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ABELIAN VARIETIES AND GALOIS EXTENSIONS OF HILBERTIAN FIELDS

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Abstract In a recent paper Moshe Jarden (Diamonds in torsion of Abelian varieties, *J. Inst. Math. Jussieu* **9**(3) (2010), 477–480) proposed a conjecture, later named the Kuykian conjecture, which states that if A is an Abelian variety defined over a Hilbertian field K, then every intermediate field of $K(A_{\mathrm{tor}})/K$ is Hilbertian. We prove that the conjecture holds for Galois extensions of K in $K(A_{\mathrm{tor}})$.

Keywords: Abelian varieties; Hilbertian fields; Galois groups

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Secondary 12E25; 14Kxx

1. Introduction

In his article 'Diamonds in torsion of Abelian varieties', Moshe Jarden made the following conjecture:

Conjecture ([5]). *Let K be a Hilbertian field, A an Abelian variety defined over K, and M an extension of K in $K(A_{\mathrm{tor}})$. Then M is Hilbertian.*

In the same article he proved that the conjecture is true if K is a number field, and in a later paper written by Fehm *et al.* [2], the class of Hilbertian fields K for which the conjecture holds was greatly extended. In each case, specific properties of the fields considered had to be used to verify the conjecture.

In this paper, we apply a group-theoretic approach which enables us to prove Jarden's conjecture for all Hilbertian fields provided that M/K is a Galois extension. This approach requires the introduction of a special type of group, the *Galois–Hilbertian* group.

Definition 1. A profinite group G is called *Galois–Hilbertian* if for every closed normal subgroup H of G the following property holds: if K is a Hilbertian field and L/K is a G/H-Galois extension, then L is Hilbertian.

Remark 2. Suppose G is Galois–Hilbertian and H is a closed normal subgroup of G. Since each quotient of G/H is also a quotient of G, it follows that G/H is also Galois–Hilbertian. Thus, if K is a Hilbertian field and L/K is a G-Galois extension, then L^H is a Hilbertian field. We see then that *every* Galois extension of K in L is Hilbertian.

There are several well-known examples of Galois–Hilbertian groups:

(i) Finite groups are Galois–Hilbertian since every finite extension of a Hilbertian field is Hilbertian [3, Proposition 12.3.5].

(ii) Abelian groups are Galois–Hilbertian since every Abelian extension of a Hilbertian field is Hilbertian. This result was originally due to Kuyk [3, Theorem 16.11.3].

(iii) A profinite group is *small* if for each positive integer n, the group has only finitely many open subgroups of index n [3, p. 328]. Small groups are Galois–Hilbertian since every quotient of a small group is small and every Galois extension of a Hilbertian field with a small Galois group is Hilbertian [3, Remark 16.10.3(d), Proposition 16.11.1]. In fact, if L is a Galois extension of a Hilbertian field K with $\mathrm{Gal}(L/K)$ small, then any extension of K in L is Hilbertian [5, Lemma 4].

The main result of this paper is the following.

Theorem. *For each n every closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{Z}_p)$ is Galois–Hilbertian.*

It is known that the maximal separable extension of K in $K(A_{\mathrm{tor}})$ has a Galois group over K which is a closed subgroup of $\prod_p \mathrm{GL}_{2\dim(A)}(\mathbb{Z}_p)$, so we can conclude that every Galois extension of a Hilbertian field K in $K(A_{\mathrm{tor}})$ is Hilbertian.

The strategy employed is to show that extensions of Galois–Hilbertian groups by Galois–Hilbertian groups are again Galois–Hilbertian, and then we will see that any closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{Z}_p)$ can be expressed as an extension of a Galois–Hilbertian group by a Galois–Hilbertian group. The latter result utilizes a theorem of Larsen and Pink [6] which describes special properties of subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$. We also state and apply several results regarding properties of p-adic analytic groups and products of finite simple groups, and we make use of Haran's diamond theorem to make several important reductions.

Throughout this paper we will consider many homomorphisms between profinite topological groups. In particular, we will use the following homomorphisms, all of which are continuous:

- inclusion of a closed subgroup into a group;
- projection from a closed subgroup of a direct product of groups to any subproduct of the groups;
- the canonical isomorphisms from the first, second, and third isomorphism theorems;
- the canonical isomorphism $\prod_i G_i / \prod_i N_i \cong \prod_i (G_i/N_i)$, where each N_i is normal in G_i.

