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Borel and Hausdorff hierarchies in topological spaces of Choquet games and their effectivization

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What parts of the classical descriptive set theory done in Polish spaces still hold for more general topological spaces, possibly T_0 or T_1, but not T_2 (i.e. not Hausdorff)? This question has been addressed by Selivanov in a series of papers centred on algebraic domains. And recently it has been considered by de Brecht for quasi-Polish spaces, a framework that contains both countably based continuous domains and Polish spaces. In this paper, we present alternative unifying topological spaces, that we call *approximation spaces*. They are exactly the spaces for which player Nonempty has a stationary strategy in the Choquet game. A natural proper subclass of approximation spaces coincides with the class of quasi-Polish spaces. We study the Borel and Hausdorff difference hierarchies in approximation spaces, revisiting the work done for the other topological spaces. We also consider the problem of effectivization of these results.

1. Introduction

The primary setting of the descriptive set theory, including the study of Borel and Hausdorff hierarchies, is that of Polish spaces. These are spaces homeomorphic to complete metric spaces that have a countable dense subset, for example the Cantor space, the Baire space, the real line and its intervals. The question of what parts of the classical descriptive set theory still hold for non-Polish spaces, specifically for general T_0 topological spaces, has not been yet systematically studied. Major progress has been made by Selivanov in his investigations centred mainly in algebraic domains (directed complete partial orders (dcpo) with a countable base of compact elements) in an ongoing series of papers on this topic that started in 1978. Recently, de Brecht (2013) presented the theory of *quasi-Polish* spaces, a unifying framework for Polish spaces and countably based domains (i.e. ω-continuous domains, or dcpo with a countable basis). De Brecht characterized quasi-Polish spaces in terms of the Choquet topological games, and he proved that a descriptive set theory can be nicely developed in such spaces.

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In this paper, we consider alternative unifying topological spaces that we call *approximation spaces*. Not only they contain all Polish spaces and all continuous domains, but a natural subclass of approximation spaces coincides with the class of quasi-Polish spaces. Approximation spaces can be viewed as the 'à la domain' version of the 'à la Polish' unifying framework of de Brecht. These spaces can also be characterized in terms of Choquet games. We study the Borel and Hausdorff difference hierarchies in approximation spaces, revisiting the work done for the other topological spaces. We also consider the problem of effectivization of these results.

The paper is organized as follows. Section 2 presents the preliminary material. We recall the needed notions about the Borel and Hausdorff hierarchies in a T_0 (possibly not T_2) topological context. We give an overview of the needed material on domains with the Scott topology and quasi-Polish spaces. We also present some prerequisites on the Banach–Mazur and Choquet topological games.

Section 3 is devoted to the class of approximation spaces. We prove that both, Polish spaces and continuous domains, are approximation spaces. Indeed, we show that all quasi-Polish spaces are approximation spaces. Theorem 3.9 characterizes approximation spaces in terms of Choquet games. Theorem 3.11 proves that quasi-Polish spaces and *convergent approximation spaces* are the same class.

In the context of Polish spaces, the Baire property asserts that any countable intersection of dense open sets is dense. Thus, countable intersections of open sets, the $\mathbf{G}_\delta$ sets, constitute the $\mathbf{\Pi}^0_2$ level of the Borel hierarchy. In the context of T_0 but not T_2 spaces this is not true anymore: the $\mathbf{\Pi}^0_2$ level consists of countable intersections of *Boolean combinations* of open sets. Then it is natural to consider the $\mathbf{\Pi}^0_2$ Baire property which asserts that any countable intersection of dense differences of open sets is dense. As shown by de Brecht (2013), the usual $\mathbf{G}_\delta$ Baire property and Hausdorff–Kuratowski's theorem both hold for quasi-Polish spaces. Consequently, these two results are ensured for convergent approximation spaces. Theorem 3.13 proves that, in fact, all approximation spaces satisfy the $\mathbf{\Pi}^0_2$ Baire property. Theorem 3.15 extends Hausdorff's theorem to spaces having a countable basis and such that every closed subspace is an approximation space: the $\mathbf{\Delta}^0_2$ class coincides with the difference hierarchy. This result was previously obtained by Selivanov for ω-algebraic domains, and then for ω-continuous domains (Selivanov 2005; 2008). De Brecht (2013) proved that the full Hausdorff–Kuratowski's theorem holds for quasi-Polish spaces; hence, it holds for convergent approximation spaces. We do not know whether it holds for all approximation spaces.

In Section 4, we revisit a part of the work by Selivanov (2005, 2008) on domains that does not apply to Polish spaces: his characterization of the classes of the Hausdorff hierarchy in terms of alternating trees, and his proof of non-existence of ambiguous sets in this hierarchy. We check that the assumption of ω-algebraicity or ω-continuity can be replaced, mutatis mutandis, by that of continuity.

Section 5 is devoted to effectivization. The definition of approximation spaces admits a straightforward definition of an effective version. We make the first steps in developing the effective theory. We first recall the notions of effective topological space and effective domains. We also include the known machinery of effective Borel codes. Theorem 5.17 proves a weak effective version of Hausdorff's theorem in effective approximation spaces.

We obtain this proof as an adaptation of the work by Selivanov (2003) for the Baire space. A general effective version of Hausdorff's theorem is still an open question.

2. Preliminary definitions and results

We write $\mathbb{N}$ for the set of natural numbers, $\mathcal{P}(\mathbb{N})$ for the set of all subsets of $\mathbb{N}$ and $\mathcal{P}_{<\omega}(\mathbb{N})$ for the set of all finite subsets of $\mathbb{N}$. Finite sequences of elements of a set X are denoted by $(x_1, x_2, \dots, x_n)$. Concatenation of sequences u, v and element x are written simply as uv, ux. We use Greek letters to denote ordinals. We write ω for the first infinite ordinal, ω_1 for the first uncountable ordinal and ω_1^{CK} for the least not computable ordinal (the Church–Kleene ordinal). For any two ordinals α, β, $\alpha \sim \beta$ means that they have the same parity.

2.1. *Borel and Hausdorff hierarchies in general topological spaces*

All the material of this subsection on the Borel and Hausdorff hierarchies in general topological spaces has first appeared in Selivanov (2005). To make the presentation self-contained, we reproduce here some of the proofs.

2.1.1. *The Borel hierarchy.* In general topological spaces, an open set may possibly not be a countable union of closed sets, cf. Remark 2.13 infra. In order to get the expected inclusion $\mathbf{\Sigma}^0_1(E) \subseteq \mathbf{\Sigma}^0_2(E)$, one has to distort the usual definition of Borel spaces given in metric spaces. This leads to define the hierarchy of Borel sets in a general setting as follows.

Definition 2.1 (Borel sets). Let E be a topological space.

1. *Borel subsets of E* are those sets obtained from open sets by iterated complementation and countable unions and intersections.
2. The *Borel classes* $\mathbf{\Sigma}^0_\alpha(E)$, $\mathbf{\Pi}^0_\alpha(E)$, $\mathbf{\Delta}^0_\alpha(E)$, where $\alpha \geqslant 1$ varies over countable ordinals, are inductively defined as follows:

$$\begin{array}{rrcl} & \mathbf{\Sigma}^0_1(E) & = & \textit{open subsets of } E \\ \text{if } \alpha \geqslant 2 & \mathbf{\Sigma}^0_\alpha(E) & = & \textit{countable unions of Boolean combinations of sets in } \bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta(E) \\ & \mathbf{\Pi}^0_\alpha(E) & = & \{E \setminus X \mid X \in \mathbf{\Sigma}^0_\alpha(E)\} \\ & \mathbf{\Delta}^0_\alpha(E) & = & \mathbf{\Sigma}^0_\alpha(E) \cap \mathbf{\Pi}^0_\alpha(E). \end{array}$$

3. The class $\mathbf{G}_\delta(E)$ (respectively $\mathbf{F}_\sigma(E)$) is the family of countable intersections of open sets (respectively unions of closed sets). In general, it is a proper subclass of $\mathbf{\Pi}^0_2(E)$ (respectively $\mathbf{\Sigma}^0_2(E)$).

The following result follows from the elementary set theory.

Proposition 2.2.

1. $\mathbf{\Sigma}^0_\beta(E) \cup \mathbf{\Pi}^0_\beta(E) \subseteq \mathbf{\Delta}^0_\alpha(E)$ for any $\alpha > \beta \geqslant 1$.

2. Each one of the Borel classes $\mathbf{\Sigma}^0_\alpha(E)$, $\mathbf{\Pi}^0_\alpha(E)$, $\mathbf{\Delta}^0_\alpha(E)$ is closed under finite unions and intersections and continuous inverse images. The $\mathbf{\Sigma}^0_\alpha(E)$ (respectively $\mathbf{\Pi}^0_\alpha(E)$) classes are closed under countable unions (respectively intersections).

As expected, the above definition is equivalent to the usual one for metric spaces. Also, in the general case, the distortion can be done solely for $\mathbf{\Sigma}^0_2(E)$.

Proposition 2.3.

1. $\mathbf{\Sigma}^0_2(E)$ coincides with the family of countable unions of differences of sets in $\mathbf{\Sigma}^0_1(E)$, i.e. sets of the form $\bigcup_{n\in\mathbb{N}} U_n \setminus V_n$, where the U_n, V_n's are open. Moreover, if $\mathcal{B}$ is a countable topological basis then one can take the U_n's in $\mathcal{B}$.
2. If $\alpha \geqslant 3$ then $\mathbf{\Sigma}^0_\alpha(E)$ is the family of countable unions of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(E)$. If E is metrizable then this also holds for $\alpha = 2$.

Proof.

1. Observe that a Boolean combination of open sets is a finite union of differences of two open sets. For the last assertion, use that U_n is a union of sets in $\mathcal{B}$.
2. It suffices to prove that the difference $X \setminus Y$ of two sets in $\mathbf{\Sigma}^0_\beta(E)$, with $\beta < \alpha$, is equal to a countable union of sets in $\mathbf{\Pi}^0_\gamma(E)$ with $\gamma < \alpha$.

In case $\beta + 1 < \alpha$ then, as the intersection of a $\mathbf{\Sigma}^0_\beta(E)$ and a $\mathbf{\Pi}^0_\beta(E)$ set, $X \setminus Y$ is $\mathbf{\Pi}^0_{\beta+1}(E)$ and we are done. In case $\alpha = \beta + 1$, since $\alpha \geqslant 3$, we have $\beta \geqslant 2$ and X is of the form $X = \bigcup_{i\in\mathbb{N}} U_i \setminus V_i$, where U_i, V_i are in $\bigcup_{\gamma<\beta} \mathbf{\Sigma}^0_\gamma(E)$. Thus, $X \setminus Y = \bigcup_{i\in\mathbb{N}}(U_i \setminus V_i) \setminus Y = \bigcup_{i\in\mathbb{N}} U_i \cap (E \setminus (V_i \cup Y))$. Now, $U_i \in \mathbf{\Sigma}^0_\gamma(E)$, with $\gamma < \beta$, hence $U_i \in \mathbf{\Pi}^0_{\gamma+1}(E)$ where $\gamma + 1 \leqslant \beta$. Also, $V_i \cup Y \in \mathbf{\Sigma}^0_\beta(E)$ hence $E \setminus (V_i \cup Y) \in \mathbf{\Pi}^0_\beta(E)$. Therefore, $X \setminus Y$ is a countable union of sets in $\mathbf{\Pi}^0_\beta(E)$. Finally, in a metric space, the topological closure $\overline{X}$ of any set X is $\mathbf{G}_\delta$ since $\overline{X} = \bigcap_{n\in\mathbb{N}}\{z \mid \exists x \in X\ d(z,x) < 2^{-n}\}$. Going to complements, any open set is $\mathbf{F}_\sigma$. Then, any difference of two open sets hence also any $\mathbf{\Sigma}^0_2(E)$ set is also $\mathbf{F}_\sigma$, i.e. a countable union of $\mathbf{\Pi}^0_1(E)$ sets. □

2.1.2. *The Hausdorff difference hierarchy.* Recall the Hausdorff difference infinitary operation, cf. Kuratowski (1966) and Kechris (1995).

Definition 2.4. Let α be an ordinal.

1. The *difference operation* D_α maps an α-sequence of subsets $(A_\beta)_{\beta<\alpha}$ of a space E to the subset $D_\alpha((A_\beta)_{\beta<\alpha}) = \bigcup\limits_{\beta<\alpha,\ \beta\not\sim\alpha} A_\beta \setminus \cup_{\gamma<\beta} A_\gamma$.
2. We let *co-*$D_\alpha((A_\beta)_{\beta<\alpha}) = E \setminus D_\alpha((A_\beta)_{\beta<\alpha})$.
3. For a class of subsets $\mathcal{A}$, we let $\mathbf{D}_\alpha(\mathcal{A})$ (respectively *co-*$\mathbf{D}_\alpha(\mathcal{A})$) be the class of all subsets $D_\alpha((A_\beta)_{\beta<\alpha})$ (respectively *co-*$D_\alpha((A_\beta)_{\beta<\alpha})$), where $A_\beta \in \mathcal{A}$ for all $\beta < \alpha$.

Remark 2.5. In particular, $\mathbf{D}_2(\mathcal{A})$ (respectively *co-*$\mathbf{D}_2(\mathcal{A})$) is the family of sets $A_1 \setminus A_0$ (respectively $A_0 \cup (E \setminus A_1)$) with $A_0, A_1 \in \mathcal{A}$.

Proposition 2.6.

1. If $\varnothing \in \mathcal{A}$ then $\mathbf{D}_\beta(\mathcal{A}) \subseteq \mathbf{D}_\alpha(\mathcal{A})$ for all $\beta < \alpha$.

Fig. 1. In grey: $D_1(A_0)$, $D_2(A_0, A_1)$, $D_3(A_0, A_1, A_2)$, $D_4(A_0, A_1, A_2, A_3)$, where $A_0 \subset A_1 \subset A_2 \subset A_3$. In white (including the unbounded complement of the largest disk): $co\text{-}D_1(A_0)$, $co\text{-}D_2(A_0, A_1)$, $co\text{-}D_3(A_0, A_1, A_2)$, $co\text{-}D_4(A_0, A_1, A_2, A_3)$.

2. If $E \in \mathcal{A}$ then $co\text{-}\mathbf{D}_\alpha(\mathcal{A}) \subseteq \mathbf{D}_{\alpha+1}(\mathcal{A})$. In particular, if $\emptyset, E \in \mathcal{A}$ and $\beta < \alpha$ then $\mathbf{D}_\beta(\mathcal{A}) \cup co\text{-}\mathbf{D}_\beta(\mathcal{A}) \subseteq \mathbf{D}_\alpha(\mathcal{A}) \cap co\text{-}\mathbf{D}_\alpha(\mathcal{A})$.
3. If $\mathcal{A}$ is closed under countable unions then, for α countable, in the definition of $\mathbf{D}_\alpha(\mathcal{A})$, one can restrict to monotone increasing α-sequences.

