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Cut cotorsion pairs

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Abstract

We present the concept of cotorsion pairs cut along subcategories of an abelian category. This provides a generalization of complete cotorsion pairs, and represents a general framework to find approximations restricted to certain subcategories. We also exhibit some connections between cut cotorsion pairs and Auslander–Buchweitz approximation theory, by considering relative analogs for Frobenius pairs and Auslander–Buchweitz contexts. Several applications are given in the settings of relative Gorenstein homological algebra, chain complexes, and quasi-coherent sheaves, as well as to characterize some important results on the Finitistic Dimension Conjecture, the existence of right adjoints of quotient functors by Serre subcategories, and the description of cotorsion pairs in triangulated categories as co-t-structures.

Introduction

Given classes of objects $\mathcal{A}$ and $\mathcal{B}$ in an abelian category $\mathcal{C}$, it is not always possible for these classes to form a complete cotorsion pair $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$. For example, if $\mathcal{A} = \mathcal{GP}(R)$ denotes the class of Gorenstein projective modules over a ring R, and $\mathcal{B} = \mathcal{P}(R)^\wedge$ the class of R-modules with finite projective dimension. In general, the pair $(\mathcal{GP}(R), \mathcal{P}(R)^\wedge)$ is not a complete cotorsion pair over an arbitrary ring R. However, by using Auslander–Buchweitz approximation theory, it is known that every R-module with finite Gorenstein projective dimension has a Gorenstein projective precover whose kernel has finite projective dimension (see [1, 3]). Moreover, the equalities $\mathcal{GP}(R) = {}^{\perp_1}(\mathcal{P}(R)^\wedge) \cap \mathcal{GP}(R)^\wedge$ and $\mathcal{P}(R)^\wedge = (\mathcal{GP}(R))^{\perp_1} \cap \mathcal{GP}(R)^\wedge$ also hold true. Hence, along the class $\mathcal{GP}(R)^\wedge$ of R-modules with finite Gorenstein projective dimension, $(\mathcal{GP}(R), \mathcal{P}(R)^\wedge)$ can be regarded, in some sense, as a complete cotorsion pair.

The first main goal of this article is to specify a meaning under which $\mathcal{A}$ and $\mathcal{B}$ form a complete cotorsion pair restricted to another class $\mathcal{S}$ of objects in $\mathcal{C}$. Specifically, orthogonality relations between $\mathcal{A}$ and $\mathcal{B}$, and the existence of special $\mathcal{A}$-precovers and special $\mathcal{B}$-preenvelopes, will be restricted to objects in $\mathcal{S}$. These "local" properties will be formally presented in the concept of *complete cotorsion pair cut along $\mathcal{S}$* (or *complete cut cotorsion pair*, for short). Many properties of this concept derive in a general language for cotorsion theory and relative homological algebra, which in particular covers some well-known results on complete cotorsion pairs in abelian categories.

A recent approach to the idea of relativizing cotorsion pairs was proposed in [3], under the name of $\mathcal{S}$-*cotorsion pairs*, where the authors consider cotorsion pairs $(\mathcal{A}, \mathcal{B})$ relative to a thick subcategory $\mathcal{S} \subseteq \mathcal{C}$, and such that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$. In that work, it was established an interplay between relative cotorsion pairs, left Frobenius pairs, and left weak Auslander–Buchweitz contexts. Specifically, the latter two concepts are in one-to-one correspondence, while left weak AB-contexts coincide with the class of cotorsion pairs $(\mathcal{F}, \mathcal{G})$ relative to the smallest thick subcategory containing $\mathcal{F}$, and where $\mathcal{G}$ is injective relative to $\mathcal{F}$. On the other hand, the cut cotorsion pairs proposed in this article are a generalization of

$\mathcal{S}$-cotorsion pairs, in the sense that for the former concept it is not required that $\mathcal{S}$ is thick or $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$ either. So, it is natural to think of a more general version of the just mentioned interplay. Our second main goal will be to present relative versions of Frobenius pairs and weak AB-contexts, which we shall call *cut Frobenius pairs* and *cut weak AB-contexts*, so that the previous interplay can be extended to the context of cut cotorsion pairs.

Cut cotorsion pairs, cut Frobenius pairs, and cut weak AB-contexts are useful to describe several situations related to approximation theory. We shall support this claim presenting several examples in the context of relative Gorenstein homological algebra, in part motivated by the behavior of Gorenstein projective and projective modules mentioned at the very beginning, but also for a better understanding of the new concepts and results. More complex examples are exhibited at the end of this article, for particular abelian categories such as chain complexes and quasi-coherent sheaves. Moreover, some applications are given with the purpose to describe some well-known results in the study of finitistic dimensions of rings, right adjoints of Serre quotients, and cotorsion pairs and co-t-structures in triangulated categories.

Organization. In Section 1, we recall some preliminary notions from relative homological algebra. Among these, the most important are the concepts of Frobenius pairs, cotorsion pairs, and Gorenstein objects relative to GP-admissible pairs. Section 2 is devoted to present the main concept of our research: complete left and right cotorsion pairs $(\mathcal{A}, \mathcal{B})$ cut along subcategories $\mathcal{S} \subseteq \mathcal{C}$. We give in Proposition 2.3 some examples of such pairs coming from left Frobenius pairs. Moreover, in Proposition 2.23, we show how to construct a complete cut cotorsion pair from classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$ satisfying a series of mild conditions. One of these conditions will be key to motivate and understand the concepts of Frobenius pairs and weak AB contexts cut along subcategories presented in Section 3. GP-admissible pairs $(\mathcal{X}, \mathcal{Y})$ satisfying certain conditions are the main source to obtain Frobenius pairs cut along Gorenstein objects relative to $(\mathcal{X}, \mathcal{Y})$, as we show in Proposition 3.8. On the other hand, in the case where $\mathcal{X}$ and $\mathcal{Y}$ have some closure properties, we see in Example 3.16 that it is possible to obtain three different types of weak AB contexts cut along the class of objects with finite $\mathcal{X}$-resolution dimension. The most important result in this section is Theorem 3.18, where it is shown that it is not possible to obtain non-trivial weak AB contexts from Gorenstein objects relative to hereditary complete cotorsion pairs. In Section 4, we prove two correspondence theorems between cut Frobenius pairs, cut weak AB contexts, and certain complete cut cotorsion pairs. More specifically, in Theorem 4.6, we establish a one-to-one correspondence (up to equivalence relations) between Frobenius pairs and weak AB contexts cut along a certain $\mathcal{S} \subseteq \mathcal{C}$. We also obtain in Theorem 4.12 another bijective correspondence between weak AB contexts cut along $\mathcal{S}$ and certain complete cotorsion pairs $(\mathcal{F}, \mathcal{G})$ cut along the smallest thick subcategory containing $\mathcal{F}$. Finally, in Section 5, we present detailed examples of complete cut cotorsion pairs, cut Frobenius pairs, and cut weak AB contexts related to relative Gorenstein homological algebra, chain complexes, quasi-coherent sheaves, the Finitistic Dimension Conjecture, Serre subcategories, and extriangulated categories.

Conventions. Throughout, $\mathcal{C}$ will always denote an abelian category (not necessarily with enough projective and injective objects), unless otherwise specified. The main examples of such categories considered in this article will be:

- $\mathsf{Mod}(R)$ = left R-modules over an associative ring R with identity. For simplicity, all modules over R will be left R-modules.
- $\mathsf{mod}(\Lambda)$ = finitely generated modules over an Artin algebra Λ.
- $\mathsf{Ch}(\mathcal{C})$ = chain complexes of objects in $\mathcal{C}$. For the case where $\mathcal{C} = \mathsf{Mod}(R)$, the corresponding category of chain complexes of R-modules will be denoted by $\mathsf{Ch}(R)$. Objects in $\mathsf{Ch}(\mathcal{C})$ are denoted as $X_\bullet$, X_m denotes the m-th component of $X_\bullet$ in $\mathcal{C}$, and $Z_m(X_\bullet)$ denotes the m-th cycle of $X_\bullet$ in $\mathcal{C}$.
- $\mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})$ = right $\mathfrak{A}$-modules. Here, $\mathfrak{A}$ is a skeletally small additive category. A right $\mathfrak{A}$-module is a contravariant additive functor $\mathfrak{A} \longrightarrow \mathsf{Mod}(\mathbb{Z})$.
- $\mathfrak{Qcoh}(X)$ = quasi-coherent sheaves over a semi-separated scheme X.

Subcategories of $\mathcal{C}$ are always assumed to be full. We shall make no distinction between the terms "classes of objects of $\mathcal{C}$" and "subcategories of $\mathcal{C}$". Given $X, Y \in \mathcal{C}$, we denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ the group of morphisms $X \to Y$. In case X and Y are isomorphic, we write $X \simeq Y$. The notation $F \cong G$, on the other hand, is reserved to denote the existence of a natural isomorphism between functors F and G. Monomorphisms and epimorphisms in $\mathcal{C}$ are denoted by using the arrows $\rightarrowtail$ and $\twoheadrightarrow$, respectively.

We shall refer to commutative grids whose rows and columns are exact sequences as *solid diagrams*.

Finally, we point out that the definitions and results presented in this article have their corresponding dual statements, which will be omitted for simplicity. Moreover, although the new concepts in Sections 2, 3, and 4 below will be stated for abelian categories, one can expect that most of them carry over to any extriangulated category after revising [26, 33, 34].

1. Preliminaries

Resolution dimensions. Let $\mathcal{B} \subseteq \mathcal{C}$ and $C \in \mathcal{C}$. The *resolution dimension of C with respect to $\mathcal{B}$* (or the *$\mathcal{B}$-resolution dimension of C*, for short), denoted $\mathrm{resdim}_{\mathcal{B}}(C)$, is the smallest integer $m \geq 0$ such that there exists an exact sequence

$$B_m \rightarrowtail B_{m-1} \to \cdots \to B_1 \to B_0 \twoheadrightarrow C,$$

where $B_k \in \mathcal{B}$ for every integer $0 \leq k \leq m$. If such m does not exist, we set $\mathrm{resdim}_{\mathcal{B}}(C) := \infty$. Dually, we have the concept of *coresolution dimension of C with respect to $\mathcal{B}$*, denoted by $\mathrm{coresdim}_{\mathcal{B}}(C)$. With respect to these two homological dimensions, we shall frequently consider the following subcategories of $\mathcal{C}$:

$$\mathcal{B}_m^{\wedge} := \{C \in \mathcal{C} : \mathrm{resdim}_{\mathcal{B}}(C) \leq m\}, \qquad \text{and} \qquad \mathcal{B}^{\wedge} := \bigcup_{m \geq 0} \mathcal{B}_m^{\wedge},$$

$$\mathcal{B}_m^{\vee} := \{C \in \mathcal{C} : \mathrm{coresdim}_{\mathcal{B}}(C) \leq m\}, \qquad \text{and} \qquad \mathcal{B}^{\vee} := \bigcup_{m \geq 0} \mathcal{B}_m^{\vee}.$$

Orthogonality with respect to extension bifunctors. In any abelian category $\mathcal{C}$, we can define the extension bifunctors $\mathrm{Ext}_{\mathcal{C}}^i(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Mod}(\mathbb{Z})$, with $i \geq 1$, in the sense of Yoneda. We shall also identify $\mathrm{Ext}_{\mathcal{C}}^0(-, -)$ with the hom bifunctor $\mathrm{Hom}_{\mathcal{C}}(-, -)$. The reader can check for instance [43] for a detailed treatise on this matter.

Given $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ and $i \geq 0$, the notation $\mathrm{Ext}_{\mathcal{C}}^i(\mathcal{A}, \mathcal{B}) = 0$ will mean that $\mathrm{Ext}_{\mathcal{C}}^i(A, B)$ vanishes for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Recall that the *right i-th orthogonal complement of $\mathcal{A}$* is defined by $\mathcal{A}^{\perp_i} := \{N \in \mathcal{C} : \mathrm{Ext}_{\mathcal{C}}^i(\mathcal{A}, N) = 0\}$, and the *total right orthogonal complement of $\mathcal{A}$* by $\mathcal{A}^{\perp} := \bigcap_{i \geq 1} \mathcal{A}^{\perp_i}$. Dually, we have the *$i$-th* and *the total left orthogonal complements $^{\perp_i}\mathcal{B}$ and $^{\perp}\mathcal{B}$ of $\mathcal{B}$*, respectively.

Relative homological dimensions. Given $\mathcal{X} \subseteq \mathcal{C}$ and $M \in \mathcal{C}$, the *relative projective dimension of M with respect to $\mathcal{X}$*, denoted $\mathrm{pd}_{\mathcal{X}}(M)$, is the smallest integer $n \geq 0$ such that $\mathrm{Ext}_{\mathcal{C}}^i(M, \mathcal{X}) = 0$ for every $i > n$. If such n does not exist, we set $\mathrm{pd}_{\mathcal{X}}(M) = \infty$. Furthermore, the *relative projective dimension of $\mathcal{Y} \subseteq \mathcal{C}$ with respect to $\mathcal{X}$* is defined as $\mathrm{pd}_{\mathcal{X}}(\mathcal{Y}) := \sup\{\mathrm{pd}_{\mathcal{X}}(Y) : Y \in \mathcal{Y}\}$. Dually, we denote by $\mathrm{id}_{\mathcal{X}}(M)$ and $\mathrm{id}_{\mathcal{X}}(\mathcal{Y})$ the *relative injective dimension of M and $\mathcal{Y}$*, respectively, *with respect to $\mathcal{X}$*. It can be seen that $\mathrm{pd}_{\mathcal{X}}(\mathcal{Y}) = \mathrm{id}_{\mathcal{Y}}(\mathcal{X})$. If $\mathcal{X} = \mathcal{C}$, we just write $\mathrm{pd}(M)$, $\mathrm{pd}(\mathcal{Y})$, $\mathrm{id}(M)$ and $\mathrm{id}(\mathcal{Y})$, for the (absolute) projective and injective dimensions.

