \mathcal{B}_g$ is a k-basis of L. Fix an ordering $<$ of $\mathcal{B}$. Consider the universal enveloping algebra $U(L)$ of L. The *standard monomials* in $\mathcal{B}$ are the elements

$$(2.2) \qquad e_{i_1}^{g_1} e_{i_2}^{g_2} \cdots e_{i_r}^{g_r} \in U(L), \text{ with } e_{i_j}^{g_j} \in \mathcal{B}_{g_j}, \ e_{i_1}^{g_1} \leq e_{i_2}^{g_2} \leq \cdots \leq e_{i_r}^{g_r}.$$

By the Poincaré–Birkoff–Witt (PBW) theorem, the standard monomials, together with 1, form a k-basis of $U(L)$. We say that the standard monomial (2.2) is of degree $g = g_1 + g_2 + \cdots + g_r$. In this situation, one can obtain a gradation of the universal enveloping algebra as follows.

Lemma 2.4 *Let G be a group and let $L = \bigoplus_{g \in G} L_g$ be a G-graded Lie k-algebra. Then the universal enveloping algebra $U(L)$ is an (associative) G-graded k-algebra. Indeed,*

$$U(L) = \bigoplus_{g \in G} U(L)_g,$$

where $U(L)_g$ is the k-span of the standard monomials of degree g.

Let k be a field, let L be a Lie k-algebra, and let $(G, <)$ be an ordered abelian group. A (descending) *separating filtration* of L is a family of subspaces $F_G L = \{F_g L\}_{g \in G}$, such that

(FL1) $F_g L \supseteq F_h L$ for all $g, h \in G$ with $g \leq h$;
(FL2) $[F_g L, F_h L] \subseteq F_{g+h} R$ for all $g, h \in G$;
(FL3) $\bigcup_{g \in G} F_g L = L$;

(FL4)   for every $x \in L$, there exists $g \in G$ such that $x \in F_g L$ and $x \notin F_h L$ for all $h \in G$ with $g < h$.

Define $F_{>g} L = \sum_{h>g} F_h L$, and $L_g = F_g L / F_{>g} L$ for all $g \in G$. Then one obtains the *associated graded Lie k-algebra*

$$\operatorname{grad}_{F_G} L = \bigoplus_{g \in G} L_g.$$

The filtration $F_G L$ of L induces a filtration $F_G U(L) = \{F_g U(L)\}_{g \in G}$ of the universal enveloping algebra $U(L)$ as follows. Define, for each $g \in G$, $g \le 0$,

$$F_g U(L) = k + \sum_{g_1 + \cdots + g_r \ge g} L_{g_1} \cdots L_{g_r},$$

and for each $g > 0$,

$$F_g U(L) = \sum_{g_1 + \cdots + g_r \ge g} L_{g_1} \cdots L_{g_r}.$$

Then $F_h U(L) \subseteq F_g U(L)$ for $g < h$, and $F_g U(L) \cdot F_h U(L) \subseteq F_{g+h} U(L)$ for all $g, h \in G$.

An easy but important example for us is the following. It will be used in Section 6.

Example 2.5   Let L be a Lie k-algebra generated by two elements $u, v \in L$. Define $FL_r = 0$ for all $r \ge 0$, $FL_{-1} = ku + kv$, and, for $n \le -1$,

$$F_{n-1} L = \sum_{n_1 + n_2 + \cdots + n_r \ge (n-1)} \left[F_{n_1} L, \left[F_{n_2} L, \ldots \right] \cdots \right].$$

Observe that, for each $n \in \mathbb{Z}$, there exists $\mathcal{B}_n \subseteq L$ whose classes give a basis of $L_n = F_n L / F_{n+1} L$ such that $\bigcup_{n \in \mathbb{Z}} \mathcal{B}_n$ is a basis of L.

The next lemma will be used in Sections 6, 7.

Lemma 2.6   *Let k be a field and L be a Lie k-algebra. The following hold true.*

(i) *Suppose that there exists a basis $\mathcal{B}_g = \{e_i^g\}_{i \in I_g}$ of L_g for each $g \in G$ such that $\bigcup_{g \in G} \mathcal{B}_g$ is a basis of L. Then the filtration is separating, and there exists an isomorphism of G-graded k-algebras*

$$U(\operatorname{grad}_{F_G}(L)) \cong \operatorname{grad}_{F_G}(U(L)).$$

Hence the filtration induces a valuation $v \colon U(L) \to G \cup \{\infty\}$.

(ii) *If $U(L)$ is an Ore domain, then $U(\operatorname{grad}_{F_G}(L))$ is an Ore domain.*

Proof   (i) This can be proved in the same way as [41, Proposition 1] or [3, Lemma 2.1.2].
(ii) This can be proved by Lemma 2.2(iv).   ∎

2.3   Free Group Algebras in Division Rings

Our work can be regarded as an application of some techniques on the existence of free group algebras in division rings. In this section, we gather together the version of those results that we will use.

We begin with [39, Theorem 3.2]. It tells us a way to obtain a free group algebra from a free algebra in case the division ring is the Ore ring of fractions of a graded Ore domain.

Theorem 2.7 *Let G be an orderable group and let k be a commutative ring. Let $A = \bigoplus_{g \in G} A_g$ be a G-graded k-algebra. Let X be a subset of A consisting of homogeneous elements where we denote by $g_x \in G$ the degree of $x \in X$, i.e., $x \in A_{g_x}$. Suppose that the following three conditions are satisfied.*

(i) *There exists a strict total ordering $<$ of G such that $(G, <)$ is an ordered group and $1 < g_x$ for all $x \in X$.*
(ii) *The k-subalgebra of A generated by X is the free k-algebra on X.*
(iii) *A is a left Ore domain with left Ore ring of fractions $Q_{cl}(A)$.*

Then the k-subalgebra of $Q_{cl}(A)$ generated by $\{1 + x, (1 + x)^{-1}\}_{x \in X}$ is the free group k-algebra on the set $\{1 + x\}_{x \in X}$.

The next proposition is [40, Proposition 2.5(4')]. It shows that (under some circumstances) the existence of a free group algebra in the graded ring induced by a valuation on a division ring D, implies the existence of a free group algebra in D.

Proposition 2.8 *Let Z be a commutative ring and R a Z-algebra. Let $v \colon R \to \mathbb{Z} \cup \{\infty\}$ be a valuation. Let X be a subset of elements of R such that the map $X \to \operatorname{grad}_v(R)$, $x \mapsto x + R_{>v(x)}$, is injective. Moreover, assume that*

(i) *the elements of X are invertible in R,*
(ii) *the Z_0-subalgebra of $\operatorname{grad}_v(R)$ generated by $\{x + R_{>v(x)}, x^{-1} + R_{>v(x^{-1})}\}_{x \in X}$ is the free group Z_0-algebra on $\{x + R_{>v(x)}\}_{x \in X}$.*

Then the Z-subalgebra of R generated by $\{x, x^{-1}\}_{x \in X}$ is the free group Z-algebra on X, where $Z_0 = Z_{\geq 0}/Z_{>0}$.

The next theorem is [40, Theorem 3.2]. It tells us that, sometimes, in order to find a free group algebra in division ring D it is enough to find a free algebra on the graded ring induced by a valuation on D.

Theorem 2.9 *Let D be a division ring with prime subring Z. Let $v \colon D \to \mathbb{R} \cup \{\infty\}$ be a nontrivial valuation. Let X be a subset of D satisfying the following three conditions.*

(i) *The map $X \to \operatorname{grad}_v(D)$, $x \mapsto x + D_{>v(x)}$, is injective.*
(ii) *For each $x \in X$, $v(x) > 0$.*
(iii) *The Z_0-subalgebra of $\operatorname{grad}_v(D)$ generated by the set $\{x + D_{>v(x)}\}_{x \in X}$ is the free Z_0-algebra on the set $\{x + D_{>v(x)}\}_{x \in X}$, where $Z_0 = Z_{\geq 0}/Z_{>0} \subseteq D_0$.*

Then, for any central subfield k, the k-subalgebra of D generated by $\{1 + x, (1 + x)^{-1}\}_{x \in X}$ is the free group k-algebra on $\{1 + x\}_{x \in X}$.

3 Nilpotent Lie Algebras with Involutions

Let k be a field and let L be a Lie k-algebra. A k-linear map $* \colon L \to L$ is a *k-involution* if for all $x, y \in L$, $[x, y]^* = [y^*, x^*]$, $x^{**} = x$. The main example of a k-involution in a Lie k-algebra is what we call the *principal involution*. It is defined by $x \mapsto -x$ for all $x \in L$.

The *Heisenberg Lie k-algebra* is the Lie k-algebra with presentation

$$H = \langle x, y : [[y, x], x] = [[y, x], y] = 0 \rangle.$$

The Heisenberg Lie k-algebra can also be characterized as the unique Lie k-algebra of dimension three such that $[H, H]$ has dimension one and $[H, H]$ is contained in the center of H; see [20, Section 4.III].

Let k be a field of characteristic different from 2. In this section, we first find all the k-involutions of H; second, we show that there are essentially three involutions on H, and then we show that any nilpotent Lie k-algebra with involution contains a k-subalgebra isomorphic to H invariant under the involution such that the restriction of the involution to H is one of those three.

Lemma 3.1 *Let k be a field of characteristic different from two. Let $H = \langle x, y : [x, [y, x]] = [y, [y, x]] = 0 \rangle$ be the Heisenberg Lie k-algebra and $z = [y, x]$. Then any k-involution $\tau: H \to H$ is of one of the following forms:*

(i) $\begin{cases} \tau(x) = ax + by + cz, \\ \tau(y) = dx - ay + fz, \\ \tau(z) = z, \end{cases}$ *where $a, b, c, d, f \in k$ satisfy* $\begin{cases} a^2 + bd = 1, \\ (a+1)c + bf = 0, \\ dc + (1-a)f = 0. \end{cases}$

(ii) $\begin{cases} \tau(x) = x + cz, \\ \tau(y) = y + fz, \\ \tau(z) = -z, \end{cases}$ *where $c, f \in k$.*

(iii) $\begin{cases} \tau(x) = -x, \\ \tau(y) = -y, \\ \tau(z) = -z. \end{cases}$

Proof Let $*: H \to H$ be a k-involution on H. Note that $Z(H)$, the center of H, is the one-dimensional k-subspace generated by $z = [y, x]$. Since $z^* \in Z(H)$ and $(z^*)^* = z$, we obtain that $z^* = z$ or $z^* = -z$.

Suppose that $x^* = ax + by + cz$ and $y^* = dx + ey + fz$ where $a, b, c, d, e, f \in k$.

Case 1: $z^* = z$. Then

$$z = [y, x] = [y, x]^* = [x^*, y^*] = [ax + by + cz, dx + ey + fz]$$
$$= [by, dx] + [ax, ey] = (bd - ae)z.$$

Thus,

(3.1) $$bd - ae = 1.$$

From

$$x = (x^*)^* = (ax + by + cz)^* = a(ax + by + cz) + b(dx + ey + fz) + cz$$
$$= (a^2 + bd)x + (ab + be)y + (ac + bf + c)z,$$

we get

(3.2)
$$a^2 + bd = 1,$$

(3.3)
$$b(a + e) = 0,$$

(3.4)
$$ac + bf + c = 0.$$

From

$$y = (y^*)^* = (dx + ey + fz)^* = d(ax + by + cz) + e(dx + ey + fz) + fz$$
$$= d(a + e)x + (e^2 + db)y + (cd + ef + f)z,$$

we obtain

(3.5)
$$e^2 + bd = 1,$$

(3.6)
$$d(a + e) = 0,$$

(3.7)
$$cd + ef + f = 0.$$

From (3.2) and (3.5), we obtain that $a = \pm e$.

Suppose that $a = e$. Then (3.2) and (3.1) imply that $a = e = 0$. Thus, this case is contained in the case $a = -e$.

Suppose now that $a = -e$. Then (3.2), (3.5), and (3.1) are in fact the same equation. Also equations (3.3) and (3.6) do not give any new information. Thus, (3.4) and (3.7) are equal to

(3.8)
$$\begin{cases} (a + 1)c + bf = 0 \\ dc + (1 - a)f = 0. \end{cases}$$

Observe that, by (3.2), $\det \begin{pmatrix} a+1 & b \\ d & 1-a \end{pmatrix} = -a^2 - bd + 1 = 0$.

Therefore $x^* = ax + by + cz$, $y^* = dx - ay + fz$, $z^* = z$, where a, b, c, d, f satisfy (3.2) and (3.8). Hence, (i) is proved.

Case 2: $z^* = -z$. $-z = [y, x]^* = [x^*, y^*] = [ax + by + cz, dx + ey + fz] = [by, dx] + [ax, ey] = (bd - ae)z$. Thus

(3.9)
$$ae - bd = 1.$$

From $x = (x^*)^* = (ax + by + cz)^* = a(ax + by + cz) + b(dx + ey + fz) - cz = (a^2 + bd)x + (ab + eb)y + (ac + bf - c)z$, we get

(3.10)
$$a^2 + bd = 1,$$

(3.11)
$$b(a + e) = 0,$$

(3.12)
$$ac + bf - c = 0.$$

From $y = (y^*)^* = (dx + ey + fz)^* = d(ax + by + cz) + e(dx + ey + fz) - fz = d(a + e)x + (e^2 + db)y + (cd + ef - f)z$, we obtain

(3.13)
$$e^2 + bd = 1,$$

(3.14)
$$d(a + e) = 0,$$

(3.15)
$$cd + ef - f = 0.$$

From (3.10) and (3.13), we obtain that $a = \pm e$.

It is not possible that $a = -e$, because (3.10) and (3.9) would imply that $1 = -1$.

Suppose now that $a = e$. Then (3.9), (3.10), and (3.13) imply that $a^2 = 1$. Hence, $a = e = \pm 1$. Now (3.11) and (3.14) imply that $b = d = 0$.

If $a = -1$, we obtain, by (3.12) and (3.15), that $f = c = 0$. Hence, we obtain (iii), *i.e.*, $x^* = -x$, $y^* = -y$, $z^* = -z$.

If $a = 1$, (3.12) and (3.15) do not give any new information. Hence, we obtain (ii), *i.e.*, $x^* = x + cz$, $y^* = y + fz$, $z^* = -z$, where $c, f \in k$. ∎

Let k be a field. Let $\tau, \eta \colon L \to L$ be two k-involutions of a Lie k-algebra L. We say that τ is *equivalent* to η if there exists an isomophism of Lie k-algebras $\varphi \colon L \to L$ such that $\varphi^{-1} \tau \varphi = \eta$.

Lemma 3.2 *Let k be a field of characteristic different from two and let H be the Heisenberg Lie k-algebra. Any k-involution $\tau \colon H \to H$ is equivalent to one of the following involutions $\eta \colon H \to H$.*

(i) *The involution $\eta \colon H \to H$ defined by $\eta(x) = x$, $\eta(y) = -y$, $\eta(z) = z$. More precisely, any k-involution in Lemma 3.1(i) is equivalent to η just defined.*

(ii) *The involution $\eta \colon H \to H$ defined by $\eta(x) = x$, $\eta(y) = y$, $\eta(z) = -z$. More precisely, any k-involution in Lemma 3.1(ii) is equivalent to η just defined.*

(iii) *The principal involution $\eta \colon H \to H$ defined by $\eta(x) = -x$, $\eta(y) = -y$, $\eta(z) = -z$.*

Furthermore, we exhibit explicit isomorphisms $\varphi \colon H \to H$ which prove that $\varphi^{-1} \tau \varphi = \eta$, where τ is any involution in Lemma 3.1(i) and (ii).

Proof Clearly the involution from Lemma 3.1(iii) is the same as the one in (iii).

First we prove (ii). Let $f \mapsto f^*$ be any involution in Lemma 3.1(ii). Suppose that $c, f \in k$ and that $x^* = x + cz$, $y^* = y + fz$ and $z^* = -z$. Define $X = \frac{1}{2}(x + x^*) = \frac{1}{2}(2x + cz)$, $Y = \frac{1}{2}(y + y^*) = \frac{1}{2}(2y + fz)$, and $Z = z$. Note that X, Y, Z form a k-basis of H and that $[Y, X] = [y, x] = z = Z$. Thus, there exists an isomorphism $\varphi \colon H \to H$ sending $x \mapsto X$, $y \mapsto Y$ and $z \mapsto Z$. Moreover, $X^* = X$, $Y^* = Y$ and $Z^* = -Z$, as desired.