Proof.

1. If $\beta \sim \alpha$ then $D_\beta((A_\gamma)_{\gamma<\beta}) = D_\alpha((A'_\delta)_{\delta<\alpha})$, where $A'_\delta = A_\delta$ for $\delta < \beta$ and $A'_\delta = \emptyset$ for $\delta \geqslant \beta$. If $\beta \not\sim \alpha$ then $D_\beta((A_\gamma)_{\gamma<\beta}) = D_\alpha((A'_\delta)_{\delta<\alpha})$, where $A'_{\delta+1} = A_\delta$ for $\delta < \beta$ and $A'_\delta = \emptyset$ for $\delta = 0$ or δ limit or $\delta \geqslant \beta$.
2. Observe that $D_\alpha((A_\beta)_{\beta<\alpha}) = D_\alpha((A'_\beta)_{\beta<\alpha})$, where $A'_\beta = \bigcup_{\gamma\leqslant\beta} A_\gamma$.
3. Letting $A_\alpha = E$, we have $co\text{-}D_\alpha((A_\beta)_{\beta<\alpha}) = D_{\alpha+1}((A_\beta)_{\beta\leqslant\alpha})$.

□

Definition 2.7. For any $0 < \beta < \omega_1$, the αth level of the *difference hierarchy over* $\mathbf{\Sigma}^0_\beta(E)$ is $\mathbf{D}_\alpha(\mathbf{\Sigma}^0_\beta(E))$. The difference hierarchy over $\mathbf{\Sigma}^0_1(E)$ is simply called the difference hierarchy and denoted by $\mathbf{D}_\alpha(E)$.

Remark 2.8.

1. Using item 2 of Proposition 2.6, we can graphically represent sets in the first levels of the difference hierarchy as in Figure 1.
2. This graphical representation makes it clear that $\mathbf{D}_\alpha(E)$ is not closed under finite union nor finite intersection: for instance, if $A_0 \subset A_1 \subset A_2$ then

$$\begin{array}{lrcl} & D_2(\emptyset, A_0) \cup D_2(A_1, A_2) & = & D_3(A_0, A_1, A_2) \\ \textit{hence} & co\text{-}D_2(\emptyset, A_0) \cap co\text{-}D_2(A_1, A_2) & = & co\text{-}D_3(A_0, A_1, A_2) \\ \textit{and} & D_3(\emptyset, A_0, E) \cap D_3(A_1, A_2, E) & = & D_4(A_0, A_1, A_2, E)\ . \end{array}$$

Proposition 2.9. $\bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(\mathbf{\Sigma}^0_\beta(E)) \subseteq \mathbf{\Delta}^0_{\beta+1}(E)$.

Proof. If the A_γ's, $\gamma < \alpha$ are in $\mathbf{\Sigma}^0_\beta(E)$ then so are the $\bigcup_{\delta<\gamma} A_\delta$'s. Thus, $D_\alpha((A_\gamma)_{\gamma<\alpha})$ is a countable union of differences of sets in $\mathbf{\Sigma}^0_\beta(E)$ hence is in $\mathbf{\Sigma}^0_{\beta+1}(E)$. By Proposition 2.6, we see that $E \setminus D_\alpha((A_\gamma)_{\gamma<\alpha}) = D_{\alpha+1}((A_\gamma)_{\gamma<\alpha}, E) \in \mathbf{\Pi}^0_{\beta+1}(E)$. □

Proposition 2.10. Let D, E be topological spaces and $f : D \to E$ be continuous. If $Y \subseteq E$ is in some Hausdorff class $\mathrm{D}_\alpha(E)$, $\alpha < \omega_1$, then $f^{-1}(\mathcal{Y})$ is in $\mathrm{D}_\alpha(D)$.

2.1.3. *The Borel and Hausdorff hierarchies may collapse.* As it is well known, the Borel and Hausdorff hierarchies are proper in uncountable Polish spaces: $\mathbf{\Sigma}^0_\alpha(E) \subsetneq \mathbf{\Sigma}^0_\beta(E)$ and $\mathbf{D}_\alpha(\mathbf{\Sigma}^0_\xi(E)) \subsetneq \mathbf{D}_\beta(\mathbf{\Sigma}^0_\xi(E))$ when $\alpha < \beta$. The same for the effective hierarchies (with $\alpha, \beta < \omega_1^{\mathrm{CK}}$) relative to some fixed enumeration of a countable basis of open sets. However, this is not true in general topological spaces. For instance, if the space is T_2 and countable then $\mathbf{\Sigma}^0_2(E) = \mathcal{P}(E)$. However, (de Brecht 2013) proves the non-collapse of the Borel and Hausdorff difference hierarchies in uncountable quasi-Polish spaces, a class containing Polish spaces and ω-continuous domains.

2.2. *Domains*

The domain theory refers to the field initiated by Dana Scott in the late 1960s to specify denotational semantics for functional programming languages. The theory formalizes the ideas of approximation and convergence via some partially ordered sets called *domains*.

Example 2.11. Some examples of Scott topologies on partially ordered sets.

1. **Scott topology on** $(\mathcal{P}(\mathbb{N}), \subseteq)$**.** For $A \in \mathcal{P}_{<\omega}(\mathbb{N})$, let $O_A = \{X \in \mathcal{P}(\mathbb{N}) \mid X \supseteq A\}$. The Scott topology on $\mathcal{P}(\mathbb{N})$ is has the O_A's, for $A \in \mathcal{P}_{<\omega}(\mathbb{N})$, as a topological basis. This is the topology of 'positive information'; in contrast, the Cantor topology on $\mathbf{2}^\omega$ gives positive and negative information.
2. **Scott topology on** $(\mathcal{P}_\infty(\mathbb{N}), \subseteq)$**.** Consider the family $\mathcal{P}_\infty(\mathbb{N})$ of infinite subsets of $\mathbb{N}$ as a topological subspace of $\mathcal{P}(\mathbb{N})$.
3. **Scott topology on the family** $(X^{\leqslant\omega}, \leqslant_{\mathbf{pref}})$ **of finite or infinite** X**-sequences.** We suppose X is any set with at least two elements. For any $s \in X^{<\omega}$, let $\mathcal{B}_s = \{u \in X^{\leqslant\omega} \mid u \textit{ extends } s\}$. The Scott topology on the space $X^{\leqslant\omega}$ is that which admits the $\mathcal{B}_s$'s, $s \in X^{<\omega}$, as a topological basis.
4. **Scott topology on the right extended real line** $(\overrightarrow{\mathbb{R}}, \leqslant)$**.** Let $\overrightarrow{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The open sets of the topology on $\overrightarrow{\mathbb{R}}$ are $\overrightarrow{\mathbb{R}}$ and the semi-intervals $]x, +\infty]$, for $x \in \mathbb{R}$.
5. **Extended real line** $(\widetilde{\mathbb{R}}, \leqslant)$ **with duplicated rationals.** Let $\widetilde{\mathbb{R}} = \mathbb{R} \cup (\mathbb{Q} \times \{+\}) \cup \{+\infty\}$ and $\leqslant$ be the following total order: $+\infty$ is a maximum element, and for all $x, y \in \mathbb{R}$ and $q \in \mathbb{Q}$, $q <_{\widetilde{\mathbb{R}}} (q, +)$; $x <_{\widetilde{\mathbb{R}}} y$ if and only if $x < y$; and $x <_{\widetilde{\mathbb{R}}} (q, +) <_{\widetilde{\mathbb{R}}} y$ if and only if $x < q < y$. The Scott topology on $\widetilde{\mathbb{R}}$ is that for which the $[(q, +), +\infty]$'s, $q \in \mathbb{Q}$, are a topological basis.

The following properties are straightforward.

Proposition 2.12. The Scott topologies of spaces in Examples 2.11 are not T_2 but are T_0, i.e. they satisfy Kolmogorov's axiom: given two distinct points, one of them has a neighbourhood which does not contain the other one (but this may not be symmetric).

Remark 2.13.

1. As noticed in Selivanov (2005), the finite levels of the Scott Borel hierarchy on $\mathcal{P}(\mathbb{N})$ do not coincide with the corresponding ones on the Cantor space $\mathbf{2}^\omega$: $\mathbf{\Sigma}^0_n(\mathcal{P}(\mathbb{N})) \subsetneq$

$\mathbf{\Sigma}^0_n(\mathbf{2}^\omega) \subsetneq \mathbf{\Sigma}^0_{n+1}(\mathcal{P}(\mathbb{N}))$ for all $n \in \mathbb{N}$. The same is true with the effective Borel hierarchy (cf. Section 5.3). For instance, $\mathcal{X} = \mathcal{P}(\mathbb{N}) \setminus \{\mathbb{N}\}$, defined by the formula $\exists x\ (x \notin X)$, is $\Sigma^0_1(\mathbf{2}^\omega)$ and $\Sigma^0_2(\mathcal{P}(\mathbb{N}))$ but neither Scott open nor closed. However, the infinite levels of the Borel hierarchy on $\mathcal{P}(\mathbb{N})$ and $\mathbf{2}^\omega$ coincide.

2. The only subsets of $\mathcal{P}(\mathbb{N})$ that are both open and F_σ are $\emptyset$ and $\mathcal{P}(\mathbb{N})$. Indeed, suppose $\mathcal{O}$ is open and $\mathcal{X}$ is F_σ and $\mathcal{O}, \mathcal{X}$ are different from $\emptyset, \mathcal{P}(\mathbb{N})$. Then there exist non-empty finite subsets X, Y of $\mathbb{N}$ such that $O_X \subseteq \mathcal{O}$ and $\mathcal{P}(\mathbb{N}) \setminus O_Y \subseteq \mathcal{X}$. Observe that the set $X \cup Y$ is in $\mathcal{O} \setminus \mathcal{X}$, showing $\mathcal{O} \neq \mathcal{X}$.

2.2.1. *The Scott topology on dcpo's.*

We briefly recall the main definitions and notions, and refer the reader to classical papers and books, for instance Abramsky and Jung (1994), Edalat (1997) and Gierz *et al.* (2003).

Definition 2.14.

1. A *dcpo* is a partially ordered set $(D, \sqsubseteq)$ such that every non-empty directed subset S has a least upper bound (denoted by $\sqcup S$). A dcpo is pointed if it has a least element $\bot$.
2. The *Scott topology* on a dcpo is the topology that admits as closed sets all sets X satisfying conditions
   - — X is a downset: $x \in X \wedge y \sqsubseteq x \Rightarrow y \in X$.
   - — X is closed under suprema of directed subsets of D.

   Then, $O \subseteq D$ is open in the Scott topology if it satisfies the following conditions.
   - — O is an upset: $x \in O \wedge x \sqsubseteq y \Rightarrow y \in O$.
   - — Every directed set with supremum in O has an element in O.

Example 2.15. For every $x \in D$, the set $U_x = \{z \mid z \not\sqsubseteq x\}$ is Scott open.

Proposition 2.16.

1. The Scott topology on a dcpo is T_0, i.e. if $x \neq y$ then there exists an open set which contains only one of the two points x, y. It is T_1 (respectively T_2) if and only if the order on D is trivial.
2. Let $x, y \in D$. The order on D can be recovered from the topology as the specialization order: $x \sqsubseteq y$ if and only if every Scott open set containing x also contains y if and only $x \in \overline{\{y\}}$ (where $\overline{\{y\}}$ is the topological closure of $\{y\}$).
3. A function $f : D \to E$ between two dcpo's is continuous with respect to the Scott topologies if and only if it is monotone increasing and preserves suprema of directed subsets: if $S \subseteq D$ is directed then $f(\sqcup S) = \sqcup f(S)$.

2.2.2. *The Scott topology on domains.*

Definition 2.17.

1. Let $(D, \sqsubseteq)$ be a dcpo. The *approximation (or way-below) relation* on D is defined as follows: let $x, y \in D$, $x \ll y$ if, for all directed subset S, $y \sqsubseteq \sqcup S$ implies $x \sqsubseteq s$ for some $s \in S$. We say x approximates, or is way-below, y.

2. An element $x \in D$ is *compact (or finite)* if $x \ll x$. The set of compact elements is denoted by $K(D)$.
3. $\twoheaduparrow x = \{y \mid x \ll y\}$ and $\twoheaddownarrow x = \{y \mid y \ll x\}$.

Proposition 2.18. Let $(D, \sqsubseteq)$ be a dcpo and $x, x', y, y' \in D$.

1. $x \ll y \Rightarrow x \sqsubseteq y$, and $(x' \sqsubseteq x \ll y \sqsubseteq y') \Rightarrow x' \ll y'$.
   If x is compact then $\forall u, v\ ((u \sqsubseteq x \sqsubseteq v) \Leftrightarrow (u \ll x \ll v))$.
2. An element x is compact if and only if $\uparrow x = \{y \mid x \sqsubseteq y\}$ is Scott open.

Proposition 2.19. Let $(D, \sqsubseteq)$ be a dcpo. The following conditions are equivalent.

i. For every $x \in D$, the set $\twoheaddownarrow x = \{z \in D \mid z \ll x\}$ is directed and $x = \sqcup \twoheaddownarrow x$.
ii. There exists $B \subseteq D$ such that, for every $x \in D$, $B \cap \twoheaddownarrow x$ is directed and $x = \sqcup(B \cap \twoheaddownarrow x)$.

D is a *continuous domain* if these conditions hold. Any set B satisfying condition (ii) is called a basis. D is an ω-*continuous domain* if (ii) holds for some countable set B. D is an algebraic (respectively ω-algebraic) domain if $K(D)$ is a basis (and is countable).

Example 2.20. All spaces in Example 2.11 are dcpo's with the Scott topology. The spaces $\mathcal{P}(\mathbb{N})$, $X^{\leqslant\omega}$, $\widetilde{\mathbb{R}}$ are ω-algebraic domains. Their sets of compact elements are respectively $\mathcal{P}_{<\omega}(\mathbb{N})$, $X^{<\omega}$ and $\mathbb{Q} \times \{+\}$. The way-below relation on any of these three spaces is the restriction of the partial order $\leqslant$ to $K(D) \times D$. The space $\overrightarrow{\mathbb{R}}$ is an ω-continuous domain but is not algebraic: there is no compact element and its way-below relation is the strict order $<$. The space $\mathcal{P}_{\infty}(\mathbb{N})$ is not continuous: its way-below relation is empty.

Let us recall classical results in continuous domains.