Since all of the groups that we consider are closed subgroups of profinite groups, they are compact. Thus, we are only considering continuous maps from compact spaces to Hausdorff spaces, and so the homomorphisms are closed maps. We will use this fact often when we consider a closed subgroup of one group as a closed subgroup of another group by means of one or more of the homomorphisms listed above.

2. Results

Proposition 3. *Every Galois–Hilbertian extension of a Galois–Hilbertian group is Galois–Hilbertian.*

Proof. Suppose we have a closed normal subgroup G' of G such that G' and G/G' are Galois–Hilbertian. Let H be any closed normal subgroup of G. The groups $G'/(G' \cap H)$ and $(G/G')/(G'H/G')$ are both Galois–Hilbertian since they are quotients of Galois–Hilbertian groups. We also have the canonical isomorphisms

$$G'/(G' \cap H) \cong G'H/H$$

and

$$(G/G')/(G'H/G') \cong G/G'H \cong (G/H)/(G'H/H).$$

Let $\bar{G}' = G'H/H$ and $\overline{G/H} = (G/H)/(G'H/H)$. We now view G/H as an extension of the Galois–Hilbertian group $\overline{G/H}$ by the Galois–Hilbertian group $\bar{G}'$. Now let K be a Hilbertian field and L an extension of K with $G/H = \mathrm{Gal}(L/K)$. Then $L^{\bar{G}'}$ is Hilbertian since K is Hilbertian and $\overline{G/H} \cong \mathrm{Gal}(L^{\bar{G}'}/K)$ is Galois–Hilbertian. Also, L is Hilbertian since $L^{\bar{G}'}$ is Hilbertian and $\bar{G}' \cong \mathrm{Gal}(L/L^{\bar{G}'})$ is Galois–Hilbertian. Thus, G is Galois–Hilbertian. □

We now introduce a class of groups called *k-stage* groups. We prove that k-stage groups are Galois–Hilbertian and then use them to construct other Galois–Hilbertian groups.

Definition 4. A topological group G is called *one-stage* if it is trivial or a direct product of finite simple groups. A topological group G is called k-stage for $k \geqslant 2$ if it has a closed normal subgroup G' such that G' is one-stage and G/G' is $(k-1)$-stage.

Remark 5. If a group G is k-stage, then it is $(k+1)$-stage, for G always has the subgroup $\{e\}$, and $G/\{e\}$ is k-stage. Therefore, G is also j-stage for any $j \geqslant k$.

We now establish that certain closed subgroups of k-stage groups are also k-stage groups, but first we will require several results regarding closed subgroups of direct products of finite simple groups.

Lemma 6. *Let $G = \prod_{i \in I} S_i$ be a direct product of finitely many finite simple groups, and let H be a subgroup of G. Suppose the projection of H on each of the factors of G is surjective. Then there is a subset J of I such that $H \cong \prod_{j \in J} S_j$.*

Proof. We proceed by induction. The result is trivial if $|I| = 1$, so suppose that the result holds for any direct product of k finite simple groups for some $k \geqslant 1$, and let $|I| = k+1$. Choose some $i \in I$, set $I' = I \setminus \{i\}$, and let $G' = \prod_{i' \neq i} S_{i'}$, so $G = S_i \times G'$. Let $\pi_i : G \to S_i$ and $\pi' : G \to G'$ be the projection maps. Then $H' = \pi'(H)$ is a subgroup of G'

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whose projection on each of the factors of G' is surjective. By induction, there is some $J' \subset I'$ such that

$$H' \cong \prod_{j \in J'} S_j. \tag{1}$$

The map $h \mapsto (\pi_i(h), \pi'(h))$ embeds H into $S_i \times H'$, so H is a subgroup of $S_i \times H'$ which projects surjectively to each factor. Let $N' = H \cap \mathrm{Ker}(\pi_i)$ and $N_i = H \cap \mathrm{Ker}(\pi')$. By Goursat's lemma, $H'/N' \cong S_i/N_i$. Since S_i is simple, either N_i is trivial or $N_i = S_i$. If N_i is trivial, then $H \cong H'$. If $N_i = S_i$, then $H' = N'$, so $H = S_i \times H'$. In both cases, the conclusion of the lemma follows from (1). □