Now we prove (i). Let $h \mapsto h^*$ be any involution from Lemma 3.1(i). Let $a, b, c, d, f \in k$ satisfy the conditions in Lemma 3.1(i). Hence, $x^* = ax + by + cz$, $y^* = dx - ay + fz$, $z^* = z$. We consider three cases:

(I) $b \neq 0$,
(II) $d \neq 0$,
(III) $b = d = 0$.

(I) Suppose $b \neq 0$. Define $X = \frac{1+a}{2}x + \frac{b}{2}y + \frac{c}{2}z$, $Y = \frac{1-a}{2}x - \frac{b}{2}y - \frac{c}{2}z$, and $Z = -\frac{b}{2}z$. Note that X, Y, Z is a k-basis of H and that $[Y, X] = Z$. Thus there exists an isomorphism $\varphi \colon H \to H$ sending $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$. Note that $X^* = X$, $Y^* = -Y$ and $Z^* = Z$, as desired.

(II) Suppose now that $d \neq 0$. Define $X = \frac{d}{2}x + \frac{1-a}{2}y + \frac{f}{2}z$, $Y = \frac{d}{2}x - \frac{a+1}{2}y + \frac{f}{2}z$, and $Z = -\frac{d}{2}z$. Note that X, Y, Z is a k-basis of H and that $[Y, X] = Z$. Thus there exists an isomorphism $\varphi \colon H \to H$ given by $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$. Note that $X^* = X$, $Y^* = -Y$ and $Z^* = Z$, as desired.

(III) Suppose that $b = d = 0$. Then $a^2 = 1$ and either $c = 0$ or $f = 0$. In both cases define $X = -\frac{1+a}{2}x + \frac{1-a}{2}y + \frac{f-c}{2}z$, $Y = \frac{a-1}{2}x + \frac{1+a}{2}y + \frac{c-f}{2}z$, $Z = -az$. It is not

difficult to show that X, Y, Z is a k-basis of H and that $[Y, X] = Z$. Thus, there exists an isomorphism $\varphi: H \to H$ given by $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$. Note that $X^* = X$, $Y^* = -Y$ and $Z^* = Z$, as desired. ∎

The following two results are the Lie algebra version of [14, Lemma 2.3, Proposition 2.4]. The proofs are analogous to the ones given there for groups.

Lemma 3.3 *Let k be a field of characteristic different from two, and let L be a finitely generated nilpotent Lie k-algebra of class 2 with involution $*: L \to L$, $f \mapsto f^*$. Then L contains a $*$-invariant Heisenberg Lie k-subalgebra H and the restriction of $*$ to H is one of the involutions in Lemma 3.2. More precisely, there exist $x, y \in L$ such that $[y, x] \neq 0$, $[y, [y, x]] = [x, [y, x]] = 0$, and either $x^* = x$ and $y^* = -y$, $x^* = x$ and $y^* = y$, or $x^* = -x$ and $y^* = -y$.*

Proof Let C denote the center of L. It follows from the nilpotency class of L that L/C is a finitely generated torsion-free abelian Lie k-algebra and the involution induces an automorphism of k-vector spaces $\varphi: L/C \to L/C$, $f + C \mapsto f^* + C$. Notice that L/C has dimension at least two, because L is not abelian. Since φ^2 is the identity, φ is diagonalizable. There exist $u_1, \ldots, u_n \in L$ such that $\{u_1 + C, \ldots, u_n + C\}$ is a basis of L/C consisting of eigenvectors with eigenvalues ± 1. Since L is not abelian, we can suppose that $[u_1, u_2] \neq 0$. Hence, there exist $z_1, z_2 \in C$ such that $u_i^* = \varepsilon_i u_i + z_i$, where $\varepsilon_i \in \{1, -1\}$, for $i = 1, 2$.

Suppose that $u_1^* = -u_1 + z_1$ and $u_2^* = u_2 + z_2$. Let H be the subalgebra with basis $x = \frac{1}{2}(u_1 - u_1^*) = u_1 - \frac{1}{2}z_1$, $y = \frac{1}{2}(u_2 + u_2^*) = u_2 + \frac{1}{2}z_2$, and $z = [y, x]$. Now proceed as in the following case.

Suppose that $u_1^* = u_1 + z_1$ and $u_2^* = -u_2 + z_2$. Let H be the subalgebra with basis $x = \frac{1}{2}(u_2 - u_2^*) = u_2 - \frac{1}{2}z_2$, $y = \frac{1}{2}(u_1 + u_1^*) = u_2 + \frac{1}{2}z_2$, and $z = [y, x]$. Clearly z commutes with x and y, because it is an element of C. Now $x^* = -x$, $y^* = y$, and $z^* = [y, x]^* = [x^*, y^*] = [-x, y] = z$. Thus, $*$, when restricted to H, is the involution (i) in Lemma 3.1.

Suppose that $u_1^* = u_1 + z_1$ and $u_2^* = u_2 + z_2$. Let H be the subalgebra with basis $x = \frac{1}{2}(u_1 + u_1^*) = u_1 + \frac{1}{2}z_1$, $y = \frac{1}{2}(u_2 + u_2^*) = u_2 + \frac{1}{2}z_2$ and $z = [y, x]$. Clearly z commutes with x and y, because it is an element of C. Now $x^* = x$, $y^* = y$ and $z^* = [y, x]^* = [x^*, y^*] = [x, y] = -z$. Thus, $*$, when restricted to H, is the involution (ii) in Lemma 3.1.

Suppose that $u_1^* = -u_1 + z_1$ and $u_2^* = -u_2 + z_2$. Let H be the subalgebra with basis $x = \frac{1}{2}(u_1 - u_1^*) = u_1 - \frac{1}{2}z_1$, $y = \frac{1}{2}(u_2 - u_2^*) = u_2 - \frac{1}{2}z_2$ and $z = [y, x]$. Clearly, z commutes with x and y, because it is an element of C. Now $x^* = -x$, $y^* = -y$, and $z^* = [y, x]^* = [x^*, y^*] = [-x, -y] = -z$. Thus $*$, when restricted to H, is the involution (iii) in Lemma 3.1 ∎

Theorem 3.4 *Let k be a field of characteristic different from two, and let L be a non-abelian nilpotent Lie k-algebra with involution $*: L \to L$, $f \mapsto f^*$. Then L contains a $*$-invariant Heisenberg Lie k-subalgebra H such that the restriction to H is one of the involutions in Lemma 3.2. More precisely, there exist $x, y \in L$ such that $[y, x] \neq 0$, $[y, [y, x]] = [x, [y, x]] = 0$, and either $x^* = x$ and $y^* = -y$, $x^* = x$ and $y^* = y$, or $x^* = -x$ and $y^* = -y$.*

Proof By taking the subalgebra of L generated by two noncommuting elements and their images by $*$, we can assume that L is finitely generated.

We shall argue by induction on the nilpotency class c of L, the case $c = 2$ having been dealt with in Lemma 3.3.

Suppose that $c > 2$ and let C denote the center of L. Then L/C is a nonabelian finitely generated nilpotent Lie algebra of class $c - 1$ with an involution induced by $*$. By the induction hypothesis, there exist $x, y \in L$ such that $\{x + C, y + C\}$ generate a $*$-invariant Heisenberg Lie subalgebra of L/C. Moreover, $x^* + C = \varepsilon x + C$ and $y^* + C = \eta y + C$ with $\varepsilon, \eta \in \{1, -1\}$ and $z = [y, x] \notin C$, $[y, z], [x, z] \in C$. It follows that M, the subalgebra of L generated by $\{x, y, C\}$, is a $*$-invariant subalgebra of L of nilpotency class at most 3.

If M has class 2, then the result follows from Lemma 3.3.

Suppose that M has class 3. Then $[x, z] \neq 0$ or $[y, z] \neq 0$. Say $[x, z] \neq 0$. We shall show that the k-subalgebra K generated by $\{x, x^*, z, z^*\}$ is a $*$-invariant subalgebra of L of class 2. It will be enough to show that $[\alpha, \beta]$ lies in the center of K for all $\alpha, \beta \in \{x, x^*, z, z^*\}$. For each $n \geq 1$, let $\gamma_n(M)$ denote the n-th term in the lower central series of M. Now, $z = [y, x] \in \gamma_2(M)$, so $z^* \in \gamma_2(M)$, because the terms in the lower central series are fully invariant subgroups of G. It follows that for every $\alpha \in K$ and $\beta \in \{z, z^*\}$ we have $[\alpha, \beta] \in \gamma_3(M)$, which is a central subalgebra of M; hence, $[\alpha, \beta]$ is central in K. Finally, that $[x, x^*] = 0$ follows from the fact that $x^* - x \in C$. So K is indeed a $*$-invariant subalgebra of L of class 2. Hence, Lemma 3.3 applies. ∎

4 Free Group Algebras in the Ore Ring of Fractions of Universal Enveloping Algebras of Nilpotent Lie Algebras

Let k be a field of characteristic zero and let H be the Heisenberg Lie k-algebra. In this section, we find various free (group) k-subalgebras in $\mathfrak{D}(H)$, the Ore ring of fraction of the universal enveloping algebra $U(H)$ of H. For that, our main tool is the result by G. Cauchon [4, Théorème]. The technique for obtaining suitable free algebras, from the paper of Cauchon, was developed in [10, Section 3]. In some cases, we then use Theorem 2.7 to obtain free group algebras in $\mathfrak{D}(H)$. If L is a nilpotent Lie k-algebra, applying Theorem 3.4, the foregoing implies the existence of free group algebras in $\mathfrak{D}(L)$, the Ore ring of fractions of the universal enveloping algebra $U(L)$ of L.

Let k be a field of characteristic different from 2. Let $K = k(t)$ be the field of fractions of the polynomial ring $k[t]$ in the variable t. Let σ be a k-automorphism of K of *infinite order*. We will consider the skew polynomial ring $K[p; \sigma]$. The elements of $K[p; \sigma]$ are "right polynomials" of the form $\sum_{i=0}^{n} p^i a_i$, where the coeficients a_i are in K. The multiplication is determined by

$$ap = p\sigma(a) \quad \text{for all } a \in K.$$

It is known that $K[p; \sigma]$ is a noetherian domain, and therefore it has an Ore division ring of fractions $D = K(p; \sigma)$.

Since σ is an automorphism of K, $\sigma(t) = \frac{at+b}{ct+d}$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k)$ defines a homography h of the projective line $\Delta = \mathbb{P}_1(k) = k \cup \{\infty\}$, $h: \Delta \to \Delta$, $z \mapsto h(z) = \frac{az+b}{cz+d}$.

We denote by $\mathcal{H} = \{h^n : n \in \mathbb{Z}\}$ the subgroup of the projective linear group $PGL_2(k)$ generated by h. The group $\mathcal{H}$ acts on Δ. If $z \in \Delta$, we denote by $\mathcal{H} \cdot z = \{h^n(z) : n \in \mathbb{Z}\}$ the orbit of z under the action of $\mathcal{H}$.

Theorem 4.1 (Cauchon's Theorem)   *Let α and β be two elements of k such that the orbits $\mathcal{H} \cdot \alpha$ and $\mathcal{H} \cdot \beta$ are infinite and different. Let s and u be the two elements of D defined by*

$$s = (t - \alpha)(t - \beta)^{-1} \quad and \quad u = (1 - p)(1 + p)^{-1}.$$

If the characteristic of k is different from 2, then the k-subalgebra Ω of D generated by $\xi = s$, $\eta = usu^{-1}$, ξ^{-1}, and η^{-1} is the free group k-algebra on the set $\{\xi, \eta\}$.

We will need the following consequence of Cauchon's Theorem.

Proposition 4.2   *Let k be a field of characteristic zero and let $K = k(t)$ be the field of fractions of the polynomial ring $k[t]$. Let $\sigma \colon K \to K$ be the automorphism of k-algebras determined by $\sigma(t) = t - 1$. Consider the skew polynomial ring $K[p; \sigma]$ and its Ore division ring of fractions $K(p; \sigma)$. Set $s = (t - \frac{5}{6})(t - \frac{1}{6})^{-1}$, $u = (1 - p^2)(1 + p^2)^{-1}$, and $u_1 = (1 - p^3)(1 + p^3)^{-1}$. The following hold true.*

  (i) *The k-subalgebra of $K(p; \sigma)$ generated by $\{s, s^{-1}, usu^{-1}, us^{-1}u^{-1}\}$ is the free group k-algebra on the set $\{s, usu^{-1}\}$.*

 (ii) *The k-subalgebra of $K(p; \sigma)$ generated by $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ is the free k-algebra on the set $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$.*

(iii) *The k-subalgebra of $K(p; \sigma)$ generated by $\{s + s^{-1}, u_1(s + s^{-1})u_1^{-1}\}$ is the free k-algebra on the set $\{s + s^{-1}, u_1(s + s^{-1})u_1^{-1}\}$.*

Proof   We will apply Cauchon's Theorem to the skew polynomial ring $K[p^2; \sigma^2]$, where $\sigma^2 \colon K \to K$ is given by $\sigma^2(t) = t - 2$.

Let $\alpha = \frac{5}{6} \in k$ and $\beta = \frac{1}{6}$. Let $\mathcal{H}$ be defined as above. Consider the orbits $\mathcal{H} \cdot \alpha = \{\frac{5}{6} - 2n : n \in \mathbb{Z}\}$, $\mathcal{H} \cdot \beta = \{\frac{1}{6} - 2n : n \in \mathbb{Z}\}$, which are infinite and different.

Then, by Cauchon's Theorem, $s = (t - \alpha)(t - \beta)^{-1}$ and $u = (1 - p^2)(1 + p^2)^{-1}$ are such that the k-algebra generated by $\xi = s, \eta = usu^{-1}$, ξ^{-1} and η^{-1} is the free group k-algebra on the free generators $\{\xi, \eta\}$. Thus, (i) is proved.

By Corollary 4.4, the k-algebra generated by $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ is the free k-algebra on the set $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$. Thus, (ii) is proved.

In order to prove (iii), apply Cauchon's Theorem to the skew polynomial ring $K[p^3; \sigma^3]$, where $\sigma^3 \colon K \to K$ is given by $\sigma^3(t) = t - 3$. Then proceed as in (i) and (ii). ∎

The following lemma is well known. For example, it appears in [10, Section 3.3].

Lemma 4.3   *Let k be a field of characteristic zero. Let $K = k(t)$ be the field of fractions of the polynomial ring $k[t]$ and $\sigma \colon K \to K$ be the automorphism of k-algebras determined by $\sigma(t) = t - 1$. Consider the skew polynomial ring $K[p; \sigma]$ and its Ore ring of fractions $K(p; \sigma)$. Let*

$$H = \langle x, y : [[y, x], x] = [[y, x], y] = 0 \rangle$$

be the Heisenberg Lie k-algebra, set $z = [y, x]$ and consider the universal enveloping algebra $U(H)$ of H. The following hold true.

(i) *Set $I = U(H)(z-1)$, the ideal of $U(H)$ generated by $z-1$. The set $\mathfrak{S} = U(H)\backslash I$ is a left Ore subset of $U(H)$.*

(ii) *There exists a surjective k-algebra homomorphism*

$$\Phi \colon \mathfrak{S}^{-1} U(H) \longrightarrow K(p; \sigma)$$

such that $\Phi(y) = p$, $\Phi(x) = p^{-1}t$, and $\Phi(z) = 1$.

Proof First note that

$$p(p^{-1}t) - (p^{-1}t)p = t - p^{-1}p(t-1) = 1.$$

Hence, there exists a k-algebra homomorphism $\Phi \colon U(H) \to K(p; \sigma)$ such that $\Phi(y) = p$, $\Phi(x) = p^{-1}t$, and $\Phi(z) = 1$. The ideal I is clearly contained in the kernel of Φ. Now note that $U(H)/I$ is the first Weyl algebra, which is a simple k-algebra. Thus, I is the kernel of Φ. The subset $\mathfrak{S}$ is an Ore subset of $U(H)$ by [25, Lemma 13]. By the universal property of the Ore localization, Φ can be extended to a k-algebra homomorphism $\Phi \colon \mathfrak{S}^{-1} U(H) \to K(p; \sigma)$. Note that $\mathfrak{S}^{-1} U(H)$ is a local ring with maximal ideal $\mathfrak{S}^{-1}I$. It induces an embedding of division rings $\mathfrak{S}^{-1}U(H)/\mathfrak{S}^{-1}I \to K(p; \sigma)$. Now Φ is surjective because $\Phi(yx) = t$ and $\Phi(y) = p$. ∎

The next result is [10, Corollary 3.2]. It will allow us to obtain free algebras generated by symmetric elements from free group algebras.