Proposition 2.21. Let $(D, \sqsubseteq)$ be a continuous domain with basis B.

1. **(Interpolation property).** If $M \subseteq D$ is finite and $a \ll x$ for each $a \in M$ then there exists $x' \in B$ such that $M \ll x' \ll x$.
2. $x \ll y$ if and only if y is interior to the upper cone $\uparrow x = \{z \mid x \sqsubseteq z\}$.
3. A set O is open if and only if $O = \bigcup_{x \in O} \twoheaduparrow x$ if and only if $O = \bigcup_{x \in O \cap B} \twoheaduparrow x$. In particular, the family of sets $\twoheaduparrow z$, where z varies in B, is a basis of the Scott topology on D.

2.3. *Quasi-Polish spaces*

Quasi-Polish spaces, developed by de Brecht (2013), are a unifying framework of Polish spaces and ω-continuous domains. We recall here the definition and main results.

Definition 2.22.

1. Giving up the symmetry axiom of metrics, a quasi-metric on a space E is defined as a function $d : E^2 \to [0, +\infty[$ such that, for all $x, y, z \in E$,

$$x = y \Leftrightarrow d(x, y) = d(y, x) = 0, \quad d(x, z) \leqslant d(x, y) + d(y, z)\ .$$

   The topology associated to d is the one generated by the open balls.
2. A sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy if $\lim_{n \to +\infty} \sup_{p \geqslant n} d(x_n, x_p) = 0$.
3. A quasi-metric space (E, d) is complete if every Cauchy sequence is convergent relative to the metric $\widehat{d}$ such that $\widehat{d}(x, y) = \max\{d(x, y), d(y, x)\}$.

4. Topological spaces associated to complete quasi-metrics with a countable topological basis are called quasi-Polish.

Example 2.23. The Scott topology on $\mathcal{P}(\mathbb{N})$ is quasi-Polish for the quasi-metric such that $d(X,Y) = \sum_{n \in X \setminus Y} 2^{-n}$.

Theorem 2.24 (Künzi 1983). A quasi-metric space (E,d) has a countable topological basis if and only if the metric space $(E,\widehat{d})$ is separable (i.e. has a countable dense subset).

The following theorem sums up some of the main results in de Brecht (2013).

Theorem 2.25 (de Brecht 2013).

1. A space is quasi-Polish if and only if it is homeomorphic to some $\mathbf{\Pi}^0_2$ subspace of $\mathcal{P}(\mathbb{N})$ endowed with the Scott topology.
2. Polish spaces and ω-continuous domains are quasi-Polish.
3. Every quasi-Polish space E satisfies the following properties:
   - (Baire property). The intersection of a sequence of dense open sets is a dense set.
   - (Hausdorff–Kuratowski's property). $\bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(\mathbf{\Sigma}^0_\beta(E)) = \mathbf{\Delta}^0_{\beta+1}(E)$ for all $0 < \beta < \omega_1$.

Remark 2.26. Two examples of subspaces of $\mathcal{P}(\mathbb{N})$ illustrate the above theorem.

1. The subspace $C = \{X \in \mathcal{P}(\mathbb{N}) \mid \forall i \in \mathbb{N}\ (2i \in X \Leftrightarrow 2i+1 \notin X)\} = \bigcap_{i \in \mathbb{N}} (O_{2i} \Delta O_{2i+1})$ is $\mathbf{\Pi}^0_2$ in $\mathcal{P}(\mathbb{N})$ and is homeomorphic to the Cantor space $\mathbf{2}^\omega$ (and is therefore Polish).
2. The subspace $\mathcal{P}_\infty(\mathbb{N}) = \bigcap_{i \in \mathbb{N}} \bigcup_{j \geqslant i} O_{\{j\}}$ is $\mathbf{\Pi}^0_2$ in $\mathcal{P}(\mathbb{N})$ hence is quasi-Polish. It has a countable basis but is not Polish since it is T_0 and not T_2. Although $(\mathcal{P}_\infty(\mathbb{N}), \subseteq)$ is a dcpo, it is not a continuous domain since its way-below relation is empty.

2.4. *Topological games*

The Choquet topological games have been used to characterize Polish spaces (Choquet 1969) and quasi-Polish spaces (de Brecht 2013). We shall use them to characterize approximation spaces (cf. Definition Section 3.1 infra). Choquet games are a variant of Banach–Mazur games. The subtlety of this variant is best understood by confronting definitions and properties of both classes of games. Also, the interest of de Brecht's Theorem 2.36 and our Corollary 3.12 about convergent Markov/stationary strategies in the Choquet game is highlighted by the counterpart (and more powerful) results by Galvin and Telgársky for Banach–Mazur games (cf. Corollary 2.34, Remark 2.35).

2.4.1. *Banach–Mazur and Choquet games.* Let us recall the classical definitions.

Definition 2.27 (Choquet 1969; Galvin and Telgárky 1986). Let X be a topological space.

1. In the Banach–Mazur game $\mathrm{BM}(X)$ two players, Empty and Nonempty, alternate turns for ω rounds. On round 0 (respectively $i+1$), Empty moves first, choosing a non-empty open subset $U_0 \subseteq X$ (respectively $U_{i+1} \subseteq V_i$). Then, Nonempty responds with a non-empty open set $V_0 \subseteq U_0$ (respectively $V_{i+1} \subseteq U_{i+1}$). After all the rounds have been played, Nonempty wins if $\bigcap_{i \in \mathbb{N}} V_i \neq \emptyset$. Otherwise, Empty wins.

2. The Choquet game $\mathrm{Ch}(X)$ is the variant of the Banach–Mazur game $\mathrm{BM}(X)$, where at round i Empty picks a pair (x_i, U_i) such that U_i is open and $x_i \in U_i$, and Nonempty picks an open set V_i such that $x_i \in V_i \subseteq U_i$.
3. A winning strategy for a player (in any of the above games) is a function that takes a partial play of the game ending with a move by its opponent and returns a move to play, such that the player wins any play of the game that follows the strategy.
4. A winning strategy for Nonempty is convergent if, when he follows it, the V_i's are a basis of neighbourhoods of some $x \in \bigcap_{i\in\mathbb{N}} V_i$.

Note 2.28. The denominations 'Banach–Mazur' and 'Choquet' are the most commonly used (and are historically accurate). However, in Kechris (1995) these games are called 'Choquet' and 'strong Choquet', respectively.

Remark 2.29. Every winning strategy for Nonempty in the Choquet game $\mathrm{Ch}(X)$ yields one in the Banach–Mazur game $\mathrm{BM}(X)$.

The following theorem sums up some known results around these topological games.

Theorem 2.30. Let X be a topological space.

1. (Oxtoby 1957, cf. Kechris 1995, Theorem 8.11) X has the Baire property (i.e. the intersection of countably many dense open sets is dense) if and only if player Empty has no winning strategy in the Banach–Mazur game $\mathrm{BM}(X)$.
2. (Choquet 1969, cf. Kechris 1995, Theorem 8.18) X is Polish if and only if X has a countable basis, is T_1 and regular and player Nonempty has a winning strategy in the Choquet game $\mathrm{Ch}(X)$.
3. (de Brecht 2013, Theorem 51) X is quasi-Polish if and only if X has a countable basis, and player Nonempty has a convergent winning strategy in the Choquet game $\mathrm{Ch}(X)$.

2.4.2. *Markov and stationary strategies.* The pioneer work of Schmidt (1966) and Choquet (1969) considered strategies of a very simple form. Then Galvin and Telgárky (1986) obtained deep results for other strategies.

Definition 2.31. A winning strategy is stationary (respectively Markov) if it depends only on the last move of the opponent (respectively and on the ordinal rank of the round).

Theorem 2.32 (Galvin and Telgárky 1986, Theorems 5 and 7).
Let $(S, \leqslant)$ be a non-empty partially ordered set. Let R be a family of monotone (non-strictly) decreasing sequences in S. In the game $G(S, \leqslant, R)$, player I starts and plays elements $a_0, a_1, \ldots$ of S, player II plays elements $b_0, b_1, \ldots$ of S in such a way that $a_{i+1} \leqslant b_i$ and $b_i \leqslant a_i$ for all i. Player II wins if and only if $(a_0, a_1, \ldots) \in R$. Suppose R is such that for all monotone (non-strictly) decreasing sequences $(x_i)_{i\in\mathbb{N}}$, $(y_i)_{i\in\mathbb{N}}$ in S,

$$(\forall i\ \exists j\ y_j \leqslant x_i) \wedge (\forall j\ \exists k\ x_k \leqslant y_j) \wedge (x_i)_{i\in\mathbb{N}} \in R \ \Rightarrow\ (y_i)_{i\in\mathbb{N}} \in R\,.$$

1. If player II has a winning strategy in $G(S, \leqslant, R)$ then he has one that depends only on the last move of I and the last move of II.
2. If player II has a Markov winning strategy in $G(S, \leqslant, R)$ then he has a stationary one.

Corollary 2.33 (Galvin and Telgárky 1986, Corollaries 9 and 14).

1. If player Nonempty has a winning (respectively and convergent) strategy in the Banach–Mazur game $\mathrm{BM}(X)$ then he has one which depends only on the last move of Empty and the last move of Nonempty (respectively and is convergent).
2. If player Nonempty has a Markov (respectively and convergent) winning strategy in $\mathrm{BM}(X)$ then he has a stationary (respectively and convergent) one.
3. (Debs 1984, 1985). There exists a T_2 (even completely regular) space X such that player Nonempty has no stationary winning strategy in $\mathrm{BM}(X)$ but has a winning strategy which depends only on the last two moves of Empty.

Remark 2.34. Theorem 2.32 does not apply to the Choquet game because the players do not play in the same partially ordered set. For simple winning strategies for Nonempty in the Choquet game $\mathrm{Ch}(X)$ in particular spaces X, see Section 5 of Bennett and Lutzer (2009).

The proof of item 3 of Theorem 2.30 given by de Brecht (Theorem 51 in de Brecht 2013) yields more:

Theorem 2.35 (de Brecht 2013).
If X is quasi-Polish then player Nonempty has a Markov convergent winning strategy in the Choquet game $\mathrm{Ch}(X)$.

We shall improve this last result (replacing Markov by stationary), cf. Corollary 3.12.

3. Approximation spaces: the spaces of Choquet games

3.1. *Approximation spaces*

We introduce another class of topological spaces: approximation spaces. They include all continuous domains, all Polish spaces and, in fact, all quasi-Polish spaces. The definition is based on an *approximation relation* which formalizes a containment relation between basic open sets. This containment relation ensures that inclusion-decreasing chains have a non-empty intersection; however, this intersection may not be reduced to a singleton set. An example of an approximation relation is obtained by lifting to basic open sets the way-below relation in a dcpo. We borrowed the notation $\ll$ from this particular example.

Theorem 3.11 proves that a subclass of second-countable approximation spaces coincides with the class of quasi-Polish spaces. This gives an 'à la domain' characterization of quasi-Polish spaces. Whether the notion of approximation space captures a substantial part of the rich theory developed by de Brecht for quasi-Polish spaces is a question still to be investigated. In the next sections, we just show two pleasant properties: a $\mathbf{\Pi}^0_2$ Baire property and Hausdorff's characterization of $\mathbf{\Delta}^0_2$.

Definition 3.1. Let E be a topological space.

— An *approximation relation* for E is a binary relation $\ll$ on some topological basis $\mathcal{B}$ such that, for all $U, V, T \in \mathcal{B}$,

1. if $U \ll V$ then $V \subseteq U$,

2. if $U \subseteq T$ and $U \ll V$ then $T \ll V$ (in particular, $\ll$ is transitive),
3. for all $x \in U$, there exists $W \in \mathcal{B}$ such that $x \in W$ and $U \ll W$,
4. for every sequence $(U_i)_{i\in\mathbb{N}}$ of sets in $\mathcal{B}$ such that $U_i \ll U_{i+1}$ for all i, the intersection set $\bigcap_{i\in\mathbb{N}} U_i$ is non-empty.

— *Convergent approximation relations* are obtained by strengthening condition (4) to

(4^+) Every sequence $(U_i)_{i\in\mathbb{N}}$ of sets in $\mathcal{B}$ such that $U_i \ll U_{i+1}$ for all i, is a neighbourhood basis of some $x \in \bigcap_{i\in\mathbb{N}} U_i$ (i.e. each open set containing x also contains some U_i).

— *An approximation relation is hereditary* if, for every closed subset C of E,

$$\ll_C \;=\; \{(C \cap U, C \cap V) \mid U \ll V \textit{ and } C \cap U, C \cap V \neq \varnothing\}$$

is an approximation relation for the subspace C.

— A space E is an *approximation space* (respectively *convergent approximation space*, respectively *hereditary approximation space*) if it admits an approximation (respectively convergent approximation, respectively hereditary approximation) relation.

Example 3.2. Every subspace D of the Scott domain $\mathcal{P}(\mathbb{N})$ which is an upset (i.e. if $X \subseteq Y$ and $X \in D$ then $Y \in D$) is a trivial approximation space: containment is an approximation relation on the basis $\{D \cap O_A \mid \alpha \in \mathcal{P}_{<\omega}(\mathbb{N})\}$, where $O_A = \{X \mid A \subseteq X\}$ since $\mathbb{N}$ belongs to all $D \cap O_A$'s. In particular, the dcpo $(\mathcal{P}_\infty(\mathbb{N}), \subseteq)$ is an approximation space which is neither Polish nor a continuous domain. In $\mathcal{P}(\mathbb{N})$, inclusion is a convergent approximation relation since $\bigcup_{i\in\mathbb{N}} A_i \in \bigcap_{i\in\mathbb{N}} O_{A_i}$ and the O_{A_i}'s converge to $\bigcup_{i\in\mathbb{N}} A_i$.

Approximation relations exist on all topological basis or on none.

Lemma 3.3. Let $\mathcal{B}$ and $\mathcal{C}$ be topological basis of a space E. If there exists an (respectively convergent; respectively hereditary) approximation relation on $\mathcal{B}$ then there exists one on $\mathcal{C}$.