Lemma 7. *Let $(H_x, \alpha_{yx})_{x,y \in X}$ be a projective system of finite groups such that the homomorphism $\alpha_{yx} : H_y \to H_x$ is surjective for all $y \geqslant x$. For each $x \in X$ we assume that $H_x = \prod_{i \in I_x} S_i$ is a direct product of finitely many finite non-Abelian simple groups. Then $H = \varprojlim H_x$ is a direct product of non-Abelian simple groups, each isomorphic to a group belonging to the set $\bigcup_{x \in X} \{S_i | i \in I_x\}$.*

Proof. We may assume that $0 \notin I_x$ and set $I'_x = 0 \cup I_x$. Given $x \leqslant y$ in X, we define a map $\beta_{yx} : I'_y \to I'_x$ in the following way. First we set $\beta_{yx}(0) = 0$. Next let $j \in I_y$. Then either $\alpha_{yx}(S_j)$ is trivial or $\alpha_{yx}(S_j)$ is a non-Abelian simple subgroup of H_x. In the former case we set $\beta_{yx}(j) = 0$. In the latter case there exists a unique $i \in I_x$ such that $\alpha_{yx}(S_j) = S_i$ [4, p. 51, Satz 9.12(b)]. We set $\beta_{yx}(j) = i$ and note that α_{yx} maps S_j isomorphically onto S_i.

Now let $I^0_{yx} = \{j \in I_y | \beta_{yx}(j) = 0\}$ and $I_{yx} = I_y \setminus I^0_{yx}$. Then β_{yx} maps I_{yx} bijectively onto I_x. Also, $\prod_{j \in I^0_{yx}} S_j = \mathrm{Ker}(\alpha_{yx})$ and α_{yx} maps $\prod_{j \in I_{yx}} S_j$ isomorphically onto $H_x = \prod_{i \in I_x} S_i$.

If $z \in X$ and $z \geqslant y$, then the uniqueness in the first paragraph of the proof implies that $\beta_{yx} \circ \beta_{zy} = \beta_{zx}$. Moreover, $\beta_{xx} : I'_x \to I'_x$ is the identity map. It follows that $(I'_x, \beta_{yx})_{x,y \in X}$ is a projective system of finite sets. Let $I' = \varprojlim I'_x$ and $I = I' \setminus \{0\}$. Thus, I' is a profinite space and I is an open subset of I'.

For each $x \in X$ let $\beta_x : I' \to I'_x$ be the inverse limit of the maps $\beta_{yx} : I'_y \to I'_x$ with $y \geqslant x$. Also, let $I^x = \varprojlim_{y \geqslant x} I_{yx}$. Then I^x is a finite subset of I and β_x maps I^x bijectively onto I_x and $\beta_x(I' \setminus I^x) = \{0\}$. If $y \geqslant x$, then $I^x \subset I^y$.

Again, for each $x \in X$ let $\alpha_x : H \to H_x$ be the inverse limit of the homomorphisms $\alpha_{yx} : H_y \to H_x$ with $y \geqslant x$. For each $i \in I^x$ we set $i_x = \beta_x(i)$. Set $S_i = \varprojlim_{y \geqslant x} S_{i_y}$. Since each of the maps $\alpha_{yx} : S_{i_y} \to S_{i_x}$ is an isomorphism, so is the map $\alpha_x : S_i \to S_{i_x}$. In particular, S_i is a finite simple non-Abelian subgroup of H. Moreover, S_i is normal in H because S_{i_x} is normal in H_x for each $i \in I^x$.

It follows that α_x maps $\langle S_i | i \in I^x \rangle$ isomorphically onto $H_x = \prod_{i \in I_x} S_i$. Thus, $\langle S_i | i \in I^x \rangle = \prod_{i \in I^x} S_i$. Since for each $x \in X$ the group H_x is generated by the groups S_i with $i \in I_x$, the group H is generated by the groups S_i, for $i \in I$. It follows that $H = \prod_{i \in I} S_i$, as claimed. □

Lemma 8. *Let $G = \prod_{i \in I} S_i$ be a direct product of finite non-Abelian simple groups, and let H be a closed subgroup of G. Suppose that the projection of H to each of the factors*

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of G is surjective. Then H is isomorphic to a direct product of simple groups, each belonging to the set $\{S_i | i \in I\}$.