Lemma 4.4 *Let G be the free group on the set of two elements $\{x, y\}$. Let k be a field and consider the group algebra $k[G]$. Then the k-algebra generated by $x + x^{-1}$ and $y + y^{-1}$ inside $k[G]$ is free on $\{x + x^{-1}, y + y^{-1}\}$.*

Now we are ready to present the main result of this section. It will be used throughout the paper: parts (i), (ii), and (iii) in Sections 5 and 6 and parts (iv) and (v) in Section 7.

Theorem 4.5 *Let k be a field of characteristic zero. Let H be the Heisenberg Lie k-algebra. Let $U(H)$ be the universal enveloping algebra of H, and let $\mathfrak{D}(H)$ be the Ore division ring of fractions of $U(H)$. Set $z = [y, x]$, $V = \frac{1}{2}(xy + yx)$, and consider the following elements of $\mathfrak{D}(H)$:*

$$S = \left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1},$$

$$T = (z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1},$$

$$S_1 = z^{-1}\left(\left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1} + \left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right)\right)z^{-1},$$

$$S_2 = z\left(\left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1} + \left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right)\right)z,$$

$$T_1 = (z + y^2)^{-1}(z - y^2)S_1(z + y^2)(z - y^2)^{-1},$$

$$T_2 = (z^2 + y^3)^{-1}(z^2 - y^3)S_1(z^2 + y^3)(z^2 - y^3)^{-1},$$

$$T_3 = (z + y^2)^{-1}(z - y^2)S_2(z + y^2)(z - y^2)^{-1},$$
$$T_4 = (z^2 + y^3)^{-1}(z^2 - y^3)S_2(z^2 + y^3)(z^2 - y^3)^{-1}.$$

The following hold true.

(i) *The k-subalgebra of $\mathfrak{D}(H)$ generated by $\{S, S^{-1}, T, T^{-1}\}$ is the free group k-algebra on the set $\{S, T\}$.*

(ii)   (a) *The elements S_1, S_1^2, T_1, and T_1^2 are symmetric with respect to the involutions in Lemma 3.2(ii) and (iii).*

    (b) *The k-subalgebra of $\mathfrak{D}(H)$ generated by $\{S_1, T_1\}$ is the free k-algebra on the set $\{S_1, T_1\}$.*

    (c) *The k-subalgebra of $\mathfrak{D}(H)$ generated by*

$$\{1 + S_1, (1 + S_1)^{-1}, 1 + T_1, (1 + T_1)^{-1}\}$$

is the free group k-algebra on the set $\{1 + S_1, 1 + T_1\}$.

    (d) *The k subalgebra of $\mathfrak{D}(H)$ generated by $\{S_1^2, T_1^2\}$ is the free k-algebra on the set $\{S_1^2, T_1^2\}$.*

    (e) *The k-subalgebra of $\mathfrak{D}(H)$ generated by $\{1+S_1^2, (1+S_1^2)^{-1}, 1+T_1^2, (1+T_1^2)^{-1}\}$ is the free group k-algebra on the set $\{1 + S_1, 1 + T_1\}$.*

(iii)   (a) *The elements S_1, S_1^2, T_2, and T_2^2 are symmetric with respect to the involution in Lemma 3.2(i).*

    (b) *The k-subalgebra of $\mathfrak{D}(H)$ generated by $\{S_1, T_2\}$ is the free k-algebra on the set $\{S_1, T_2\}$.*

    (c) *The k-subalgebra of $\mathfrak{D}(H)$ generated by*

$$\{1 + S_1, (1 + S_1)^{-1}, 1 + T_2, (1 + T_2)^{-1}\}$$

is the free group k-algebra on the set $\{1 + S_1, 1 + T_2\}$.

    (d) *The k-subalgebra of $\mathfrak{D}(H)$ generated by $\{S_1^2, T_2^2\}$ is the free k-algebra on the set $\{S_1^2, T_2^2\}$.*

    (e) *The k-subalgebra of $\mathfrak{D}(H)$ generated by*

$$\{1 + S_1^2, (1 + S_1^2)^{-1}, 1 + T_2^2, (1 + T_2^2)^{-1}\}$$

is the free group k-algebra on the set $\{1 + S_1^2, 1 + T_2^2\}$.

(iv)   (a) *The elements S_2, S_2^2, T_3, and T_3^2 are symmetric with respect to the involutions in Lemma 3.2(ii) and (iii).*

    (b) *The k-subalgebra of $\mathfrak{D}(H)$ generated by $\{S_2^2, T_3^2\}$ is the free k-algebra on the set $\{S_2^2, T_3^2\}$.*

(v)   (a) *The elements S_2, S_2^2, T_4, and T_4^2 are symmetric with respect to the involution in Lemma 3.2(i).*

    (b) *The k-subalgebra of $\mathfrak{D}(H)$ generated by $\{S_2^2, T_4^2\}$ is the free k-algebra on the set $\{S_2^2, T_4^2\}$.*

Proof   Consider the surjective k-algebra homomorphism $\Phi\colon \mathfrak{S}^{-1}U(H) \to K(p;\sigma)$ given in Lemma 4.3. Then $\Phi(y) = p$, $\Phi(x) = p^{-1}t$, and $\Phi(z) = 1$.

Recall that in $K(p;\sigma)$, we have $tp = p(t-1)$. Thus,

$$\Phi(V) = \Phi\left(\frac{1}{2}(xy + yx)\right) = \frac{1}{2}(p^{-1}tp + pp^{-1}t) = \frac{1}{2}(t - 1 + t) = t - \frac{1}{2},$$

$$\Phi\left(V - \frac{1}{3}z\right) = t - \frac{1}{2} - \frac{1}{3} = t - \frac{5}{6}, \quad \Phi\left(V + \frac{1}{3}z\right) = t - \frac{1}{2} + \frac{1}{3} = t - \frac{1}{6},$$

$$\Phi(z + y^2) = 1 + p^2, \qquad\qquad \Phi(z - y^2) = 1 - p^2,$$

$$\Phi(z^2 + y^3) = 1 + p^3, \qquad\qquad \Phi(z^2 - y^3) = 1 - p^3.$$

Hence, the elements $V - \frac{1}{3}z$, $V + \frac{1}{3}z$, $z + y^2$, $z - y^2$, $z^2 + y^3$, $z^2 - y^3$ are invertible in $\mathfrak{S}^{-1}U(H)$.

Thus, $S, S^{-1}, T, T^{-1}, S_1, S_1^{-1}, T_1, T_1^{-1}, T_2, T_2^{-1} \in \mathfrak{S}^{-1}U(H)$. Moreover, following the notation of Proposition 4.2,

$$\Phi(S) = \left(t - \frac{5}{6}\right)\left(t - \frac{1}{6}\right)^{-1} = s,$$

$$\Phi(S_1) = \left(t - \frac{5}{6}\right)\left(t - \frac{1}{6}\right)^{-1} + \left(t - \frac{1}{6}\right)\left(t - \frac{5}{6}\right)^{-1} = s + s^{-1},$$

$$\Phi(S_2) = \left(t - \frac{5}{6}\right)\left(t - \frac{1}{6}\right)^{-1} + \left(t - \frac{1}{6}\right)\left(t - \frac{5}{6}\right)^{-1} = s + s^{-1},$$

$$\Phi\left((z + y^2)^{-1}(z - y^2)\right) = (1 + p^2)^{-1}(1 - p^2) = (1 - p^2)(1 + p^2)^{-1} = u,$$

$$\Phi\left((z + y^2)(z - y^2)^{-1}\right) = (1 + p^2)(1 - p^2)^{-1} = u^{-1},$$

$$\Phi\left((z^2 + y^3)(z^2 - y^3)^{-1}\right) = (1 + p^3)(1 - p^3)^{-1} = u_1^{-1}.$$

Hence, $\Phi(T) = usu^{-1}$, $\Phi(T_1) = u(s + s^{-1})u^{-1}$, $\Phi(T_2) = u_1(s + s^{-1})u_1^{-1}$, $\Phi(T_3) = u(s + s^{-1})u^{-1}$, and $\Phi(T_4) = u_1(s + s^{-1})u_1^{-1}$.

We proceed to show that the elements in statements (i)–(v) generate free (group) algebras. That they are symmetric will be proved below.

(i) By Proposition 4.2(i), the set $\{s, s^{-1}, usu^{-1}, us^{-1}u^{-1}\}$ generates a free group k-algebra. Therefore, the k-subalgebra of $\mathfrak{S}^{-1}U(H)$ generated by $\{S, S^{-1}, T, T^{-1}\}$ is the free group k-algebra on the set $\{S, T\}$.

(ii) By Proposition 4.2(ii), the set $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ are the free generators of a free k-algebra. Therefore the k-subagebra generated by $\{S_1, T_1\}$ is the free k-algebra on $\{S_1, T_1\}$. This implies that the k-subalgebra generated by $\{S_1^2, T_1^2\}$ is the free k-algebra on $\{S_1^2, T_1^2\}$.

Consider H as a $\mathbb{Z}$-graded Lie k-algebra as in Example 2.3(i). Then $U(H)$ is graded according to Lemma 2.4. The k-algebra $U(H)$ is an Ore domain. Recall that given a $\mathbb{Z}$-graded k-algebra that is an Ore domain, localizing at the set of nonzero homogeneous elements yields a graded division ring. Thus, if we localize at the set $\mathcal{H}$ of homogeneous elements of $U(H)$, we get that $\mathcal{H}^{-1}U(H)$ is a graded division ring. Notice that $z, V - \frac{1}{3}z, V + \frac{1}{3}z$ are homogeneous of degree -2. Therefore, S_1 is homogeneous of degree 4, and S_1^2 is homogeneous of degreee 8. Notice that $z + y^2, z - y^2$ are homogeneous of degree -2. Therefore, T_1 is homogeneous of degree 4, and T_1^2 is homogeneous of degree 8. By Theorem 2.7, the k-subalgebra generated by the set $\{1 + S_1, (1 + S_1)^{-1}, 1 + T_1, (1 + T_1)^{-1}\}$ is the free group k-algebra on the set $\{1 + S_1, 1 + T_1\}$. Also, by Theorem 2.7, the k-subalgebra generated by the

set $\{1 + S_1^2, (1 + S_1^2)^{-1}, 1 + T_1^2, (1 + T_1^2)^{-1}\}$ is the free group k-algebra on the set $\{1 + S_1^2, 1 + T_1^2\}$, as desired.

(iii) By Proposition 4.2(iii), the set $\{s + s^{-1}, u_1(s + s^{-1})u_1^{-1}\}$ are the free generators of a free k-algebra. Therefore the k-subagebra generated by $\{S_1, T_2\}$ is the free k-algebra on $\{S_1, T_2\}$. This implies that the k-subalgebra generated by $\{S_1^2, T_2^2\}$ is the free k-algebra on $\{S_1^2, T_2^2\}$.

Consider H as a $\mathbb{Z}$-graded Lie k-algebra as in Example 2.3(ii). Then $U(H)$ is graded according to Lemma 2.4. The k-algebra $U(H)$ is an Ore domain. Recall that given a $\mathbb{Z}$-graded k-algebra that is an Ore domain, localizing at the set of nonzero homogeneous elements yields a graded division ring. Thus, if we localize at the set $\mathcal{H}$ of homogeneous elements of $U(H)$, we get that $\mathcal{H}^{-1}U(H)$ is a graded division ring. Notice that z, $V - \frac{1}{3}z$, $V + \frac{1}{3}z$ are homogeneous of degree -3. Therefore, S_1 is homogeneous of degree 6 and S_1^2 is homogeneous of degree 12. Notice that $z^2 + y^3$, $z^2 - y^3$ are homogeneous of degree -6. Therefore, T_2 is homogeneous of degree 6 and T_2^2 is homogeneous of degree 12. By Theorem 2.7, the k-subalgebra generated by the set $\{1 + S_1, (1 + S_1)^{-1}, 1 + T_2, (1 + T_2)^{-1}\}$ is the free group k-algebra on the set $\{1 + S_1, 1 + T_2\}$. Also, by Theorem 2.7, the k-subalgebra generated by the set $\{1 + S_1^2, (1 + S_1^2)^{-1}, 1 + T_2^2, (1 + T_2^2)^{-1}\}$ is the free group k-algebra on the set $\{1 + S_1^2, 1 + T_2^2\}$, as desired.

(iv) By Proposition 4.2(ii), the set $\{s + s^{-1}, u(s + s^{-1})u^{-1}\}$ are the free generators of a free k-algebra. Therefore the k-subagebra generated by $\{S_2, T_3\}$ is the free k-algebra on $\{S_2, T_3\}$. This implies that the k-subalgebra generated by $\{S_2^2, T_3^2\}$ is the free k-algebra on $\{S_2^2, T_3^2\}$.

(v) By Proposition 4.2(iii), the set $\{s + s^{-1}, u_1(s + s^{-1})u_1^{-1}\}$ are the free generators of a free k-algebra. Therefore the k-subagebra generated by $\{S_2, T_4\}$ is the free k-algebra on $\{S_2, T_4\}$. This implies that the k-subalgebra generated by $\{S_2^2, T_4^2\}$ is the free k-algebra on $\{S_2^2, T_4^2\}$.

Now we prove that the elements considered in the statements of (ii), (iii), (iv), and (v) are symmetric. Consider first the principal involution, that is, the one in Lemma 3.2(iii):

$$V^* = \frac{1}{2}(xy + yx)^* = \frac{1}{2}(xy + yx) = V,$$

$$\left(V - \frac{1}{3}z\right)^* = V + \frac{1}{3}z, \qquad \left(V + \frac{1}{3}z\right)^* = V - \frac{1}{3}z,$$

$$S_1^* = \left(z^{-1}\left(\left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1} + \left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right)\right)z^{-1}\right)^*$$

$$= z^{-1}\left(\left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right) + \left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1}\right)z^{-1}$$

$$= S_1,$$

$$\left((z + y^2)^{-1}(z - y^2)\right)^* = (-z - y^2)(-z + y^2)^{-1}$$

$$= (z + y^2)(z - y^2)^{-1},$$

$$T_1^* = ((z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1})^*$$

$$= (z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1}$$

$$= T_1.$$

Similarly, $S_2^* = S_2$ and $T_3^* = T_3$.

Now consider the involution in Lemma 3.2(ii):

$$V^* = \frac{1}{2}(xy + yx)^* = \frac{1}{2}(xy + yx) = V,$$

$$\left(V - \frac{1}{3}z\right)^* = V + \frac{1}{3}z, \qquad \left(V + \frac{1}{3}z\right)^* = V - \frac{1}{3}z,$$

$$S_1^* = \left(z^{-1}\left(\left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1} + \left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right)\right)z^{-1}\right)^*$$

$$= z^{-1}\left(\left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right) + \left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1}\right)z^{-1}$$

$$= S_1,$$

$$((z + y^2)^{-1}(z - y^2))^* = (-z - y^2)(-z + y^2)^{-1}$$
$$= (z + y^2)(z - y^2)^{-1},$$

$$T_1^* = ((z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1})^*$$
$$= (z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1}$$
$$= T_1.$$

Similarly, $S_2^* = S_2$ and $T_3^* = T_3$.