Resolving subcategories. Let $\mathcal{P}$ and $\mathcal{I}$ denote the subcategories of projective and injective objects in $\mathcal{C}$, respectively. It is said that a subcategory $\mathcal{X}$ is *resolving* if $\mathcal{P} \subseteq \mathcal{X}$ and if it is closed under extensions and under epi-kernels (that is, under taking kernels of epimorphisms between objects in $\mathcal{X}$). If the dual properties hold true, then $\mathcal{X}$ is said to be *coresolving*. A subcategory is *left thick* if it is closed under extensions, epi-kernels, and direct summands. *Right thick subcategories* are defined dually. Finally, a subcategory is *thick* if it is both left and right thick. For $\mathcal{X} \subseteq \mathcal{C}$, we shall denote by $\mathsf{Thick}(\mathcal{X})$ the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{X}$.

Approximations. Let $\mathcal{X} \subseteq \mathcal{C}$. An $\mathcal{X}$-*precover* of $C \in \mathcal{C}$ is a morphism $f \colon X \to C$ with $X \in \mathcal{X}$ such that the induced homomorphism $\mathrm{Hom}_{\mathcal{C}}(X', f) \colon \mathrm{Hom}_{\mathcal{C}}(X', X) \to \mathrm{Hom}_{\mathcal{C}}(X', C)$ is epic for every $X' \in \mathcal{X}$. An $\mathcal{X}$-precover $f \colon X \to C$ is *special* if it is epic and $\mathrm{Ker}(f) \in \mathcal{X}^{\perp_1}$. The dual concept is called (*special*) $\mathcal{X}$-*preenvelope*.

Cotorsion pairs. Two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{C}$ form a *cotorsion pair* $(\mathcal{X}, \mathcal{Y})$ if they are complete with respect to the orthogonality relation defined by the vanishing of the functor $\mathrm{Ext}^1_{\mathcal{C}}(-, -)$ (see for instance [14, 15, 17, 21]). For the purpose of this article, it comes handy to split this concept as follows.

Definition 1.1. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$. The pair $(\mathcal{X}, \mathcal{Y})$ is a **left cotorsion pair** in $\mathcal{C}$ if $\mathcal{X} = {}^{\perp_1}\mathcal{Y}$. If in addition, for every $C \in \mathcal{C}$ there exists a short exact sequence $Y \rightarrowtail X \twoheadrightarrow C$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, then $(\mathcal{X}, \mathcal{Y})$ is a **complete left cotorsion pair**. Dually, we have the notions of (**complete**) **right cotorsion pairs** in $\mathcal{C}$. Finally, $(\mathcal{X}, \mathcal{Y})$ is a (**complete**) **cotorsion pair** in $\mathcal{C}$ if it is both a (complete) left and right cotorsion pair. A pair $(\mathcal{X}, \mathcal{Y})$ is called **hereditary** if $\mathrm{Ext}^i_{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = 0$ for every $i \geq 1$.*

Remark 1.2.
  (1) *If $(\mathcal{X}, \mathcal{Y})$ is a complete left cotorsion pair in $\mathcal{C}$, then every object of $\mathcal{C}$ has a special $\mathcal{X}$-precover.*
  (2) *$(\mathcal{P}, \mathcal{C})$ is a complete left cotorsion pair if, and only if, $\mathcal{C}$ has enough projective objects. Dually, $(\mathcal{C}, \mathcal{I})$ is a complete right cotorsion pair if, and only if, $\mathcal{C}$ has enough injective objects.*
  (3) *If $(\mathcal{X}, \mathcal{Y})$ is a hereditary cotorsion pair, then $\mathcal{X}$ is resolving and $\mathcal{Y}$ is coresolving. Moreover, in an abelian category $\mathcal{C}$ with enough projective (resp., injective) objects, $(\mathcal{X}, \mathcal{Y})$ is hereditary if, and only if, $\mathcal{X}$ is resolving (resp., $\mathcal{Y}$ is coresolving).*

Example 1.3. *There are some well-known important examples of hereditary complete cotorsion pairs:*

  (1) *The **flat** or **Enochs' cotorsion pair** in $\mathsf{Mod}(R)$ given by $(\mathcal{F}(R), (\mathcal{F}(R))^{\perp_1})$, where $\mathcal{F}(R)$ denotes the subcategory of all flat R-modules.*
  (2) *From [12, Coroll. 4.2], [42, Lem. 4.25] and [11, Lem. A.1], we have the non-affine version of the previous example, for quasi-compact and semi-separated schemes X, given by the pair $(\mathfrak{F}(X), (\mathfrak{F}(X))^{\perp_1})$ in $\mathfrak{Qcoh}(X)$, where $\mathfrak{F}(X)$ denotes the subcategory of quasi-coherent flat sheaves over X.*
  (3) *From [14], if R is an Iwanaga–Gorenstein ring, we have the pairs $(\mathcal{GP}(R), \mathcal{P}(R)^{\wedge})$ and $(\mathcal{I}(R)^{\wedge}, \mathcal{GI}(R))$ in $\mathsf{Mod}(R)$, where $\mathcal{GP}(R)$ and $\mathcal{GI}(R)$ denote the subcategories of Gorenstein projective and Gorenstein injective R-modules.*
      Similar assertions hold for the subcategories $\mathcal{DP}(R)$ and $\mathcal{DI}(R)$ of Ding projective and Ding injective R-modules, provided that R is a Ding–Chen ring (see Gillespie's [19] for details).

Frobenius pairs. The concept of left and right Frobenius pairs was introduced in [3, Def. 2.5] from the notion of (co)generators in Auslander–Buchweitz approximation theory. Given $\mathcal{X}, \omega \subseteq \mathcal{C}$, recall that ω is said to be a *relative cogenerator in $\mathcal{X}$* if $\omega \subseteq \mathcal{X}$ and if for every $X \in \mathcal{X}$ there exists a short exact sequence $X \rightarrowtail W \twoheadrightarrow X'$ where $W \in \omega$ and $X' \in \mathcal{X}$.

Definition 1.4. *A pair $(\mathcal{X}, \omega)$ of subcategories of $\mathcal{C}$ is a **left Frobenius pair** if the following conditions hold true:*

  (lFp1) *$\mathcal{X}$ is left thick.*
  (lFp2) *ω is closed under direct summands.*
  (lFp3) *ω is an $\mathcal{X}$-injective (that is, $\mathrm{id}_{\mathcal{X}}(\omega) = 0$) relative cogenerator in $\mathcal{X}$.*

*The notions of **relative generator** and **right Frobenius pair** in $\mathcal{C}$ are dual.*

We summarize in the following result the most important properties of Frobenius pairs that will be used in the sequel. The proofs of these properties can be found in [3, Thms. 2.8, 2.11, 2.16, 3.6 3.7, Props. 2.7, 2.13 & 3.5].

Proposition 1.5. *Let $(\mathcal{X}, \omega)$ be a pair of subcategories of $\mathcal{C}$. The following assertions hold.*

(1) *If ω is $\mathcal{X}$-injective, then $\omega^\wedge$ is $\mathcal{X}$-injective.*

(2) *If in addition, ω is a relative cogenerator in $\mathcal{X}$ and closed under direct summands, then $\omega = \mathcal{X} \cap \omega^\wedge$.*

(3) *If $\mathcal{X}$ is closed under extensions, $0 \in \mathcal{X}$, and ω is a relative cogenerator in $\mathcal{X}$, then for any $C \in \mathcal{C}$ with $\mathrm{resdim}_\mathcal{X}(C) = n < \infty$, there exist short exact sequences $K \rightarrowtail X \twoheadrightarrow C$ and $C \rightarrowtail H \twoheadrightarrow X'$ in $\mathcal{C}$ with $X, X' \in \mathcal{X}$, $\mathrm{resdim}_\omega(K) = n - 1$ and $\mathrm{resdim}_\omega(H) \leq n$.*

(4) *If $(\mathcal{X}, \omega)$ is a left Frobenius pair, then $\mathcal{X}^\wedge = \mathsf{Thick}(\mathcal{X})$.*

(5) *If $\mathcal{X}$ is closed under extensions, and ω is closed under direct summands and an $\mathcal{X}$-injective relative cogenerator in $\mathcal{X}$, then $\omega^\wedge = \mathcal{X}^\perp \cap \mathcal{X}^\wedge$.*

(6) *If $\mathcal{X}$ is left thick, $\mathcal{Y} \subseteq \mathcal{X}^\wedge$ is right thick and $\omega = \mathcal{X} \cap \mathcal{Y}$ is an $\mathcal{X}$-injective relative cogenerator in $\mathcal{X}$, then $\mathcal{X}^\wedge \cap \mathcal{X}^{\perp_1} = \omega^\wedge$.*

(7) *If $(\mathcal{X}, \omega)$ is a left Frobenius pair, then $(\mathcal{X}, \omega^\wedge)$ is a complete cotorsion pair in the exact subcategory $\mathcal{X}^\wedge$. If in addition, ω is an $\mathcal{X}$-projective relative generator in $\mathcal{X}$, then $(\omega, \mathcal{X}^\wedge)$ is also a complete cotorsion pair in $\mathcal{X}^\wedge$.*

Relative Gorenstein objects. Most of our examples in this article will be built from Gorenstein objects relative to certain pairs $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{C}$ (see Definition 1.7 below). Before specifying how these Gorenstein objects are defined, recall that a chain complex $X_\bullet = (X_m)_{m \in \mathbb{Z}} \in \mathsf{Ch}(\mathcal{C})$ is said to be $\mathrm{Hom}_\mathcal{C}(-, \mathcal{Y})$-*acyclic* if the complex of abelian groups $\mathrm{Hom}_\mathcal{C}(X_\bullet, Y) = (\mathrm{Hom}_\mathcal{C}(X_m, Y))_{m \in \mathbb{Z}}$ is exact for every $Y \in \mathcal{Y}$. $\mathrm{Hom}_\mathcal{C}(\mathcal{Y}, -)$-*acyclic complexes* are defined dually. The following concept is due to [4, Def. 3.2].

Definition 1.6. *Let $(\mathcal{X}, \mathcal{Y})$ be a pair of subcategories of $\mathcal{C}$. An object $C \in \mathcal{C}$ is $(\mathcal{X}, \mathcal{Y})$-**Gorenstein projective** if C is the 0-th cycle of an exact and $\mathrm{Hom}_\mathcal{C}(-, \mathcal{Y})$-acyclic complex $X_\bullet \in \mathsf{Ch}(\mathcal{C})$ where $X_m \in \mathcal{X}$ for every $m \in \mathbb{Z}$. Dually $(\mathcal{X}, \mathcal{Y})$-**Gorenstein injective objects** are defined as 0-cycles of exact and $\mathrm{Hom}_\mathcal{C}(\mathcal{X}, -)$-acyclic complexes with components in $\mathcal{Y}$.*

Following [4], let us denote by $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$ and $\mathcal{GI}_{(\mathcal{X}, \mathcal{Y})}$ the subcategories of $(\mathcal{X}, \mathcal{Y})$-Gorenstein projective and $(\mathcal{X}, \mathcal{Y})$-Gorenstein injective objects of $\mathcal{C}$, respectively. For example, $\mathcal{GP}_{(\mathcal{P}, \mathcal{P})}$ and $\mathcal{GI}_{(\mathcal{I}, \mathcal{I})}$ are precisely the subcategories of Gorenstein projective and Gorenstein injective objects of $\mathcal{C}$, which we shall write as $\mathcal{GP}$ and $\mathcal{GI}$, for simplicity. Moreover, Definition 1.6 also covers the following examples of relative Gorenstein projective and injective objects:

- Ding projective and Ding injective modules, in the sense of [19, Defs. 3.2 & 3.7], by setting $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}(R), \mathcal{F}(R))$ and $(\mathcal{X}, \mathcal{Y}) = (\mathrm{FP}\text{-}\mathcal{I}(R), \mathcal{I}(R))$, respectively. Here, $\mathrm{FP}\text{-}\mathcal{I}(R)$ stands for the subcategory of FP-injective (or absolutely pure) R-modules.
- Gorenstein AC-projective and Gorenstein AC-injective modules, in the sense of [5, §5 & §8], by setting the pairs $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}(R), \mathcal{L}(R))$ and $(\mathcal{X}, \mathcal{Y}) = (\mathrm{FP}_\infty\text{-}\mathcal{I}(R), \mathcal{I}(R))$, respectively. Here, $\mathcal{L}(R)$ and $\mathrm{FP}_\infty\text{-}\mathcal{I}(R)$ denote the subcategories of level and FP_∞-injective (or absolutely clean) R-modules (see [5, Def. 2.6]). These subcategories of relative Gorenstein modules will be denoted by $\mathcal{GP}_{\mathrm{AC}}(R)$ and $\mathcal{GI}_{\mathrm{AC}}(R)$, for simplicity.
- Gorenstein flat sheaves over a noetherian and semi-separated scheme X, by setting $(\mathcal{X}, \mathcal{Y}) = (\mathfrak{F}(X), \mathfrak{F}(X) \cap (\mathfrak{F}(X))^{\perp_1})$. See Murfet and Salarian's [35, Thm. 4.18]. The subcategory of Gorenstein flat sheaves over X will be denoted by $\mathfrak{GF}(X)$. In particular, the latter holds in the affine case $X = \mathrm{Spec}(R)$ provided that R is a commutative noetherian ring.

Many useful properties of Gorenstein objects relative to $(\mathcal{X}, \mathcal{Y})$ are obtained in the case where $(\mathcal{X}, \mathcal{Y})$ is a GP-admissible or a GI-admissible pair [4, Defs. 3.1 & 3.6]. We recall this notion for further referring.

Definition 1.7. *A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{C}$ is **GP-admissible** if the following conditions are satisfied:*

(GPa1) $\mathrm{pd}_{\mathcal{Y}}(\mathcal{X}) = 0$.
(GPa2) $\mathcal{C}$ *has enough $\mathcal{X}$-objects, that is, for every $C \in \mathcal{C}$ there exists an epimorphism $X \twoheadrightarrow C$ with $X \in \mathcal{X}$.*
(GPa3) $\mathcal{X}$ *and $\mathcal{Y}$ are closed under finite coproducts, and $\mathcal{X}$ is closed under extensions.*
(GPa4) $\mathcal{X} \cap \mathcal{Y}$ *is a relative cogenerator in $\mathcal{X}$.*

*A pair $(\mathcal{X}, \mathcal{Y})$ satisfying the dual conditions is called **GI-admissible**.*

Example 1.8.

