PROBLEMS FOR SOLUTION

P. 153. Kuratowski has shown that at most 14 distinct sets can be constructed from a subset A of a topological space X by successive applications in any order of the closure operator f and the complementation operator g. Let us call sets thus constructed relatives of A. Defining the rim of A to be $r(A) = A \cap fg(A)$, prove the following :

(i) If the rim of A is nowhere dense then A has at most 10 relatives;

(ii) If the rim of A is dense in a regular closed subset of X, then A has at most 12 relatives.

K.P. Shum, University of Alberta

P. 154. Let n identical weighted coins, each falling heads with probability x, be tossed, and let $p_i(x)$ be the probability that exactly i of them fall heads. Evaluate

$$f_n = \min_{0 \le x \le 1} \; \max_{i = 0, 1, \ldots, n} \; p_i(x) .$$

W. Moser, McGill University

P. 155. If $a_1 < a_2 < \ldots < a_k \le n$ is a sequence of positive integers such that $[a_i, a_j] > n$ for all $i \ne j$, show that $\sum_{i=1}^{k} \frac{1}{a_i} < 2$, where $[a_i, a_j]$ denotes the least common multiple of a_i and a_j

Anonymous

SOLUTIONS

P. 145. If two disjoint subsets of a metric space have the property that every function lipschitz on each is lipschitz on their union, then every function continuous on each is continuous on their union. Prove this and give an example to show that this is false if the sets are not disjoint.

J.B. Wilker, Pahlavi University, Iran

Solution by the proposer

The function which is 0 on one set and 1 on the other is Lipschitz on each. If it is to be Lipschitz on their union then the two sets must be bounded apart. They are therefore separated and a subset of the union is open if and only if it is open in each. Under these circumstances a function continuous on each of the sets is necessarily continuous on their union.

For an example of intersecting sets for which the Lipschitz property holds and the continuity property fails, consider the following subsets of the Euclidean plane:

$$A = \{(x, y) : 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad x + y > 0\} \quad \text{and}$$

$$B = \{(x, y) : x = 0, \quad 0 \leq y \leq 1\} .$$

It is not hard to show that the Lipchitz property holds. However, the continuity property fails because of the function

$$f(x, y) = \begin{cases} \dfrac{y}{x+y} & \text{if } x + y > 0 \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

Also solved by J. Marsden.

P. 147. Let p be a prime with $p \equiv 1 \pmod 3$. Prove that $(x + 1)^p - x^p - 1 \equiv 0 \pmod{p^3}$ has at least two solutions in the range $1 \leq x \leq p - 1$.

H. A. Heilbronn, University of Toronto

Solution by the proposer

If $p > 3$, in $Z[x]$

$$(x + 1)^p - x^p - 1 \equiv -x^{2p} - x^p - 1 \qquad \text{mod } (x^2 + x + 1)$$

and

$$x^{2p} + x^p + 1 \equiv x^2 + x + 1 \qquad \text{mod } (x^3 - 1) .$$

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As $x^3 - 1 = (x-1)(x^2 + x + 1)$

$$(x+1)^p - x^p - 1 \equiv 0 \qquad \mod (x^2 + x + 1) .$$

If $p \equiv 1 \pmod 3$, and p is prime, and therefore odd,

$$\frac{d}{dx}\left[(x+1)^p - x^p - 1\right] = p((x+1)^{p-1} - x^{p-1}) \equiv 0 \qquad \mod (x^2 + x + 1) .$$

The coefficients of $(x+1)^p - x^p - 1$ are all divisible by p. Hence for $p \equiv 1 \pmod 3$,

$$(x+1)^p - x^p - 1 = p(x^2 + x + 1)^2 g(x) ,$$

where $g(x)$ is a polynomial in $Z[x]$. The polynomial $x^2 + x + 1$ has exactly two zeros mod p, namely the third roots of 1 mod p.

This completes the solution.

Also solved by W. J. Blundon, L. Carlitz and R. Breusch.

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