Proof. For each finite subset J of I let $G_J = \prod_{j \in J} S_j$ and let H_J be the projection of H to G_J. Lemma 6 gives a subset J' of J such that $H_J \cong \prod_{j \in J'} S_j$. Thus, for each $j \in J'$, there is a subgroup S'_j of H_j which is isomorphic to S_J such that $H_J = \prod_{j \in J'} S'_j$. By Lemma 7, $H = \varprojlim H_J$ is a direct product of non-Abelian simple groups, each isomorphic to a group belonging to the set $\{S_i | i \in I\}$. □

Lemma 9. *For each prime p, let G_p be a pro-p group, and let H be a closed subgroup of $\prod_p G_p$ such that the projection $\pi_p : H \to G_p$ is surjective for each p. Then $H = \prod_p G_p$.*

Proof. If G is finite, then $|G| = \prod_p p^{n_p}$. Since $H/\mathrm{Ker}(\pi_p) \cong G_p$, we have that p^{n_p} divides $|H|$ for each p. Hence, $|H| = |G|$, so $H = G$.

If G is an infinite group, then consider an open normal subgroup N of the form $\prod_p N_p$, where N_p is an open normal subgroup of G_p for each p and $N_p = G_p$ for almost all p. Then $G/N \cong \prod_p G_p/N_p$, and the projection of HN/N onto G_p/N_p is surjective for each p. Since N_p has finite index in G_p for each p, G/N is finite. Thus, by the previous paragraph, $HN/N = G/N$; hence, $HN = G$. Since the intersection of all the N as above is trivial and H is a closed subgroup of G, we have that $H = G$ [3, Lemma 1.2.2(b)]. □

Lemma 10. *Let H be a closed subgroup of $G_1 \times G_2$, where G_1 is a direct product of non-Abelian finite simple groups and G_2 is Abelian. If the projection of H to each of G_1 and G_2 is surjective, then $H = G_1 \times G_2$.*

Proof. Let H, G_1, and G_2 be as above, and let $\pi_1 : H \to G_1$ and $\pi_2 : H \to G_2$ be the projection maps, which are surjective. Thus, $\mathrm{Ker}(\pi_1)$ and $\mathrm{Ker}(\pi_2)$ are normal subgroups of G_2 and G_1, respectively. It is known that every closed normal subgroup of a direct product of finite simple non-Abelian groups is itself a direct product of a subcollection of those groups [3, Lemma 18.3.9], so $G_1/\mathrm{Ker}(\pi_2)$ is isomorphic to a direct product of non-Abelian finite simple groups, which is necessarily non-Abelian unless $\mathrm{Ker}(\pi_2) = G_1$. By Goursat's lemma,

$$G_1/\mathrm{Ker}(\pi_2) \cong G_2/\mathrm{Ker}(\pi_1).$$

Since $G_2/\mathrm{Ker}(\pi_1)$ is Abelian, it follows that $\mathrm{Ker}(\pi_2) = G_1$ and $\mathrm{Ker}(\pi_1) = G_2$. Thus, $H = G_1 \times G_2$. □

Lemma 11. *Let H be a closed subgroup of a direct product of finite simple groups which projects surjectively to each factor. Then H is a direct product of finite simple groups.*

Proof. Let H be a closed subgroup of $G = \prod_{i \in I} G_i$, where the G_i are finite simple groups. Let A be the set of indices i for which G_i is Abelian, and let $R = I \setminus A$. Let $\pi_A : H \to \prod_{i \in A} G_i$ and $\pi_R : H \to \prod_{j \in R} G_j$ be the projection maps. First we write $\prod_{i \in A} G_i$ as $\prod_p (\mathbb{Z}/p\mathbb{Z})^{\alpha_p}$, where α_p is a (possibly infinite) cardinal, by grouping all of the Abelian finite simple groups of order p together for each prime p, and we let $\pi_p : H \to (\mathbb{Z}/p\mathbb{Z})^{\alpha_p}$ be the projection map. Then $\pi_A(H)$ is a closed subgroup of the direct product of the

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pro-p groups $\pi_p(H)$. From Lemma 9 we get that $\pi_A(H) = \prod_p \pi_p(H)$. In addition, $\pi_p(H)$ is a closed subgroup of a $\mathbb{Z}/p\mathbb{Z}$-vector space, so it is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\beta_p}$ for some $\beta_p \leqslant \alpha_p$ [3, Lemma 22.7.3]. Hence, $\pi_A(H)$ is a direct product of finite simple groups. From Lemma 8 we also have that $\pi_R(H) \cong \prod_{j\in R'} G_j$, where R' is some subset of R. Thus, we can view H as a closed subgroup of $\prod_{j\in R'} G_j \times \prod_p \pi_p(H)$ which projects surjectively to each of the two factors. We conclude from Lemma 10 that $H = \prod_{j\in R'} G_j \times \prod_p \pi_p(H)$, and so we have that H is a direct product of finite simple groups. □