(1) *Every hereditary complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is a GP-admissible pair, and also induces the GP-admissible pair $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$.*
(2) *The pairs $(\mathcal{P}(R), \mathcal{F}(R))$, $(\mathcal{P}(R), \mathcal{L}(R))$ and $(\mathfrak{F}(X), (\mathfrak{F}(X))^{\perp_1})$ are GP-admissible for any ring R and any noetherian and semi-separated scheme X. Dually, the pairs $(\mathrm{FP}\text{-}\mathcal{I}(R), \mathcal{I}(R))$ and $(\mathrm{FP}_{\infty}\text{-}\mathcal{I}(R), \mathcal{I}(R))$ are clearly GI-admissible.*

We summarize in the following result the most important properties of GP-admissible pairs and relative Gorenstein objects that will be used in the sequel. The proofs of these properties can be found in [4, Thms. 3.30, 3.32 & 3.34, Corolls. 3.15, 3.17, 3.25, 3.33 & 4.10].

Proposition 1.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathcal{C}$.*

(1) *If $(\mathcal{X}, \mathcal{Y})$ is a hereditary pair, then so is $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \mathcal{Y}^{\wedge})$. If in addition $\mathcal{X}$ is closed under extensions and the intersection $\omega := \mathcal{X} \cap \mathcal{Y}$ is closed under finite coproducts and a relative cogenerator in $\mathcal{X}$, then $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$ is left thick.*

If $(\mathcal{X}, \mathcal{Y})$ is a GP-admissible pair, then the following assertions hold:

(2) *The pair $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \mathcal{Y})$ is GP-admissible and $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$ is left thick.*
(3) *The subcategory ω is closed under extensions and a relative $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$-injective cogenerator in $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$.*
(4) *If ω is closed under direct summands, then $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \omega)$ is a left Frobenius pair and $\omega = \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y}$.*
(5) *If $\mathcal{Y}^{\wedge}$, ω and $\mathcal{X} \cap \mathcal{Y}^{\wedge}$ are closed under direct summands, then $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y}^{\wedge} = \omega = \mathcal{X} \cap \mathcal{Y}^{\wedge}$.*

In the particular case where $(\mathcal{X}, \mathcal{Y})$ is a right complete hereditary cotorsion pair, one has that $\mathcal{X} = \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$.

2. Complete cut cotorsion pairs and cotorsion cuts

We present the following concept of cotorsion pairs relative to subcategories.

Definition 2.1. *Let $\mathcal{S}, \mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$. We say that $(\mathcal{A}, \mathcal{B})$ is a **left cotorsion pair cut along $\mathcal{S}$** if the following conditions are satisfied:*

(lccp1) $\mathcal{A}$ *is closed under direct summands.*
(lccp2) $\mathcal{A} \cap \mathcal{S} = {}^{\perp_1}\mathcal{B} \cap \mathcal{S}$.

A left cotorsion pair $(\mathcal{A}, \mathcal{B})$ *cut along* $\mathcal{S}$ *is* **complete** *if in addition the following holds:*

(lccp3)   *For every* $S \in \mathcal{S}$, *there exists a short exact sequence* $B \rightarrowtail A \twoheadrightarrow S$ *with* $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$.

Dually, we say that $(\mathcal{A}, \mathcal{B})$ *is a* (**complete**) **right cotorsion pair cut along** $\mathcal{S}$ *if it satisfies the dual conditions, labeled as* (rccp1), (rccp2) *(and* (rccp3)*). Finally,* $(\mathcal{A}, \mathcal{B})$ *is a* (**complete**) **cotorsion pair cut along** $\mathcal{S}$ *if it is both a (complete) left and right cotorsion pair cut along* $\mathcal{S}$.

In case there is no need to refer to the subcategory $\mathcal{S}$, *we shall simply say that* $(\mathcal{A}, \mathcal{B})$ *is a (complete) left and/or right* **cut cotorsion pair**.

If $(\mathcal{A}, \mathcal{B})$ *is a complete (left or right) cotorsion pair cut along* $\mathcal{S}$, *we may sometimes refer to* $\mathcal{S}$ *as a* (**left or right**) **cotorsion cut for** $(\mathcal{A}, \mathcal{B})$. *If* $\mathcal{A}$ *is closed under direct summands, we shall denote by* $\mathrm{lCuts}(\mathcal{A}, \mathcal{B})$ *the class of left cotorsion cuts for* $(\mathcal{A}, \mathcal{B})$. *Similarly, we shall denote by* $\mathrm{rCuts}(\mathcal{A}, \mathcal{B})$ *the class of right cotorsion cuts for* $(\mathcal{A}, \mathcal{B})$ *provided that* $\mathcal{B}$ *is closed under direct summands, and by* $\mathrm{Cuts}(\mathcal{A}, \mathcal{B})$ *the class of cotorsion cuts for* $(\mathcal{A}, \mathcal{B})$ *provided that* $\mathcal{A}$ *and* $\mathcal{B}$ *are both closed under direct summands.*

Remark 2.2.

(1) *Notice that the previous definition coincides with Definition 1.1 by taking* $\mathcal{S} = \mathcal{C}$. *Furthermore, in case* $\mathcal{A}$ *and* $\mathcal{B}$ *are contained in a thick subcategory* $\mathcal{S} \subseteq \mathcal{C}$, $(\mathcal{A}, \mathcal{B})$ *is a complete left cotorsion pair cut along* $\mathcal{S}$ *if, and only if,* $(\mathcal{A}, \mathcal{B})$ *is a left* $\mathcal{S}$-*cotorsion pair in the sense of* [3, Def. 3.4]. *This implies that several of our results proved below will recover some facts from the theory of (relative) cotorsion pairs appearing in* [3].

(2) *For certain* $\mathcal{A} \subseteq \mathcal{C}$ *closed under direct summands, it is possible to find another subcategory* $\mathcal{B} \subseteq \mathcal{C}$ *such that* $\mathrm{lCuts}(\mathcal{A}, \mathcal{B}) = \emptyset$. *Consider for instance* $\mathcal{A}$ *the subcategory of all objects in* $\mathcal{C}$ *isomorphic to 0, and* $\mathcal{B} := \mathcal{C} - \mathcal{A}$ *the subcategory of nonzero objects in* $\mathcal{C}$. *Notice that there is no subcategory* $\mathcal{S} \subseteq \mathcal{C}$ *satisfying condition* (lccp3). *Dually, one can find a pair* $(\mathcal{A}, \mathcal{B})$ *with* $\mathcal{B}$ *closed under direct summands for which* $\mathrm{rCuts}(\mathcal{A}, \mathcal{B}) = \emptyset$. *Nevertheless, for any two subcategories* $\mathcal{A}$ *and* $\mathcal{B}$ *of objects in* $\mathcal{C}$ *closed under direct summands, one has* $\mathrm{Cuts}(\mathcal{A}, \mathcal{B}) \neq \emptyset$. *Indeed, if* $\mathcal{A}$ *is a subcategory closed under direct summands, a sufficient condition to have* $\mathrm{lCuts}(\mathcal{A}, \mathcal{B}) \neq \emptyset$ *is that* $\mathcal{B}$ *is a pointed subcategory of* $\mathcal{C}$ *(that is,* $0 \in \mathcal{B}$*). Similarly,* $\mathrm{rCuts}(\mathcal{A}, \mathcal{B}) \neq \emptyset$ *if* $\mathcal{A}$ *is pointed and* $\mathcal{B}$ *is closed under direct summands. It suffices to take* $\mathcal{S} := \{0\}$.

Now let us give some examples of complete cut cotorsion pairs which are not necessarily complete cotorsion pairs. Frobenius pairs and relative Gorenstein objects will be the main source to construct our first examples.

Proposition 2.3. *Let* $(\mathcal{X}, \omega)$ *be a left Frobenius pair in* $\mathcal{C}$. *The following assertions hold:*

(1) $(\mathcal{X}, \omega^\wedge)$ *is a complete cotorsion pair cut along* $\mathcal{X}^\wedge$ *and* $\omega^\wedge$.
(2) $(\omega, \mathcal{X}^{\perp_1})$ *is a complete cotorsion pair cut along* $\omega^\wedge$.
(3) $(\omega, \mathcal{X}^{\perp_1})$ *is a complete left cotorsion pair cut along* $\mathcal{X}^\wedge$ *if, and only if,* $\mathcal{X}^\wedge = \omega^\wedge$.

Proof. First, note by (lFp1), (lFp2) and [3, Thm. 2.11 & Prop. 2.13] that $\mathcal{X}$, ω and $\omega^\wedge$ are closed under direct summands. Also, it is clear that the same holds for $\mathcal{X}^{\perp_1}$. Moreover, $\mathcal{X}^\wedge$ is thick by [3, Thm. 2.11].

(1) By the previous comments, we have that the pair $(\mathcal{X}, \omega^\wedge)$ and $\mathcal{X}^\wedge$ satisfy (lccp1) and (rccp1). Moreover, from [3, Thm. 2.8] we clearly obtain (lccp3) and (rccp3). Finally, conditions (lccp2) and (rccp2) follow from [3, Part 1. of Prop. 2.7], (lccp1), (rccp1), (lccp3) and (rccp3). Hence, $\mathcal{X}^\wedge \in \mathrm{Cuts}(\mathcal{X}, \omega^\wedge)$. The assertion $\omega^\wedge \in \mathrm{Cuts}(\mathcal{X}, \omega^\wedge)$ can be easily deduced from the previous.

(2) We already have conditions (lccp1) and (rccp1) for $(\omega, \mathcal{X}^{\perp_1})$. On the one hand, for every $C \in \omega^\wedge$ it is clear the existence of a short exact sequence $K \rightarrowtail W \twoheadrightarrow C$, where $W \in \omega$ and $K \in \omega^\wedge$. Since $\omega^\wedge \subseteq \mathcal{X}^{\perp_1}$ by [3, Part 1. of Prop. 2.7], we have that $K \in \mathcal{X}^{\perp_1}$. Thus, (lccp3) follows. On the other hand, (rccp3) is immediate. Regarding (lccp2), note that the containment $\omega \cap \omega^\wedge \subseteq$

$^{\perp_1}(\mathcal{X}^{\perp_1}) \cap \omega^\wedge$ follows by (lFp3). The converse containment follows by (lccp3) and the fact that ω is closed under direct summands. Finally, condition (rccp2) follows by (lFp3) and (rccp3).

(3) The implication ($\Leftarrow$) follows from part 2. For the direct implication, suppose that $\mathcal{X}^\wedge \in$ lCuts($\omega, \mathcal{X}^{\perp_1}$). Then, for every $C \in \mathcal{X}^\wedge$ there exists a short exact sequence $K \rightarrowtail W \twoheadrightarrow C$ with $W \in \omega$ and $K \in \mathcal{X}^{\perp_1}$. By (lFp1), (lFp3) and Proposition 1.5, we can note that $K \in \mathcal{X}^{\perp_1} \cap \mathcal{X}^\wedge = \omega^\wedge$. It then follows that $C \in \omega^\wedge$. $\square$

Remark 2.4. *Part (3) of Proposition 2.3 suggests that there should exist a left Frobenius pair $(\mathcal{X}, \omega)$ in $\mathcal{C}$ for which $\mathcal{X}^\wedge \notin$ lCuts($\omega, \mathcal{X}^{\perp_1}$). This is for instance the case of the left Frobenius pair $(\mathcal{GP}(R), \mathcal{P}(R))$ [3, Prop. 6.1]. Indeed, if R is an Iwanaga–Gorenstein ring with infinite global dimension, and $M \in \mathcal{GP}(R) - \mathcal{P}(R)$, then it is not possible to construct a short exact sequence $K \rightarrowtail P \twoheadrightarrow M$ with P projective and $K \in \mathcal{GP}(R)^{\perp_1}$.*

Recall from [4, Def. 3.3] that, for a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{C}$, the $(\mathcal{X}, \mathcal{Y})$-*Gorenstein projective dimension* of an object $C \in \mathcal{C}$, which we denote by $\mathrm{Gpd}_{(\mathcal{X}, \mathcal{Y})}(C)$, is defined as the $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$-resolution dimension of C. Note that setting $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}(R), \mathcal{P}(R))$ and $(\mathcal{X}, \mathcal{Y}) = (\mathcal{P}(R), \mathcal{F}(R))$ yields the *Gorenstein projective* and the *Ding projective dimensions* of an R-module C, which we denote by $\mathrm{Gpd}(C)$ and $\mathrm{Dpd}(C)$ for simplicity. The $(\mathcal{X}, \mathcal{Y})$-*Gorenstein injective*, *Gorenstein injective* and *Ding injective dimensions* $\mathrm{Gid}_{(\mathcal{X}, \mathcal{Y})}(C)$, $\mathrm{Gid}(C)$ and $\mathrm{Did}(C)$, are defined dually.

Concerning relative Gorenstein dimensions, we recall the following properties from [4, Coroll. 4.3].

Proposition 2.5. *Let $(\mathcal{X}, \mathcal{Y})$ be a GP-admissible pair in $\mathcal{C}$ and $\omega := \mathcal{X} \cap \mathcal{Y}$. Then, the following are equivalent for any $C \in \mathcal{C}$:*

(a) $\mathrm{Gpd}_{(\mathcal{X}, \mathcal{Y})}(C) \leq n$.

(b) There is a short exact sequence $K \rightarrowtail G \twoheadrightarrow C$ with $\mathrm{resdim}_\omega(K) \leq n - 1$ and $G \in \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$.

(c) There is an exact sequence $C \rightarrowtail H \twoheadrightarrow G'$ with $\mathrm{resdim}_\omega(H) \leq n$ and $G' \in \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$.