Proof. Let $\ll$ be an approximation relation on $\mathcal{B}$. Consider the relation $\lll$ on $\mathcal{C}$ such that, for any $C, D \in \mathcal{C}$,

$$(*) \qquad C \lll D \iff \exists U, V \in \mathcal{B}\,(C \supseteq U \ll V \supseteq D)\,.$$

Let us check that $\lll$ satisfies conditions (1) to (4) of Definition 3.1. Conditions (1) and (2) are straightforward. We now look at Condition (3). Suppose $x \in C \in \mathcal{C}$. Let $U \in \mathcal{B}$ be such that $x \in U \subseteq C$. Applying condition (3) for $\ll$, there exists $V \in \mathcal{B}$ such that $x \in V$ and $U \ll V$. Let $D \in \mathcal{C}$ be such that $x \in D$ and $D \subseteq V$. Then $C \lll D$ so that condition (3) holds for $\lll$. Finally, we check Condition (4). Suppose $C_i \lll C_{i+1}$ for all i. Let $U_i, V_i \in \mathcal{B}$ be such that $C_i \supseteq U_i \ll V_i \supseteq C_{i+1}$. In particular, $V_i \supseteq U_{i+1} \ll V_{i+1}$ hence $V_i \ll V_{i+1}$ by condition (2) for $\ll$. Applying condition (4) for $\ll$, we see that $\bigcap_{i\in\mathbb{N}} C_i = \bigcap_{i\in\mathbb{N}} V_i \neq \varnothing$ so that condition (4) holds for $\lll$.

In case $\ll$ is convergent then the V_i's are a neighbourhood basis of some $x \in \bigcap_{i\in\mathbb{N}} V_i$. Hence, so are the C_{i+1}'s and $\lll$ is convergent. Suppose, $\ll$ is hereditary. To see that $\lll$ is also hereditary, observe that, for any closed subset F of E, the relation $\lll_F$ is obtained from $\ll_F$ via condition $(*)$. □

3.2. *Approximation spaces versus quasi-Polish spaces*

Proposition 3.4. Polish spaces and continuous domains are convergent approximation spaces.

Proof. Case of Polish spaces. Let $\mathcal{B}$ be the basis consisting of open balls centred in some fixed countable dense set and having rational radius. For $U, V \in \mathcal{B}$, let $U \ll V$ if and only if $\overline{V} \subseteq U$ and $diam(V) \leqslant diam(U)/2\}$, where $\overline{V}$ is the topological closure of V and *diam* is the diameter. All wanted conditions on $\ll$ are straightforward.

Case of continuous domains. Let B be a basis in the sense of continuous domains. The family $\mathcal{B} = \{\Uparrow b \mid b \in B \text{ and } \Uparrow b \neq \emptyset\}$ is a topological basis. Define the relation $\ll$ on $\mathcal{B}$ as $U \ll V$ if and only if $V = \Uparrow c$ for some $c \in U$. To check condition (1), observe that if $U = \Uparrow b$ then $c \in U$ yields $b \ll c$, so that $U = \Uparrow b \supseteq \Uparrow c = V$. As for condition (2), if $W \supseteq U \ll V$ and $V = \Uparrow c$ with $c \in U$ then $c \in W$ hence $W \ll V$. As for condition (3), let $x \in U = \Uparrow b$, i.e. $b \ll x$. Using the interpolation property, let c be such that $b \ll c \ll x$ and set $W = \Uparrow c$. Then $x \in W$ and $c \in \Uparrow b = U$, so that $U \ll W$. Finally, for condition (4^+), suppose $U_i \ll U_{i+1}$ for all $i \in \mathbb{N}$ and choose $b_{i+1} \in U_i$ such that $U_{i+1} = \Uparrow b_{i+1}$. Since $b_{i+2} \in U_{i+1} = \Uparrow b_{i+1}$, we have $b_{i+1} \ll b_{i+2}$. Let x be the supremum of the b_j's for $j \geqslant 1$. Then $x \in \bigcap_{j \geqslant 1} \Uparrow b_j = \bigcap_{i \in \mathbb{N}} U_i$ which is therefore a non-empty set. Also, the $\Uparrow b_j = U_j$'s, $j \geqslant 1$, are a basis of neighbourhoods of x. ☐

Using de Brecht's Theorem 2.25 for second-countable spaces, the above Proposition is partly subsumed by the next theorem.

Theorem 3.5. Quasi-Polish spaces are convergent approximation spaces.

Proof. By de Brecht's result stated in item 1 of Theorem 2.25, it suffices to show that any $\mathbf{\Pi}^0_2$ subspace $\mathcal{A}$ of $\mathcal{P}(\mathbb{N})$ (with the Scott topology) is a convergent approximation space. In this proof, we use Greek letters to denote finite sets. For $\alpha \in \mathcal{P}_{<\omega}(\mathbb{N})$, let $O_\alpha = \{X \mid \alpha \subseteq X\}$. As a basis $\mathcal{B}$ of the subspace $\mathcal{A}$, we consider those $B_\alpha = O_\alpha \cap \mathcal{A}$, $\alpha \in \mathcal{P}_{<\omega}(\mathbb{N})$, which are non-empty. Observe that if $U \in \mathcal{B}$, there may be infinitely many α's such that $U = B_\alpha$. The family $\mathcal{A}$ is of the form $\mathcal{A} = \bigcap_{n \in \mathbb{N}}(U_n \cup F_n)$ where $U_n = \bigcup_{\alpha \in I_n} O_\alpha$ is open in $\mathcal{P}(\mathbb{N})$ and F_n is closed in $\mathcal{P}(\mathbb{N})$. Since F_n is a countable intersection of closed sets of the form $\mathcal{P}(\mathbb{N}) \setminus O_\alpha$, by merging this intersection with the global one, we can reduce to the case where $F_n = \mathcal{P}(\mathbb{N}) \setminus O_{\alpha_n}$ with $\alpha_n \in \mathcal{P}_{<\omega}(\mathbb{N})$. Then,

$$X \in \mathcal{A} \Leftrightarrow \forall n\ (\alpha_n \nsubseteq X \ \vee\ \exists \gamma \in I_n\ \gamma \subseteq X) \Leftrightarrow \forall n\ (\alpha_n \subseteq X \Rightarrow \exists \gamma \in I_n\ \gamma \subseteq X)\ .$$

Let us call clause n the clause $\alpha_n \subseteq X \Rightarrow \exists \gamma \in I_n\ \gamma \subseteq X$. Thus, a set $X \subseteq \mathbb{N}$ is in $\mathcal{A}$ if and only if it satisfies clause n for all n. We introduce two notions:

— clause n *is a* U*-clause* if $U \subseteq O_{\alpha_n}$ (i.e. the premiss $\alpha_n \subseteq X$ of clause n is satisfied by all $X \in U$),

— clause n *is* U*-solved* if $U \subseteq O_\gamma$ for some $\gamma \in I_n$. (i.e. the conclusion $\exists \gamma \in I_n\ \gamma \subseteq X$ of clause n is satisfied by all $X \in U$ with the same witness γ).

Observe that if $W \supseteq U$ then any W-clause is a U-clause and any W-solved W-clause is a U-solved U-clause. We denote by n_U the least n such that clause n is a U-unsolved

U-clause or $+\infty$ if there is no such clause. We now define the relation $\ll$ on $\mathcal{B}$: for $U, V \in \mathcal{B}$,

$$U \ll V \iff V \subseteq U \text{ and } \begin{cases} (i) & \textit{either } n_U = +\infty \\ (ii) & \textit{or } n_U < +\infty \textit{ and clause } n_U \textit{ is } V\textit{-solved} \\ (iii) & \textit{or } n_U < +\infty \textit{ and, for some } m < n_U, \textit{ clause } m \\ & \textit{is not a } U\textit{-clause and is a } V\textit{-solved } V\textit{-clause.} \end{cases}$$

We check conditions (1), (2), (3) and (4^+) of Definition 3.1. Condition (1) is trivial. Condition (2). Suppose $W \supseteq U \ll V$. If $n_W = +\infty$ then $W \ll V$ holds by condition *(i)*. So we shall suppose $n_W < +\infty$. Suppose $n_W < n_U$. Then clause n_W, being a W-clause hence a U-clause, is U-solved (by definition of n_U), hence it is also V-solved (since $U \supseteq V$), so that $W \ll V$ holds by condition *(ii)*. We now assume $n_U \leqslant n_W < +\infty$. Since n_U is finite, $U \ll V$ cannot hold by condition *(i)*. If $U \ll V$ holds by condition *(iii)* then the witnessing clause m is not a U-clause hence is not a W-clause, so that $W \ll V$ holds by condition *(iii)*. Suppose now that $U \ll V$ holds by condition *(ii)*. If $n_W = n_U$ then $W \ll V$ also holds by condition *(ii)*. Suppose $n_W > n_U$. If clause n_U were a W-clause then it would be W-solved (by definition of n_W and inequality $n_U < n_W$) hence it would be a U-solved U-clause, contradicting the definition of n_U. Thus, clause n_U is not a W-clause and $W \ll V$ holds by condition *(iii)*.

Condition (3). Let X be in $U \in \mathcal{B}$. If $n_U = +\infty$ then it suffices to set $V = U$. Suppose, now that $n_U < +\infty$. Since clause n_U is a U-clause and $X \in U$, we have $\alpha_{n_U} \subseteq X$ so that X satisfies the premiss of clause n_U. Then, X being in $\mathcal{A}$, satisfies all clauses, in particular clause n_U. So, there exists some $\gamma \in I_{n_U}$ such that $\gamma \subseteq X$. Now, $U \in \mathcal{B}$ hence $U = \mathcal{A} \cap O_\beta$ for some $\beta \in \mathcal{P}_{<\omega}(\mathbb{N})$. Set $V = U \cap O_\gamma$. Then $V = \mathcal{A} \cap O_\beta \cap O_\gamma = \mathcal{A} \cap O_{\beta\cup\gamma} \in \mathcal{B}$. Also, $X \in V$ since $X \in U$ and $\gamma \subseteq X$. Finally, V solves clause n_U, hence $U \ll V$ by condition (ii) of $\ll$.

Condition (4). Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of families in $\mathcal{B}$ such that $U_i \ll U_{i+1}$ for all i. Let β_i be any set in $\mathcal{P}_{<\omega}(\mathbb{N})$ such that $U_i = \mathcal{A} \cap O_{\beta_i}$. Set $\delta_i = \bigcup_{j\leqslant i} \beta_j$ (so that the δ_i's are increasing with i). Since $U_j \supseteq U_i$ for $j < i$, we have $U_i = \mathcal{A} \cap \bigcap_{j\leqslant i} O_{\beta_i} = \mathcal{A} \cap O_{\delta_i}$. To finish the proof, we show that the set $X = \bigcup_{i\in\mathbb{N}} \delta_i$ is in the family $\bigcap_{i\in\mathbb{N}} U_i = \mathcal{A} \cap \bigcap_{i\in\mathbb{N}} O_{\delta_i}$ (which is therefore non-empty). Clearly, $X \in \bigcap_{i\in\mathbb{N}} O_{\delta_i}$. To show that $X \in \mathcal{A}$, i.e. X satisfies clause n for all n, we argue by contradiction. Suppose clause n is the first clause not satisfied by X. Then, X satisfies clause m for all $m < n$. This means that if $\alpha_m \subseteq X$ then there is $\gamma_{X,m} \in I_m$ such that $\gamma_{X,m} \subseteq X$. Also, X satisfies the premiss of clause n (but not its conclusion), i.e. $\alpha_n \subseteq X$. Since α_n and those α_m's, $\gamma_{X,m}$'s included in X (for $m < n$) are finite, they are all included in δ_i for some i. Thus,

— for each $m < n$, clause m is a U_i-clause if and only if it is a U_{i+1}-clause (if and only if $\alpha_m \subseteq X$),
— for each $m < n$, if clause m is a U_i-clause then it is U_i-solved,
— clause n is a U_i-clause and is not U_j-solved for any $j \geqslant i$.

Then, clause n is the first U_i-unsolved U_i-clause and no clause m, $m < n$, can be a U_{i+1}-solved U_{i+1}-clause without being a U_i-clause. As a consequence, the assumed property

$U_i \ll U_{i+1}$ necessarily comes from condition *(ii)* in the definition of $\ll$, i.e. clause n is U_{i+1}-solved. Since $X \in U_{i+1}$, clause n is satisfied by X. This is a contradiction.

Condition (4^+). To show that the U_i's are a basis of neighbourhoods of X in $\mathcal{A}$, it suffices to prove that, for all $\beta \in \mathcal{P}_{<\omega}(\mathbb{N})$ such that $X \in O_\beta$, there exists i such that $\mathcal{A} \cap O_\beta \subseteq \mathcal{A} \cap O_{\delta_i}$. Since $X = \bigcup_{i \in \mathbb{N}} \delta_i \in O_\beta$ we have $\beta \subseteq \delta_i$ for some i hence $O_{\delta_i} \subseteq O_\beta$ and the wanted inclusion $\mathcal{A} \cap O_{\delta_i} \subseteq \mathcal{A} \cap O_\beta$. □

Example 3.6. Applied to the quasi-Polish space $\mathcal{P}_\infty(\mathbb{N})$ and its basis $\{\mathcal{O}_A \cap \mathcal{P}_\infty(\mathbb{N}) \mid A \in \mathcal{P}_{<\omega}(\mathbb{N})\}$ (cf. Examples 2.11 and 2.20), the previous proof gives an approximation relation which is not the containment relation (cf. Example 3.2). Clause n is $\exists j \geqslant n\ j \in X$ (i.e. A_n is empty). It is an $\mathcal{O}_A \cap \mathcal{P}_\infty(\mathbb{N})$-clause and it is $\mathcal{O}_A \cap \mathcal{P}_\infty(\mathbb{N})$-solved if and only if $n \leqslant \max A$. Then, for $A, B \in \mathcal{P}_{<\omega}(\mathbb{N})$, we have $\mathcal{O}_A \cap \mathcal{P}_\infty(\mathbb{N}) \ll \mathcal{O}_B \cap \mathcal{P}_\infty(\mathbb{N})$ if and only if $A \subseteq B$ and $\max(A) < \max(B)$, with the convention: $\max \varnothing = -1$.

The converse of Theorem 3.5 is false.

Proposition 3.7. There exists a Hausdorff convergent approximation space with a countable basis which is not a hereditary approximation space hence is not quasi-Polish.

Proof. Consider the topology τ on the reals generated by the usual open sets and the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers. A topological basis $\mathcal{B}$ is $\{]a,b[,]a,b[\setminus\mathbb{Q} \mid a < b\}$. Consider the relation $\ll$ on $\mathcal{B}$ defined by the following clauses:

$$\begin{array}{rcl}]a,b[\ \ll\ ]c,d[& \iff & a < c < d < b \quad (\text{i.e. } [c,d] \subset]a,b[) \\ ]a,b[\ \ll\ ]c,d[\setminus\mathbb{Q} & \iff & a < c < d < b \\ ]a,b[\setminus\mathbb{Q} \ \ll\ ]c,d[\setminus\mathbb{Q} & \iff & \theta(]a,b[\setminus\mathbb{Q}) \ll_{Baire} \theta(]c,d[\setminus\mathbb{Q}), \end{array}$$

where $d-c \leqslant (b-a)/2$ and θ is the usual homeomorphism (given by continued fractions) between $\mathbb{R} \setminus \mathbb{Q}$ and the Baire space $\mathbb{N}^\omega$ and $\ll_{Baire}$ is any convergent approximation relation on the family of images by θ of bounded open real intervals (cf. Propositions 3.4 infra and 3.3 supra). It is easy to check that $\ll$ makes $(\mathbb{R}, \tau)$ an approximation space. However, Theorem 3.13 infra ensures that $(\mathbb{R}, \tau)$ is not a hereditary approximation space since $\mathbb{Q}$ is closed in $(\mathbb{R}, \tau)$ and, as a subspace, has the usual topology induced by $\mathbb{R}$ hence fails the Baire property. □

Open Problem 3.8. Suppose, a space is (T_0 or T_1) hereditary approximation space and has a countable topological basis. Is it convergent (hence quasi-Polish)?