Proposition 12. *Let H be a closed subgroup of a direct product of k-stage groups such that the projection of H to each factor is surjective. Then H is k-stage.*

Proof. We proceed by induction on k. A direct product of one-stage groups is again a one-stage group, so suppose H is a closed subgroup of a one-stage group which projects surjectively to each factor. From Lemma 11, we see that H is isomorphic to a product of finite simple groups, so H is one-stage.

Now suppose that H is a closed subgroup of a product of $(k+1)$-stage groups $G = \prod_i G_i$ which projects surjectively to each factor, and any closed subgroup of a product of k-stage groups which projects surjectively to each of the factors is again a k-stage group. For each G_i we have a closed normal subgroup G_i' such that G_i' is one-stage and $\bar{G}_i = G_i/G_i'$ is k-stage.

Let $G' = \prod_i G_i'$, so $H' = H \cap G'$ is a closed normal subgroup of H. We claim that H' is one-stage. Let $\pi_i : H \to G_i$ be the projection map. Then $G_i'' = \pi_i(H')$ is a closed normal subgroup of G_i, and hence also of G_i'. Since G_i' is a direct product of finite simple groups, it follows that G_i'' is also a direct product of finite simple groups [3, Lemma 25.5.3(b)]. Now we have that $G'' = \prod_i G_i''$ is a one-stage group, and H' is a closed subgroup of G'' which projects surjectively to each factor. From the first paragraph of the proof, H' is one-stage.

Note that $H/H' \cong HG'/G'$, which is a closed subgroup of $G/G' \cong \prod_i \bar{G}_i$. Since $\pi(H) = G_i$ and $\pi_i(H') = G_i''$, the projection of HG'/G' to the ith component is $G_i/G_i'' = \bar{G}_i$. By induction, HG'/G' is k-stage, so H/H' is also k-stage. Therefore, H is a $(k+1)$-stage group. □

Lemma 13. *Every finite group G is k-stage, where $k \leqslant |G|$.*

Proof. The result is trivial if $|G| = 1$, so suppose that all groups of order at most k are k-stage. Let G be a finite group with $|G| = k+1$. If G is simple, then G is one-stage, so suppose G is not simple. Then G has a minimal normal subgroup H, and so $H \cong \prod_{i=1}^n S$ for some finite simple group S; hence, H is one-stage. Also, $|G/H| \leqslant k$, so G/H is k-stage. Therefore, G is $(k+1)$-stage. □

Corollary 14. *Let H be a closed subgroup of a direct product of groups of order bounded by k such that the projection of H to each factor is surjective. Then H is k-stage.*

Proof. Suppose each of the groups in the product has order bounded by k. By Lemma 13, all of these groups are k-stage. Thus, H is a closed subgroup of a direct product of k-stage groups which projects surjectively to each factor, so by Proposition 12, H is k-stage. □

Haran's diamond theorem is a powerful result for identifying Hilbertian fields within Galois extensions of Hilbertian fields. We will need this result to make several important reductions.

Diamond Theorem ([3, Theorem 13.8.3]). *Let K be a Hilbertian field, M_1 and M_2 Galois extensions of K, and M an intermediate field of M_1M_2/K. Suppose that $M \not\subset M_1$ and $M \not\subset M_2$. Then M is Hilbertian.*

Lemma 15. *Let $\{G_i\}_{i\in I}$ be a collection of groups with the property that whenever F is a Hilbertian field and M/F is a Galois extension with $\mathrm{Gal}(M/F) \cong G_i$ for some i, then every intermediate extension of M/F is Hilbertian. Suppose K is a Hilbertian field and L/K is a Galois extension with $\mathrm{Gal}(L/K) \cong \prod_{i\in I} G_i$. Then every intermediate extension of L/K is Hilbertian.*