Example 2.6. *We know from the previous remark that $(\mathcal{GP}(R), \mathcal{P}(R))$ is a left Frobenius pair over any ring R. So it follows by parts (1) and (2) of Proposition 2.3 that $(\mathcal{GP}(R), \mathcal{P}(R)^\wedge)$ is a complete cotorsion pair cut along $\mathcal{GP}(R)^\wedge$ and $\mathcal{P}(R)^\wedge$, and that $(\mathcal{P}(R), \mathcal{GP}(R)^{\perp_1})$ is a complete cotorsion pair cut along $\mathcal{P}(R)^\wedge$. Similar results hold for the left Frobenius pairs $(\mathcal{DP}(R), \mathcal{P}(R))$ and $(\mathcal{GP}_{\mathrm{AC}}(R), \mathcal{P}(R))$ (see [3, Coroll. 6.11 and Prop. 6.12]).*

Remark 2.7. *There are important differences between the notions of $\mathcal{S}$-cotorsion pairs $(\mathcal{A}, \mathcal{B})$ [3, Def. 3.4] and complete cotorsion pairs $(\mathcal{A}, \mathcal{B})$ cut along $\mathcal{S}$. In the former, $\mathcal{S}$ is taken as a thick subcategory of $\mathcal{C}$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{S}$. The latter containments do not occur for instance in the previous example, since Gorenstein projective R-modules may have infinite projective dimension. More examples of complete cut cotorsion pairs which are not relative cotorsion pairs are given below in Proposition 2.24, Corollary 2.25 and Example 2.26.*

We shall mention a couple of extra properties for the previous example after showing the following general result.

Proposition 2.8. *The following hold for every left Frobenius pair $(\mathcal{X}, \omega)$ in $\mathcal{C}$:*

(1) The following conditions are equivalent:

(a) $(\mathcal{X}, \omega^\wedge)$ is a complete left cotorsion pair in $\mathcal{C}$.

(b) $(\mathcal{X}, \omega^\wedge)$ is a complete cotorsion pair in $\mathcal{C}$.

(c) $\mathcal{C} = \mathcal{X}^\wedge$.

(2) For every $n \geq 0$, $(\mathcal{X}, \mathcal{X}^\perp)$ is a complete cotorsion pair cut along $\mathcal{X}_n^\wedge$.

Proof. In what follows, some of the mentioned facts come from Proposition 1.5. In the first part, let us first assume condition (a). We need to verify that $\omega^\wedge = \mathcal{X}^{\perp_1}$ and that for every $C \in \mathcal{C}$ there exists a short exact sequence of the form $C \rightarrowtail H \twoheadrightarrow X$ with $H \in \omega^\wedge$ and $X \in \mathcal{X}$. For the latter, let $C \in \mathcal{C}$ and consider a short exact sequence $K \rightarrowtail X \twoheadrightarrow C$ with $X \in \mathcal{X}$ and $K \in \omega^\wedge$. On the other hand, since ω is a relative cogenerator in $\mathcal{X}$, there is a short exact sequence $X \rightarrowtail W \twoheadrightarrow X'$ with $W \in \omega$ and $X' \in \mathcal{X}$. Taking the push-out of $W \leftarrow X \rightarrow C$ yields the following solid diagram:

$$
\begin{array}{ccccc}
K & \rightarrowtail & X & \twoheadrightarrow & C \\
\| & & \downarrow & \text{po} & \downarrow \\
K & \rightarrowtail & W & \twoheadrightarrow & H \\
& & \downarrow & & \downarrow \\
& & X' & = & X'
\end{array}
\tag{i}
$$

Note that $H \in \omega^\wedge$. Then, the right-hand column in (i) is the desired short exact sequence for the right completeness of $(\mathcal{X}, \omega^\wedge)$. In order to show $\omega^\wedge = \mathcal{X}^{\perp_1}$, note that the containment $\omega^\wedge \subseteq \mathcal{X}^{\perp_1}$ is clear since $\mathrm{id}_\mathcal{X}(\omega^\wedge) = \mathrm{id}_\mathcal{X}(\omega) = 0$. The converse containment follows from the right completeness and the fact that $\omega^\wedge$ is closed under direct summands. Hence, (a) $\Rightarrow$ (b) follows.

The implication (b) $\Rightarrow$ (c) follows by the left completeness of the cotorsion pair $(\mathcal{X}, \omega^\wedge)$ and the containment $\omega^\wedge \subseteq \mathcal{X}^\wedge$. On the other hand, we know that $(\mathcal{X}, \omega^\wedge)$ is a $\mathcal{X}^\wedge$-cotorsion pair (that is, a complete cotorsion pair in the exact subcategory $\mathcal{X}^\wedge$). Thus, the implication (c) $\Rightarrow$ (a) is clear.

Now for the assertion (2) $\mathcal{X}_n^\wedge \in \mathrm{Cuts}(\mathcal{X}, \mathcal{X}^\perp)$, we already know that $\mathcal{X}$ is closed under direct summands. In order to show (lccp2), note that the containment $\mathcal{X} \cap \mathcal{X}_n^\wedge \subseteq {}^{\perp_1}(\mathcal{X}^\perp) \cap \mathcal{X}_n^\wedge$ is clear. For the converse, if we take $C \in {}^{\perp_1}(\mathcal{X}^\perp) \cap \mathcal{X}_n^\wedge$, then there exists a short exact sequence $K \rightarrowtail X \twoheadrightarrow C$ with $X \in \mathcal{X}$ and $K \in \omega_{n-1}^\wedge$[1]. Note also that $K \in \mathcal{X}^\perp$. Then, the previous sequence splits and so $C \in \mathcal{X}$. On the other hand, for (rccp2) $\mathcal{X}^\perp \cap \mathcal{X}_n^\wedge = \mathcal{X}^{\perp_1} \cap \mathcal{X}_n^\wedge$, the containment ($\subseteq$) is clear. Now if $C \in \mathcal{X}^{\perp_1} \cap \mathcal{X}_n^\wedge$ we can find a short exact sequence $C \rightarrowtail H \twoheadrightarrow C'$ where $H \in \omega_n^\wedge$ and $C' \in \mathcal{X}$, which is split and so C is a direct summand of H. This in turn implies that $C \in \mathcal{X}^\perp$. The previous arguments also show (lccp3) and (rccp3). $\qquad\square$

Corollary 2.9. *The following hold for every GP-admissible pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$ with $\omega := \mathcal{X} \cap \mathcal{Y}$ closed under direct summands:*

(1) The following conditions are equivalent:

(a) $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \omega^\wedge)$ is a complete left cotorsion pair in $\mathcal{C}$.

(b) $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \omega^\wedge)$ is a complete cotorsion pair in $\mathcal{C}$.

(c) $\mathcal{C} = \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}^\wedge$.

(2) For every $n \geq 0$, $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^\perp)$ is a complete cotorsion pair cut along $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^\wedge$.

Proof. It follows after applying Proposition 2.8 to the pair $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \omega)$, which is left Frobenius by Proposition 1.9. $\qquad\square$

Remark 2.10. *Although in all of our examples of GP-admissible pairs $(\mathcal{X}, \mathcal{Y})$, the subcategory $\omega := \mathcal{X} \cap \mathcal{Y}$ is closed under direct summands, another proof of Corollary 2.9 can be obtained without assuming this property. Indeed, keep in mind the properties from Propositions 1.9 and 2.5, and consider the pair $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^\perp)$. The closure under direct summands of $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^\perp$ is clear, and the same property holds for $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$. Also, the following containments are clear:*

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[1]Note that for the case $n = 0$ we simply take $K = 0$.

$$\mathcal{GP}_{(\mathcal{X},\mathcal{Y})} \cap (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^{\wedge} \subseteq {}^{\perp_1}((\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^{\perp}) \cap (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^{\wedge},$$
$$(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^{\perp} \cap (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^{\wedge} \subseteq (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^{\perp_1} \cap (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^{\wedge}.$$

On the other hand, for every $C \in {}^{\perp_1}((\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^{\perp}) \cap (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^{\wedge}$ there exists a short exact sequence $K \rightarrowtail G \twoheadrightarrow C$ with $G \in \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ and $K \in \omega_{n-1}^{\wedge}$. Also, we can note that $\omega_{n-1}^{\wedge} \subseteq (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^{\perp_1}$. Then, (lccp2) *and* (lccp3) *follow. The containment*

$$(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^{\perp} \cap (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^{\wedge} \supseteq (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})^{\perp_1} \cap (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})})_n^{\wedge}$$

follows as in the proof of part (2) of Proposition 2.8. Hence, (rccp2) *follows, and* (rccp3) *is also a consequence from Propositions 1.9 and 2.5.*

Example 2.11.

   (1) *From Corollary 2.9 we can note that it is not always possible to extend a cotorsion cut associated to a pair to the whole category $\mathcal{C}$. Indeed, consider the complete cotorsion pair $(\mathcal{GP}(R), \mathcal{P}(R)^{\wedge})$ cut along $\mathcal{GP}(R)^{\wedge}$ from Example 2.6. Then, we have that $(\mathcal{GP}(R), \mathcal{P}(R)^{\wedge})$ is a complete cotorsion pair in $\mathsf{Mod}(R)$ if, and only if, $\mathsf{Mod}(R) = \mathcal{GP}(R)^{\wedge}$. The latter equality occurs, for instance, if R an Iwanaga-Gorenstein ring, but it is not true in general.*

   (2) *We can also characterize when the complete cotorsion pair $(\mathcal{P}(R), \mathcal{GP}(R)^{\perp_1})$ cut along $\mathcal{P}(R)^{\wedge}$ is a complete cotorsion pair in $\mathsf{Mod}(R)$. Specifically, the pair $(\mathcal{P}(R), \mathcal{GP}(R)^{\perp_1})$ is a complete cotorsion pair in $\mathsf{Mod}(R)$ if, and only if, $\mathcal{P}(R) = \mathcal{GP}(R)$. The latter equality occurs, for instance, over any ring with finite global dimension.*

In [3, Prop. 3.5], it is given an alternative description of relative cotorsion pairs. Following the spirit of this result, we present the following characterization for cut cotorsion pairs. Its proof is straightforward.

Proposition 2.12. *Let $\mathcal{S}, \mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$. Then, $(\mathcal{A}, \mathcal{B})$ is a complete left cotorsion pair cut along $\mathcal{S}$ if, and only if, $\mathcal{A}$ and $\mathcal{B}$ satisfy the following conditions:*

   (1) *$\mathcal{A}$ is closed under direct summands;*
   (2) *$\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{A} \cap \mathcal{S}, \mathcal{B}) = 0$; and*
   (3) *for every $S \in \mathcal{S}$ there is an exact sequence $B \rightarrowtail A \twoheadrightarrow S$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

Remark 2.13. *Regarding condition (3) in Proposition 2.12, in the case $\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{A}, \mathcal{B}) = 0$, the morphism $A \twoheadrightarrow S$ is an $\mathcal{A}$-precover.*

Getting new cotorsion cuts and pairs from old ones. In the following result, whose proof is straightforward and left to the reader, we assert that the class $\mathrm{lCuts}(\mathcal{A}, \mathcal{B})$ is closed under restrictions, arbitrary unions and intersections.

Proposition 2.14. *The following properties hold for $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$.*

   (1) **Restriction:** *If $(\mathcal{A}, \mathcal{B})$ is a (complete) left cotorsion pair cut along $\mathcal{S}$ and $\mathcal{X} \subseteq \mathcal{S}$, then $(\mathcal{A}, \mathcal{B})$ is a (complete) left cotorsion pair cut along $\mathcal{X}$.*

If $\{\mathcal{S}_i\}_{i \in I}$ is a nonempty family of subcategories of $\mathcal{C}$, then the following hold:

   (2) **Unions:** *$(\mathcal{A}, \mathcal{B})$ is a (complete) left cotorsion pair cut along $\mathcal{S} := \bigcup_{i \in I} \mathcal{S}_i$ if, and only if, $(\mathcal{A}, \mathcal{B})$ is a (complete) left cotorsion pair cut along $\mathcal{S}_i$, for every $i \in I$.*
   (3) **Intersections:** *If $(\mathcal{A}, \mathcal{B})$ is a (complete) left cotorsion pair cut along $\mathcal{S}_i$ for every $i \in I$, then $(\mathcal{A}, \mathcal{B})$ is a (complete) left cotorsion pair cut along $\mathcal{S} := \bigcap_{i \in I} \mathcal{S}_i$.*

Remark 2.15.

(1) *It is not always true that cotorsion cuts can be extended to a bigger subcategory, as shown in Example 2.11 (1).*

(2) *The converse of the intersection property does not hold in general. It suffices to consider the pair $(\mathcal{GP}(R), \mathcal{P}(R)^\wedge)$ from Example 2.6 and $\mathcal{S}_1 := \mathcal{GP}(R)^\wedge$ and $\mathcal{S}_2 := \mathsf{Mod}(R)$.*

(3) *By the union property and its dual, we can note that $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in $\mathcal{C}$ if, and only if, there exists a family $\{\mathcal{S}_i\}_{i \in I}$ of subcategories of $\mathcal{C}$ such that $\mathcal{C} = \bigcup_{i \in I} \mathcal{S}_i$ and that $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair cut along $\mathcal{S}_i$ for every $i \in I$.*

Example 2.16. *Over any ring R, $(\mathcal{GP}(R), \mathcal{GP}(R)^{\perp_1})$ is a complete cotorsion pair cut along $\mathcal{GP}(R)^\wedge$. This clearly follows by Corollary 2.9 and by the union property.*

Maximal cotorsion cuts. From the union property, it is natural to think of the possibility of finding the largest cotorsion cut for a pair $(\mathcal{A}, \mathcal{B})$. Indeed, assuming that $\mathcal{A}$ is closed under direct summands and that $0 \in \mathcal{B}$ (and so $\mathrm{lCuts}(\mathcal{A}, \mathcal{B}) \neq \emptyset$), it is possible to define the largest cut for $(\mathcal{A}, \mathcal{B})$ as a certain union.