3.3. *Characterization as spaces of Choquet games*

The next theorem asserts that approximation relations on a topological basis of X are essentially normalized strategies for player Nonempty in the Choquet game $\mathrm{Ch}(X)$.

Theorem 3.9. If X is a topological space with a well-orderable topological basis (in particular, if X has a countable basis) then X is an approximation (respectively convergent approximation) space if and only if player Nonempty has a stationary (respectively and convergent) winning strategy in the Choquet game $\mathrm{Ch}(X)$.

Proof. Suppose $\ll$ is an approximation relation on some topological basis $\mathcal{B}$ of X. Fix some well ordering of $\mathcal{B}$. Define a stationary strategy σ for player Nonempty in the Choquet game $\mathrm{Ch}(X)$ as follows. If U is a non-empty set and $x \in U$ then define $\sigma(x, U) = B$ where, letting C be least in $\mathcal{B}$ such that $x \in C \subseteq U$, applying condition (3) of Definition 3.1, B is least in $\mathcal{B}$ such that $C \ll B$ and $x \in B$. If

$$(x_0, U_0), B_0, (x_1, U_1), B_1, \ldots$$

is a play where Nonempty follows this strategy σ, there are open sets $C_0, C_1, \ldots$ in $\mathcal{B}$ such that

$$U_0 \supseteq C_0 \ll B_0 \supseteq U_1 \supseteq C_1 \ll B_1 \ldots$$

Applying condition (2), we see that $B_0 \ll B_1 \ll \ldots$. Condition (4) of Definition 3.1 ensures that $\bigcap_{i \in \mathbb{N}} U_i = \bigcap_{i \in \mathbb{N}} B_i \neq \emptyset$. Thus, σ is a winning strategy for player Nonempty. In case $\ll$ is convergent, condition (4^+) ensures that σ is convergent.

Conversely, suppose that τ is a winning stationary strategy for Nonempty in the Choquet game $\mathrm{Ch}(X)$. In particular, $x \in \tau(x, U) \subseteq U$ for all open set U and $x \in U$. Fix some basis $\mathcal{B}$ of X and define a relation $\ll$ on this basis as follows: for $B, C \in \mathcal{B}$

$$B \ll C \iff \exists D \in \mathcal{B}\, \exists x \in D\ (B \supseteq D\ \wedge\ x \in C \subseteq \tau(x, D))\ .$$

Let us check the four conditions of Definition 3.1. Condition (1) is trivial since $C \subseteq \tau(x, D) \subseteq D \subseteq B$. Condition (2) is obvious from the definition of $\ll$. As for condition (3), if $x \in B$ then $x \in \tau(x, B)$ so that, for any $C \in \mathcal{B}$ such that $x \in C \subseteq \tau(x, B)$, we have $B \ll C$. We now show condition (4). Suppose that $B_n \ll B_{n+1}$ for all n. Let $D_n \in \mathcal{B}$, $x_n \in D_n$ be such that $B_n \supseteq D_n$ and $x_n \in B_{n+1} \subseteq \tau(x_n, D_n)$. Observe that the infinite sequence

$$(x_0, D_0), \tau(x_0, D_0), (x_1, D_1), \tau(x_1, D_1), (x_2, D_2), \tau(x_2, D_2), \ldots$$

is a legal play of the Choquet game $\mathrm{Ch}(X)$ in which Nonempty follows his winning strategy τ. Since $\tau(x_n, D_n) \supseteq B_{n+1} \supseteq D_{n+1} \supseteq \tau(x_{n+1}, D_{n+1})$, we have $\bigcap_{n \in \mathbb{N}} B_n = \bigcap_{i \in \mathbb{N}} D_n \neq \emptyset$. In case τ is a convergent stationary strategy, it is clear that $\ll$ is a convergent approximation relation. ☐

Remark 3.10. The above proof fails if we consider the Banach–Mazur game in place of the Choquet game: we fail to obtain condition (3) for the relation $\ll$ associated to τ.

3.4. *Quasi-Polish and second countable convergent approximation spaces coincide*

Theorem 3.11. Quasi-Polish spaces coincide with T_0 convergent approximation spaces with a countable basis.

Proof. By Theorem 3.5, quasi-Polish spaces are convergent approximation spaces. If X is a convergent approximation space then, using Theorem 3.9, Nonempty has a convergent winning strategy in the Choquet game $\mathrm{Ch}(X)$. By de Brecht's result (cf. item 3 of Theorem 2.30), if X is also T_0 and has a countable basis then it is quasi-Polish. ☐

Putting together Theorems 3.11, 3.9 and 2.35 we can complement Corollary 2.33.

Corollary 3.12. Let X be a T_0 space with a countable basis. The following conditions are equivalent:

1. Nonempty has a convergent winning strategy in the Choquet game $\mathrm{Ch}(X)$,
2. Nonempty has a stationary convergent winning strategy in the Choquet game $\mathrm{Ch}(X)$,
3. X is a quasi-Polish space.

3.5. *The* $\mathbf{\Pi}^0_2$ *Baire property in approximation spaces*

The Baire property holds in Polish spaces and in compact T_2 (i.e. Hausdorff) spaces, and it is also true in quasi-Polish spaces (de Brecht 2013): the intersection of countably many dense open sets is dense. It trivially implies that the intersection of countably many dense $\mathbf{G}_\delta$ sets is dense. The next result is a formulation of the Baire property for spaces, where the $\mathbf{\Pi}^0_2$ and $\mathbf{G}_\delta$ classes do not coincide. As a corollary, we see that the classical Baire property on ω-continuous domains and also that on quasi-Polish spaces (cf. Theorem 2.25) can be strengthened from $\mathbf{G}_\delta$ to $\mathbf{\Pi}^0_2$.

Theorem 3.13. If X is an approximation space with a well-orderable topological basis (in particular, if X has a countable basis) then it satisfies the Baire property for $\mathbf{\Pi}^0_2(E)$ sets: *the intersection of countably many dense* $\mathbf{\Pi}^0_2(E)$ *sets is dense.*

Proof. Using Lemma 3.3, consider a well-orderable basis $\mathcal{B}$ and an approximation relation $\ll$ on $\mathcal{B}$. We fix some well order on $\mathcal{B}$, and we will speak freely of the least U in $\mathcal{B}$ satisfying a given property. Since $\mathbf{\Pi}^0_2(E)$ sets are intersections of countably many sets in *co*-$\mathbf{D}_2(E)$ (i.e. unions of an open and a closed sets), and any superset of a dense set is dense, it suffices to prove the Baire property for *co*-$\mathbf{D}_2$ sets. Suppose $X = \bigcap_{n\in\mathbb{N}} U_n \cup F_n$ where, for each n, U_n is open, F_n is closed and $U_n \cup F_n$ is dense. We prove that X is dense, i.e. X meets O for each $O \in \mathcal{B}$. Fix some $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi^{-1}(j)$ is infinite for each $j \in \mathbb{N}$. Fix some $O \in \mathcal{B}$. We inductively define a sequence of basic open sets $(O_n)_{n\in\mathbb{N}}$ such that $O \ll O_0$ and $O_n \ll O_{n+1}$ for all n.

a. Let O_0 be the least $W \in \mathcal{B}$ such that $O \ll W$ (there exists such a W : take any element x in O and apply condition (3) of Definition 3.1).

b. Suppose $n \geqslant 1$, O_{n-1} is defined and let $W \in \mathcal{B}$ be least such that $O_{n-1} \ll W$ (as above there exists such a W). If $U_{\pi(n)}$ contains some W^* such that $W \ll W^*$ then set $O_n = W^*$. Otherwise, set $O_n = W$. The following Claim finishes the proof of the Theorem.

Claim. The set $\bigcap_{n\in\mathbb{N}} O_n$ is non-empty and is included in $O \cap \left(\cap_{j\in\mathbb{N}} U_j \cup F_j\right)$.

Proof of Claim. Since $\ll$ is transitive, $O_n \ll O_{n+1}$ holds for all n's and the non-emptiness of $\bigcap_{n\in\mathbb{N}} O_n$ is ensured by condition (4) of Definition 3.1. Let $x \in \bigcap_{n\in\mathbb{N}} O_n$. By clause (a) we have $O \ll O_0$. Thus, $O_0 \subseteq O$ and $x \in O$. Let $j \in \mathbb{N}$. Suppose $x \notin U_j$, we show that $x \in F_j$. Since $x \notin U_j$ we have $O_n \nsubseteq U_j$ for all $n \in \mathbb{N}$. Let n be such that $\pi(n) = j$, clause (b) ensures that $O_n = W$ (instead of W^*) and

(*) U_j contains no $V \in \mathcal{B}$ such that $O_n \ll V$.

Since F_j is closed, to show that $x \in F_j$, it suffices to prove that x is adherent to F_j. Consider $T \in \mathcal{B}$ such that $x \in T$, we have to prove that T meets F_j. Let n be such that

$\pi(n) = j$. Since $U_j \cup F_j$ is dense and $T \cap O_n$ is non-empty (it contains x), there exists $x_n \in (U_j \cup F_j) \cap T \cap O_n$. By way of contradiction, suppose x_n is in U_j. Then let $S \in \mathcal{B}$ be such that $x_n \in S \subseteq U_j \cap T \cap O_n$. Condition (3) yields some $V \in \mathcal{B}$ such that $x_n \in V$ and $S \ll V$. Using condition (2) and inclusion $S \subseteq O_n$ we have $O_n \ll V$. Now, we also have $V \subseteq S \subseteq U_j$, contradicting (*). Thus, $x_n \in F_j$; hence, T meets F_j. $\square$

Remark 3.14. The fact that every continuous domain D satisfies the $\mathbf{\Pi}^0_2(D)$ Baire property questions on a possible relation with the Lawson topology. Recall that the Lawson topology on a continuous domain is the refinement of the Scott topology such that the complement of the uppercone set $\uparrow x$ is open, for each x in D. The Lawson topology is T_2 and compact, so it satisfies the usual Baire property with $\mathbf{G}_\delta$ sets. Also, the Scott $\mathbf{\Pi}^0_2$ class is included in the Lawson class $\mathbf{G}_\delta$. However, the $\mathbf{\Pi}^0_2$ Baire property for the Scott topology differs from the usual $\mathbf{G}_\delta$ Baire property for the Lawson topology because the notion of dense set is not the same in the two topologies.

3.6. *Hausdorff's theorem and the $\mathbf{\Pi}^0_2$ Baire property*

Hausdorff–Kuratowski's theorem establishes that for Polish spaces the inclusion in Proposition 2.9 is an equality: $\bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(\mathbf{\Sigma}^0_\beta(E)) = \mathbf{\Delta}^0_{\beta+1}(E)$. One of the properties of Polish spaces used in the proof of this result is the classical Baire property. De Brecht (2013) has proved that Hausdorff–Kuratowski's theorem holds for quasi-Polish spaces. Our next theorem gives conditions of approximation spaces to establish the case $\beta = 1$ i.e. Hausdorff's theorem $\mathbf{\Delta}^0_2(E) = \bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(E)$. The theorem pinpoints some topological properties that suffice to prove this equality. These properties are true in second countable hereditary approximation spaces. A similar result was obtained in Tang (1981) for the Scott domain $\mathcal{P}(\mathbb{N})$. Using a different proof Selivanov (2005) proved it for ω-algebraic domains.

Theorem 3.15. Let E be a topological space satisfying the following properties.

i. There exists a countable basis of open sets.
ii. Every closed subspace F of E satisfies the $\mathbf{\Pi}^0_2(E)$-Baire property: the intersection of countably many $\mathbf{\Pi}^0_2(E)$ subsets of F which are dense in F is also dense in F.

Then, $\bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(E) = \mathbf{\Delta}^0_2(E)$.

Proof. The $\subseteq$ inclusion is Proposition 2.9. For the $\supseteq$ inclusion, we follow, mutatis mutandis, Hausdorff's original proof with residues and adjoins, as exposed in Kechris (1995), Theorem 22.27 pages 176–177. For any subset $A \subseteq E$, we define by transfinite recursion a family $(F_\alpha)_{\alpha<\omega_1}$:

$$F_0 = E,\ F_{2\alpha+1} = \overline{A \cap F_{2\alpha}}\ ,\ F_{2\alpha+2} = \overline{(E \setminus A) \cap F_{2\alpha+1}}\ ,\ F_\lambda = \bigcap_{\alpha<\lambda} F_\alpha \quad \textit{if } \lambda \textit{ is limit.}$$

Applying property (i), let $(O_n)_{n\in\mathbb{N}}$ be a countable basis of open sets of E. The F_α's are a decreasing sequence of closed sets. If $F_{\xi+1} = F_\xi$ then $F_\alpha = F_\xi$ for all $\alpha \geqslant \xi$. If $F_{\alpha+1}$ is a strict subset of F_α then there is some O_n which meets F_α but not $F_{\alpha+1}$. Since there are countably many O_n's, this implies that there is some countable θ such that $F_\theta = F_\alpha$ for all $\alpha \geqslant \theta$. We shall consider the least even such θ.

Claim. If A is $\mathbf{\Delta}^0_2(E)$ then $F_\theta = \varnothing$.

Proof of Claim. Suppose $F_\theta \neq \varnothing$. Applying property (ii), the subspace F_θ is in E hence it satisfies the Baire property. Since $F_\theta = F_{\theta+2}$, we have $F_\theta = \overline{A \cap F_\theta} = \overline{(E \setminus A) \cap F_\theta}$. Thus, arguing in the subspace F_θ, the sets $A \cap F_\theta$ and $(E \setminus A) \cap F_\theta$ are $\mathbf{\Pi}^0_2(F_\theta)$ dense subsets of F_θ. Since they are disjoint, this contradicts property (ii) in F_θ.