Finally, consider the involution in Lemma 3.2(i).

$$V^* = \frac{1}{2}(xy + yx)^* = -\frac{1}{2}(xy + yx) = -V,$$

$$\left(V - \frac{1}{3}z\right)^* = -V - \frac{1}{3}z, \qquad \left(V + \frac{1}{3}z\right)^* = -V + \frac{1}{3}z,$$

$$S_1^* = \left(z^{-1}\left(\left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1} + \left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right)\right)z^{-1}\right)^*$$

$$= z^{-1}\left(\left(-V + \frac{1}{3}z\right)^{-1}\left(-V - \frac{1}{3}z\right) + \left(-V + \frac{1}{3}z\right)\left(-V - \frac{1}{3}z\right)^{-1}\right)z^{-1}$$

$$= z^{-1}\left(\left(V - \frac{1}{3}z\right)^{-1}\left(V + \frac{1}{3}z\right) + \left(V - \frac{1}{3}z\right)\left(V + \frac{1}{3}z\right)^{-1}\right)z^{-1}$$

$$= S_1,$$

$$((z^2 + y^3)^{-1}(z^2 - y^3))^* = (z^2 + y^3)(z^2 - y^3)^{-1},$$

$$T_2^* = ((z^2 + y^3)^{-1}(z^2 - y^3)S_1(z^2 + y^3)(z^2 - y^3)^{-1})^*$$
$$= (z^2 + y^3)^{-1}(z^2 - y^3)S_1(z^2 + y^3)(z^2 - y^3)^{-1}$$
$$= T_2.$$

Similarly $S_2^* = S_2$ and $T_4^* = T_4$. ∎

Theorem 4.6 *Let k be a field of characteristic zero and L be a nonabelian nilpotent Lie k-algebra. Let $U(L)$ be the universal enveloping algebra of L and $\mathfrak{D}(L)$ be the Ore ring of fractions of $U(L)$. Then, for any involution $*: L \to L$, $f \mapsto f^*$, there exist nonzero symmetric elements $U, V \in L$ such that the k-subalgebra of $\mathfrak{D}(L)$ generated by $\{U, U^{-1}, V, V^{-1}\}$ is the free group k-algebra on $\{U, V\}$.*

Proof By Theorem 3.4, there exists a ∗-invariant Heisenberg Lie k-subalgebra H generated by two elements x, y and such that ∗, when restricted to H is one of the three involutions in Lemma 3.2. Since $U(L)$ is an Ore domain, $U(H)$ is also an Ore domain. Thus, the division ring generated by $U(H)$ inside $\mathfrak{D}(L)$ is $\mathfrak{D}(H)$. By Theorem 4.5, there exist elements U and V as desired.   ∎

5 Free Group Algebras in Division Rings Generated by Universal Enveloping Algebras of Residually Nilpotent Lie Algebras

Let k be a field, L a Lie k-algebra, and $U(L)$ its universal enveloping algebra. It was proved in [8] that $U(L)$ can be embedded in a division ring. Two similar proofs of this fact were given in [21, 24]. Moreover, the division ring constructed in the foregoing three papers is the same [21, Theorem 8]. We will work with the construction of the skew field $\mathfrak{D}(L)$ that contains $U(L)$ and it is generated by $U(L)$ (as a division ring) given by Lichtman in [24]. The interested reader can also find this construction in [7, Section 2.6]. Of course, when $U(L)$ is an Ore domain, $\mathfrak{D}(L)$ is the Ore ring of fractions of $U(L)$.

In this subsection, we want to obtain free group algebras in $\mathfrak{D}(L)$, where L is a (generalization of) a residually nilpotent Lie algebra, from the ones obtained in $\mathfrak{D}(H)$, where H is the Heisenberg Lie algebra. The technique we will use is from [10, Section 4].

For that, we will need some results on the division ring $\mathfrak{D}(L)$. For example, $\mathfrak{D}(L)$ is well behaved for Lie subalgebras of L as shown in [26, Proposition 2.5]. More precisely, if N is a Lie subalgebra of L, then the natural embedding $U(N) \hookrightarrow U(L)$ can be extended to an embedding $\mathfrak{D}(N) \hookrightarrow \mathfrak{D}(L)$. Furthermore, if $\mathcal{B}_N$ is a basis of N and $\mathcal{C}$ is a set of elements of $L \backslash N$ such that $\mathcal{B}_N \cup \mathcal{C}$ is a basis of L, then the standard monomials in $\mathcal{C}$ are linearly independent over $\mathfrak{D}(N)$. Notice that if $U(L)$ is an Ore domain, then these assertions are easily verified.

Let k be a field and R be a k-algebra. Suppose that $\delta \colon R \to R$ is a k-derivation of R; that is, δ is a k-linear map such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all a, $b \in R$. We will consider the skew polynomial ring $R[x;\delta]$. The elements of $R[x;\delta]$ are "right polynomials" of the form $\sum_{i=0}^{n} x^i a_i$, where the coeficients a_i are in R. The multiplication is determined by
$$ax = xa + \delta(a) \quad \text{for all } a \in R.$$

Given $R[x;\delta]$, one can construct the *formal pseudo-differential operator ring*, denoted $R((t_x;\delta))$, consisting of the formal Laurent series $\sum_{i=n}^{\infty} t_x^i a_i$, with $n \in \mathbb{Z}$ and coefficients $a_i \in R$, satisfying $at_x^{-1} = t_x^{-1}a + \delta(a)$ for all $a \in R$. Therefore,
$$at_x = t_x a - t_x \delta(a) t_x = \sum_{i=1}^{\infty} t_x^i (-1)^{i-1} \delta^{i-1}(a),$$
for any $a \in R$.

The subset $R[[t_x;\delta]]$ of $R((t_x;\delta))$ consisting of the Laurent series of the form $\sum_{i=0}^{\infty} t_x^i a_i$ is a k-subalgebra of $R((t_x;\delta))$. The set $\mathcal{S} = \{1, t_x, t_x^2, \dots\}$ is a left denominator set of $R[[t_x;\delta]]$ such that the Ore localization $\mathcal{S}^{-1}R[[t_x;\delta]]$ is the k-algebra $R((t_x;\delta))$; see, for example, [7, Theorem 2.3.1]. If R is a domain, then a series $f \in R((t_x;\delta))$ is invertible if and only if the coefficient of the least element in

the support of f is invertible in R. Notice that there is a natural embedding $R[x;\delta] \hookrightarrow R((t_x;\delta_x))$ sending x to t_x^{-1}.

In what follows, $R[y;\delta_y][x;\delta_x]$ means polynomials of the form $\sum_{i=0}^{n} x^i f_i$ where each $f_i \in R[y;\delta_y]$ and δ_x is a k-derivation of $R[y;\delta_y]$. Also, $R((t_y;\delta_y))((t_x;\delta_x))$ is the ring of series of the form $\sum_{i=n}^{\infty} t_x^i f_i$, with $n \in \mathbb{Z}$, coefficients $f_i \in R((t_y;\delta_y))$ and δ_x is a k-derivation of $R((y;\delta_y))$.

Let k be a field. Let L a Lie k-algebra generated by two elements u, v. Let $H = \langle x, y : [[y,x],x] = [[y,x],y] = 0 \rangle$ be the Heisenberg Lie k-algebra. Suppose that there exists a Lie k-algebra homomorphism

$$L \xrightarrow{\rho} H, \quad u \longmapsto x, \quad v \longmapsto y.$$

Define $w = [v,u]$ and $z = [y,x]$. Let $N = \ker \rho$. Thus, N is a (Lie) ideal of L.

By the universal property of universal enveloping algebras, ρ can be uniquely extended to a k-algebra homomorphism $\psi \colon U(L) \to U(H)$ between the corresponding universal enveloping algebras. Note that $\ker \psi$ is the ideal of $U(L)$ generated by N. The restriction $\psi_{|U(N)}$ coincides with the augmentation map $\varepsilon \colon U(N) \to k$.

By the PBW-Theorem, the elements of $U(H)$ are uniquely expressed as finite sums $\sum_{l,m,n\geq 0} x^l y^m z^n a_{lmn}$, with $a_{lmn} \in k$. Let δ_x be the inner k-derivation of $U(H)$ determined by x, i.e., $\delta_x(f) = [f,x] = fx - xf$ for all $f \in U(H)$. It can be proved that

(5.1) $$U(H) = k[z][y][x;\delta_x],$$

$$U(H) \hookrightarrow k((t_z))((t_y))((t_x;\delta_x)), \quad z \longmapsto t_z^{-1}, \quad y \longmapsto t_y^{-1}, \quad x \longmapsto t_x^{-1}.$$

Now consider $U(L)$, the universal enveloping algebra of L. By the PBW-Theorem, the elements of $U(L)$ can be uniquely expressed as finite sums $\sum_{l,m,n\geq 0} u^l v^m w^n f_{lmn}$ with $f_{lmn} \in U(N)$. Since N is an ideal of L, the inner derivations $\delta_u, \delta_v, \delta_w$ of $U(L)$ defined by u, v, w, respectively, are such that $\delta_u(U(N)) \subseteq U(N)$, $\delta_v(U(N)) \subseteq U(N)$, $\delta_w(U(N)) \subseteq U(N)$. The k-subalgebra of $U(L)$ generated by $U(N)$ and w is $U(N)[w;\delta_w]$. Since $\delta_v(w) \in U(N) \subseteq U(N)[w;\delta_w]$, the k-subalgebra of $U(L)$ generated by $U(N)$ and $\{w,v\}$ is $U(N)[w;\delta_w][v;\delta_v]$. Furthermore, since $\delta_u(v) = w$ and $\delta_u(w) \in U(N)$,

(5.2) $$U(L) = U(N)[w;\delta_w][v;\delta_v][u;\delta_u],$$

$$U(L) \hookrightarrow U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)),$$

$$w \longmapsto t_w^{-1}, \; v \longmapsto t_v^{-1}, \; u \longmapsto t_u^{-1}, f \longmapsto \varepsilon(f), \text{ for all } f \in U(N).$$

In this setting, the next two lemmas are [10, Lemmas 4.1, 4.2].

Lemma 5.1 *There exists a commutative diagram of embeddings of k-algebras*
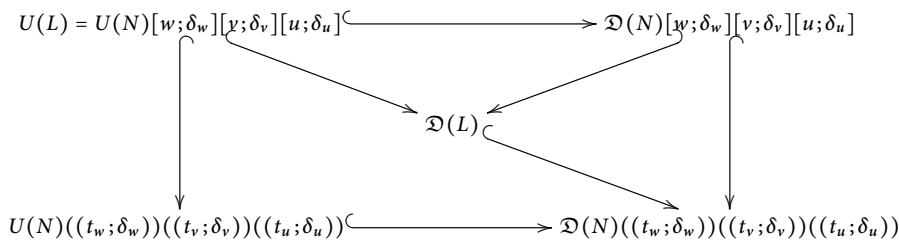
Lemma 5.2   *Let $\varepsilon\colon U(N) \to k$ denote the augmentation map. The following hold true.*

(i)   *There exists a k-algebra homomorphism*

$$\Phi_w\colon U(N)((t_w;\delta_w)) \longrightarrow k((t_z)), \quad \sum_i t_w^i f_i \longmapsto \sum_i t_z^i \varepsilon(f_i),$$

where $f_i \in U(N)$ for each i.

(ii)   *There exists a k-algebra homomorphism*

$$\Phi_v\colon U(N)((t_w;\delta_w))((t_v;\delta_v)) \longrightarrow k((t_z))((t_y)), \quad \sum_i t_v^i g_i \longmapsto \sum_i t_y^i \Phi_w(g_i),$$

where $g_i \in U(N)((t_w;\delta_w))$ for each i.

(iii)   *There exists a k-algebra homomorphism*

$$\Phi_u\colon U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)) \longrightarrow k((t_z))((t_y))((t_x;\delta_x)),$$
$$\sum_i t_u^i h_i \longmapsto \sum_i t_x^i \Phi_v(h_i),$$

where $h_i \in U(N)((t_w;\delta_w))((t_y;\delta_y))$ for each i, and extending the embeddings of (5.1) and (5.2).

Now we turn our attention to k-involutions of $U(L)$ induced from the ones in L.

Lemma 5.3   *Let k be a field and L be a Lie k-algebra generated by two elements u, v. Let $H = \langle x, y : [[y, x], x] = [[y, x], y] = 0 \rangle$ be the Heisenberg Lie k-algebra. Suppose that there exists a Lie k-algebra homomorphism $L \xrightarrow{\rho} H$, $u \mapsto x$, $v \mapsto y$. Let $N = \ker \rho$. Consider the induced k-algebra homomorphism $\psi\colon U(L) \to U(H)$. Suppose that $*\colon L \to L$ is an involution in L such that N is a $*$-invariant ideal of L and call again $*$ the induced involution on $H \cong L/N$. Then $\psi(f^*) = \psi(f)^*$ for all $f \in U(L)$.*

Proof   Define $w = [v, u]$ and $z = [y, x]$. Since $*$ is the induced involution on $H \cong L/N$, then $\psi(u^*) = \psi(u)^* = x^*$, $\psi(v^*) = \psi(v)^* = y^*$, $\psi(w^*) = \psi(w)^* = z^*$ and $\psi(f^*) = \psi(f)^* = \varepsilon(f) \in k$ for all $f \in U(N)$.

Given $\sum_{l,m,n \geq 0} u^l v^m w^n f_{lmn}$ with $f_{lmn} \in U(N)$, we have

$$\psi\left(\sum_{l,m,n \geq 0} u^l v^m w^n f_{lmn}\right)^* = \left(\sum_{l,m,n \geq 0} x^l y^m z^n \varepsilon(f_{lmn})\right)^*$$
$$= \sum_{l,m,n \geq 0} (z^*)^n (y^*)^m (x^*)^l \varepsilon(f_{lmn}).$$

On the other hand,

$$\psi\left(\left(\sum_{l,m,n \geq 0} u^l v^m w^n f_{lmn}\right)^*\right) = \psi\left(\sum_{l,m,n \geq 0} f_{lmn}^* (w^*)^n (v^*)^m (u^*)^l\right)$$
$$= \sum_{l,m,n \geq 0} (z^*)^n (y^*)^m (x^*)^l \varepsilon(f_{lmn}),$$

as desired.   ∎

It is well known that any k-involution on L can be extended to a k-involution of $U(L)$. Moreover, it was proved in [6, Proposition 5] that any k-involution of L can be

uniquely extended to a k-involution of $\mathfrak{D}(L)$. See also [10, Proposition 2.1]. With this in mind, we are ready to prove the main result of this section.

Theorem 5.4 *Let k be a field of characteristic zero, let $H = \langle x, y : [[y, x], x] = [[y, x], y] = 0 \rangle$ be the Heisenberg Lie k-algebra and let L be a Lie k-algebra generated by two elements u, v. Suppose that there exists a Lie k-algebra homomorphism*

$$L \longrightarrow H, \quad u \longmapsto x, \quad v \longmapsto y,$$

with kernel N. Let $w = [v, u]$, $V = \frac{1}{2}(uv + vu)$, and consider the following elements of $\mathfrak{D}(L)$:

$$S = \left(V - \frac{1}{3}w\right)\left(V + \frac{1}{3}w\right)^{-1}, \qquad T = (w + v^2)^{-1}(w - v^2)S(w + v^2)(w - v^2)^{-1},$$

$$S_1 = w^{-1}\left(\left(V - \frac{1}{3}w\right)\left(V + \frac{1}{3}w\right)^{-1} + \left(V - \frac{1}{3}w\right)^{-1}\left(V + \frac{1}{3}w\right)\right)w^{-1},$$

$$T_1 = (w + v^2)^{-1}(w - v^2)S_1(w + v^2)(w - v^2)^{-1},$$

$$T_2 = (w^2 + v^3)^{-1}(w^2 - v^3)S_1(w^2 + v^3)(w^2 - v^3)^{-1}.$$

Then the following hold true.

(i) *The k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S, S^{-1}, T, T^{-1}\}$ is the free group k-algebra on the set $\{S, T\}$.*

(ii) *Suppose that $*: L \to L$ is an involution in L such that N is a $*$-invariant ideal of L and that the induced involution on $H \cong L/N$ is one of the involutions in Lemma 3.2(ii) and (iii). Then the following hold true.*

(a) *The elements $S_1 S_1^*$ and $T_1 T_1^*$ are symmetric.*

(b) *The k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S_1 S_1^*, T_1 T_1^*\}$ is the free k-algebra on $\{S_1 S_1^*, T_1 T_1^*\}$.*

(c) *The k-subalgebra of $\mathfrak{D}(L)$ generated by*

$$\left\{1 + S_1 S_1^*, (1 + S_1 S_1^*)^{-1}, 1 + T_1 T_1^*, (1 + T_1 T_1^*)^{-1}\right\}$$

is the free group k-algebra on the set $\{1 + S_1 S_1^*, 1 + T_1 T_1^*\}$.

(iii) *Suppose that $*: L \to L$ is an involution in L such that N is a $*$-invariant ideal of L and that the induced involution on $H \cong L/N$ is one of the involutions in Lemma 3.2(i). Then the following hold true.*

(a) *The elements $S_1 S_1^*$ and $T_2 T_2^*$ are symmetric.*

(b) *The k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S_1 S_1^*, T_2 T_2^*\}$ is the free k-algebra on $\{S_1 S_1^*, T_2 T_2^*\}$.*

(c) *The k-subalgebra of $\mathfrak{D}(L)$ generated by*

$$\left\{1 + S_1 S_1^*, (1 + S_1 S_1^*)^{-1}, 1 + T_2 T_2^*, (1 + T_2 T_2^*)^{-1}\right\}$$

is the free group k-algebra on the set $\{1 + S_1 S_1^*, 1 + T_2 T_2^*\}$.