Definition 2.17. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ with $\mathcal{A}$ closed under direct summands and $0 \in \mathcal{B}$. The **maximal left cotorsion cut of $(\mathcal{A}, \mathcal{B})$** is the union*

$$\mathbb{S}_l(\mathcal{A}, \mathcal{B}) := \bigcup \{\mathcal{S} : \mathcal{S} \in \mathrm{lCuts}(\mathcal{A}, \mathcal{B})\}.$$

*The **maximal right cotorsion cut** and the **maximal cotorsion cut of $(\mathcal{A}, \mathcal{B})$** are defined similarly, and will be denoted by $\mathbb{S}_r(\mathcal{A}, \mathcal{B})$ and $\mathbb{S}(\mathcal{A}, \mathcal{B})$.*

The subcategory $\mathbb{S}_l(\mathcal{A}, \mathcal{B})$ has the interesting property that, under some mild conditions, it can cover the subcategory of all objects satisfying (lccp3). Let us denote this subcategory by $\mathcal{E}_l(\mathcal{A}, \mathcal{B})$, that is, $\mathcal{E}_l(\mathcal{A}, \mathcal{B})$ is formed by all the objects $C \in \mathcal{C}$ for which there is an exact sequence $B \rightarrowtail A \twoheadrightarrow C$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We can note that $\mathbb{S}_l(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{E}_l(\mathcal{A}, \mathcal{B})$, although the equality does not hold in general. Below we show that if $\mathcal{E}_l(\mathcal{A}, \mathcal{B})$ is a left cotorsion cut of $(\mathcal{A}, \mathcal{B})$, then it has to be the maximal one.

Theorem 2.18. *The following conditions are equivalent for any $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$, where $\mathcal{A}$ is closed under direct summands and $0 \in \mathcal{B}$:*

(a) $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = 0.$

(b) $\mathbb{S}_l(\mathcal{A}, \mathcal{B}) = \mathcal{E}_l(\mathcal{A}, \mathcal{B}).$

(c) $\mathcal{E}_l(\mathcal{A}, \mathcal{B}) \in \mathrm{lCuts}(\mathcal{A}, \mathcal{B}).$

(d) $\mathcal{A} = {}^{\perp_1}\mathcal{B} \cap \mathcal{E}_l(\mathcal{A}, \mathcal{B}).$

Proof. Note first that $\mathcal{A} \subseteq \mathcal{E}_l(\mathcal{A}, \mathcal{B})$ since $0 \in \mathcal{B}$. Then, $\mathcal{A} \cap \mathcal{E}_l(\mathcal{A}, \mathcal{B}) = \mathcal{A}$, and so the implication (c) $\Rightarrow$ (d) is clear. On the other hand, the implications (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a) are trivial. Thus, we only focus on proving that (a) $\Rightarrow$ (b).

Suppose that $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = 0$. Note that the containment $\mathbb{S}_l(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{E}_l(\mathcal{A}, \mathcal{B})$ is clear. Now let $X \in \mathcal{E}_l(\mathcal{A}, \mathcal{B})$. We prove that $(\mathcal{A}, \mathcal{B})$ is a complete left cotorsion pair cut along $\mathcal{S} := \mathbb{S}_l(\mathcal{A}, \mathcal{B}) \cup \{X\}$. For this, we only need to prove (lccp2). Consider the following two cases:

(1) $X \in \mathcal{A}$: Since $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = 0$ we have that $X \in {}^{\perp_1}\mathcal{B}$. Thus, $\mathcal{A} \cap \{X\} = \{X\} = {}^{\perp_1}\mathcal{B} \cap \{X\}$.

(2) $X \notin \mathcal{A}$: Since $X \in \mathcal{E}_l(\mathcal{A}, \mathcal{B})$, there exists a non-split short exact sequence $B \rightarrowtail A \twoheadrightarrow X$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. It follows that $X \notin {}^{\perp_1}\mathcal{B}$, and so $\mathcal{A} \cap \{X\} = \emptyset = {}^{\perp_1}\mathcal{B} \cap \{X\}$.

In both cases, we get the equality $\mathcal{A} \cap \{X\} = {}^{\perp_1}\mathcal{B} \cap \{X\}$. Therefore, $\mathbb{S}_l(\mathcal{A}, \mathcal{B}) \cup \{X\} \in \mathrm{lCuts}(\mathcal{A}, \mathcal{B})$, and so $\mathbb{S}_l(\mathcal{A}, \mathcal{B}) \cup \{X\} = \mathbb{S}_l(\mathcal{A}, \mathcal{B})$ by maximality, proving that $\mathbb{S}_l(\mathcal{A}, \mathcal{B}) \supseteq \mathcal{E}_l(\mathcal{A}, \mathcal{B})$. $\square$

A similar equivalence holds for the subcategory $\mathcal{E}_r(\mathcal{A}, \mathcal{B})$ of all objects satisfying (rccp3), and for $\mathcal{E}(\mathcal{A}, \mathcal{B}) := \mathcal{E}_l(\mathcal{A}, \mathcal{B}) \cap \mathcal{E}_r(\mathcal{A}, \mathcal{B})$.

Remark 2.19. *As we mentioned earlier, $\mathbb{S}_l(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{E}_l(\mathcal{A}, \mathcal{B})$. In some cases this containment is strict and nontrivial, that is, we can find subcategories $\mathcal{A}$ and $\mathcal{B}$ such that $\{0\} \subsetneq \mathbb{S}_l(\mathcal{A}, \mathcal{B}) \subsetneq \mathcal{E}_l(\mathcal{A}, \mathcal{B})$. Indeed, let us consider $\mathcal{A} = \mathcal{GP}(R)$ and $\mathcal{B} = \mathsf{Mod}(R)$. One can note that $(\mathcal{GP}(R), \mathsf{Mod}(R))$ is a complete left cotorsion pair cut along $\mathcal{P}(R)^\wedge$, and so $\mathcal{P}(R)^\wedge \subseteq \mathbb{S}_l(\mathcal{GP}(R), \mathsf{Mod}(R))$, that is, $\mathbb{S}_l(\mathcal{GP}(R), \mathsf{Mod}(R)) \neq \{0\}$. On the other hand, by Theorem 2.18 we have that*

$$\mathbb{S}_l(\mathcal{GP}(R), \mathsf{Mod}(R)) = \mathcal{E}_l(\mathcal{GP}(R), \mathsf{Mod}(R)) \Longleftrightarrow \mathrm{Ext}_R^1(\mathcal{GP}(R), \mathsf{Mod}(R)) = 0,$$

and there are rings over which the latter condition does not hold (see Example 2.11 (2)).

Compatibility between cotorsion cuts. So far the methods we have showed to obtain new cotorsion cuts are restricted to a fixed pair $(\mathcal{A}, \mathcal{B})$ of subcategories of $\mathcal{C}$. In some cases, it is possible to get new pairs along new cuts. Specifically, we show in Proposition 2.21 below an extension of the union property of Proposition 2.14 (2), in the sense that it is possible to take the union of two different complete cut cotorsion pairs along the union of their cuts, provided that certain compatibility condition between the given pairs is satisfied.

Definition 2.20. *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{C}$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed under direct summands and $0 \in \mathcal{B}_1 \cap \mathcal{B}_2$, and let $\mathcal{S}_1 \in \mathrm{lCuts}(\mathcal{A}_1, \mathcal{B}_1)$ and $\mathcal{S}_2 \in \mathrm{lCuts}(\mathcal{A}_2, \mathcal{B}_2)$ be cotorsion cuts in $\mathcal{C}$. We shall say that $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2)$ are **compatible complete left cut cotorsion pairs** if the following two conditions hold:*

(1) $\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{A}_1, \mathcal{B}_2) = 0$ and $\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{A}_2, \mathcal{B}_1) = 0$.
(2) $\mathcal{A}_1 \cap \mathcal{S}_2 = \mathcal{A}_2 \cap \mathcal{S}_1$.

Compatible complete right cut cotorsion pairs *and* **compatible complete cut cotorsion pairs** *are defined similarly.*

The following result is straightforward and its proof is left to the reader.

Proposition 2.21. *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{S}_1$ and $\mathcal{S}_2$ be as in Definition 2.20. If $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2)$ are compatible complete left cut cotorsion pairs, then $\mathcal{S}_1 \cup \mathcal{S}_2 \in \mathrm{lCuts}(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}_1 \cup \mathcal{B}_2)$.*

Example 2.22. *The complete cotorsion pairs $(\mathcal{GP}(R), \mathcal{P}(R)^\wedge)$ and $(\mathcal{P}(R), \mathcal{GP}(R)^{\perp_1})$ cut along $\mathcal{GP}(R)^\wedge$ and $\mathcal{P}(R)^\wedge$, respectively, are compatible. Indeed, conditions (1) and (2) of Definition 2.20 are clear by [14, Prop. 10.2.3], and the dual of (2), that is, the equality $\mathcal{GP}(R)^{\perp_1} \cap \mathcal{GP}(R)^\wedge = \mathcal{P}(R)^\wedge$ follows by Proposition 1.5.*

More induced cotorsion cuts and examples. Previously we showed how to construct new cotorsion cuts from a given one, or from a family of cotorsion cuts (as in Propositions 2.14 and 2.21). In the last part of this section, we give some sufficient conditions on subcategories $\mathcal{A}, \mathcal{B}, \mathcal{S} \subseteq \mathcal{C}$, without needing that $\mathcal{S} \in \mathrm{Cuts}(\mathcal{A}, \mathcal{B})$, which imply that $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair cut along $\mathcal{A}^\wedge \cap \mathcal{S}$. The subcategories $\mathcal{E}_l(\mathcal{A}, \mathcal{B})$, $\mathcal{E}_r(\mathcal{A}, \mathcal{B})$, and $\mathcal{E}(\mathcal{A}, \mathcal{B})$ will be useful to prove the following result.

Proposition 2.23. *Let $\mathcal{S}, \mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ and $\omega := \mathcal{A} \cap \mathcal{B}$ such that the following are satisfied:*

(1) $\mathcal{A}$ is closed under extensions and direct summands;
(2) $\mathcal{B}$ is closed under direct summands;
(3) $\omega \cap \mathcal{S}$ is a relative cogenerator in $\mathcal{A}$;
(4) $(\omega \cap \mathcal{S})^\wedge \subseteq \mathcal{B}$;
(5) $\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{A} \cap \mathcal{S}, \mathcal{B}) = 0$ and $\mathrm{Ext}_{\mathcal{C}}^1(\mathcal{A}, \mathcal{B} \cap \mathcal{S}) = 0$.

Then, $\mathcal{A}^\wedge \cap \mathcal{S} \in \mathrm{Cuts}(\mathcal{A}, \mathcal{B})$.

Proof. First, we show that $\mathcal{A}_n^\wedge \subseteq \mathcal{E}_r(\mathcal{A}, (\omega \cap \mathcal{S})^\wedge)$ for every $n \geq 0$ (and so $\mathcal{A}^\wedge \subseteq \mathcal{E}_r(\mathcal{A}, (\omega \cap \mathcal{S})^\wedge)$) by using induction on n.

- Initial step: $\mathcal{A}_0^\wedge \subseteq \mathcal{E}_r(\mathcal{A}, (\omega \cap \mathcal{S})^\wedge)$ follows by condition (3).
- Induction step: Suppose that $n \geq 1$ and that $\mathcal{A}_{n-1}^\wedge \subseteq \mathcal{E}_r(\mathcal{A}, (\omega \cap \mathcal{S})^\wedge)$ holds. Now consider a short exact sequence $L \rightarrowtail A \twoheadrightarrow M$ with $A \in \mathcal{A}$ and $\operatorname{resdim}_\mathcal{A}(L) = n - 1$. By the induction hypothesis, there is a short exact sequence $L \rightarrowtail K \twoheadrightarrow A'$ with $A' \in \mathcal{A}$ and $K \in (\omega \cap \mathcal{S})^\wedge$. Taking the pushout of $K \leftarrow L \rightarrow A$ yields the following solid diagram:

$$
\begin{array}{ccc}
L & \rightarrowtail A & \twoheadrightarrow M \\
\downarrow & \quad \text{po} \quad \downarrow & \parallel \\
K & \rightarrowtail E & \twoheadrightarrow M \\
\downarrow & \downarrow & \\
A' & = A' &
\end{array}
\tag{ii}
$$

Note that $E \in \mathcal{A}$ by condition (1), and so by condition (3) there exists a short exact sequence $E \rightarrowtail W \twoheadrightarrow A''$ with $W \in \omega \cap \mathcal{S}$ and $A'' \in \mathcal{A}$. Now take the pushout of $W \leftarrow E \rightarrow M$ to obtain the following solid diagram:

$$
\begin{array}{ccc}
K & \rightarrowtail E & \twoheadrightarrow M \\
\parallel & \downarrow & \quad \text{po} \quad \downarrow \\
K & \rightarrowtail W & \twoheadrightarrow F \\
& \downarrow & \downarrow \\
& A'' & = A''
\end{array}
\tag{iii}
$$

We have that $F \in (\omega \cap \mathcal{S})^\wedge$, and so the right-hand column of (iii) implies that $M \in \mathcal{E}_r(\mathcal{A}, (\omega \cap \mathcal{S})^\wedge)$.

The containment $\mathcal{A}^\wedge \subseteq \mathcal{E}_r(\mathcal{A}, (\omega \cap \mathcal{S})^\wedge)$ then holds true. Moreover, from the central row in (ii) and condition (4) we have that for every $M \in \mathcal{A}^\wedge$ with $\operatorname{resdim}_\mathcal{A}(M) \geq 1$ there is a short exact sequence $B \rightarrowtail A \twoheadrightarrow M$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Moreover, for the case $n = 0$ we can simply take $A = M$ and $B = 0$ in this sequence since $0 \in \mathcal{B}$, as $\mathcal{B}$ is closed under direct summands. In other words, we also have that the containment $\mathcal{A}^\wedge \subseteq \mathcal{E}_l(\mathcal{A}, \mathcal{B})$ holds. Hence, from the latter along with $\mathcal{A}^\wedge \subseteq \mathcal{E}_r(\mathcal{A}, (\omega \cap \mathcal{S})^\wedge)$ and condition (4) we conclude that $\mathcal{A}^\wedge \subseteq \mathcal{E}(\mathcal{A}, \mathcal{B})$. It is clear now that the pair $(\mathcal{A}, \mathcal{B})$ satisfies conditions (lccp3) and (rccp3) with respect to the subcategory $\mathcal{A}^\wedge \cap \mathcal{S}$, and we also know from the hypotheses the validity of (lccp1) and (rccp1). Finally, by condition (5) we have that $\mathcal{A} \cap (\mathcal{A}^\wedge \cap \mathcal{S}) \subseteq {}^{\perp_1}\mathcal{B} \cap (\mathcal{A}^\wedge \cap \mathcal{S})$ and that $\mathcal{B} \cap (\mathcal{A}^\wedge \cap \mathcal{S}) \subseteq \mathcal{A}^{\perp_1} \cap (\mathcal{A}^\wedge \cap \mathcal{S})$, and the converse containments follow by (lccp3) and (rccp3). Therefore, $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair cut along $\mathcal{A}^\wedge \cap \mathcal{S}$. $\square$

Let us apply the previous result to obtain another example of a complete cut cotorsion pair from Gorenstein objects relative to a GP-admissible pair.