Letting $\theta = 2\zeta$ and $B = \bigcup_{\alpha<\zeta} F_{2\alpha+1} \setminus F_{2\alpha+2}$, we claim that $A = B$. Indeed, suppose $x \in A$ and let η be least such that $x \notin F_\eta$. The inductive definition of the F_α's ensures that $\eta = 2\alpha + 2$ for some α. Therefore, $x \in F_{2\alpha+1} \setminus F_{2\alpha+2}$ hence $x \in B$. Similarly, if $x \notin A$ and η is least such that $x \notin F_\eta$ then $\eta = 2\alpha + 1$ for some α. Thus, $x \in F_{2\alpha} \setminus F_{2\alpha+1}$ hence $x \notin B$. Observe that, for λ limit, we have $(E \setminus F_\lambda) \setminus \bigcup_{\alpha<\lambda}(E \setminus F_\alpha) = \varnothing$. Thus, $A = B = \bigcup_{\alpha<\zeta}(E \setminus F_{2\alpha+2}) \setminus (E \setminus F_{2\alpha+1}) = D_{\theta+1}((E \setminus F_\alpha)_{\alpha\leqslant\theta})$ hence A is in $\mathbf{D}_{\theta+1}(E)$. ☐

Corollary 3.16. Equality $\bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(E) = \mathbf{\Delta}^0_2(E)$ holds in any hereditary approximation space having a countable basis.

Open Problem 3.17. What topological conditions ensure higher levels of Hausdorff–Kuratowski's theorem?

4. The Hausdorff hierarchy in continuous domains

Selivanov made a fine analysis of the Hausdorff difference hierarchy in ω-algebraic domains and he proved Hausdorff's theorem for these spaces (Selivanov 2005). He also showed the non-existence of ambiguous sets in the Hausdorff hierarchy, provided there is a least element in the ω-algebraic domain. As stated in Selivanov (2008, Theorem 3.4), the methods in his paper of 2005 can be pushed from ω-algebraic to ω-continuous domains.

In this section, we reconsider the question of the non-existence of ambiguous sets in the Hausdorff hierarchy for continuous domains (possibly not ω-continuous). We prove that the hypothesis of a least element considered by Selivanov can be removed for successor levels of the Hausdorff hierarchy, but not for limit levels. Although this improvement is a modest addition to Selivanov's result (proofs being, mutatis mutandis, the same), it requires to use the machinery of well-founded alternating trees developed in Selivanov (2005). The extension from ω-continuous domains to continuous domains led us to consider possibly non-countable alternating trees. In fact, the countability of the domain basis is useful only to ensure that the tree is countable, hence the rank of an alternating tree is countable. But this hypothesis appears directly as an assumption on the ordinal α in Theorems 4.7 and 4.8.

4.1. *Alternating trees*

First, we recall some simple notions about possibly uncountable trees.

Definition 4.1.

1. A *tree* is any non-empty set of finite sequences closed under prefix. The root of a tree is the empty sequence *nil*. If σ is in a tree T we let T_σ be the tree $\{\tau \mid \sigma\tau \in T\}$.
2. A tree T is *well founded* if it has no infinite branch. The ranks of the elements in a well-founded tree T are defined inductively: $rank_T(\sigma) = \sup\{rank_T(\sigma n) + 1 \mid \sigma n \in T\}$ (convention for leaves of T : $\sup \varnothing = 0$). The rank of T is that of its root.
3. An *alternating tree* is a map $f : T \to \{0,1\}$ such that T is a tree and $f(\tau) \neq f(\sigma)$ whenever σ is a son of τ (i.e. τ is a prefix of σ and the length of σ is the successor of that of τ). We say f is ε-alternating if $f(nil) = \varepsilon$.
4. An *embedding* of an alternating tree $g : S \to \{0,1\}$ into an alternating tree $f : T \to \{0,1\}$ is a monotone increasing (with respect to the prefix ordering on finite sequences) injective map $\theta : S \to T$ such that $g(\sigma) = f(\theta(\sigma))$. We write $g \leqslant f$.

The next basic result is taken from Selivanov (2005), Lemma 3.6 page 48.

Lemma 4.2. Let $f : T \to \{0,1\}$ be a well-founded alternating tree with rank α. For every $\beta < \alpha$ and every $\varepsilon \in \{0,1\}$, there exists an ε-alternating tree $g_\varepsilon \leqslant f$ with rank β.

Proof. We reproduce Selivanov's proof. Argue by induction on α. If α is finite then take a branch of length α and remove an appropriate tail and/or the root. If α is limit then, for some x, $T_{(x)}$ (cf. Definition 4.1) has rank γ such that $\beta < \gamma < \alpha$. Use the induction hypothesis with $f_{(x)} : T_{(x)} \to \{0,1\}$ such that $f_{(x)}(\sigma) = f(x\sigma)$. Finally, suppose $\alpha = \lambda + m + 1$ with λ limit and $m \in \mathbb{N}$. If $\beta < \lambda + m$ then let x such that $T_{(x)}$ has rank $\lambda + m$ and use the induction hypothesis. Suppose now $\beta = \lambda + m$. Let s with length $m+1$ such that T_s has rank λ. Let η be the cofinality of λ and let $(\lambda_\xi)_{\xi<\eta}$ be strictly increasing with supremum λ and x_ξ be such that T_{sx_ξ} has rank λ_ξ. Let $g_{\xi,\delta} \leqslant f_{sx_{\xi+1}}$ for $\delta \in \{0,1\}$ be δ-alternating with rank λ_ξ. Set $g_\varepsilon(0^p) = f(s \restriction p)$ for $p \leqslant m$ and $g_\varepsilon(0^m\xi\sigma) = g_{\xi,\delta}(\sigma)$ where δ is chosen so as to get alternation from 0^m to $0^m\xi$. ☐

4.2. *Alternating trees and the Hausdorff hierarchy*

The following notion plays the role of intervals $[p,x]$ in Selivanov (2005).

Definition 4.3. Let b, x be elements of the domain D. We let $[\![b,x]\!] = \{y \mid b \ll y \leqslant x\}$.

Proposition 4.4. Let $(D, \sqsubseteq)$ be a dcpo and $A \in \mathbf{\Sigma}^0_2(D)$. If $(x_i)_{i\in I}$ is a directed system with supremum in A then $x_i \in A$ for all i large enough. In particular, if D is a continuous domain with basis $B \subseteq D$ and $x \in A$ then there exists $b \in B$ such that $b \ll x$ and $[\![b,x]\!] \subseteq A$.

Proof. Let $A = \bigcup_{n\in\mathbb{N}} U_n \setminus V_n$, where U_n, V_n are open. Let $x = \sqcup_i x_i$. Since $x \in U_n$ and U_n is Scott open, the x_i's are in U_n for i large enough. Since $x \notin V_n$ and V_n is an upset, no x_i is in V_n. Thus, the x_i's are in $U_n \setminus V_n$ hence in A for i large enough. ☐

To keep the presentation self-contained, we prove the extensions to continuous domains of Theorems 3.10 and 3.14 in Selivanov (2005). Proofs are almost the same: compact elements are replaced by elements of some fixed basis B and the countability of the basis (which ensures the countability of the rank of alternating B-trees, cf. Definition 4.5) is replaced by the assumed countability of the considered ordinal α.

Definition 4.5. Let $(D, \sqsubseteq)$ be a dcpo and $B \subseteq D$. If A is any subset of D we let $\chi_A : A \to \{0,1\}$ be the characteristic function of A (which takes value 1 on A).

1. A B-tree is a map $f : T \to B$ where T is a tree.
2. A B-tree $f : T \to B$ is (A, ε)-alternating if the tree $\chi_A \circ f : T \to \{0,1\}$ is ε-alternating in the sense of Definition 4.1, i.e. $f(\sigma n) \in A \Leftrightarrow f(\sigma) \notin A$ for all $n \in \mathbb{N}$ and $\sigma, \sigma n \in T$.
3. Let $\preceq$ be $\sqsubseteq$ or $\ll$. A B-tree f is $\preceq$-increasing if $f(\sigma) \preceq f(\sigma n)$ for all $\sigma, \sigma n \in T$.

Proposition 4.6. Let $(D, \sqsubseteq)$ be a dcpo and $B \subseteq D$. If A is $\mathbf{\Delta}^0_2(D)$ then every A-alternating $\sqsubseteq$-increasing B-tree $f : T \to B$ is well founded.

Proof. Let $(n_i)_{i \in \mathbb{N}}$ be an infinite branch of f. The $f(n_0 \ldots n_i)$'s are $\sqsubseteq$-increasing. Let x be their supremum. Suppose $x \in A$. Proposition 4.4 establishes that $f(n_0 \ldots n_i)$ is in A for all i large enough, contradicting alternation of f. Idem if $x \notin A$. ☐

The next result is a slight variant of Selivanov (2005, Proposition 3.8 and Theorem 3.10).

Theorem 4.7. Let $(D, \sqsubseteq)$ be a dcpo, $B \subseteq D$, $A \in \mathbf{\Delta}^0_2(D)$ and $0 < \alpha < \omega_1$.

1. If $A \in \mathbf{D}_\alpha(D)$ then there is no $\ll$-increasing $(A, 1)$-alternating B-tree with rank α.
2. If D is a continuous domain with basis B, the following conditions are equivalent.
   i. $A \in \mathbf{D}_\alpha(D)$.
   ii. There is no $\sqsubseteq$-increasing $(A, 1)$-alternating B-tree with rank α.
   iii. There is no $\ll$-increasing $(A, 1)$-alternating B-tree with rank α.

Proof.

1. We argue by induction on α. Let $A = D_\alpha((A_\beta)_{\beta<\alpha})$ where the A_β's are a monotone increasing α-sequence of open sets. By way of contradiction, suppose there exists an increasing $(A, 1)$-alternating B-tree $f : T \to B$ with rank α. Since $f(nil) \in A$ we have $f(nil) \in A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma$ for some $\beta < \alpha$, $\beta \not\sim \alpha$. Since $f(nil) \ll f(\sigma)$ for all $\sigma \in T$ and A_β is open (hence, an upset), we see that the range of f is included in A_β. If $(n) \in T$ then $f((n)) \notin A$ hence $f((n)) \in A_\beta \setminus A \subseteq \bigcup_{\gamma<\beta} A_\gamma$. Since $\bigcup_{\gamma<\beta} A_\gamma$ is open (hence, an upset) we see that $f(\sigma) \in \bigcup_{\gamma<\beta} A_\gamma$ for all $\sigma \in T$, $\sigma \neq nil$. Let $A^- = D_\alpha((A_\gamma)_{\gamma<\beta})$. Then, f is an $(A^-, 0)$-alternating B-tree. Now, f has rank α and Lemma 4.2 implies that there exists $g \ll f$ which is an $(A^-, 1)$-alternating B-tree with rank β. Since A^- is in $\mathbf{D}_\beta$, the inductive hypothesis is contradicted.
2. Since $i \Rightarrow ii$ is item 1 and $ii \Rightarrow iii$ is trivial, it remains to prove $iii \Rightarrow i$. For each $b \in B$, let S_b be the family of finite sequences $(b, b_1, \ldots, b_k)$ of elements of B satisfying the following conditions:

— $b \ll b_1 \ll \cdots \ll b_k$,

— $b_i \in A \Leftrightarrow b_{i+1} \notin A$ for all $0 \leqslant i < k$ (with $b_0 = b$).

Fix some bijection θ between B and an initial segment of $\mathbb{N}$. Applying θ, transform S_b into an A-alternating B-tree $f_b : T_b \to B$ such that

$$T_b = \{nil\} \cup \{(\theta(b_1), \dots, \theta(b_k)) \mid (b, b_1, \dots, b_k) \in S_b\}$$

and $f_b(nil) = b$, $f_b((\theta(b_1), \dots, \theta(b_k))) = b_k$. Since A is assumed to be $\mathbf{\Delta}^0_2(D)$, Proposition 4.6 ensures that f_b is well founded. Suppose there is no $(A, 1)$-alternating B-tree with rank α. Then, f_b has rank $\leqslant \alpha$ if $b \notin A$ and rank $< \alpha$ if $b \in A$. For $\beta < \alpha$ define the open sets

$$A_\beta = \bigcup \{\Uparrow f_b(\sigma) \mid b \in B,\ rank_{T_b}(\sigma) \leqslant \beta,\ f_b(\sigma) \in A \Leftrightarrow \beta \not\sim \alpha\}.$$

To conclude we prove that $A = \mathbf{D}_\alpha((A_\beta)_{\beta<\alpha})$. First, we show that $A \subseteq \bigcup_{\beta<\alpha} A_\beta$. Suppose $x \in A$. Applying Proposition 4.4, we get $[\![b, x]\!] \subseteq A$ for some $b \ll x$, $b \in B$. Since $b \in A$, we have $rank(T_b) < \alpha$. *A fortiori*, $rank(T_b) \leqslant \beta$ with $\beta < \alpha$ and $\beta \not\sim \alpha$. Since $f_b(nil) = b$ we have $\Uparrow b \subseteq A_\beta$ hence $x \in A_\beta$. For $\beta < \alpha$, let $A'_\beta = A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma$. Since $A \subseteq \bigcup_{\beta<\alpha} A_\beta$, to show $A = \mathbf{D}_\alpha((A_\beta)_{\beta<\alpha})$ it suffices to prove that $\beta \not\sim \alpha \Rightarrow A'_\beta \subseteq A$ and $\beta \sim \alpha \Rightarrow A'_\beta \subseteq D \setminus A$.

Case $\beta \not\sim \alpha$. By way of contradiction, suppose $A'_\beta \nsubseteq A$ and let $x \in A'_\beta \setminus A$. By Proposition 4.4, we have $[\![c, x]\!] \subseteq D \setminus A$ for some $c \ll x$, $c \in B$. Now, since $x \in A_\beta$, there exist $b \in B$ and $\sigma \in T_b$ such that $x \in \Uparrow f_b(\sigma)$, $rank_{T_b}(\sigma) \leqslant \beta$, $f_b(\sigma) \in A$. Since $c, f_b(\sigma) \ll x$, the interpolation property gives an $e \in B$ such that $c, f_b(\sigma) \ll e \ll x$. Since $e \in [\![c, x]\!]$ we have $e \notin A$. Let $\sigma = (\theta(b_1), \dots, \theta(b_k))$ where $(b, b_1, \dots, b_k) \in S_b$. Since $b_k = f_b(\sigma) \in A$, $f_b(\sigma) \ll e$ and $e \notin A$, the sequence $(b, b_1, \dots, b_k, e) \in S_b$. Hence, $\sigma\theta(e) \in T_b$. Now, $rank(\sigma\theta(e)) < rank(\sigma) \leqslant \beta$. Since $\beta \not\sim \alpha$, there is some $\gamma < \beta$ such that $\gamma \sim \alpha$ and $rank(\sigma\theta(e)) \leqslant \gamma$. Summing up, we have $\Uparrow e \subseteq A_\gamma$. Since $e \ll x$ we get $x \in A_\gamma$ which contradicts $x \in A'_\beta$.