Proof Define $z = [y, x] \in H$. Consider the embedding $U(H) \hookrightarrow k((t_z))((t_y))((t_x; \delta_x))$ given in (5.1). Since $k((t_z))((t_y))((t_x; \delta_x))$ is a division k-algebra and $U(H)$ is an Ore domain, it extends to an embedding $\mathfrak{D}(H) \hookrightarrow k((t_z))((t_y))((t_x; \delta_x))$.

Consider the embedding

$$U(L) \hookrightarrow U(N)((t_w ; \delta_w))((t_v ; \delta_v))((t_u ; \delta_u))$$

given in (5.2). Let $\Phi_u : U(N)((t_w ; \delta_w))((t_v ; \delta_v))((t_u ; \delta_u)) \to k((t_z))((t_y))((t_x ; \delta_x))$ be the homomorphism given in Lemma 5.2.

Define the following elements in $\mathfrak{D}(H)$:

$$V_H = \frac{1}{2}(xy + yx)$$

$$S_H = \left(V_H - \frac{1}{3}z\right)\left(V_H + \frac{1}{3}z\right)^{-1},$$

$$T_H = (z + y^2)^{-1}(z - y^2)S_H(z + y^2)(z - y^2)^{-1}.$$

$$S_{1H} = z^{-1}\left(\left(V_H - \frac{1}{3}z\right)\left(V_H + \frac{1}{3}z\right)^{-1} + \left(V_H - \frac{1}{3}z\right)^{-1}\left(V_H + \frac{1}{3}z\right)\right)z^{-1},$$

$$T_{1H} = (z + y^2)^{-1}(z - y^2)S_{1H}(z + y^2)(z - y^2)^{-1},$$

$$T_{2H} = (z^2 + y^3)^{-1}(z^2 - y^3)S_{1H}(z^2 + y^3)(z^2 - y^3)^{-1}.$$

Claim 1: The elements $V - \frac{1}{3}w$, $V + \frac{1}{3}w$, $w + v^2$, $w - v^2$, $w^2 + v^3$, and $w^2 - v^3$ are all invertible in $U(N)((t_w ; \delta_w))((t_v ; \delta_v))((t_u ; \delta_u))$.

We proceed to prove claim 1. We begin with the element $w + v^2 = t_w^{-1} + t_v^{-2}$. As a series in t_v, this element is invertible in $U(N)((t_w ; \delta_w))((t_v ; \delta_v))$ if and only if the coefficient of t_v^{-2} is invertible in the ring of coefficients $U(N)((t_w ; \delta_w))$. The coefficient is 1, which is clearly invertible. Similarly, it can be proved that $w - v^2$, $w^2 + v^3$, and $w^2 - v^3$ are invertible. Now we show that $V + \frac{1}{3}w$ is invertible in $U(N)((t_w ; \delta_w))((t_v ; \delta_v))((t_u ; \delta_u))$. First, we obtain an expression of $V + \frac{1}{3}w$ as a series in t_u.

$$(5.3) \qquad V + \frac{1}{3}w = \frac{1}{2}(uv + vu) + \frac{1}{3}w = \frac{1}{2}(uv + [v, u] + uv) + \frac{1}{3}w$$
$$= \frac{1}{2}w + uv + \frac{1}{3}w = \frac{5}{6}w + uv = \frac{5}{6}t_w^{-1} + t_u^{-1}t_v^{-1}.$$

Thus, as a series in t_u, the coefficient of the least element in the support of $V + \frac{1}{3}w^3$ is t_v^{-1}, which is invertible in $U(N)((t_w ; \delta_w))((t_v ; \delta_v))$. Hence, $V + \frac{1}{3}w^3$ is invertible in $U(N)((t_w ; \delta_w))((t_v ; \delta_v))((t_u ; \delta_u))$. The case of $V - \frac{1}{3}w$ is shown analogously, and the claim is proved.

(i) By Claim 1, S and T are invertible in $U(N)((t_w ; \delta_w))((t_v ; \delta_v))((t_u ; \delta_u))$, and we have $\Phi_u(V) = V_H$, $\Phi_u(S) = S_H$, and $\Phi_u(T) = T_H$. By Theorem 4.5, the k-algebra generated by $\{S_H, S_H^{-1}, T_H, T_H^{-1}\}$ is the free group k-algebra on the set $\{S_H, T_H\}$. By Lemma 5.1, V, S and T belong to $\mathfrak{D}(L)$. Therefore, the elements S and T are nonzero and invertible in $\mathfrak{D}(L)$, and the k-subalgebra generated by $\{S, S^{-1}, T, T^{-1}\}$ is the free group k-algebra on the set $\{S, T\}$.

(ii)(a) This is clear.

(ii)(b) We will prove in detail the result for the involution in Lemma 3.2(ii); the other case can be shown similarly. Let $n_u, n_v, n_w, \in N$ be such that

$$u^* = u + n_u, \quad v^* = v + n_v, \quad w^* = -w + n_w.$$

Claim 2: The elements

$$\left(V + \frac{1}{3}w\right)^*, \ \left(V - \frac{1}{3}w\right)^*, \ (w + v^2)^*, \ (w - v^2)^*$$

belong to and are invertible in $U(N)((t_w \ ; \delta_w))((t_v \ ; \delta_v))((t_u \ ; \delta_u))$.

From Claim 2, it follows that we have the elements

$$S_1^*, T_1^* \in U(N)\big((t_w \ ; \delta_w)\big)\big((t_v \ ; \delta_v)\big)\big((t_u \ ; \delta_u)\big).$$

By Lemma 5.3, $\Phi_u(Z^*) = \Phi_u(Z)^*$, where Z is any of the elements in Claim 2. Thus, by Theorem 4.5(ii)(a),

(5.4) $\Phi_u(S_1^*) = \Phi_u(S_1)^* = S_{1H}^* = S_{1H}$ and $\Phi_u(T_1^*) = \Phi_u(T_1)^* = T_{1H}^* = T_{1H}.$

Hence $\Phi_u(S_1 S_1^*) = S_{1H}^2$ and $\Phi_u(T_1 T_1^*) = T_{1H}^2$. By Theorem 4.5(ii)(d), the k-algebra generated by $\{S_{1H}^2, T_{1H}^2\}$ is the free algebra on $\{S_{1H}^2, T_{1H}^2\}$. Therefore, the result follows.

We proceed to prove Claim 2.

$$\begin{aligned}
\text{(5.5)} \quad \left(V + \frac{1}{3}w\right)^* &= \left(\frac{1}{2}(uv + vu) + \frac{1}{3}w\right)^* \\
&= \frac{1}{2}\big((u + n_u)(v + n_v) + (v + n_v)(u + n_u)\big) + \frac{1}{3}(-w + n_w) \\
&= \frac{1}{2}(uv + vu + un_v + n_v u + n_u v + vn_u + n_u n_v + n_v n_u) \\
&\quad - \frac{1}{3}w + \frac{1}{3}n_w \\
&= \frac{1}{2}\big(uv + uv + [v, u] + un_v + un_v + [n_v, u] + vn_u + vn_u \\
&\quad + [n_u, v] + n_u n_v + n_v n_u\big) - \frac{1}{3}w + \frac{1}{3}n_w \\
&= u(v + n_v) + vn_u + \frac{1}{6}w + f_1 \\
&= t_u^{-1}(t_v^{-1} + n_v) + t_v^{-1}n_u + \frac{1}{6}t_w^{-1} + f_1,
\end{aligned}$$

where $f_1 \in U(N)$.

$$\begin{aligned}
\text{(5.6)} \quad \left(V - \frac{1}{3}w\right)^* &= \left(\frac{1}{2}(uv + vu) - \frac{1}{3}w\right)^* \\
&= \frac{1}{2}\big((u + n_u)(v + n_v) + (v + n_v)(u + n_u)\big) - \frac{1}{3}(-w + n_w) \\
&= \frac{1}{2}(uv + vu + un_v + n_v u + n_u v + vn_u + n_u n_v + n_v n_u) \\
&\quad + \frac{1}{3}w - \frac{1}{3}n_w \\
&= \frac{1}{2}\big(uv + uv + [v, u] + un_v + un_v + [n_v, u] + vn_u + vn_u \\
&\quad + [n_u, v] + n_u n_v + n_v n_u\big) + \frac{1}{3}w - \frac{1}{3}n_w
\end{aligned}$$

$$= u(v + n_v) + v n_u + \frac{5}{6} w + f_2$$

$$= t_u^{-1}(t_v^{-1} + n_v) + t_v^{-1} n_u + \frac{5}{6} t_w^{-1} + f_2,$$

where $f_2 \in U(N)$. Note that the element $(t_v^{-1} + n_v)$ is invertible in $U(N)((t_w \, ; \delta_w))$ $((t_v \, ; \delta_v))$. Thus, $(V + \frac{1}{3})^*$ and $(V - \frac{1}{3})^*$ are invertible in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$ $((t_u \, ; \delta_u))$.

There exist $f_3, f_4 \in U(N)$ such that

$$(5.7) \qquad (w + v^2)^* = -w + n_w + (v + n_v)^2$$

$$= v^2 + v n_v + n_v v + n_v^2 - w + n_w$$

$$= v^2 + 2 v n_v - w + [n_v, v] + n_v^2 + n_w$$

$$= t_v^{-2} + 2 t_v^{-1} n_v - t_w^{-1} + f_3,$$

$$(5.8) \qquad (w - v^2)^* = -w + n_w - (v + n_v)^2$$

$$= -v^2 - v n_v - n_v v - n_v^2 - w + n_w$$

$$= -v^2 - 2 v n_v - w - [n_v, v] - n_v^2 + n_w$$

$$= -t_v^{-2} - 2 t_v^{-1} n_v - t_w^{-1} + f_4.$$

The elements $(w + v^2)^*, (w - v^2)^*$ are invertible, because the coefficient of t_v^{-2} is ± 1, which is clearly invertible. And the claim is proved.

(ii)(c) By Theorem 4.5(ii)(d), the k-subalgebra generated by

$$\left\{ 1 + S_{1H}^2, (1 + S_{1H})^{-1}, 1 + T_{1H}^2, (1 + T_{1H})^{-1} \right\}$$

is the free group k-algebra on the set $\{1 + S_{1H}^2, 1 + T_{1H}^2\}$. Moreover, by (5.4), $\Phi_u(1 + S_1 S_1^*) = 1 + S_{1H}^2$ and $\Phi_u(1 + T_1 T_1^*) = 1 + T_{1H}^2$. Therefore, it is enough to prove that the elements $1 + S_1 S_1^*$ and $1 + T_1 T_1^*$ are invertible in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))((t_u \, ; \delta_u))$.

By (5.3), $V - \frac{1}{3} w$ and $V + \frac{1}{3} w$ are series of the form $t_u^{-1} t_v^{-1}(1 + h_1)$, where h_1 is a series on positive powers of t_u with coefficients in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$. Hence, $(V - \frac{1}{3} w)^{-1}$ and $(V + \frac{1}{3} w)^{-1}$ are series of the form $t_v t_u (1 + h_2)$, where h_2 is a series on positive powers of t_u with coefficients in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$. Using that $w^{-1} = t_w$, we obtain that S_1 is a series of the form $2 t_w^2 + h_3$, where h_3 is a series on positive powers of t_u with coefficients in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$.

By (5.5) and (5.6), $(V - \frac{1}{3} w)^*$ and $(V + \frac{1}{3} w)^*$ are series of the form $t_u^{-1}(t_v^{-1} + n_v)(1 + h_4)$, where h_4 is a series on positive powers of t_u with coefficients in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$. Hence, $((V - \frac{1}{3} w)^*)^{-1}$ and $((V + \frac{1}{3} w)^*)^{-1}$ are series of the form $(t_v^{-1} + n_v)^{-1} t_u (1 + h_5)$, where h_5 is a series on positive powers of t_u with coefficients in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$. Using that $(w^*)^{-1} = (-t_w^{-1} + n_w)^{-1}$, we obtain that S_1^* is a series of the form $2 t_w^2 + h_6$, where h_6 is a series on positive powers of t_u with coefficients in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$. From these considerations, it follows that $1 + S_1 S_1^*$ is a series of the form $1 + 4 t_w^4 + h_7$, where h_7 is a series on positive powers of t_u with coefficients in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$. Now $1 + 4 t_w^4$ is invertible in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))$, and $1 + S_1 S_1^*$ is therefore invertible in $U(N)((t_w \, ; \delta_w))((t_v \, ; \delta_v))((t_u \, ; \delta_u))$.

Clearly, $(w + v^2)$ and $(w - v^2)$ are series of the form $\pm t_v^{-2}(1 + g_1)$, where g_1 is a series on positive powers of t_v and coefficients in $U(N)((t_w \; ; \delta_w))$. Thus, $(w + v^2)^{-1}$ and $(w - v^2)^{-1}$ are series of the form $\pm t_v^2(1 + g_2)$, where g_2 is a series on positive powers of t_v and coefficients in $U(N)((t_w \; ; \delta_w))$. Using that S_1 is a series of the form $2t_w^2 + h_3$, where h_3 is as stated above, we obtain that T_1 is a series of the form $2t_w^2 + g_3 + h_8$, where g_3 is a series on positive powers of t_v and coefficients in $U(N)((t_w \; ; \delta_w))$ and h_8 is a series on positive powers of t_u with coefficients in $U(N)((t_w \; ; \delta_w))((t_v \; ; \delta_v))$.

By (5.7) and (5.8), $(w + v^2)^*$ and $(w - v^2)^*$ are series of the form $\pm t_v^{-2}(1 + g_4)$, where g_4 is a series on positive powers of t_v and coefficients in $U(N)((t_w \; ; \delta_w))$. Thus, $((w + v^2)^*)^{-1}$ and $((w - v^2)^*)^{-1}$ are series of the form $\pm t_v^2(1 + g_5)$, where g_5 is a series on positive powers of t_v and coefficients in $U(N)((t_w \; ; \delta_w))$. Using that S_1^* is a series of the form $2t_w^2 + h_6$, where h_6 is as stated above, we obtain that T_1^* is a series of the form $2t_w^2 + g_6 + h_9$, where g_6 is a series on positive powers of t_v and coefficients in $U(N)((t_w \; ; \delta_w))$ and h_9 is a series on positive powers of t_u with coefficients in $U(N)((t_w \; ; \delta_w))((t_v \; ; \delta_v))$. Therefore $1 + T_1 T_1^*$ is a series of the form $1 + 4t_w^4 + g_7 + h_{10}$, where g_7 is a series on positive powers of t_v and coefficients in $U(N)((t_w \; ; \delta_w))$ and h_{10} is a series on positive powers of t_u with coefficients in $U(N)((t_w \; ; \delta_w))((t_v \; ; \delta_v))$. Now $1 + T_1 T_1^*$ is invertible, because the series $1 + 4t_w^4 + g_7$ is invertible in $U(N)((t_w \; ; \delta_w))((t_v \; ; \delta_v))$, since $1 + 4t_w^4$ is invertible in $U(N)((t_w \; ; \delta_w))$.

(iii) Suppose that the induced involution on L/N is the one on Lemma 3.2(iii).

The result follows very much like (ii) from the following claim which can be shown as Claim 2.