Proposition 2.24. *Let $(\mathcal{X}, \mathcal{Y})$ be a GP-admissible pair in $\mathcal{C}$ and $\omega := \mathcal{X} \cap \mathcal{Y}$, such that $\mathcal{Y}^\wedge$, ω and $\mathcal{X} \cap \mathcal{Y}^\wedge$ are closed under direct summands. Then, $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \mathcal{Y}^\wedge)$ is a complete cotorsion pair cut along $\mathcal{X}^\wedge$. Moreover,*

$$
\mathcal{GP}_{(\mathcal{X},\mathcal{Y})} \cap \mathcal{Y}^\wedge = \omega = \mathcal{X} \cap \mathcal{Y}^\wedge = \mathcal{Y} \cap \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}.
\tag{iv}
$$

Proof. We have by hypothesis that $\mathcal{Y}^\wedge$ is closed under direct summands. On the other hand, by [4, Coroll. 3.33] we also have that $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ is closed under extensions and direct summands. We then have that conditions (1) and (2) in Proposition 2.23 are valid. Now let us show that $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})} \cap \mathcal{Y}^\wedge \cap \mathcal{X}^\wedge$ is a relative cogenerator in $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$. By [4, Thm. 3.34 (c) & (d)] we know that the equality (iv) holds. Then, $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})} \cap \mathcal{Y}^\wedge \cap \mathcal{X}^\wedge = \omega \cap \mathcal{X}^\wedge = \omega$, and so condition (3) in Proposition 2.23 follows as well by [4, Coroll. 3.25 (a)], while condition (4) is clear. Finally, the orthogonality relations in condition (5) of Proposition 2.23 are a consequence of [4, Coroll. 3.15]. $\qquad\square$

Corollary 2.25. *Let $(\mathcal{X}, \mathcal{Y})$ be a hereditary complete cotorsion pair in $\mathcal{C}$ and $\omega := \mathcal{X} \cap \mathcal{Y}$. Then, $(\mathcal{GP}_{(\mathcal{X},\omega)}, \omega^\wedge)$ and $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{Y})$ are complete cotorsion pairs cut along $\mathcal{X}^\wedge$. Moreover,*

$$\mathcal{GP}_{(\mathcal{X},\omega)} \cap \omega^\wedge = \omega = \mathcal{X} \cap \omega^\wedge. \tag{v}$$

Proof. Recall from Example 1.8 (1) that $(\mathcal{X}, \omega)$ is a GP-admissible pair. Moreover, it is clear that $\mathcal{X}$ and ω are closed under direct summands, while the same holds for $\omega^\wedge$ by Proposition 1.5. Then, $\mathcal{X} \cap \omega^\wedge$ is also closed under direct summands, and Proposition 2.24 implies that $(\mathcal{GP}_{(\mathcal{X},\omega)}, \omega^\wedge)$ is a complete cotorsion pair cut along $\mathcal{X}^\wedge$ and the equality (v).

For the second pair $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{Y})$, the subcategories $\mathcal{A} := \mathcal{GP}_{(\mathcal{X},\omega)}$, $\mathcal{B} := \mathcal{Y}$ and $\mathcal{S} := \mathcal{X}^\wedge$ satisfy the conditions in Proposition 2.23. Indeed, the first four conditions are clear. For (5), it suffices to note that $\mathcal{Y} \cap \mathcal{X}^\wedge = \omega^\wedge$ and $\mathcal{GP}_{(\mathcal{X},\omega)} \cap \mathcal{X}^\wedge = \mathcal{X}$, which follow from the assumptions and Proposition 1.5. Finally, for the equality (v) we have that

$$\mathcal{GP}_{(\mathcal{X},\omega)} \cap \omega^\wedge = \mathcal{GP}_{(\mathcal{X},\omega)} \cap \omega^\wedge \cap \mathcal{X}^\wedge = \mathcal{X} \cap \omega^\wedge = \mathcal{X} \cap \mathcal{Y} \cap \mathcal{X}^\wedge = \omega \cap \mathcal{X}^\wedge = \omega. \qquad\square$$

Example 2.26. *For any ring R, $(\mathcal{DP}(R), \mathcal{F}(R)^\wedge)$ is a complete cotorsion pair cut along $\mathcal{P}(R)^\wedge$ in $\mathsf{Mod}(R)$. Indeed, we know from Example 1.8 (2) that the pair $(\mathcal{P}(R), \mathcal{F}(R))$ is GP-admissible, and $\mathcal{GP}_{(\mathcal{P}(R),\mathcal{F}(R))}$ is precisely the subcategory $\mathcal{DP}(R)$ of Ding projective R-modules. Moreover, it is clear that $\mathcal{P}(R) \cap \mathcal{F}(R) = \mathcal{P}(R)$, $\mathcal{F}(R)^\wedge$ and $\mathcal{P}(R) \cap \mathcal{F}(R)^\wedge = \mathcal{P}(R)$ are closed under direct summands. Then by Proposition 2.24 we get the desired result.*

Note also that this example cannot be obtained by using Corollary 2.25 since in general there is no hereditary complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathsf{Mod}(R)$ such that $\mathcal{X} = \mathcal{P}(R)$ and $\mathcal{X} \cap \mathcal{Y} = \mathcal{F}(R)$. This is possible for example for the trivial cotorsion pair $(\mathcal{P}(R), \mathsf{Mod}(R))$ over a left perfect ring R.

Remark 2.27. *Several of the examples of relative Gorenstein pairs are cut along $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$, but this is not always the case. For instance, the pair $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \mathcal{Y})$ from Corollary 2.25 is another example of a complete cut cotorsion pair that cannot be extended to a bigger subcategory. In this case, one can note that $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{Y})$ is a complete cotorsion pair cut along $\mathcal{GP}_{(\mathcal{X},\omega)}^\wedge$ if, and only if, $\mathcal{GP}_{(\mathcal{X},\omega)} = \mathcal{X}$.*

Intersections of the form $\omega \cap \mathcal{S}$ considered in Proposition 2.23 are not a mere technicality to obtain new cut cotorsion pairs. They are in fact an important component of the notions of cut Frobenius pairs and cut Auslander–Buchweitz contexts, which will be presented and studied in detail in the next section.

3. Cut Frobenius pairs and cut Auslander–Buchweitz contexts

As mentioned in the introduction, one of the main purposes of this article is to describe an interplay between cut Frobenius pairs, cut Auslander–Buchweitz contexts, and complete cut cotorsion pairs. This interplay will be the main topic of the next section. For the moment, we can note the following relation between certain Gorenstein complete cut cotorsion pairs and left Frobenius pairs.

Proposition 3.1. *Let $(\mathcal{X}, \mathcal{Y})$ be a GP-admissible pair in $\mathcal{C}$, with $\omega := \mathcal{X} \cap \mathcal{Y}$ closed under direct summands. Then, the following conditions are equivalent:*

(a) $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \mathcal{Y})$ is a left Frobenius pair.

(b) $\mathcal{Y} \subseteq \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$.

(c) $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \mathcal{Y}^{\wedge})$ is a complete cotorsion pair cut along $\mathcal{GP}^{\wedge}_{(\mathcal{X},\mathcal{Y})}$ with $\mathcal{Y} \subseteq \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$.

Moreover, if any of the previous conditions is satisfied, then $\mathcal{Y} = \omega$.

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b) are trivial, while (a) $\Rightarrow$ (c) follows by Proposition 1.5. Then, we only focus on proving (b) $\Rightarrow$ (a), for which one needs Proposition 1.9. So let us assume that the containment $\mathcal{Y} \subseteq \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ is satisfied. We know that $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ is left thick, that is, condition (IFp1) holds. By hypothesis, we also have that ω is closed under direct summands. On the other hand, we have that $\omega = \mathcal{Y} \cap \mathcal{GP}_{(\mathcal{X},\mathcal{Y})} = \mathcal{Y}$, and then (IFp2) follows. Finally, the fact that $\mathcal{Y} = \omega$ is a $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$-injective relative cogenerator in $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ follows also by Proposition 1.9. $\square$

Remark 3.2. *For any hereditary complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$, with $\omega := \mathcal{X} \cap \mathcal{Y}$, one has that $(\mathcal{X}, \omega)$ is a GP-admissible pair with $\omega \subseteq \mathcal{GP}_{(\mathcal{X},\omega)}$. Then, by Proposition 3.1, $(\mathcal{GP}_{(\mathcal{X},\omega)}, \omega^{\wedge})$ is a complete cotorsion pair cut along $\mathcal{GP}^{\wedge}_{(\mathcal{X},\omega)}$. In particular, we have another way to obtain the first pair appearing in Corollary 2.25, since $\mathcal{X}^{\wedge} \subseteq \mathcal{GP}^{\wedge}_{(\mathcal{X},\omega)}$.*

Some technical lemmas. Before giving the definition of cut Frobenius pairs, we need to prove some preliminary results. The idea is to find conditions under which $\omega^{\wedge}$ is closed under extensions, and for this the Induction Principle will be a frequently used argument.

Lemma 3.3. *Let $\omega, \mathcal{S} \subseteq \mathcal{C}$ such that ω is closed under extensions and $\omega \cap \mathcal{S}$ is a relative generator in ω. Let $C \in \mathcal{C}$ for which there exists an exact sequence*

$$E_n \overset{f_n}{\rightarrowtail} W_{n-1} \to \cdots \to W_1 \overset{f_1}{\to} W_0 \overset{f_0}{\twoheadrightarrow} C, \tag{i}$$
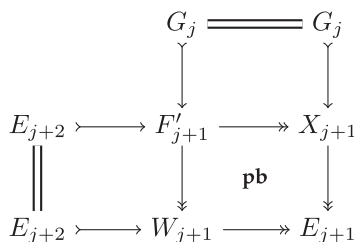
for some $n \geq 1$, with $E_{j+1} := \mathrm{Ker}(f_j)$ and $W_j \in \omega$ for every $0 \leq j \leq n-1$. Then, there exist short exact sequences

$$G_j \rightarrowtail X_{j+1} \twoheadrightarrow E_{j+1}, \qquad \text{and} \qquad X_{j+1} \rightarrowtail F_j \twoheadrightarrow X_j, \tag{ii}$$

where $X_0 := C$, $F_j \in \omega \cap \mathcal{S}$ and $G_j \in \omega$ for every $0 \leq j \leq n-1$.

Proof. Let us prove this result by induction on j.

- Initial step: For the case $j = 0$, since $\omega \cap \mathcal{S}$ is a relative generator in ω, there is a short exact sequence $G_0 \rightarrowtail F_0 \twoheadrightarrow W_0$ with $G_0 \in \omega$ and $F_0 \in \omega \cap \mathcal{S}$. Taking the pullback of $E_1 \to W_0 \leftarrow F_0$ yields the result.

- Induction step: Now suppose that for $1 \leq j \leq n-2$ there are short exact sequences $G_j \rightarrowtail X_{j+1} \twoheadrightarrow E_{j+1}$ and $X_{j+1} \rightarrowtail F_j \twoheadrightarrow X_j$, with $F_j \in \omega \cap \mathcal{S}$ and $G_j \in \omega$. Consider also the $(j+1)$-th splicer from the resolution (i), namely $E_{j+2} \rightarrowtail W_{j+1} \twoheadrightarrow E_{j+1}$. Taking the pullback of $W_{j+1} \to E_{j+1} \leftarrow X_{j+1}$ yields the following solid diagram:

$$
\begin{array}{ccc}
G_j & =\!=\!= & G_j \\
\downarrow & & \downarrow \\
E_{j+2} \rightarrowtail & F'_{j+1} \twoheadrightarrow & X_{j+1} \\
\| & \downarrow \quad \mathbf{pb} & \downarrow \\
E_{j+2} \rightarrowtail & W_{j+1} \twoheadrightarrow & E_{j+1}
\end{array}
$$

Since ω is closed under extensions, $F'_{j+1} \in \omega$. By using again that $\omega \cap \mathcal{S}$ is a generator in ω, we have an exact sequence $G_{j+1} \rightarrowtail F_{j+1} \twoheadrightarrow F'_{j+1}$ with $F_{j+1} \in \omega \cap \mathcal{S}$ and $G_{j+1} \in \omega$. Taking the pullback of $E_{j+2} \to F'_{j+1} \leftarrow F_{j+1}$ yields the result.

Lemma 3.4. *Let $\omega \subseteq \mathcal{C}$ be closed under extensions. If $W \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence with $W \in \omega$ and $C \in \omega^{\wedge}$, then $B \in \omega^{\wedge}$ and* $\mathrm{resdim}_{\omega}(B) \leq \mathrm{resdim}_{\omega}(C)$.

Proof. For $\mathrm{resdim}_{\omega}(C) = 0$, the result follows since ω is closed under extensions. So we may assume that $\mathrm{resdim}_{\omega}(C) \geq 1$. Then, there exists an exact sequence $W' \rightarrowtail W_0 \twoheadrightarrow C$ with $W_0 \in \omega$ and $\mathrm{resdim}_{\omega}(W') = \mathrm{resdim}_{\omega}(C) - 1$. Taking the pullback of $B \to C \leftarrow W_0$ gives the desired inequality. □

We are now ready to prove the main technical lemma of this section, where we give sufficient conditions so that the subcategory $\omega^{\wedge}$ is closed under extensions and direct summands. It is not enough to assume that ω is closed under extensions and direct summands. In addition, we need an auxiliary subcategory $\mathcal{S}$ so that $\omega \cap \mathcal{S}$ is a ω-projective relative generator in ω. Such $\mathcal{S}$ will play the role of a suitable cut to propose a relative version for the concept of left Frobenius pair in Definition 3.6 below.