Proof. Let K, L, and $\{G_i\}_{i\in I}$ be as above, and let K' be an extension of K in L, so $K' = L^H$ for some closed subgroup H of $\prod_{i\in I} G_i$. Suppose there are two indices i_1 and i_2 such that H contains neither G_{i_1} nor G_{i_2}. Let $G_1' = \prod_{i\neq i_1} G_i$, and then $L^{G_{i_1}} L^{G_1'} = L$, and K' is an extension of K which is contained in neither $L^{G_{i_1}}$ nor $L^{G_1'}$. Thus, by Haran's diamond theorem, K' is Hilbertian. If, on the other hand, H contains all but possibly one G_i, say G_{i_1}, then H contains the subgroup G_1' above. Thus, $K' \subset L^{G_1'}$. Since $\mathrm{Gal}(L^{G_1'}/K) \cong G_{i_1}$, we have that K' is Hilbertian. □

Proposition 16. *Every k-stage group is Galois–Hilbertian.*

Proof. We proceed by induction on k. If G is a one-stage group and H is any closed normal subgroup of G, then G/H is a direct product of finite simple groups [3, Lemma 25.5.3(d)]. Finite groups satisfy the hypotheses of the groups in Lemma 15, so in particular we see that every G/H-extension of a Hilbertian field is Hilbertian. Thus, G is Galois–Hilbertian.

Now suppose G is $(k+1)$-stage and all k-stage groups are Galois–Hilbertian, so G has a closed normal subgroup G' such that G' is one-stage and G/G' is k-stage. Thus, G' and G/G' are Galois–Hilbertian. It follows from Proposition 3 that G is Galois–Hilbertian. □

Lemma 17. *Every Abelian profinite extension of a k-stage group is Galois–Hilbertian.*

Proof. Abelian groups and k-stage groups are Galois–Hilbertian, so the result follows from Proposition 3. □

A group G is called *p-adic analytic* if G is an analytic manifold over $\mathbb{Q}_p$ such that the functions $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ for x and y in G are analytic. We will be particularly interested in p-adic analytic groups which are also pro-p groups. There are many equivalent characterizations of pro-p p-adic analytic groups. We will adopt one of these characterizations and say that a pro-p group G is p-adic analytic if it is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some n [1, p. 97]. There are several properties of p-adic analytic groups that will prove useful. In particular, every closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ is finitely generated [3, Lemma 22.14.4], so p-adic analytic groups are finitely generated; hence, they are small [3, Lemma 16.10.2]. Hence, closed subgroups and quotients of pro-p p-adic analytic groups are pro-p p-adic analytic groups [1, Theorem 9.6]. We now introduce another class of groups, which we term $\mathbb{Z}$-*analytic*.

Definition 18. A topological group is called $\mathbb{Z}$-*analytic* if it is isomorphic to $\prod_p G_p$, where for each prime p, G_p is a pro-p p-adic analytic group.

Proposition 19. *Every closed subgroup of a* $\mathbb{Z}$*-analytic group is* $\mathbb{Z}$*-analytic.*

Proof. Suppose H is a closed subgroup of a $\mathbb{Z}$-analytic group $G = \prod_p G_p$. Let π_p be the projection of H to G_p. Then $\pi_p(H)$ is a closed subgroup of G_p, so $\pi_p(H)$ is a pro-p p-adic analytic group. Thus, H is a closed subgroup of the $\mathbb{Z}$-analytic group $\prod_p \pi_p(H)$. It follows from Lemma 9 that $H = \prod_p \pi_p(H)$, so H is $\mathbb{Z}$-analytic. □

Proposition 20. *Every* $\mathbb{Z}$*-analytic group is Galois–Hilbertian.*

Proof. We use the notation from the proof of Proposition 19. Let H be any closed normal subgroup of G, and then $H = \prod_p \pi_p(H)$ is a $\mathbb{Z}$-analytic group. Also, $G/H \cong \prod_p(G_p/\pi_p(H))$, and since quotients of pro-p p-adic analytic groups by closed normal subgroups are again pro-p p-adic analytic groups, we have that G/H is $\mathbb{Z}$-analytic. Since $G_p/\pi_p(H)$ is a small group for each p, the groups $G_p/\pi_p(H)$ satisfy the hypotheses of the groups in Lemma 15. Thus, every G/H-extension of a Hilbertian field is Hilbertian, so G is Galois–Hilbertian. □

Remark 21. It is worth noting that Proposition 19 would not necessarily be true if the groups G_p were not pro-p groups. In particular, this means that a closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{Z}_p)$ is not necessarily a direct product of closed subgroups of the $\mathrm{GL}_n(\mathbb{Z}_p)$. If this were the case, then the Kuykian conjecture would follow from a slight modification of Lemma 15. The proof of Proposition 19 requires the result of Lemma 9, which depends on the fact that quotients of the factors in the direct product have orders that are pairwise relatively prime. This is not the case in general.