Claim 3: The elements

$$\left(V + \frac{1}{3}w\right)^*, \; \left(V - \frac{1}{3}w\right)^*, \; (w^2 + v^3)^*, \; (w^3 - v^3)^*$$

belong to and are invertible in $U(N)((t_w \; ; \delta_w))((t_v \; ; \delta_v))((t_u \; ; \delta_u))$:

$$
\begin{aligned}
(w^2 + v^3)^* &= (-w + n_w)^2 + (-v + n_v)^3 \\
&= w^2 - w n_w - n_w w + n_w^2 - v^3 + v^2 n_v + v n_v v + n_v v^2 \\
&\quad - v n_v^2 - n_v^2 v - n_v v n_v + n_v^3 \\
&= -v^3 + 3v^2 n_v - v(3n_v^2 + [n_v, v]) - [n_v^2, v] + [n_v, v^2] \\
&\quad - [n_v, v]n_v + n_v^3 + w^2 - 2w n_w + [n_w, w] + n_w^2 \\
&= -t_v^{-3} + 3t_v^{-2} n_v - t_v^{-1}(3n_v^2 + [n_v, v]) + t_w^{-2} - 2t_w^{-1} + f_5,
\end{aligned}
$$

$$
\begin{aligned}
(w^2 - v^3)^* &= (-w + n_w)^2 - (-v + n_v)^3 \\
&= w^2 - w n_w - n_w w + n_w^2 - (-v^3 + v^2 n_v + v n_v v + n_v v^2 - v n_v^2 \\
&\quad - n_v^2 v - n_v v n_v + n_v^3) \\
&= t_v^{-3} - 3t_v^{-2} n_v + t_v^{-1}(3n_v^2 + [n_v, v]) + t_w^{-2} - 2t_w^{-1} + f_6.
\end{aligned}
$$

The elements $(w^2 + v^3)^*, (w^2 - v^3)^*$ are invertible, because the coefficient of t_v^{-3} is ± 1, which is clearly invertible. ∎

Corollary 5.5   *Let k be a field of characteristic zero and let K be a residually nilpotent Lie k-algebra. Let $u, v \in K$ be such that $[v, u] \neq 0$ and denote by L the Lie k-subalgebra of K generated by $\{u, v\}$.*

Let $w = [v, u]$, $V = \frac{1}{2}(uv + vu)$, and consider the following elements of $\mathfrak{D}(L)$:

$$S = \left(V - \frac{1}{3}w\right)\left(V + \frac{1}{3}w\right)^{-1},$$

$$T = (w + v^2)^{-1}(w - v^2)S(w + v^2)(w - v^2)^{-1},$$

$$S_1 = w^{-1}\left(\left(V - \frac{1}{3}w\right)\left(V + \frac{1}{3}w\right)^{-1} + \left(V - \frac{1}{3}w\right)^{-1}\left(V + \frac{1}{3}w\right)\right)w^{-1},$$

$$T_1 = (w + v^2)^{-1}(w - v^2)S_1(w + v^2)(w - v^2)^{-1},$$

$$T_2 = (w^2 + v^3)^{-1}(w^2 - v^3)S_1(w^2 + v^3)(w^2 - v^3)^{-1}.$$

Then the following hold true.

(i)   *The Lie k-algebra $L/[[L, L], L]$ is isomorphic to H, the Heisenberg Lie k-algebra.*

(ii) *The k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S, S^{-1}, T, T^{-1}\}$ is the free group k-algebra on the set $\{S, T\}$.*

(iii) *Suppose that L is invariant under $*$ and that the induced involution on $L/[[L, L], L]$ is one of the involutions in Lemma 3.2.*

   (a) *If the induced involution on $L/[[L, L], L]$ is one of the involutions in Lemma 3.2(ii) and (iii), then the following hold true.*

      (a.1) *The elements $S_1 S_1^*$ and $T_1 T_1^*$ are symmetric.*

      (a.2) *The k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S_1 S_1^*, T_1 T_1^*\}$ is the free k-algebra on $\{S_1 S_1^*, T_1 T_1^*\}$.*

      (a.3) *The k-subalgebra of $\mathfrak{D}(L)$ generated by*

$$\left\{1 + S_1 S_1^*, (1 + S_1 S_1^*)^{-1}, 1 + T_1 T_1^*, (1 + T_1 T_1^*)^{-1}\right\}$$

      is the free group k-algebra on the set $\{1 + S_1 S_1^*, 1 + T_1 T_1^*\}$.

   (b) *If the induced involution on $L/[[L, L], L]$ is one of the involutions in Lemma 3.2(i), then the following hold true.*

      (b.1) *The elements $S_1 S_1^*$ and $T_2 T_2^*$ are symmetric.*

      (b.2) *The k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S_1 S_1^*, T_2 T_2^*\}$ is the free k-algebra on $\{S_1 S_1^*, T_2 T_2^*\}$.*

      (b.3) *The k-subalgebra of $\mathfrak{D}(L)$ generated by*

$$\left\{1 + S_1 S_1^*, (1 + S_1 S_1^*)^{-1}, 1 + T_2 T_2^*, (1 + T_2 T_2^*)^{-1}\right\}$$

      is the free group k-algebra on the set $\{1 + S_1 S_1^*, 1 + T_2 T_2^*\}$.

Proof   Define $N = [[L, L], L]$. Since L is residually nilpotent and not abelian, $[v, u] \in [L, L]\backslash N$. Thus, L/N is not abelian. Moreover, L/N is a noncommutative 3-dimensional Lie k-algebra with basis $\{\overline{u}, \overline{v}, \overline{w}\}$, the classes of u, v and w in L/N. Moreover, $[L/N, L/N] = k\overline{w}$, which is contained in the center of L/N. Therefore, L/N is the Heisenberg Lie k-algebra.

By Theorem 5.4, the result holds for $\mathfrak{D}(L)$. Since $\mathfrak{D}(L) \hookrightarrow \mathfrak{D}(K)$, the result follows.   ∎

Corollary 5.6 *Let k be a field of characteristic zero and let L be a nonabelian residually nilpotent Lie k-algebra endowed with an involution* $*: L \to L$*. Then there exist symmetric elements* $A, B \in \mathfrak{D}(L)$ *such that the k-subalgebra of* $\mathfrak{D}(L)$ *generated by* $\{A, A^{-1}, B, B^{-1}\}$ *is the free group k-algebra on* $\{A, B\}$*.*

Proof Let N be the $*$-invariant ideal $[L, [L, L]]$. The Lie k-algebra L/N is nilpotent but not abelian, and $*$ induces an involution on L/N. By Theorem 3.4, there exists an invariant Heisenberg Lie k-subalgebra H of L/N such that the restriction of the involution is one of involutions of Lemma 3.2. Let $a + N$, $b + N$ be the generators of H. Let M be the Lie k-subalgebra of L generated by $N \cup \{a, b\}$. Then $*$ induces an involution $*: M \to M$ by restriction, $M/N \cong H$, and the induced involution on M/N is one of the involutions of Lemma 3.2. Apply Theorem 5.4(ii) and (iii) to obtain that $\mathfrak{D}(M)$ satisfies the desiered result. Now observe that $\mathfrak{D}(M) \subseteq \mathfrak{D}(L)$. ∎

6 Free Group Algebras in the Ore Ring of Fractions of Universal Enveloping Algebras that are Ore Domains

The main results in this section are Theorems 6.1, 6.4, and 6.5. They all have a similar but technical proof. Thanks to the results in Section 2.3, the method can be seen as an improvement of the technique originally used in the proof of [25, Theorem 2] and that was also used to show [10, Theorem 5.2].

6.1 On Conjecture (GA)

Theorem 6.1 *Let k be a field of characteristic zero and L be a Lie k-algebra whose universal enveloping algebra* $U(L)$ *is an Ore domain. Let* $u, v \in L$ *be such that the Lie subalgebra generated by them is of dimension at least three.*

Define $w = [v, u]$*,* $V = \frac{1}{2}(uv + vu)$*, and consider the following elements of* $\mathfrak{D}(L)$ *the Ore ring of fractions of* $U(L)$*:*

$$S = \left(V - \frac{1}{3}w\right)\left(V + \frac{1}{3}w\right)^{-1} \quad and \quad T = (w + v^2)^{-1}(w - v^2)S(w + v^2)(w - v^2)^{-1}.$$

Then the k-subalgebra of $\mathfrak{D}(L)$ *generated by* $\{S, S^{-1}, T, T^{-1}\}$ *is the free group k-algebra on* $\{S, T\}$*.*

Proof Let L_1 be the Lie k-subalgebra of L generated by u and v. Since $U(L)$ is an Ore domain, $U(L_1)$ is also an Ore domain and $\mathfrak{D}(L_1) \subseteq \mathfrak{D}(L)$. Thus, we can suppose that L is generated by u and v.

Consider the filtration $F_{\mathbb{Z}}L = \{F_n L\}_{n \in \mathbb{Z}}$ of L given in Example 2.5. It induces a filtration $F_{\mathbb{Z}}U(L) = \{F_n U(L)\}_{n \in \mathbb{Z}}$ on $U(L)$ as shown in Section 2.2. Moreover, by Lemma 2.6(i), there exists an isomorphism of $\mathbb{Z}$-graded k-algebras

$$(6.1) \qquad U(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) \cong \mathrm{grad}_{F_{\mathbb{Z}}}(U(L)),$$

which induces a valuation $v: U(L) \to \mathbb{Z} \cup \{\infty\}$ as in Section 2.1. It can be extended to a valuation $v: \mathfrak{D}(L) \to \mathbb{Z} \cup \{\infty\}$ [7, Proposition 9.1.1]. We recall that the filtration it induces is $F_{\mathbb{Z}}\mathfrak{D}(L) = \{F_n \mathfrak{D}(L)\}_{n \in \mathbb{Z}}$, where $F_n \mathfrak{D}(L) = \{f \in \mathfrak{D}(L) : v(f) \geq n\}$.

In what follows, the two objects in (6.1) will be identified. Consider u, v and $w = [v, u]$. Note that $v(u) = v(v) = -1$ and $v(w) = -2$, because L is not two-dimensional. Denote by $\bar{u}, \bar{v}$ the class of $u, v \in U(L)_{-1}$ and also the class of u and v in L_{-1}. Denote by $\bar{w}$ the class of w in $U(L)_{-2}$ and in L_{-2}. By Lemma 2.2(iv), $U(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ is an Ore domain. Let $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ be its Ore ring of fractions.

Now, $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$ is a (negatively) graded Lie k-algebra that is not abelian ($w \in L_{-2} \backslash L_{-1}$). Thus, $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$ is a nonabelian residually nilpotent Lie k-algebra. Observe that $[\bar{v}, \bar{u}] = \bar{w}$ as elements of $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$.

Now define $\bar{V} = \frac{1}{2}(\overline{uv} + \overline{vu})$,

$$\bar{S} = \left(\bar{V} - \frac{1}{3}\bar{w}\right)\left(\bar{V} + \frac{1}{3}\bar{w}\right)^{-1}, \qquad \bar{T} = (\bar{w} + \bar{v}^2)^{-1}(\bar{w} - \bar{v}^2)\bar{S}(\bar{w} + \bar{v}^2)(\bar{w} - \bar{v}^2)^{-1}.$$

Then Corollary 5.5(ii) shows that the k-subalgebra of $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ generated by $\{\bar{S}, \bar{S}^{-1}, \bar{T}, \bar{T}^{-1}\}$ is the free group k-algebra on $\{\bar{S}, \bar{T}\}$. Let $\mathcal{H}$ be the set of homogeneous elements of $\mathrm{grad}_{F_{\mathbb{Z}}}(U(L))$. From (6.1), and Lemma 2.2, we obtain the following commutative diagram

$$\mathrm{grad}_{F_{\mathbb{Z}}}(U(L)) \cong U(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) \hookrightarrow \mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) ,$$

$$\mathcal{H}^{-1}\mathrm{grad}_{F_{\mathbb{Z}}}(U(L)) \cong \mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L))$$

where the diagonal arrow is obtained from the universal property of the Ore localization. Note that $\bar{V}, \bar{V} - \frac{1}{3}\bar{w}, \bar{V} + \frac{1}{3}\bar{w}, \bar{w} + \bar{v}^2, \bar{w} - \bar{v}^2$ are homogeneous elements of degree -2 in $\mathrm{grad}_{F_{\mathbb{Z}}}(U(L))$. Thus $\bar{S}, \bar{S}^{-1}, \bar{T}, \bar{T}^{-1}$ are in fact homogeneous elements of degree zero in $\mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L))$.

Now observe that S and T are elements of $\mathfrak{D}(L)$ such that $v(S) = v(T) = 0$ and $\bar{S} = S + \mathfrak{D}(L)_{>0}$, $\bar{T} = T + \mathfrak{D}(L)_{>0}$ in $\mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L))$. By Proposition 2.8, the k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S, S^{-1}, T, T^{-1}\}$ is the free group k-algebra on $\{S, T\}$. ∎

When the Lie subalgebra generated by u and v is of dimension two, we cannot apply the methods developed thus far, but we have the following consequence of Cauchon's Theorem.

Proposition 6.2 *Let k be a field of characteristic zero. Let M be the nonabelian two dimensional Lie k-algebra. Thus, M has a basis $\{e, f\}$ such that $[e, f] = f$. Define $s = (e - \frac{1}{3})(e + \frac{1}{3})^{-1}$ and $u = (1 - f)(1 + f)^{-1}$. Consider the embedding $U(M) \hookrightarrow \mathfrak{D}(M)$. Then the k-algebra generated by the set $\{S = s, S^{-1}, T = usu^{-1}, T^{-1}\}$ is the free group k-algebra on $\{S, T\}$.*

Proof Since $[e, f] = ef - fe = f$, $ef = f(e + 1)$. Thus, $U(M)$ can be seen as a skew polynomial k-algebra, $U(M) = k[e][f; \sigma]$, where $\sigma(e) = e + 1$.

According to Cauchon's Theorem, if we define $s = (e - \frac{1}{3})(e + \frac{1}{3})^{-1}$ and $u = (1 - f)(1 + f)^{-1}$, the k-subalgebra generated by $\{s, s^{-1}, usu^{-1}, us^{-1}u^{-1}\}$ is the free group k-algebra on $\{s, usu^{-1}\}$. ∎

Combining Theorem 6.1 and Proposition 6.2, we obtain the following result, which is [25, Theorem 4].

Theorem 6.3 *Let k be a field of characteristic zero. Let L be a noncommutative Lie k-algebra such that $U(L)$ is an Ore domain. Then there exist elements $S, T \in \mathfrak{D}(L)$ such that the k-subalgebra of $\mathfrak{D}(L)$ generated by $\{S, S^{-1}, T, T^{-1}\}$ is the free group k-algebra on $\{S, T\}$. More precisely, let $u, v \in L$ such that $[v, u] \neq 0$. Then*

(i) *if the Lie k-subalgebra of L generated by $\{u, v\}$ is of dimension greater than two, then one can choose S and T as defined in Theorem 6.1;*

(ii) *if the Lie k-subalgebra of L generated by $\{u, v\}$ is of dimension exactly two, then one can choose S and T as defined in Proposition 6.2.*

6.2 On Involutional Versions of Conjecture (GA)

Now we turn our attention to involutions and the existence of free group algebras generated by symmetric elements.

Theorem 6.4 *Let k be a field of characteristic zero and L be a Lie k-algebra such that $U(L)$ is an Ore domain. Let $*: L \to L$ be a k-involution. Suppose that there exists an element $a \in L$ such that $[a^*, a] \neq 0$ and the Lie k-subalgebra generated by $\{a, a^*\}$ is of dimension at least 3.*

Define $u = a + a^*$, $v = a^* - a$, $w = [v, u]$ and $V = \frac{1}{2}(uv + vu)$, and consider the following elements of $\mathfrak{D}(L)$:

$$S_1 = w^{-1}\left(\left(V - \frac{1}{3}w\right)\left(V + \frac{1}{3}w\right)^{-1} + \left(V - \frac{1}{3}w\right)^{-1}\left(V + \frac{1}{3}w\right)\right)w^{-1},$$
$$T_2 = (w^2 + v^3)^{-1}(w^2 - v^3)S_1(w^2 + v^3)(w^2 - v^3)^{-1}.$$

Then the k-subalgebra of $\mathfrak{D}(L)$ generated by

$$\{1 + S_1 S_1^*, (1 + S_1 S_1^*)^{-1}, 1 + T_2 T_2^*, (1 + T_2 T_2^*)^{-1}\}$$

is the free group k-algebra on the set $\{1 + S_1 S_1^*, 1 + T_2 T_2^*\}$.

Proof Let L_1 be the Lie k-subalgebra of L generated by u and v.

Since $U(L)$ is an Ore domain, $U(L_1)$ is also an Ore domain. Moreover, $\mathfrak{D}(L_1) \subseteq \mathfrak{D}(L)$. Thus, we can suppose that L is generated by u and v.

Consider the filtration $F_{\mathbb{Z}}L = \{F_n L\}_{n \in \mathbb{Z}}$ of L defined by $F_r L = 0$ for all $r \geq 0$, $F_{-1}L = ku$, $F_{-2}L = kv + F_{-1}L$, $F_{-3}L = k[v, u] + F_{-2}L$ and for $n \leq -3$,

$$F_{n-1}L = \sum_{n_1 + n_2 + \cdots + n_r \geq (n-1)} \left[F_{n_1}L, [F_{n_2}L, \dots] \cdots\right].$$

Observe that, for each $n \in \mathbb{Z}$, there exists $\mathcal{B}_n \subseteq L$ whose classes give a basis of $L_n = F_n L / F_{n+1} L$ such that $\bigcup_{n \in \mathbb{Z}} \mathcal{B}_n$ is a basis of L. This filtration on L induces a filtration $F_{\mathbb{Z}}U(L) = \{F_n U(L)\}_{n \in \mathbb{Z}}$ on $U(L)$ as shown in Section 2.2. Moreover, by Lemma 2.6, there exists an isomorphism of $\mathbb{Z}$-graded k-algebras

(6.2) $$U\left(\mathrm{grad}_{F_{\mathbb{Z}}}(L)\right) \cong \mathrm{grad}_{F_{\mathbb{Z}}}\left(U(L)\right),$$

which induces a valuation $v: U(L) \to \mathbb{Z} \cup \{\infty\}$ as in Section 2.1. In what follows, the two objects in (6.2) will be identified via the isomorphism given in either [41, Proposition 1] or [3, Lemma 2.1.2]. This isomorphism sends the class of an element of $\mathcal{B}_n$ in L_n to its class in $U(L)_n$.