Case $\beta \sim \alpha$. The proof that $A'_\beta \subseteq D \setminus A$ is similar. □

4.3. *Ambiguous sets in the Hausdorff hierarchy*

We now come to the question of whether there are ambiguous sets in the Hausdorff hierarchy. Item 1 of the next Theorem was obtained by Selivanov (2005) for ω-algebraic domains.

Theorem 4.8. Let D be a continuous domain and $0 \leqslant \alpha < \omega_1$.

1. If D has a least element $\bot$ then $\mathbf{D}_\alpha(D) \cap co\text{-}\mathbf{D}_\alpha(D) = \bigcup_{\beta<\alpha} \mathbf{D}_\beta(D) \cup co\text{-}\mathbf{D}_\beta(D)$.
2. $\mathbf{D}_{\alpha+1}(D) \cap co\text{-}\mathbf{D}_{\alpha+1}(D) = \bigcup_{\beta\leqslant\alpha} \mathbf{D}_\beta(D) \cup co\text{-}\mathbf{D}_\beta(D)$ for all $1 \leqslant \alpha < \omega_1$.
3. In general, equality $\mathbf{D}_\alpha(D) \cap co\text{-}\mathbf{D}_\alpha(D) = \bigcup_{\beta<\alpha} \mathbf{D}_\beta(D) \cup co\text{-}\mathbf{D}_\beta(D)$ fails for $\alpha = 1$ and for α limit.

Proof.

1. Inclusion right to left comes from Proposition 2.6. Inclusion left to right is proved by induction on α. Case $\alpha = 0$ is trivial since both members are empty.
   Suppose, $\alpha = \beta + 1$ and $A \in \mathbf{D}_\alpha(D) \cap co\text{-}\mathbf{D}_\alpha(D)$. Towards a contradiction, suppose $A \notin \mathbf{D}_\beta(D) \cup co\text{-}\mathbf{D}_\beta(D)$, i.e. neither A nor $D \setminus A$ is in $\notin \mathbf{D}_\beta(D)$. Theorem 4.7 proves

the existence, for $\varepsilon = 0, 1$, of an increasing (A, ε)-alternating B-tree $f_\varepsilon : T_\varepsilon \to B$ with rank β. Observe that every domain basis contains $\perp$, hence $\perp \in B$. Let $\tau = 0$ if $\perp \in A$ and $\tau = 1$ otherwise, so that $f_\tau(nil) \neq \perp$. Let $S = \{0\sigma \mid \sigma \in T_\tau\}$ and $g : S \to B$ be such that $g(nil) = \perp$ and $g(0\sigma) = f_\varepsilon(\sigma)$. Then g is an increasing $(A, 1-\tau)$-alternating B-tree with rank $\beta + 1 = \alpha$. Applying again Theorem 4.7, if $\tau = 0$ this contradicts $A \in \mathbf{D}_\alpha(D)$, if $\tau = 1$ this contradicts $A \in co\text{-}\mathbf{D}_\alpha(D)$.

Suppose, now α is a limit ordinal and $(\alpha_n)_{n\in\mathbb{N}}$ is increasing with α as supremum. Towards a contradiction, suppose $A \notin \bigcup_{\beta<\alpha} \mathbf{D}_\beta(D) \cup co\text{-}\mathbf{D}_\beta(D)$. Then, for every n and $\varepsilon = 0, 1$, there is a monotone increasing (A, ε)-alternating B-tree $f_{n,\varepsilon} : T_{n,\varepsilon} \to B$ with rank α_n. Let $\tau \in \{0, 1\}$ be as above. Set $S = \{n\sigma \mid \sigma \in T_{n,\varepsilon}\}$ and $g : S \to B$ be such that $g(nil) = \perp$ and $g(n\sigma) = f_{n,\varepsilon}(\sigma)$. Then g is a monotone increasing (A, τ)-alternating B-tree with rank α. As above, this gives a contradiction.

2. Let $A = D_{\alpha+1}((A_\beta)_{\beta\leqslant\alpha})$ and $D \setminus A = D_{\alpha+1}((E_\beta)_{\beta\leqslant\alpha})$, where the A_β, E_β's are $\alpha + 1$ increasing sequences of open sets. Set $D^+ = D \cup \{\perp_\alpha\}$ and extend the order of D by setting $\perp_\alpha < x$ for all $x \in A_\alpha$. Set $A^+ = A \cup \{\perp_\alpha\}$. Then, in D^+, we have $A^+ = D_{\alpha+1}((A^*_\beta)_{\beta\leqslant\alpha})$ where $A^*_\alpha = A_\alpha \cup \{\perp_\alpha\}$ and $A^*_\beta = A_\beta$ for $\beta < \alpha$. Also, $D^+ \setminus A^+ = D \setminus A$. Observe that the A^*_β's and E_β's are open in D^+. Thus, A^+ is ambiguous at level $\alpha+1$ in D^+. Though $\perp_\alpha$ is not a least element in D^+, it is smaller than A_α hence smaller than all elements labelling an $(A, 1)$-alternating B-tree with rank α. Arguing as in item 1, we see that A^+ must have level at most α in D^+. If $D^+ \setminus A^+ = D \setminus A$ has level α in D^+ then $D \setminus A$ is obtained via open sets not containing $\perp_\alpha$ hence $D \setminus A$ has level α in D. Suppose, now that A^+ has level α in D^+. Then, $A^+ = D_\alpha((C_\delta)_{\delta<\alpha})$ where the C_δ's are open in D^+. Since $\perp_\alpha \in A^+$ there is some $\beta < \alpha$ with parity different from α such that $\perp_\alpha \in C_\beta$ hence $C_\beta \supset \bigcup_{\delta<\alpha} A_\delta$. This yields $A = D_\alpha((C^-_\gamma)_{\gamma<\alpha})$ where $C^-_\gamma = C_\gamma$ for $\gamma < \beta$ and $C^-_\gamma = C_\gamma \cap D$ for $\gamma \geqslant \beta$. So, A has level β in D. Finally, observe that the argument breaks down if $\alpha = 0$.
3. In $\mathbf{2}^{\leqslant\omega}$, the set $0\mathbf{2}^{\leqslant\omega}$ is in $\mathbf{D}_1(D) \cap co\text{-}\mathbf{D}_1(D)$ but not in $\mathbf{D}_0(D) \cap co\text{-}\mathbf{D}_0(D)$. Let α be a countable limit ordinal. Let $(a_\beta)_{\beta<\alpha}$ be a strictly decreasing α-sequence of reals with no lower bound and which is continuous: $a_\lambda = \inf_{\delta<\lambda} a_\delta$ for all limit $\lambda < \alpha$. In $\overrightarrow{\mathbb{R}}$ (cf. Example 2.11), consider the set $A = \mathbf{D}_\alpha((A_\beta)_{\beta<\alpha})$ where $A_\beta =]a_\beta, +\infty]$. Then, $\overrightarrow{\mathbb{R}} \setminus A = \mathbf{D}_\alpha((A^*_\beta)_{\beta<\alpha})$ where $A^*_0 = \varnothing$ and $A^*_{\beta+1} = A_\beta$ and $A^*_\lambda = A_\lambda$ for λ limit. Thus, A is ambiguous at level α. It is easy to check that A is not in a lesser level.

□

5. Effective Borel and Hausdorff hierarchies

5.1. *Effective topological spaces*

The first step to deal with the effective Borel hierarchy is the definition of effective topological space. We follow the book of Weihrauch (2000), Definition 3.2.1, page 63.

Definition 5.1. An *effective topological space* is a pair $(E, (O_n)_{n\in\mathbb{N}})$, where E is a topological space admitting a countable basis and $(O_n)_{n\in\mathbb{N}}$ is an enumeration (not necessarily injective) of some topological basis of E.

Remark 5.2. For the notion of *computable topological space*, one also requires that the equivalence relation $\{(m, n) \mid O_m = O_n\}$ be computably enumerable. For instance, this is the case if $n \mapsto O_n$ is injective, which is usually true. We shall not need this notion.

Definition 5.3. An effective approximation space is a triple $(E, (O_n)_{n\in\mathbb{N}}, \ll)$ such that $(O_n)_{n\in\mathbb{N}}$ enumerates a topological basis $\mathcal{B}$, the relation $\{(i, j) \mid O_i \ll O_j\}$ is computably enumerable and $\ll$ is an approximation relation on $\mathcal{B}$ (cf. Definition 3.1).

5.2. *Effective ω-continuous domains*

Definition 5.4. An ω-continuous domain is *effective* if it admits a basis $B = \{b_n \mid n \in \mathbb{N}\}$ such that $\{(i, j) \mid b_j \ll b_i\}$ is computably enumerable.

Example 5.5. $(\mathcal{P}_\infty(\mathbb{N}), \subseteq)$ is not a continuous domain (cf. Example 3.6) but it is an effective approximation space. Other spaces in Example 2.11 are effective ω-continuous domains.

Proposition 5.6. Every effective ω-continuous domain is an effective approximation space (hence, an effective topological space).

Proof. Immediate from the proof of Proposition 3.4. ☐

5.3. *Borel codes*

There are several ways to code Borel sets by elements in the Baire space $\mathbb{N}^\omega$, cf. Moschovakis (1979/2009) Sections 3H and 7B and Marker (2002) Section 7. We choose a coding adapted to the context of effective topological spaces.

Definition 5.7. Let $(E, (O_n)_{n\in\mathbb{N}})$ be an effective topological space. We code Borel sets by well-founded trees (cf. Definition 4.1). To any σ in a well-founded tree T we attach a Borel subset $[\![\sigma]\!]$ of E by induction on the rank.

i. If $rank_T(nil) = 0$ (i.e. the tree T is reduced to its root) then $[\![nil]\!]_T = \emptyset$.
ii. If $rank_T(\sigma) = 0$ and $\sigma \neq nil$ and σ has last element n then $[\![\sigma]\!]_T = O_n$.
iii. If $rank_T(\sigma) = 1$ then $[\![\sigma]\!]_T = \bigcup_{n\in\mathbb{N}:\sigma n\in T} O_n$.
iv. If $rank_T(\sigma) \geqslant 2$ then $[\![\sigma]\!]_T = \bigcup_{n\in\mathbb{N}:\sigma\frown 2n,\sigma\frown 2n+1\in T} [\![\sigma\frown 2n]\!]_T \setminus [\![\sigma\frown 2n+1]\!]_T$.

The Σ-Borel and Π-Borel sets coded by T are $[\![T]\!]_\Sigma = [\![nil]\!]_T$ and $[\![T]\!]_\Pi = E \setminus [\![nil]\!]_T$.

Observing that the above inductive definition of $[\![\sigma]\!]_T$ follows exactly that of the Borel hierarchy, a straightforward induction on the ordinal α shows the following result.

Proposition 5.8. Let $(E, (O_n)_{n\in\mathbb{N}})$ be an effective topological space. A subset of E is in $\mathbf{\Sigma}^0_\alpha(E)$ (respectively $\mathbf{\Pi}^0_\alpha(E)$) if and only if it is of the form $[\![T]\!]_\Sigma$ (respectively $[\![T]\!]_\Pi$) for some well-founded tree with rank at most α.

5.4. *The effective Borel hierarchy*

Borel codes lead to a definition of the effective Borel hierarchy. First, we recall some classical results about computable ordinals which imply that there is a huge latitude to represent them: from computability with very low resource complexity up to hyperarithmeticity.

Proposition 5.9. There exists an ordinal ω_1^{CK} (the Church–Kleene ordinal) such that, for every countable ordinal α, the following properties are equivalent.

i. $\alpha < \omega_1^{\mathrm{CK}}$,
ii. α is the rank of some computable well-founded tree,
iii. α is the rank of some hyperarithmetical (i.e. Δ_1^1) well-founded tree (Spector 1955),
iv. α is the order type of some computable linear order on $\mathbb{N}$, i.e. α is computable,
v. α is the order type of some hyperarithmetical linear order on $\mathbb{N}$, i.e. α is Δ_1^1,
vi. α is the order type of some linear order on $\mathbb{N}$ which is computable in real time and logarithmic space (Dehornoy 1986; Grigorieff 1990).

Moreover, in (iv)–(vi), one can suppose that, for each $n \in \mathbb{N}$, the set of elements with rank exactly n and that with ranks in $\{\omega\alpha + n \mid \alpha < \omega_1^{\mathrm{CK}}\}$ are computable.

Remark 5.10. The last assertion in Proposition 5.9 is a simple trick in Ershov (1968). If $(\mathbb{N}, R)$ has type α then the lexicographic product of $(\mathbb{N}, R)$ and $(\mathbb{N}, <)$ has type $\omega\alpha$ and the (computable) set $\mathbb{N} \times \{n\}$ consists of all elements of rank $\equiv n \bmod \omega$.

The definition of the effective version of Borel hierarchy for countably based spaces, and the basic properties appeared previously in Selivanov (2008).

Definition 5.11 (effective Borel hierarchy). Let $(E, (O_n)_{n\in\mathbb{N}})$ be an effective topological space and α an ordinal such that $1 \leqslant \alpha < \omega_1^{\mathrm{CK}}$.

1. The *effective Borel classes* $\Sigma_\alpha^0(E)$, $\Pi_\alpha^0(E)$, $\Delta_\alpha^0(E)$, are defined as follows:
   — A set is in the class $\Sigma_\alpha^0(E)$ (respectively $\Pi_\alpha^0(E)$) if and only if it is of the form $[\![T]\!]_\Sigma$ (respectively $[\![T]\!]_\Pi$) for some well-founded tree T with rank at most α such that both T and the rank order relation on T (i.e. $\{(s,t) \in T \times T \mid rank_T(s) \leqslant rank_T(t)\}$) are computable.
   — The class $\Delta_\alpha^0(E) = \Sigma_\alpha^0(E) \cap \Pi_\alpha^0(E)$.
2. A sequence $(X_n)_{n\in\mathbb{N}}$ of subsets of E is uniformly $\Sigma_\alpha^0(E)$ if there exists a computable sequence of well-founded trees $(T_n)_{n\in\mathbb{N}}$ with ranks at most α such that $X_n = [\![T_n]\!]_\Sigma$. Idem with uniformly $\Pi_\alpha^0(E)$.
3. G_δ and F_σ are the classes of intersections of uniformly $\Sigma_1^0(E)$ sequences (respectively unions of uniformly $\Pi_1^0(E)$ sequences).

Remark 5.12.