Lemma 3.5. *Let $\omega, \mathcal{S} \subseteq \mathcal{C}$ such that ω is closed under extensions and $\omega \cap \mathcal{S}$ is an ω-projective relative generator in ω. Then, the following assertions hold true:*

(1) $\omega \cap \mathcal{S}$ is an $\omega^{\wedge}$-projective relative generator in $\omega^{\wedge}$. Moreover, for any $C \in \omega^{\wedge}$ with $\mathrm{resdim}_{\omega}(C) \geq 1$, there exists an exact sequence $K \rightarrowtail F \twoheadrightarrow C$ such that $F \in \omega \cap \mathcal{S}$ and $\mathrm{resdim}_{\omega}(K) = \mathrm{resdim}_{\omega}(C) - 1$.
(2) $\omega^{\wedge}$ is closed under extensions.
(3) If ω is closed under direct summands, then so is $\omega^{\wedge}$.
(4) If ω is closed under isomorphisms and $\mathcal{S}$ is closed under epi-kernels and mono-cokernels, then $\omega^{\wedge} \cap \mathcal{S} = (\omega \cap \mathcal{S})^{\wedge}$.

Proof.

(1) First, we show that $\omega \cap \mathcal{S}$ is a relative generator in $\omega^{\wedge}$. We use induction on $n := \mathrm{resdim}_{\omega}(C)$ for $C \in \omega^{\wedge}$.

- Initial step: This is clear.
- Induction step: For the case $n \geq 1$, we have an exact sequence

$$W_n \rightarrowtail W_{n-1} \to \cdots \to W_1 \to W_0 \twoheadrightarrow C,$$

with $W_k \in \omega$ for every $0 \leq k \leq n$. By Lemma 3.3 and its notation therein, we have that $X_n \in \omega$ since $G_{n-1}, E_n := W_n \in \omega$ and ω is closed under extensions. Glueing together the splicer sequences (ii) in Lemma 3.3 gives rise to the exact sequence

$$X_n \rightarrowtail F_{n-1} \to \cdots \to F_1 \to F_0 \twoheadrightarrow C,$$

with $F_k \in \omega \cap \mathcal{S}$ for every $0 \leq k \leq n - 1$. Thus, for the short exact sequence $X_1 \rightarrowtail F_0 \twoheadrightarrow C$ we have $F_0 \in \omega \cap \mathcal{S}$ and $X_1 \in \omega^{\wedge}$, with $\mathrm{resdim}_{\omega}(X_1) = n - 1$; that is, $\omega \cap \mathcal{S}$ is a relative generator in $\omega^{\wedge}$.

Now by [32, dual of Lem. 2.13 (a) Rmk. 1.2 (a)], we have that

$$\mathrm{pd}_{\omega^{\wedge}}(\omega \cap \mathcal{S}) = \mathrm{id}_{\omega \cap \mathcal{S}}(\omega^{\wedge}) = \mathrm{id}_{\omega \cap \mathcal{S}}(\omega) = \mathrm{pd}_{\omega}(\omega \cap \mathcal{S}) = 0,$$

and so $\omega \cap \mathcal{S}$ is $\omega^{\wedge}$-projective.
(2) Suppose we are given a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ with $A, C \in \omega^{\wedge}$. Let us use induction on $n := \mathrm{resdim}_{\omega}(A)$ to show that $B \in \omega^{\wedge}$.

- Initial step: If $\text{resdim}_\omega(A) = 0$, the result follows by Lemma 3.4.
- Induction step: We may assume that $\text{resdim}_\omega(A) \geq 1$. Suppose also that for any short exact sequence $A' \rightarrowtail B' \twoheadrightarrow C'$ with $C' \in \omega^\wedge$ and $\text{resdim}_\omega(A') \leq n - 1$, one has that $B' \in \omega^\wedge$. By part (1) there is a short exact sequence $C' \rightarrowtail P \twoheadrightarrow C$ with $P \in \omega \cap S$ and $C' \in \omega^\wedge$. We take the pullback of $B \to C \leftarrow P$ to obtain the following solid diagram:

$$
\begin{array}{ccc}
C' & =\!=\!= & C' \\
\downarrow & & \downarrow \\
A \rightarrowtail E & \longrightarrow\!\!\!\!\rightarrow & P \\
\| & \downarrow \quad \mathbf{pb} & \downarrow \\
A \rightarrowtail B & \longrightarrow\!\!\!\!\rightarrow & C
\end{array}
$$

Since $\text{pd}_{\omega^\wedge}(\omega \cap S) = 0$, the central row in this diagram splits, and then $E \simeq A \oplus P$. Now by part (1) again, consider a short exact sequence $A' \rightarrowtail W_0 \twoheadrightarrow A$ with $W_0 \in \omega \cap S$ and $\text{resdim}_\omega(A') = \text{resdim}_\omega(A) - 1$. By taking the direct sum of this sequence and the identity complex on P, we get the short exact sequence $A' \rightarrowtail W_0 \oplus P \twoheadrightarrow A \oplus P$, with $W_0 \oplus P \in \omega$ since ω is closed under extensions. Taking the pullback of $C' \to A \oplus P \leftarrow W_0 \oplus P$ and using the induction hypothesis gives the result.

(3) Given an object $C = C_1 \oplus C_2 \in \omega^\wedge$ with $n := \text{resdim}_\omega(C)$, the proof that $C_1, C_2 \in \omega^\wedge$ follows by induction on n and using an argument similar to the one appearing in [6, Proof of Prop. 5.3].

(4) The containment $\omega^\wedge \cap S \supseteq (\omega \cap S)^\wedge$ is clear since S is closed under mono-cokernels. Now suppose we are given an object $S \in \omega^\wedge \cap S$, that is, an $S \in S$ with a finite ω-resolution $W_n \rightarrowtail W_{n-1} \to \cdots \to W_0 \twoheadrightarrow S$. If $n = 0$, the result follows since ω is closed under isomorphisms. So we may assume that $n \geq 1$. Since S is closed under epi-kernels, from Lemma 3.3 there are short exact sequences $X_{j+1} \rightarrowtail F_j \twoheadrightarrow X_j$ with $X_j \in S$ and $F_j \in \omega \cap S$ for all $0 \leq j \leq n - 1$. Moreover, $X_n \in \omega \cap S$ since $G_{n-1}, E_n := W_n \in \omega$ and ω is closed under extensions. Then, $X_n \rightarrowtail F_{n-1} \to \cdots \to F_0 \twoheadrightarrow S$ is a $(\omega \cap S)$-resolution of S. $\qquad \square$

Cut Frobenius pairs. We are now ready to present the concept of Frobenius pairs cut along subcategories.

Definition 3.6. *Let $\mathcal{X}, \omega, S \subseteq \mathcal{C}$. We say that $(\mathcal{X}, \omega)$ is a **left Frobenius pair cut along S** if the following conditions are satisfied:*

- (lcFp1) $\mathcal{X}$ *is left thick.*
- (lcFp2) $(\mathcal{X} \cap S, \omega \cap S)$ *is a left Frobenius pair in $\mathcal{C}$.*
- (lcFp3) $\omega \cap S$ *is an ω-projective relative generator in ω.*
- (lcFp4) ω *is closed under extensions and direct summands.*

*Dually, we have the notion of **right Frobenius pairs $(v, \mathcal{Y})$ cut along S**.*

Remark 3.7. *For any left Frobenius pair $(\mathcal{X}, \omega)$ cut along S, we can note the following by Lemma 3.5 and [3, Prop. 2.7 (2)]:*

(1) $\omega \cap S$ is closed under extensions and finite coproducts, and it is an $\omega^\wedge$-projective relative generator in $\omega^\wedge$.

(2) $\omega^\wedge$ is closed under extensions and direct summands.

(3) ω is closed under isomorphisms, and so $\omega^\wedge \cap S = (\omega \cap S)^\wedge$ whenever S is closed under epi-kernels and mono-cokernels.

Of course any left Frobenius pair $(\mathcal{X}, \omega)$, with ω closed under finite coprodutos, is a left Frobenius pair cut along $\mathcal{C}$, but not every relative left Frobenius pair is an absolute left Frobenius pair, as we show in Example 3.9 below.

Proposition 3.8. *Let $(\mathcal{X}, \mathcal{Y})$ be a GP-admissible pair in $\mathcal{C}$ such that:*

(1) $\mathcal{X}$ is closed under direct summands;
(2) $\mathcal{Y}$ is closed under extensions and direct summands; and
(3) $\mathcal{X} \cap \mathcal{Y}$ is a relative generator in $\mathcal{Y}$.

Then, $(\mathcal{C}, \mathcal{Y}^\wedge)$ is a left Frobenius pair cut along $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$. Moreover, the following equality holds true:

$$\mathcal{GP}_{(\mathcal{X},\mathcal{Y})} \cap \mathcal{Y}^\wedge = \mathcal{X} \cap \mathcal{Y} = \mathcal{X} \cap \mathcal{Y}^\wedge. \tag{iii}$$

Proof. Let us verify each condition in Definition 3.6, although we need to do this in a specific order. Condition (lcFp1) is clearly satisfied. On the other hand, since $\mathcal{Y}$ is closed under extensions and direct summands (by (2)), and $\mathcal{X} \cap \mathcal{Y}$ is an $\mathcal{Y}$-projective relative generator in $\mathcal{Y}$ (by (3) and (GPa1) in Definition 1.7), we have from Lemma 3.5 (2) and (3) that $\mathcal{Y}^\wedge$ is closed under extensions and direct summands, that is, condition (lcFp4) holds. Moreover, since also $\mathcal{X} \cap \mathcal{Y}$ and $\mathcal{X} \cap \mathcal{Y}^\wedge$ are closed under direct summands, we obtain the equality (iii) from Proposition 1.9. This, along with Lemma 3.5 (1), implies that $\mathcal{Y}^\wedge \cap \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ is a $\mathcal{Y}^\wedge$-projective relative generator in $\mathcal{Y}^\wedge$, that is, we have condition (lcFp3). Finally, condition (lcFp2), that is, that $(\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \mathcal{Y}^\wedge \cap \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}) = (\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}, \mathcal{X} \cap \mathcal{Y})$ is a left Frobenius pair in $\mathcal{C}$, follows by Proposition 1.9. □

Example 3.9. *For any ring R, if $\mathcal{Y}$ is a subcategory of $\mathsf{Mod}(R)$ closed under finite coproducts and containing $\mathcal{P}(R)$, then $(\mathsf{Mod}(R), \mathcal{Y}^\wedge)$ is a left Frobenius pair cut along $\mathcal{GP}_{(\mathcal{P}(R),\mathcal{Y})}$, since $(\mathcal{P}(R), \mathcal{Y})$ is a GP-admissible pair satisfying the conditions of the previous proposition. In particular, we have that $(\mathsf{Mod}(R), \mathcal{P}(R)^\wedge)$ and $(\mathsf{Mod}(R), \mathcal{F}(R)^\wedge)$ are left Frobenius pairs cut along $\mathcal{GP}(R)$ and $\mathcal{DP}(R)$, respectively.*

In a more general sense, consider a hereditary complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$. Then, $(\mathcal{C}, \mathcal{Y}^\wedge)$ is a left Frobenius pair cut along $\mathcal{X}$. Note in this case that $\mathcal{X} = \mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ by Proposition 1.9. On the other hand, for the subcategory $\omega := \mathcal{X} \cap \mathcal{Y}$, we have that $(\mathcal{C}, \omega^\wedge)$ is a left Frobenius pair cut along $\mathcal{GP}_{(\mathcal{X},\omega)}$.

Below we establish necessary and sufficient conditions under which the relative Frobenius pair from the previous example is left Frobenius in $\mathsf{Mod}(R)$. This will be a consequence of the following general result.

Proposition 3.10. *Let $\mathcal{X}, \omega \subseteq \mathcal{C}$ such that $\omega^\wedge \subseteq \mathcal{X}$, $\mathcal{X}$ is left thick, and ω is closed under direct summands and a relative cogenerator in $\mathcal{X}$. Then, the following statements are equivalent:*

(a) $(\mathcal{X}, \omega^\wedge)$ is a left Frobenius pair in $\mathcal{C}$.
(b) $\omega = \mathcal{X}^\perp \cap \mathcal{X}$.

Moreover, if any of the above equivalent conditions holds, then $\omega = \omega^\wedge$.

Proof. The implication (a) $\Rightarrow$ (b) is straightforward. Now suppose that $\omega = \mathcal{X}^\perp \cap \mathcal{X}$. This implies that $(\mathcal{X}, \omega)$ is a left Frobenius pair. By Proposition 1.5, we have that $\omega^\wedge = \mathcal{X}^\perp \cap \mathcal{X}$, and so $\omega^\wedge$ is closed under direct summands. Finally, it is clear that $\omega^\wedge \subseteq \mathcal{X}$ is an $\mathcal{X}$-injective relative cogenerator in $\mathcal{X}$. □

An immediate consequence of the previous result is the following.

Corollary 3.11. *The following assertions are equivalent for any subcategory $\mathcal{Y}$ of R-modules:*

(a) $(\mathsf{Mod}(R), \mathcal{Y}^\wedge)$ is a left Frobenius pair in $\mathsf{Mod}(R)$.
(b) $\mathcal{Y}^\wedge = \mathcal{I}(R)$.

If any of these conditions holds true and $\mathcal{P}(R) \subseteq \mathcal{Y}$, then R is a quasi-Frobenius ring.

From the previous two results, we can note that left Frobenius pairs $(\mathcal{X}, \omega)$ that induce a left Frobenius pair of the form $(\mathcal{X}, \omega^\wedge)$ are scarce. As a matter of fact, what we expect from a left Frobenius pair $(\mathcal{X}, \omega)$ is that $(\mathcal{X}, \omega^\wedge)$ is a complete cut cotorsion pair. This is precisely the case of Gorenstein left Frobenius pairs $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \mathcal{Y})$ in Proposition 3.1. We shall explore this in more detail in Section 4.