Lemma 22. *Every* $\mathbb{Z}$*-analytic extension of a Galois–Hilbertian group is Galois–Hilbertian.*

Proof. $\mathbb{Z}$-analytic groups are Galois–Hilbertian, and so the result follows from Proposition 3. □

Corollary 23. *Every* $\mathbb{Z}$*-analytic extension of an Abelian extension of a* k*-stage group is Galois–Hilbertian.*

Proof. By Lemma 17 we know that an Abelian extension of a k-stage group is Galois–Hilbertian. Thus, by Lemma 22 we have that a $\mathbb{Z}$-analytic extension of an Abelian extension of a k-stage group is Galois–Hilbertian. □

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Proposition 24. *For each n there exists some k such that every closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{F}_p)$ is a $\mathbb{Z}$-analytic extension of an Abelian extension of a k-stage group.*

Proof. Let H be a closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{F}_p)$. Consider the subgroups $H_p = \pi_p(H)$, where π_p is projection to the pth factor. We simultaneously view H_p as a subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ and $\prod_p \mathrm{GL}_n(\mathbb{F}_p)$ in the natural way. Larsen and Pink showed that there are normal subgroups $\Gamma_{p,1} \supseteq \Gamma_{p,2} \supseteq \Gamma_{p,3}$ of H_p satisfying the following properties [6, Theorem 0.2]:

(i) $[H_p : \Gamma_{p,1}]$ is bounded by a constant $J'(n)$ which depends only on n.
(ii) $\Gamma_{p,1}/\Gamma_{p,2}$ is a direct product of finite simple groups.
(iii) $\Gamma_{p,2}/\Gamma_{p,3}$ is Abelian with order not divisible by p.
(iv) $\Gamma_{p,3}$ is a p-group.

We wish to show that there are closed normal subgroups A_1 of H and A_2 of H/A_1 such that A_1 is $\mathbb{Z}$-analytic, A_2 is Abelian, and $(H/A_1)/A_2$ is k-stage.

First, we define A_1 as

$$A_1 = H \cap \prod_p \Gamma_{p,3}.$$

Each $\Gamma_{p,3}$ is a pro-p p-adic analytic group, so $\prod_p \Gamma_{p,3}$ is $\mathbb{Z}$-analytic. From Proposition 19 we have that every closed subgroup of a $\mathbb{Z}$-analytic group is $\mathbb{Z}$-analytic, so A_1 is $\mathbb{Z}$-analytic.

Now we let $B = \prod_p \Gamma_{p,2}$ and

$$A_2 = (H \cap B)/A_1,$$

so A_2 is then a closed normal subgroup of H/A_1. Note that

$$A_2 = \frac{(H \cap B)}{(H \cap B) \cap \prod_p \Gamma_{p,3}} \cong \frac{(H \cap B)(\prod_p \Gamma_{p,3})}{\prod_p \Gamma_{p,3}} \leqslant \frac{B}{\prod_p \Gamma_{p,3}} \cong \prod_p (\Gamma_{p,2}/\Gamma_{p,3}).$$

Since the groups $\Gamma_{p,2}/\Gamma_{p,3}$ are Abelian, so is $\prod_p(\Gamma_{p,2}/\Gamma_{p,3})$, and we see that A_2 is Abelian.

It only remains to prove that $(H/A_1)/A_2$ is k-stage. Since H is a subgroup of $\prod_p H_p$ and B is a normal subgroup of $\prod_p H_p$, we have that

$$(H/A_1)/A_2 \cong H/(H \cap B) \cong HB/B.$$

Now HB/B is a closed subgroup of the group $\prod_p H_p/B$, and the latter group is isomorphic to $\prod_p(H_p/\Gamma_{p,2})$. For each p, $H_p/\Gamma_{p,2}$ is an extension of the group $H_p/\Gamma_{p,1}$ by the group $\Gamma_{p,1}/\Gamma_{p,2}$, and $\Gamma_{p,1}/\Gamma_{p,2}$ is a direct product of finite simple groups. Thus, by definition, $H_p/\Gamma_{p,2}$ is k-stage for some $k \leqslant |H_p/\Gamma_{p,1}| + 1 \leqslant J'(n) + 1$, so we can let $k = J'(n) + 1$. By Proposition 12, $\prod_p(H_p/\Gamma_{p,2})$ is k-stage, and since H projects surjectively to each of the factors of $\prod_p H_p$, we see that HB/B projects surjectively to each of the factors of $\prod_p(H_p/\Gamma_{p,2})$. Thus, by Proposition 12, HB/B is k-stage, and so $(H/A_1)/A_2$ is k-stage. □