1. If $\preceq$ is an order on $\mathbb{N}$ isomorphic to the ordinal α then the set of sequences $(n_1, \ldots, n_k)$ such that $n_1 \succeq n_2 \ldots \succeq n_k$ is a well-founded tree with rank α and computable rank order relation (i.e. it satisfies the above condition (a)).

2. The requirement that the rank order relation on T is computable (in condition 1) allows to get the usual definition of $\Sigma^0_\alpha(E)$ for finite α's. For instance, a subset X of E is $\Sigma^0_2(E)$ (respectively $\Sigma^0_3(E)$) if and only if there exists computably enumerable sets $A, B \subseteq \mathbb{N}^2$ (respectively $A, B \subseteq \mathbb{N}^3$) such that $X = \cup_{i\in\mathbb{N}} \left(\cup_{j:(i,j)\in A} O_j\right) \setminus \left(\cup_{j:(i,j)\in B} O_j\right)$ (respectively, $X = \cup_{i\in\mathbb{N}} \cap_{j\in\mathbb{N}} \left(E \setminus \cup_{k:(i,j,k)\in A} O_k\right) \cup \left(\cup_{k:(i,j,k)\in B} O_k\right)$).

Proposition 5.13. All assertions in Proposition 2.2 hold for the effective Borel classes with the proviso that $1 \leqslant \alpha < \omega_1^{\mathrm{CK}}$ and countable unions (respectively intersections) are relative to uniformly $\Sigma^0_\alpha(E)$ (respectively $\Pi^0_\alpha(E)$) sequences of sets.

5.5. *Hausdorff codes and the effective Hausdorff hierarchy*

The definition of the effective difference hierarchy for countably based spaces and their basic properties appeared previously in Selivanov (2008).

Definition 5.14. Let $(E, (O_n)_{n\in\mathbb{N}})$ be an effective topological space.

1. A Hausdorff α-code for a set X in $\mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\beta(E))$ is a triple $(\preceq, P, (T_n)_{n\in\mathbb{N}})$ such that
   - — $\preceq$ is a well order of type α on $\mathbb{N}$ or on a finite initial segment of $\mathbb{N}$,
   - — $P = \{n \in dom(\preceq) \mid \varphi(n) \sim \alpha\}$, where φ is the unique isomorphism from $(dom(\preceq), \preceq)$ onto the ordinal α,
   - — $(T_n)_{n\in\mathbb{N}}$ is a family of well-founded trees with ranks at most β,
   - — $X = \cup_{p\in P} [\![T_p]\!]_\Sigma \setminus \cup_{\varphi(q)<\varphi(p)} [\![T_q]\!]_\Sigma$.
2. The effective Hausdorff classes $\mathrm{D}_\alpha(\Sigma^0_\beta(E))$ are defined as follows: for $1 \leqslant \alpha < \omega_1^{\mathrm{CK}}$, a set X is in $\mathrm{D}_\alpha(\Sigma^0_\beta(E))$ if and only if it admits an α-code $(\preceq, P, (T_n)_{n\in\mathbb{N}})$ such that $\preceq$ and P are computable and the T_n's and the rank relations on the T_n's are uniformly computable, (i.e. $\{(n, s) \mid s \in T_n\}$ and $\{(n, s, t) \mid s, t \in T_n, rank_{T_n}(s) \leqslant rank_{T_n}(t)\}$ are computable).

The effective class $\mathrm{D}_\alpha(\Sigma^0_1(E))$ is also denoted by $\mathrm{D}_\alpha(E)$.

Proposition 5.15. Propositions 2.6, 2.9 and 2.10 hold with the effective Hausdorff classes and effectively continuous maps.

5.6. *Does Hausdorff's theorem fully effectivize?*

Open Problem 5.16. Equality $\cup_{\alpha<\omega_1^{\mathrm{CK}}} \mathrm{D}_\alpha(E) = \Delta^0_2(E)$ holds in computable Polish spaces, (cf. Selivanov 2003, pages 76–79 for the Baire space). Is this also true for more general spaces including effective ω-continuous domains endowed with the Scott topology?

Contrary to what was the case with the other results, the proof of Theorem 3.15 does not effectivize. The reason is that although the topological closure of a Δ^0_2 set is closed, hence $\boldsymbol{\Pi}^0_1$, it may not be Π^0_1. For instance, let X be any countable $\Delta^0_2(\mathbb{N})$ subset of $\mathbb{N}$. In the real line, X is $\Delta^0_2(\mathbb{R})$ and closed hence $\boldsymbol{\Pi}^0_1(\mathbb{R})$, but X is not $\Pi^0_1(\mathbb{R})$. Idem in the Baire space with the set $\{f \in \mathbb{N}^\omega \mid f(0) \in X\}$. This difficulty is mentioned in Selivanov (2005), page 53, lines 6–7, with open sets: in the proof of his Theorem 3.10 page 50, open sets

A_β's are defined using some Δ^0_2 set A: this is a stumbling block to get Σ^0_1 sets. For the Baire space $\mathbb{N}^\omega$, Selivanov (2003) uses a proof different from Hausdorff's original one. We adapt it to get the following weak effective version of Hausdorff's theorem.

Theorem 5.17. Let $(E, (O_n)_{n\in\mathbb{N}}, \ll)$ be an effective topological approximation space. Then $(F_\sigma \cap G_\delta)(E) \subseteq \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} \mathrm{D}_\alpha(E) \subseteq \Delta^0_2(D)$. In particular, if $(F_\sigma \cap G_\delta)(E) = \Delta^0_2(E)$ then the effective Hausdorff's theorem holds.

Proof. Without loss of generality, we suppose that the family $(O_n)_{n\in\mathbb{N}}$ is effectively closed under finite union: $O_{i_1} \cup \ldots \cup O_{i_k} = O_{\lambda(\{i_1,\ldots,i_k\})}$ for some computable function $\lambda : \mathcal{P}_{<\omega}(\mathbb{N}) \setminus \{\varnothing\} \to \mathbb{N}$. Using Remark 5.12, for $\varepsilon = 0, 1$, let $(I^\varepsilon_n)_{n\in\mathbb{N}}$ be a family of subsets of $\mathbb{N}$ such that $\{(n,i) \mid i \in I^\varepsilon_n\}$ is computably enumerable and

$$(*) \qquad A = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in I^1_n} O_i \quad , \quad E \setminus A = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in I^0_n} O_i \, .$$

We can suppose that the I^ε_n's are closed under the above function λ. Let R be the computably enumerable set $R = \{(i,j) \mid O_i \ll O_j\}$. Let $R^{(t)}$, $I^{\varepsilon,t}_n$ be the finite parts of R and I^t_n obtained after t steps of enumeration. Let $F_\varepsilon : \mathbb{N}^2 \to \mathbb{N}$ be the function such that

$$F_\varepsilon(m,t) \;=\; \max\{p \mid 0 \leqslant p \leqslant t \textit{ and } \forall q < p \ (m, \lambda(I^{\varepsilon,t}_q)) \in R^{(t)}\}.$$

In particular,

$$(\dagger) \qquad O_m \;\subseteq\; \bigcap_{q<F_\varepsilon(m,t)} \; \bigcup_{i\in I^{\varepsilon,t}_q} O_i \;\subseteq\; \bigcap_{q<F_\varepsilon(m,t)} \; \bigcup_{i\in I^{\varepsilon}_q} O_i.$$

We define a family $\mathcal{T}$ of finite sequences of integers in $\mathbb{N}$. The empty sequence is in $\mathcal{T}$. A sequence $(t_0, \ldots, t_k)$ is in $\mathcal{T}$ if and only if the following conditions are satisfied:

a. $m_0 < \cdots < m_k$ and $t_0 < \cdots < t_k$,

b. $(m_{\ell+1}, m_\ell) \in R^{(t_{\ell+1})}$ for all $\ell < k$. In particular, this says that the sequence of subsets $(O_{m_\ell})_{\ell=0,\ldots,k}$ is decreasing.

c. For all $\ell \leqslant k$, $F_0(m_\ell, t_\ell) \neq F_1(m_\ell, t_\ell)$. For all $\ell < k$, if $F_\varepsilon(m_\ell, t_\ell) < F_{1-\varepsilon}(m_\ell, t_\ell)$ then $F_{1-\varepsilon}(m_{\ell+1}, t_{\ell+1}) < F_\varepsilon(m_{\ell+1}, t_{\ell+1})$.

It is clear that, as a family of finite sequences, $\mathcal{T}$ is a tree and is computable.

Claim 1. The tree $\mathcal{T}$ is well founded (i.e. it has no infinite branch).

Proof of Claim 1. Else consider an infinite branch $(m_\ell, t_\ell)_{\ell\in\mathbb{N}}$. Condition (3) of the Definition 5.11 and condition (b) of the definition of $\mathcal{T}$ imply that the set $\bigcap_{\ell\in\mathbb{N}} O_{m_\ell}$ contains at least one element x. Observe that condition (c) implies that $F_\varepsilon(m_\ell, t) \geqslant \lfloor \ell/2 \rfloor$. Using ($\dagger$) for even and odd n's, we see that $x \in \bigcap_{q<\lfloor \ell/2 \rfloor} \bigcup_{i\in I^\varepsilon_q} O_i$ for all ℓ and for $\varepsilon = 0, 1$. Thus, x is in both A and $E \setminus A$, a contradiction. Let us say that a pair $(m,t) \in X$ has type ε if $F_\varepsilon(m,t) > F_{1-\varepsilon}(m,t)$.

Claim 2. Suppose, $x \in A$ and $(m,t) \in X$ has type 0 and $x \in O_m$. Then, there exists $(p,u) \in X$ such that (p,u) has type 1 and $x \in O_p$ and $(p,m) \in R$, $m < p$ and $t \leqslant u$. Switching types 0 and 1, the same is true with $x \notin A$.

Proof of Claim 2. We treat the sole case $x \in A$, the other one being trivial modification. Since $x \notin E \setminus A$, condition $(*)$ insures that there exists n such that $x \notin \bigcup_{i\in I^0_n} O_i$. Since $x \in A$, using again $(*)$, we see that $x \in \bigcap_{r\leqslant n} \bigcup_{i\in I^1_r} O_i$. Choose $i_1, \ldots, i_n$, p such that $x \in O_{i_r}$ and

$(i_r, p) \in R$ for all $r \leqslant n$. Let $u > t, n, i_1, \ldots, i_n$. Then, $F_0(p,u) < n$ whereas $F_1(p,u) \geqslant n$ hence (p,u) has type 1 and satisfies Claim 2. We extend à la Kleene–Brouwer the computable well-founded reverse prefix partial ordering on $\mathcal{T}$ into a computable total well-ordering $\preceq$ on $\mathcal{T}$: let $\sigma, \tau \in \mathcal{T}$,

— If the sequences σ and τ are prefix comparable, then we $\preceq$-compare them relative to the *reverse* prefix partial order.

— If the sequences σ and τ are not prefix comparable, then we compare the first elements on which they differ relative to the usual order on $\mathbb{N}$.

Using Ershov's trick, cf. Remark 5.10, we consider the set $\mathcal{S} = \mathcal{T} \times (\omega + 2)$ well-ordered lexicographically using $\preceq$ on $\mathcal{T}$ and the ordering on the ordinal $\omega + 2 = \{0, 1, 2, \ldots, \omega, \omega + 1\}$. In other words, $\mathcal{S}$ is obtained from $\mathcal{T}$ by replacing each element of $\mathcal{T}$ by a chain of $\omega + 2$ copies of that element. We also denote by $\preceq$ the well ordering on $\mathcal{S}$. The following Claim is straightforward.

Claim 3. $(\mathcal{S}, \preceq)$ is a computable well ordering and its order type is an even ordinal. Moreover, the parity of the rank relative to $\preceq$ of an element $(\sigma, \gamma) \in \mathcal{S}$ is equal to the parity of the ordinal γ. In particular, this parity function is computable. Attach to any element $(\sigma, \theta) \in \mathcal{S}$ an open set $U(\sigma, \theta)$ as follows: if σ is the empty sequence then $U(\sigma, \theta) = \emptyset$. Else, if the last pair (m_k, t_k) of σ has type $\varepsilon \in \{0, 1\}$ then

$$\forall n < \omega \ U(\sigma, n) = \emptyset, \quad U(\sigma, \omega + \varepsilon) = O_{m_k}, \quad U(\sigma, \omega + (1 - \varepsilon)) = \emptyset\,.$$

If α is the rank of the element (σ, θ) in T then let $A_\alpha = U(\sigma, \theta)$. The family $(A_\alpha)_{\alpha < \xi}$ is an effective family of open sets in the sense of Definition 5.14. Let ξ be the (limit) ordinal to which $(\mathcal{S}, \preceq)$ is isomorphic. The following Claim finishes the proof.

Claim 4. $D_\xi((A_\alpha)_{\alpha<\xi}) = A$.

Proof of Claim 4. Suppose $x \in A$. By Claim 2 there exists a pair (m, t) such that the one element sequence $\tau = ((m, t))$ is in $\mathcal{T}$, has type 1 and $x \in U(\tau, \omega + 1)$. Thus, we can consider the least ordinal α which is the rank of some $(\sigma, \gamma) \in \mathcal{S}$ such that $x \in U(\sigma, \gamma)$. First, we show that α is odd. Since $U(\sigma, \gamma)$ is not empty, its rank is of the form $(\omega + 2)\delta + \omega + \varepsilon$, where ε is the type of σ. If $\varepsilon = 0$ then Claim 2 would allow to extend σ to $\sigma\ (p, u) \in \mathcal{T}$ such that $x \in O_p$ and $\sigma\ (p, u)$ has type 1. Since $\sigma\ (p, u)$ extends σ it has lesser rank in $\mathcal{T}$. Thus, $x \in U(\sigma\ (p, u), \delta)$ where $(\sigma\ (p, u), \delta)$ has lesser rank than (σ, γ), a contradiction. By definition of α, we have $A_\alpha = U(\sigma, \gamma)$ hence $x \in A_\alpha$. The choice of α insures that x is in no A_β for $\beta < \alpha$. Thus, $x \in A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$ with α odd. This shows that $x \in \mathrm{D}((A_\mu)_{\mu<\xi})$. A similar argument shows that if $x \notin A$ then $x \in A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$ with α even hence $x \notin \mathrm{D}((A_\mu)_{\mu<\xi})$. Summing up, we see that $A = \mathrm{D}((A_\mu)_{\mu<\xi})$. □

Corollary 5.18. In every effective ω-continuous domain D,

$$G_\delta(D) \cap F_\sigma(D) \subseteq \bigcup_{\alpha<\omega_1} \mathrm{D}_\alpha(D) \subseteq \Delta^0_2(D).$$

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