Note that each $F_n L$ is invariant under $*$, because $u^* = u$ and $v^* = -v$. Hence, $*$ induces an involution on $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$ and hence on $U(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$. Moreover, each $F_n U(L)$ is invariant under $*$, and thus $*$ also induces an involution on $\mathrm{grad}_{F_{\mathbb{Z}}}(U(L))$. Therefore, the isomorphism given in (6.2) is an isomorphism of k-algebras with involution; that is, if Φ is the isomorphism of (6.2), then $\Phi(f^*) = \Phi(f)^*$.

Observe that $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$ is a residually nilpotent Lie k-algebra. Define $N = \bigoplus_{n \geq 4} L_n$. Then $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ is isomorphic to the Heisenberg Lie k-algebra H. Moreover, N is invariant under the involution $*$, and the induced involution in $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ is the one in Lemma 3.2(i).

The valuation $v: U(L) \to \mathbb{Z} \cup \{\infty\}$ can be extended to a valuation $v: \mathfrak{D}(L) \to \mathbb{Z} \cup \{\infty\}$ [7, Proposition 9.1.1].

Consider u, v, and $w = [v, u]$. Note that $v(u) = -1, v(v) = -2$, and $v(w) = -3$. Denote by $\overline{u}, \overline{v}, \overline{w}$ the class of $u \in U(L)_{-1}, v \in U(L)_{-2}, w \in U(L)_{-3}$ and also the class of u in $L_{-1}, v \in L_{-2}$, and $w \in L_{-3}$, respectively. By Lemma 2.2(iv), $U(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ is an Ore domain. Let $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ be its Ore ring of fractions. Observe that $[\overline{v}, \overline{u}] = \overline{w}$ as elements of $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$. Define $\overline{V} = \frac{1}{2}(\overline{uv} + \overline{vu})$, and consider the following elements of $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$:

$$\overline{S}_1 = \overline{w}^{-1}\left(\left(\overline{V} - \frac{1}{3}\overline{w}\right)\left(\overline{V} + \frac{1}{3}\overline{w}\right)^{-1} + \left(\overline{V} - \frac{1}{3}\overline{w}\right)^{-1}\left(\overline{V} + \frac{1}{3}\overline{w}\right)\right)\overline{w}^{-1},$$
$$\overline{T}_2 = (\overline{w}^2 + \overline{v}^3)^{-1}(\overline{w}^2 - \overline{v}^3)\overline{S}_1(\overline{w}^2 + \overline{v}^3)(\overline{w}^2 - \overline{v}^3)^{-1}.$$

By Corollary 5.5(iii)(a.2), the k-subalgebra of $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ generated by $\{\overline{S}_1 \overline{S}_1^*, \overline{T}_2 \overline{T}_2^*\}$ is the free k-algebra on $\{\overline{S}_1 \overline{S}_1^*, \overline{T}_2 \overline{T}_2^*\}$. Let $\mathcal{H}$ be the set of homogeneous elements of $\mathrm{grad}_{F_{\mathbb{Z}}}(U(L))$. From (6.2) and Lemma 2.2, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{grad}_{F_{\mathbb{Z}}}(U(L)) \cong U(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) & \hookrightarrow & \mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) \,, \\
\downarrow \uparrow & \nearrow & \\
\mathcal{H}^{-1}\,\mathrm{grad}_{F_{\mathbb{Z}}}(U(L)) \cong \mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L)) & &
\end{array}
$$

where the diagonal arrow is obtained from the universal property of the Ore localization. Note that $\overline{V}, \overline{V} - \frac{1}{3}\overline{w}, \overline{V} + \frac{1}{3}\overline{w}$ are homogeneous elements of degree -3, and the elements $\overline{w}^2 + \overline{v}^3, \overline{w}^2 - \overline{v}^3$ are homogeneous elements of degree -3 in $\mathrm{grad}_{F_{\mathbb{Z}}}(U(L))$. Thus, $\overline{S}_1, \overline{S}_1^*, \overline{T}_2, \overline{T}_2^*$ are in fact homogeneous elements of degree -6 in $\mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L))$.

Now observe that S_1, S_1^*, T_2 and T_2^* are elements of $\mathfrak{D}(L)$ such that $v(S_1) = v(S_1^*) = v(T_2) = v(T_2^*) = 6$; hence, $v(S_1 S_1^*) = 12, v(T_2 T_2^*) = 12$ and $\overline{S}_1 \overline{S}_1^* = S_1 S_1^* + \mathfrak{D}(L)_{>12}$, $\overline{T}_2 \overline{T}_2^* = T_2 T_2^* + \mathfrak{D}(L)_{>12}$ in $\mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L))$.

Now, by Theorem 2.9, the result follows. ∎

In the case where $[x, x^*] = 0$ for all $x \in L$, we are able to prove the following theorem.

Theorem 6.5 *Let k be a field of characteristic zero and L be a Lie k-algebra such that $U(L)$ is an Ore domain. Let $*: L \to L$ be a k-involution. Suppose that $[x, x^*] = 0$ for all $x \in L$, but there exist elements $x, y \in L$ such that $[y, x] \neq 0$ and the k-subspace of L spanned by $\{x, x^*, y, y^*\}$ is not the Lie k-subalgebra generated by $\{x, x^*, y, y^*\}$. Then there exist symmetric elements $A, B \in \mathfrak{D}(L)$ such that the k-subalgebra generated by $\{A, A^{-1}, B, B^{-1}\}$ is the free group k-algebra on $\{A, B\}$.*

Proof Let L_1 be the Lie k-subalgebra of L generated by $\{x, x^*, y, y^*\}$. Since $U(L)$ is an Ore domain, $U(L_1)$ is also an Ore domain. Moreover, $\mathfrak{D}(L_1) \subseteq \mathfrak{D}(L)$. Thus, we can suppose that L is generated by $\{x, x^*, y, y^*\}$. Let V be the k-subspace spanned by $\{x, x^*, y, y^*\}$. Consider the filtration $F_{\mathbb{Z}}L = \{F_n L\}_{n \in \mathbb{Z}}$ of L defined by $F_r L = 0$ for all $r \geq 0$, $F_{-1}L = V$, $F_{-2}L = [V, V] + F_{-1}L$, and for $n \leq -2$,

$$F_{n-1}L = \sum_{n_1 + n_2 + \cdots + n_r \geq (n-1)} \left[F_{n_1}L, [F_{n_2}L, \ldots] \cdots \right].$$

Note that $F_n L$ is invariant under $*$ for all $n \in \mathbb{Z}$. Thus, the involution on L induces an involution on $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$. Now define $N = \bigoplus_{n \leq -3} L_n$. Then N is an ideal of $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$ such that $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ is a nonabelian nilpotent Lie k-algebra because $[V, V]$ is not contained in V by assumption. Moreover, N is invariant under $*$, and thus the involution on $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$ induces an involution on $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$, again denoted by $*$. By Theorem 3.4, there exist $u, v \in \mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ such that they generate a $*$-invariant Heisenberg Lie k-subalgebra of $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ and the restriction to it is one of the involutions in Lemma 3.2. Note that $F_{-1}L = L_{-1}$. Also $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N \cong L_{-1} \oplus L_{-2}$ as k-vector spaces, and the induced product $[L_{-1}, L_{-2}] = 0$. Thus, we can choose $u, v \in L_{-1} = F_{-1}L$.

Suppose that the involution on $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ is like the one in Lemma 3.2(i), *i.e.*, $u^* = u$, $v^* = -v$ and $w^* = w$, where $w = [v, u]$. Then take $u_1 = u + v$, $v_1 = u - v \in L_{-1}$. Note that $u_1^* = v_1$ and $[u_1, v_1] = 2[v, u] \neq 0$, a contradiction to our assumption that $[x, x^*] = 0$ for all $x \in L$. Hence the involution on the Heisenberg subalgebra of $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ generated by u, v is of the type in either Lemma 3.2(ii) or Lemma 3.2(iii).

Let L_2 be the Lie k-subalgebra of L generated by $\{u, v\}$. Since $U(L)$ is an Ore domain, $U(L_2)$ is also an Ore domain. Moreover, $\mathfrak{D}(L_2) \subseteq \mathfrak{D}(L)$. Thus, we may suppose that L is generated by $\{u, v\}$. Let V be the k-subspace spanned by $\{u, v\}$. Consider the filtration $F_{\mathbb{Z}}L = \{F_n L\}_{n \in \mathbb{Z}}$ of L defined by $F_r L = 0$ for all $r \geq 0$, $F_{-1}L = V$, $F_{-2}L = [V, V] + F_{-1}L$, and for $n \leq -2$,

$$F_{n-1}L = \sum_{n_1 + n_2 + \cdots + n_r \geq (n-1)} [F_{n_1}L, [F_{n_2}L, \ldots] \cdots].$$

Note that $F_n L$ is invariant under $*$ for all $n \in \mathbb{Z}$. Thus the involution on L induces an involution on $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$. Define now $N = \bigoplus_{n \leq -3} L_n$. Then N is an ideal of $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$ such that $\mathrm{grad}_{F_{\mathbb{Z}}}(L)/N$ is the Heisenberg Lie k-algebra and the involution induced on it is of type either Lemma 3.2(ii) or Lemma 3.2(iii).

Observe that for each $n \in \mathbb{Z}$, there exists $\mathcal{B}_n \subseteq L$ whose classes give a basis of $L_n = F_n L / F_{n+1} L$ such that $\bigcup_{n \in \mathbb{Z}} \mathcal{B}_n$ is a basis of L. This filtration on L induces a filtration $F_{\mathbb{Z}} U(L) = \{F_n U(L)\}_{n \in \mathbb{Z}}$ on $U(L)$, as shown in Section 2.2. Moreover, by Lemma 2.6, there exists an isomorphism of $\mathbb{Z}$-graded k-algebras

$$(6.3) \qquad\qquad U(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) \cong \mathrm{grad}_{F_{\mathbb{Z}}}(U(L)),$$

which induces a valuation $v \colon U(L) \to \mathbb{Z} \cup \{\infty\}$ as in Section 2.1. In what follows, the two objects in (6.3) will be identified via the isomorphism given in either [41, Proposition 1] or [3, Lemma 2.1.2]. This isomorphism sends the class of an element of $\mathcal{B}_n$ in L_n to its class in $U(L)_n$.

The valuation $v \colon U(L) \to \mathbb{Z} \cup \{\infty\}$ can be extended to a valuation $v \colon \mathfrak{D}(L) \to \mathbb{Z} \cup \{\infty\}$ [7, Proposition 9.1.1].

Consider u, v, and $w = [v, u]$. Note that $v(u) = v(v) = -1$ and $v(w) = -2$, because L is not two-dimensional. Denote by $\overline{u}, \overline{v}$ the class of $u, v \in U(L)_{-1}$ and also the class of u and v in L_{-1}. Denote by $\overline{w}$ the class of w in $U(L)_{-2}$ and in L_{-2}. By Lemma 2.2(iv), $U(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ is an Ore domain. Let $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ be its Ore ring of fractions.

Observe that $[\overline{v}, \overline{u}] = \overline{w}$ as elements of $\mathrm{grad}_{F_{\mathbb{Z}}}(L)$. Define $\overline{V} = \frac{1}{2}(\overline{uv} + \overline{vu})$, and consider the following elements of $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$:

$$\overline{S}_1 = \overline{w}^{-1}\left(\left(\overline{V} - \frac{1}{3}\overline{w}\right)\left(\overline{V} + \frac{1}{3}\overline{w}\right)^{-1} + \left(\overline{V} - \frac{1}{3}\overline{w}\right)^{-1}\left(\overline{V} + \frac{1}{3}\overline{w}\right)\right)\overline{w}^{-1},$$
$$\overline{T}_1 = (\overline{w} + \overline{v}^2)^{-1}(\overline{w} - \overline{v}^2)\overline{S}_1(\overline{w} + \overline{v}^2)(\overline{w} - \overline{v}^2)^{-1}.$$

By Corollary 5.5(iii)(a.2), the k-subalgebra of $\mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L))$ generated by $\{\overline{S}_1\overline{S}_1^*, \overline{T}_1\overline{T}_1^*\}$ is the free k-algebra on $\{\overline{S}_1\overline{S}_1^*, \overline{T}_2\overline{T}_2^*\}$. Let $\mathcal{H}$ be the set of homogeneous elements of $\mathrm{grad}_{F_{\mathbb{Z}}}(U(L))$. From (6.3) and Lemma 2.2, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{grad}_{F_{\mathbb{Z}}}(U(L)) \cong U(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) \lhook\joinrel\longrightarrow & \mathfrak{D}(\mathrm{grad}_{F_{\mathbb{Z}}}(L)) \ , \\ \Big\uparrow & \nearrow \\ \mathcal{H}^{-1}\,\mathrm{grad}_{F_{\mathbb{Z}}}(U(L)) \cong \mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L)) \end{array}$$

where the diagonal arrow is obtained from the universal property of the Ore localization. Note that $\overline{V}, \overline{V} - \frac{1}{3}\overline{w}, \overline{V} + \frac{1}{3}\overline{w}$ are homogeneous elements of degree -3 and the elements $\overline{w} + \overline{v}^2, \overline{w} - \overline{v}^2$ are homogeneous elements of degree -2 in $\mathrm{grad}_{F_{\mathbb{Z}}}(U(L))$. Thus, $\overline{S}_1, \overline{S}_1^*, \overline{T}_1, \overline{T}_1^*$ are in fact homogeneous elements of degree -4 in $\mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L))$.

Now observe that S_1, S_1^*, T_2 and T_2^* are elements of $\mathfrak{D}(L)$ such that $v(S_1) = v(S_1^*) = v(T_2) = v(T_2^*) = 4$; hence, $v(S_1 S_1^*) = 8$, $v(T_2 T_2^*) = 8$, and $\overline{S}_1\overline{S}_1^* = S_1 S_1^* + \mathfrak{D}(L)_{>8}$, $\overline{T}_2\overline{T}_2^* = T_2 T_2^* + \mathfrak{D}(L)_{>8}$ in $\mathrm{grad}_{F_{\mathbb{Z}}}(\mathfrak{D}(L))$.

Defining $A = 1 + S_1 S_1^*$ and $B = 1 + T_1 T_1^*$, the result follows from Theorem 2.9. ∎

As a corollary, we obtain a generalization of [10, Theorem 5.2], where the existence of a free k-algebra was proved. We recall that the principal involution on a Lie k-algebra L is defined by $L \to L$, $f \mapsto -f$.

Corollary 6.6 *Let k be a field of characteristic zero and L be a Lie k-algebra such that its universal enveloping algebra $U(L)$ is an Ore domain. Let $\mathfrak{D}(L)$ be its Ore ring of fractions. Let $u, v \in L$ be such that the Lie subalgebra generated by them is of dimension at least three. Then there exist symmetric elements $A, B \in \mathfrak{D}(L)$ with respect to the principal involution such that the k-subalgebra generated by $\{A, A^{-1}, B, B^{-1}\}$ is the free group k-algebra on $\{A, B\}$.*

7 Free Group Algebras in the Malcev–Neumann Division Ring of Fractions of a Residually Torsion-free Nilpotent Group

In this section, for a group G and elements $x, y \in G$, then (y, x) denotes the commutator $(y, x) = y^{-1}x^{-1}yx$. Also, if A, B are subgroups of G, (B, A) denotes the subgroup of G generated by the commutators (y, x) with $y \in B$, $x \in A$.