An immediate consequence of Corollary 23 and Proposition 24 is the following result:

Corollary 25. *For each n every closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{F}_p)$ is Galois–Hilbertian.*

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Now we are ready to prove the main result:

Theorem 26. *For each n every closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{Z}_p)$ is Galois–Hilbertian.*

Proof. First we make a small reduction. If H is a closed subgroup of $A \times B$, where $A = \mathrm{GL}_n(\mathbb{Z}_2)$ and $B = \prod_{p>2} \mathrm{GL}_n(\mathbb{Z}_p)$, then consider the projection $\pi_B : H \to B$. If $H_0 = \mathrm{Ker}(\pi_B)$, then H_0 is a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_2)$, so H_0 is finitely generated, and hence, small. Thus, H_0 is Galois–Hilbertian. Now $\pi_B(H)$ is a closed subgroup of $\prod_{p>2} \mathrm{GL}_n(\mathbb{Z}_p)$, so if $\pi_B(H)$ is Galois–Hilbertian, then H is Galois–Hilbertian by Proposition 3. Thus we need only consider closed subgroups of $\prod_{p>2} \mathrm{GL}_n(\mathbb{Z}_p)$.

Let H be a closed subgroup of $\prod_{p>2} \mathrm{GL}_n(\mathbb{Z}_p)$, K a Hilbertian field, and L/K a Galois extension with $\mathrm{Gal}(L/K) \cong H$. For each prime $p > 2$ let

$$N_p = \{M \in \mathrm{GL}_n(\mathbb{Z}_p) : M \equiv I (\mathrm{mod}\ p)\}$$

be the principal congruence subgroup. Let $N' = \prod_p N_p$ and let $N = H \cap N'$. Note that

$$H/N \cong HN'/N' < \prod_{p>2} \mathrm{GL}_n(\mathbb{Z}_p)/N' \cong \prod_{p>2} \mathrm{GL}_n(\mathbb{F}_p).$$

Thus, we can view H/N as a closed subgroup of $\prod_p \mathrm{GL}_n(\mathbb{F}_p)$. From Corollary 25 we see that H/N is Galois–Hilbertian.

It is known that N_p is a pro-p p-adic analytic group for each $p > 2$ [3, Lemma 22.14.2], so $\prod_{p>2} N_p$ is a $\mathbb{Z}$-analytic group, and N is a closed subgroup of $\prod_{p>2} N_p$. Thus, by Proposition 19, N is also $\mathbb{Z}$-analytic. Now we see that H is a $\mathbb{Z}$-analytic extension of the Galois–Hilbertian group H/N, so by Lemma 22, H is Galois–Hilbertian. □

Immediately, we have the following theorem.

Theorem 27. *Let K be a Hilbertian field and A an Abelian variety defined over K. If M is a Galois extension of K in $K(A_{\mathrm{tor}})$, then M is Hilbertian.*

Proof. For each prime p, we have $A_{p^\infty} = \bigcup_{n=1}^{\infty} A_{p^n}$, where A_{p^n} is the set of p^n-torsion points of A, and we have $A_{\mathrm{tor}} = \bigcup_p A_{p^\infty}$. Let K_p be the maximal separable extension of K in $K(A_{p^\infty})$, and let K' be the compositum of the K_p. Since $K(A_{\mathrm{tor}})$ is the compositum of all the $K(A_{p^\infty})$, we have that K' is the maximal separable extension of K in $K(A_{\mathrm{tor}})$. Thus, if M is a Galois extension of K in $K(A_{\mathrm{tor}})$, then $M \subset K'$. For each p, $\mathrm{Gal}(K_p/K)$ is a closed subgroup of $\mathrm{GL}_{2\dim(A)}(\mathbb{Z}_p)$ (proof of [5, Lemma 6]), so $\mathrm{Gal}(K'/K)$ is a closed subgroup of $\prod_p \mathrm{GL}_{2\dim(A)}(\mathbb{Z}_p)$. By Theorem 26, $\mathrm{Gal}(K'/K)$ is Galois–Hilbertian, so M is Hilbertian. □

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