Let R be a ring and $(G, <)$ be an ordered group. Suppose that $R[G]$ is the group ring of G over R. We define a new ring, denoted $R((G ;<))$ and called *Malcev–Neumann series ring*, in which $R[G]$ embeds. As a set,

$$R((G ;<)) = \left\{ f = \sum_{x \in G} a_x x : a_x \in R, \ \mathrm{supp}(f) \text{ is well ordered} \right\},$$

where $\mathrm{supp}(f) = \{x \in G : a_x \neq 0\}$. Addition and multiplication are defined extending the ones in $R[G]$. That is, given $f = \sum_{x \in G} a_x x$ and $g = \sum_{x \in G} b_x x$, elements of $R((G ;<))$, one has

$$f + g = \sum_{x \in G} (a_x + b_x)x \quad \text{and} \quad fg = \sum_{x \in G} \left(\sum_{yz=x} a_y b_z \right) x.$$

It was shown, independently, in [31, 34] that if R is a division ring, then $R((G ;<))$ is a division ring.

If k is a field, the division subring of $k((G ;<))$ generated by the group ring $k[G]$ will be called the *Malcev–Neumann division ring of fractions* of $k[G]$ and will be denoted by $k(G)$. It is important to observe the following. For a subgroup H of G, $k((H ;<))$ and $k(H)$ can be regarded as division subrings of $k((G ;<))$ and $k(G)$, respectively, in the natural way. We remark that $k(G)$ does not depend on the order $<$ of G; see [19]. When the group ring $k[G]$ is an Ore domain, then $k(G)$ is the Ore ring of fractions of $k[G]$.

An *involution* on a group G is a map $*: G \to G$, $x \mapsto x^*$, that satisfies

$$(xy)^* = y^*x^* \quad \text{and} \quad (x^*)^* = x \text{ for all } x, y \in G.$$

In other words, $*$ is an anti-automorphism of order two.

Suppose that G is a group endowed with an involution $*: G \to G$, $x \mapsto x^*$, k is a field, and $k[G]$ is the group k-algebra. The map $*: k[G] \to k[G]$ defined by $(\sum_{x \in G} a_x x)^* = \sum_{x \in G} a_x x^*$ is a k-involution on $k[G]$.

If $(G, <)$ is an ordered group, we remark that the k-involution on the group algebra $k[G]$ induced the involution $*$ on G extends uniquely to a k-involution on the Malcev–Neumann division ring of fractions $k(G)$ of $k[G]$; see [13, Theorem 2.9].

Let G be a group. If H is a subgroup of G, we denote by $\sqrt{H}$ the subset of G defined by

$$\sqrt{H} = \{x \in G : x^n \in H, \text{ for some } n \geq 1\}.$$

We shall denote the n-th term of the lower central series of G by $\gamma_n(G)$. That is, we set $\gamma_1(G) = G$ and, for $n \geq 1$, define $\gamma_{n+1}(G) = (G, \gamma_n(G))$.

A group G is *residually torsion-free nilpotent* if for each $g \in G$, there exists a normal subgroup N_g of G such that $g \notin N_g$ and G/N_g is torsion-free nilpotent. Equivalently, $\bigcap_{n \geq 1} \sqrt{\gamma_n(G)} = \{1\}$. It is well known that any residually torsion-free nilpotent group is orderable; see, for example, [15, Theorem IV.6].

Let G be a residually torsion-free nilpotent group, let k be a field of characteristic zero, and consider the group algebra $k[G]$. Consider an involution on G and its extension to the Malcev–Neumann division ring of fractions $k(G)$ of $k[G]$. The aim of this section is to prove that there exist symmetric elements in $k(G)$ that generate a noncommutative free group k-algebra. For that we will need the following discussion.

Let G be a torsion-free nilpotent group. An $\mathcal{N}$-*series* of G is a sequence $\{H_i\}_{i \geq 1}$,

$$G = H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq \cdots$$

of normal subgroups of G that satisfies the following three conditions

$$(H_i, H_j) \subseteq H_{i+j}, \quad \bigcap_{i \geq 1} H_i = \{1\}, \quad G/H_i \text{ is torsion free for all } i \geq 1.$$

The $\mathcal{N}$-series induces a *weight function* $w: G \to \mathbb{N} \cup \{\infty\}$ defined by $w(g) = i$ if $g \in H_i \setminus H_{i+1}$ and $w(1) = \infty$.

Let k be a field of characteristic zero, G torsion-free nilpotent group with an $\mathcal{N}$-series $\{H_i\}_{i \geq 1}$, and consider the group ring $k[G]$. The $\mathcal{N}$-series defines the *cannonical filtration*, $F_{\mathbb{Z}} k[G] = \{F_n k[G]\}_{n \in \mathbb{Z}}$, induced by $\{H_i\}_{i \geq 1}$, which is defined by $F_n k[G] = k[G]$, for all $n \leq 0$, and for $n \geq 1$, $F_n k[G]$ is the k-vector space spanned by the set

$$\left\{(h_1 - 1)(h_2 - 1)\cdots(h_s - 1) : s \geq 1, \sum_{j=1}^{s} w(h_j) \geq n\right\}.$$

Note that $F_1 k[G]$ is the augmentation ideal of $k[G]$ and that $F_n k[G] \cdot F_m k[G] \subseteq F_{n+m} k[G]$.

For each $i \geq 1$, H_i/H_{i+1} is an abelian group. Denote the operation additively. More precisely, if $x_i, x_i' \in H_i$, $\widetilde{x_i}$ denotes the class $x_i H_{i+1}$. Then $\widetilde{x_i} + \widetilde{x_i'} = \widetilde{x_i x_i'}$ in H_i/H_{i+1}. The abelian group $L(G) = \bigoplus_{i \geq 1} H_i/H_{i+1}$ can be endowed with a $\mathbb{Z}$-graded Lie $\mathbb{Z}$-algebra structure with the following product on homogeneous elements $[\widetilde{x_i}, \widetilde{x_j}] = \widetilde{(x_i, x_j)} \in H_{i+j}/H_{i+j+1}$, for $x_i \in H_i$, $x_j \in H_j$, and then extending by bilinearity. Then $k \otimes_{\mathbb{Z}} L(G)$ is a Lie k-algebra with universal enveloping algebra $U(k \otimes_{\mathbb{Z}} L(G))$.

In [23, Theorem 2.3], Lichtman proved a more general version of [37], in a similar way as Quillen's result is shown in [35, Chapter VIII]. Lichtman's result implies that that there exists an isomorphism of $\mathbb{Z}$-graded k-algebras

(7.1)
$$\Theta: U(k \otimes_{\mathbb{Z}} L(G)) \longrightarrow \mathrm{grad}_{F_{\mathbb{Z}}}(k[G])$$
$$\widetilde{x_i} \longmapsto (x_i - 1) + F_{i+1} k[G].$$

Let $\mathbb{H} = \langle a, b : (b, (b, a)) = (a, (b, a)) = 1 \rangle$ be the Heisenberg group. Define $c = (b, a)$. Consider the following *main involutions* of $\mathbb{H}$ that are defined on the generators and extended accordingly:

(a) $a^* = a$, $b^* = b^{-1}$ and $c^* = c$.
(b) $a^* = a$, $b^* = b$ and $c^* = c^{-1}$.
(c) $a^* = a^{-1}$, $b^* = b^{-1}$ and $c^* = c^{-1}$.

Proposition 7.1 *Let k be a field of characteristic zero. Let*

$$\mathbb{H} = \langle a, b : (b, (b, a)) = (a, (b, a)) = 1 \rangle$$

be the Heisenberg group and $c = (b, a)$. Consider the group k-algebra $k[\mathbb{H}]$ and its Ore ring of fractions $k(\mathbb{H})$. Consider the elements of $k(H)$

$$\mathcal{V} = \frac{1}{2}((a-1)(b-1) + (b-1)(a-1)),$$

$$\mathcal{S}_2 = (c-1)\left(\left(\mathcal{V} - \frac{1}{3}(c-1)\right)\left(\mathcal{V} + \frac{1}{3}(c-1)\right)^{-1}\right.$$
$$\left. + \left(\mathcal{V} - \frac{1}{3}(c-1)\right)^{-1}\left(\mathcal{V} + \frac{1}{3}(c-1)\right)\right)(c-1),$$

$$\mathcal{T}_3 = \left((c-1) + (b-1)^2\right)^{-1}\left((c-1) - (b-1)^2\right)$$
$$\times \mathcal{S}_2\left((c-1) + (b-1)^2\right)\left((c-1) - (b-1)^2\right)^{-1},$$

$$\mathcal{T}_4 = \left((c-1)^2 + (b-1)^3\right)^{-1}\left((c-1)^2 - (b-1)^3\right)$$
$$\times \mathcal{S}_2\left((c-1)^2 + (b-1)^3\right)\left((c-1)^2 - (b-1)^3\right)^{-1}.$$

The following statements hold true.

(i) *Suppose that $*: \mathbb{H} \to \mathbb{H}$ is one of the main involutions* (b) *or* (c) *above. Then the k-subalgebra of $k(\mathbb{H})$ generated by*

$$\{1 + \mathcal{S}_2\mathcal{S}_2^*, (1 + \mathcal{S}_2\mathcal{S}_2^*)^{-1}, (1 + \mathcal{T}_3\mathcal{T}_3^*), (1 + \mathcal{T}_3\mathcal{T}_3^*)^{-1}\}$$

is the free group k-algebra on the set $\{1 + \mathcal{S}_2\mathcal{S}_2^*, (1 + \mathcal{T}_3\mathcal{T}_3^*)\}$.

(ii) *Suppose that $*: \mathbb{H} \to \mathbb{H}$ is the main involution* (i) *above. Then the k-subalgebra of $k(\mathbb{H})$ generated by*

$$\left\{1 + \mathcal{S}_2\mathcal{S}_2^*, (1 + \mathcal{S}_2\mathcal{S}_2^*)^{-1}, (1 + \mathcal{T}_4\mathcal{T}_4^*), (1 + \mathcal{T}_4\mathcal{T}_4^*)^{-1}\right\}$$

is the free group k-algebra on the set $\{1 + \mathcal{S}_2\mathcal{S}_2^*, (1 + \mathcal{T}_4\mathcal{T}_4^*)\}$.

Proof (i) Consider the following $\mathcal{N}$-series of $\mathbb{H}$:

$$H_1 = \mathbb{H} \supseteq H_2 = (c) \supseteq H_3 = \{1\}.$$

If we set $x = aH_2, y = bH_2 \in H_1/H_2$ and $z = cH_3 \in H_2/H_3$, then the $\mathbb{Z}$-graded Lie $\mathbb{Z}$-algebra $L(H)$ has as $\mathbb{Z}$-basis the elements x, y, z with products $[y, x] = z$, $[y, z] = [x, z] = 0$. Hence the $\mathbb{Z}$ graded Lie k-algebra $k \otimes_{\mathbb{Z}} L(\mathbb{H})$ is the Heisenberg Lie k-algebra H with the $\mathbb{Z}$-grading given in Example 2.3(iii). The isomorphism (7.1) implies that the cannonical filtration induced by the $\mathcal{N}$-series is in fact a valuation, because the graded ring is a domain. Since $k[\mathbb{H}]$ is an Ore domain, the valuation can be extended to a valuation $v: k(\mathbb{H}) \to \mathbb{Z} \cup \{\infty\}$. If we let $\mathcal{H}$ be the homogeneous

elements of $\mathrm{grad}_{F_{\mathbb{Z}}}(k[\mathbb{H}])$, Lemma 2.1(iii) implies that there exists an isomorphism of $\mathbb{Z}$-graded k-algebras

(7.2) $$\mathrm{grad}_v(k(\mathbb{H})) \cong \mathcal{H}^{-1}\mathrm{grad}_{F_{\mathbb{Z}}}(k[\mathbb{H}]) \cong \mathcal{H}^{-1}U(k \otimes_{\mathbb{Z}} L(\mathbb{H})).$$

Observe that $\mathcal{H}^{-1}U(k \otimes_{\mathbb{Z}} L(\mathbb{H})) \hookrightarrow \mathfrak{D}(k \otimes_{\mathbb{Z}} L(\mathbb{H}))$. Now note that

$$\mathcal{V}, \mathcal{V} \pm (c-1), (c-1), (b-1)^2, (c-1) \pm (b-1)^2 \in F_2 k[\mathbb{H}] \backslash F_3 k[\mathbb{H}].$$

Hence, the classes of these elements in $\mathrm{grad}_{F_{\mathbb{Z}}}(k[\mathbb{H}])$ are homogeneous of degree two. It implies that the class of $\mathcal{S}_2$ and $\mathcal{T}_3$ in $\mathrm{grad}_v(k(\mathbb{H}))$ are homogeneous of degree four. Moreover, their image under the isomorphism (7.2) are the elements $S_2, T_3 \in \mathfrak{D}(k \otimes_{\mathbb{Z}} L(\mathbb{H}))$ given in Theorem 4.5(iv).

Since each H_i is invariant under the involution $*$, it induces a k-involution in the Lie k-algebra $k \otimes_{\mathbb{Z}} L(\mathbb{H})$. Hence, the isomorphism (7.2) is an isomorphism of $*$-algebras, i.e., $\Phi(f^*) = \Phi(f)^*$. Note that the induced involution on $k \otimes_{\mathbb{Z}} L(\mathbb{H})$ is one of the involutions in Lemma 3.2(ii) or (iii). By Theorem 4.5(iv)(a), the elements S_2, T_2 are symmetric with respect to the induced involution on $\mathfrak{D}(k \otimes_{\mathbb{Z}} L(\mathbb{H}))$. Hence, the image of the classes of $\mathcal{S}_2^*$ and $\mathcal{T}_3^*$ are also S_2 and T_3, respectively. The classes of the elements $\mathcal{S}_2\mathcal{S}_2^*, \mathcal{T}_3\mathcal{T}_3^*$ in $\mathrm{grad}_v(k(\mathbb{H}))$ are homogeneous of degree 8. Moreover, they generate a free algebra in $\mathrm{grad}_v(k(\mathbb{H}))$, because S_2^2 and T_3^2 generate a free algebra in $\mathfrak{D}(k \otimes_k L(\mathbb{H}))$ by Theorem 4.5(iv)(b). Now the result follows by Theorem 2.9.

(ii) It follows in the same way as (i). Now one has to consider the $\mathcal{N}$-series

$$H_1 = G \supseteq H_2 = \langle b, c \rangle \supseteq H_3 = \langle c \rangle \supseteq H_4 = \{1\}.$$

Then again, $k \otimes_{\mathbb{Z}} L(H)$ is the Heisenberg Lie k-algebra, but with the gradation given in Example 2.3(iv). Then the isomorphism in (7.2) (with a different gradation) sends $\mathcal{S}_2$ and $\mathcal{T}_4$ to the elements S_2 and T_4 in Theorem 4.5(v). ∎

The next result is [14, Proposition 2.4].

Proposition 7.2 *Let G be a nonabelian torsion-free nilpotent group with involution $*$. Then G contains a $*$-invariant Heisenberg subgroup $\mathbb{H}$ such that the induced involution is one of the main involutions of $\mathbb{H}$. More precisely, there exist $x, y \in G$ such that $(x, y) \neq 1$, $(x, (x, y)) = (y, (x, y)) = 1$, $x^* = x^{\pm 1}$, $y^* = y^{\pm 1}$.*

Recall that given a group G and a field k such that $k[G]$ is an Ore domain; then $k[N]$ is an Ore domain for any subgroup N of G. Hence, if G is a torsion-free nilpotent group and $\mathbb{H}$ is a subgroup of G, then $k(\mathbb{H})$ is embedded in $k(G)$. This fact, together with Propositions 7.2 and 7.1, imply the following result.

Theorem 7.3 *Let G be a nonabelian torsion-free nilpotent group with an involution $*: G \to G$ and k be a field of characteristic zero. Consider the group ring $k[G]$ and its Ore ring of fractions $k(G)$. Then there exist nonzero symmetric elements $A, B \in k(G)$ such that the k-subalgebra generated by $\{A, A^{-1}, B, B^{-1}\}$ is the free group k-algebra on the set $\{A, B\}$.*

Theorem 7.4 *Let G be a residually torsion-free nilpotent group with an involution $*: G \to G$ and let k be a field of characteristic zero. Consider the group ring $k[G]$ and its*

Malcev–Neumann division ring of fractions $k(G)$. *Then there exist nonzero symmetric elements* $A, B \in k(G)$ *such that the k-subalgebra generated by* $\{A, A^{-1}, B, B^{-1}\}$ *is the free group k-algebra on the set* $\{A, B\}$.

Proof As noted in [14, Section 3], the argument used there can also be used to prove the existence of free group algebras generated symmetric elements in $k(G)$ using the existence of free group algebras generated by symmetric elements in Ore ring of fractions $k(L)$, where L is a torsion-free nilpotent group. This fact has already been proved in Theorem 7.3. ∎

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