Let us now present one more example of cut Frobenius pairs in the context of quiver representations.

Example 3.12. *Several facts in this example are extracted from [48, Ex. 5.3]. Let Λ be the quotient path k-algebra given by the quiver*

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \xleftarrow{\gamma} 3$$
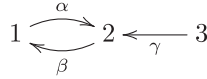
with relations $\alpha\beta = 0 = \beta\alpha$. In the category $\mathsf{mod}(\Lambda)$ of finitely generated (left) Λ-modules, the indecomposable projective Λ-modules are:

$$\begin{smallmatrix}1\\2\end{smallmatrix}, \qquad\qquad \begin{smallmatrix}2\\1\end{smallmatrix} \qquad\qquad \text{and} \qquad\qquad \begin{smallmatrix}3\\2\\1\end{smallmatrix}.$$

On the other hand, the indecomposable injective Λ-modules are:

$$\begin{smallmatrix}3\\2\\1\end{smallmatrix}, \qquad\qquad \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \qquad\qquad \text{and} \qquad\qquad 3.$$
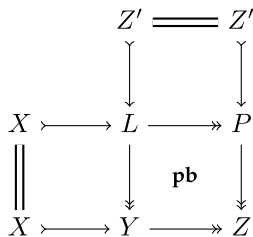
It follows that the Auslander–Reiten quiver of $\mathsf{mod}(\Lambda)$ is given by:
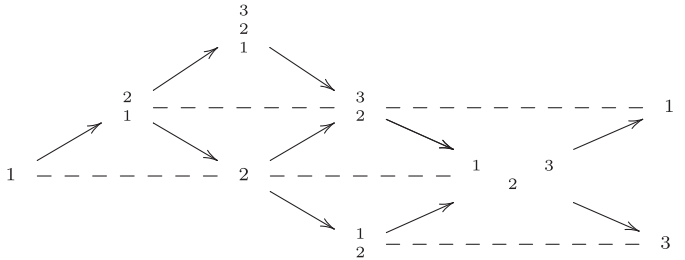
where the two vertices 1 represent the same simple module.

Now let $\mathcal{X} := \mathrm{add}(\begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix})$ *be the subcategory of direct summands of finite coproducts of copies of the Λ-module $\begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$. Then, $\mathcal{X}$ is closed under extensions and a Frobenius subcategory of $\mathsf{mod}(\Lambda)$. Moreover, the subcategory of the projective-injective objects in $\mathcal{X}$ is $\mathcal{P}(\mathcal{X}) = \mathrm{add}(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix})$. Indeed, by using Auslander-Reiten theory, it can be shown that $\mathrm{Ext}^i_\Lambda(\mathcal{X}, \mathcal{P}(\mathcal{X})) = 0$ for every $i \geq 1$.*

We assert that $(\mathsf{mod}(\Lambda), \mathcal{P}(\mathcal{X}))$ *is a left Frobenius pair cut along $\mathcal{X}$. Note first that $\mathcal{X}$ is not a resolving subcategory of $\mathsf{mod}(\Lambda)$ since it does not contain all indecomposable projective Λ-modules. On the one hand, it is easy to see that conditions $(\mathsf{lcFp1})$, $(\mathsf{lcFp3})$ and $(\mathsf{lcFp4})$ in Definition 3.6 hold true as $\mathcal{P}(\mathcal{X}) \subseteq \mathcal{P}(\Lambda)$, and $\mathcal{X}$ is closed under extensions and direct summands. Since $\mathcal{X}$ is also a Frobenius subcategory of $\mathsf{mod}(\Lambda)$, we only need to verify that $\mathcal{X}$ is closed under epi-kernels in order to conclude $(\mathsf{lcFp2})$. So suppose we are given an exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathsf{mod}(\Lambda)$ with $Y, Z \in \mathcal{X}$. Using that $\mathcal{X}$ has enough projective objects, taking the pullback of $Y \to Z \leftarrow P$ yields a solid diagram*

$$\begin{array}{ccc}
Z' & = & Z' \\
\downarrow & & \downarrow \\
X \rightarrowtail L & \twoheadrightarrow & P \\
\| & \text{pb} & \downarrow \\
X \rightarrowtail Y & \twoheadrightarrow & Z
\end{array}$$

with $P \in \mathcal{P}(\mathcal{X})$ and $Z' \in \mathcal{X}$. Note that $L \in \mathcal{X}$. Moreover, since $\mathcal{P}(\mathcal{X}) \subseteq \mathcal{P}(\Lambda)$ the central row in the previous diagram splits, and then $X \in \mathcal{X}$.

Cut Auslander–Buchweitz contexts. In this part, we introduce the concept of cut Auslander–Buchweitz contexts and present some examples related to hereditary complete cotorsion pairs. Let us recall the following from [3, Def. 5.1].

Definition 3.13. *A pair* $(\mathcal{A}, \mathcal{B})$ *of subcategories of* $\mathcal{C}$ *with* $\omega := \mathcal{A} \cap \mathcal{B}$ *is a **left weak Auslander–Buchweitz context** (or a **left weak AB context**, for short) if the following three conditions are satisfied:*

(lABc1) $(\mathcal{A}, \omega)$ *is a left Frobenius pair in* $\mathcal{C}$.
(lABc2) $\mathcal{B}$ *is right thick.*
(lABc3) $\mathcal{B} \subseteq \mathcal{A}^{\wedge}$.

*The notion of **right weak AB context** is dual.*

Let us now propose the following generalization of the previous definition.

Definition 3.14. *Let* $\mathcal{A}, \mathcal{B}, \mathcal{S} \subseteq \mathcal{C}$ *and* $\omega := \mathcal{A} \cap \mathcal{B}$. *We say that* $(\mathcal{A}, \mathcal{B})$ *is a **left weak AB context cut along** $\mathcal{S}$ if the following three conditions are satisfied:*

(lcABc1) $(\mathcal{A}, \omega)$ *is a left Frobenius pair cut along* $\mathcal{S}$.
(lcABc2) $\mathcal{B} \cap \mathcal{S}$ *is right thick.*
(lcABc3) $\mathcal{B} \cap \mathcal{S} \subseteq (\mathcal{A} \cap \mathcal{S})^{\wedge}$.

*Dually, we have the concept of **right weak AB context cut along** $\mathcal{S}$.*

Remark 3.15. *We can have a first approach to the relation between cut AB contexts and relative cotorsion pairs, which will be analyzed in more detail in Section 4. For any left weak AB context* $(\mathcal{A}, \mathcal{B})$ *cut along* $\mathcal{S}$, *one has that* $(\mathcal{A} \cap \mathcal{S}, \mathcal{B} \cap \mathcal{S})$ *is a left weak AB context in* $\mathcal{C}$. *Then,* $(\mathcal{A} \cap \mathcal{S}, \mathcal{B} \cap \mathcal{S})$ *is a relative* $\mathsf{Thick}(\mathcal{A} \cap \mathcal{S})$-*cotorsion pair with* $\mathrm{id}_{\mathcal{A} \cap \mathcal{S}}(\mathcal{B} \cap \mathcal{S}) = 0$ *and* $(\mathcal{A} \cap \mathcal{B} \cap \mathcal{S})^{\wedge} = \mathcal{B} \cap \mathcal{S}$. *See [3, Def. 3.4 & Prop. 5.5] for details.*

Example 3.16. *Let* $(\mathcal{X}, \mathcal{Y})$ *be a GP-admissible pair in* $\mathcal{C}$ *with* $\omega := \mathcal{X} \cap \mathcal{Y}$, *such that* $\mathcal{X}$ *is closed under epi-kernels and direct summands, and such that* $\mathcal{Y}$ *is right thick.*

(1) $(\mathcal{X}, \mathcal{Y})$ *is a left weak AB context cut along* $\mathcal{X}^{\wedge}$: *It is easy to check that* $(\mathcal{X}, \mathcal{Y})$ *satisfies conditions* (lcABc1) *and* (lcABc3) *in Definition 3.14. Moreover, the subcategory* $\mathcal{Y} \cap \mathcal{X}^{\wedge}$ *is right thick since* $\mathcal{X}^{\wedge}$ *is thick by Proposition 1.5 and* $\mathcal{Y}$ *is right thick by assumption. Thus,* $(\mathcal{X}, \mathcal{Y})$ *satisfies also* (lcABc2).

(2) $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \mathcal{Y})$ *is a left weak AB context cut along* $\mathcal{X}^{\wedge}$: *Let us first see that the pair* $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y})$ *is left Frobenius cut along* $\mathcal{X}^{\wedge}$. *By Proposition 1.9 we have that* $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$ *is left thick and that* $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y} = \omega$ *and* $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{X}^{\wedge} = \mathcal{X}$. *Indeed, the first equality and the containment* $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{X}^{\wedge} \supseteq \mathcal{X}$ *are clear. Now let* $C \in \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{X}^{\wedge}$. *By Proposition 1.5 we can find a short exact sequence* $K \rightarrowtail X \twoheadrightarrow C$ *with* $X \in \mathcal{X}$ *and* $K \in \omega^{\wedge}$. *This sequence splits since* $C \in \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$, *and so* $C \in \mathcal{X}$. *We then have that* $(\mathcal{X}, \omega) = (\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{X}^{\wedge}, (\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y}) \cap \mathcal{X}^{\wedge})$ *is a left Frobenius pair. From the previous equalities we can also note that* $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y}) \cap \mathcal{X}^{\wedge}$ *is a* $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y})$-*projective relative generator in* $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y}$, *and that* $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y}$ *is closed under extensions and direct summands. Hence, we have that the pair* $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y})$ *satisfies* (lcABc1). *Finally, conditions* (lcABc2) *and* (lcABc3) *are clear.*

(3) $(\mathcal{GP}_{(\mathcal{X}, \omega)}, \mathcal{Y})$ *is a left weak AB context cut along* $\mathcal{X}^{\wedge}$: *Note that* $\mathrm{pd}_{\omega}(\mathcal{X}) = 0$, ω *is closed under finite coproducts, and* ω *is a relative cogenerator in* $\mathcal{X}$. *Then, it follows by Proposition 1.9 that* $\mathcal{GP}_{(\mathcal{X}, \omega)}$ *is left thick. Using again Proposition 1.5, we have that* $\mathcal{GP}_{(\mathcal{X}, \omega)} \cap \mathcal{X}^{\wedge} = \mathcal{X}$ *and* $(\mathcal{GP}_{(\mathcal{X}, \omega)} \cap \mathcal{Y}) \cap \mathcal{X}^{\wedge} = \omega$. *Thus,* $(\mathcal{GP}_{(\mathcal{X}, \omega)} \cap \mathcal{X}^{\wedge}, (\mathcal{GP}_{(\mathcal{X}, \omega)} \cap \mathcal{Y}) \cap \mathcal{X}^{\wedge}) = (\mathcal{X}, \omega)$ *is a left Frobenius pair in* $\mathcal{C}$. *The rest of the proof follows easily.*

Remark 3.17. *Note that hereditary complete cotorsion pairs provide with a wide range of GP-admissible pairs $(\mathcal{X}, \mathcal{Y})$ satisfying the assumptions in the previous example. Let us now exhibit a GP-admissible pair $(\mathcal{X}, \mathcal{Y})$, with $\mathcal{X}$ closed under epi-kernels and direct summands, and with $\mathcal{Y}$ right thick, such that $(\mathcal{X}, \mathcal{Y})$ is not a hereditary complete cotorsion pair. This is the case of the subcategories $\mathcal{X} = \mathcal{GP}_{\mathrm{AC}}(R)$ of Gorenstein AC-projective R-modules, and $\mathcal{Y} = \mathcal{L}(R)^\wedge$ of R-modules of finite level dimension, provided that R is not an AC-Gorenstein ring (see Gillespie's [20, Thm. 6.2]). Indeed, by [5, Lem. 8.6 Prop. 8.10] we know that $(\mathcal{GP}_{\mathrm{AC}}(R), (\mathcal{GP}_{\mathrm{AC}}(R))^{\perp_1})$ is a hereditary complete cotorsion pair in $\mathsf{Mod}(R)$. Moreover, it is clear that $\mathrm{pd}_{\mathcal{L}(R)^\wedge}(\mathcal{GP}_{\mathrm{AC}}(R)) = 0$ and that $\mathcal{L}(R)^\wedge$ is closed under finite coproducts. On the other hand, it is straightforward to check that $\mathcal{GP}_{\mathrm{AC}}(R) \cap \mathcal{L}(R)^\wedge = \mathcal{P}(R)$. Thus, in particular, we have from the previous example that $(\mathcal{GP}_{\mathrm{AC}}(R), \mathcal{L}(R)^\wedge)$ is a left weak AB-context cut along $\mathcal{GP}_{\mathrm{AC}}(R)^\wedge$.*

In Example 3.16, we obtained the left weak AB-context $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{Y})$ cut along $\mathcal{X}^\wedge$, from a GP-admissible pair $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{X}$ left thick and $\mathcal{Y}$ right thick. With a couple of extra assumptions on $\mathcal{X}$ and $\mathcal{Y}$, we are able to characterize $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{Y})$ as a left weak AB-context that is *absolute* (that is, cut along the whole category $\mathcal{C}$).

Theorem 3.18. *Let $(\mathcal{X}, \mathcal{Y})$ be a GP-admissible pair in $\mathcal{C}$ such that $\omega := \mathcal{X} \cap \mathcal{Y}$ is closed under direct summands, $^{\perp_1}\mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{Y}$ is right thick. Then, the following conditions are equivalent:*

(a) $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{Y})$ is a left weak AB-context in $\mathcal{C}$.
(b) $\mathcal{Y} \subseteq \mathcal{X}^\wedge$ and $\mathcal{X}$ is left thick.
(c) $(\mathcal{X}, \mathcal{Y})$ is a left weak AB-context in $\mathcal{C}$.

Moreover, if any of the previous conditions holds, then $\mathcal{X} = \mathcal{GP}_{(\mathcal{X},\omega)}$.

Proof. Let us show first that (a) $\Rightarrow$ (b). Since $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{Y})$ is a left weak AB-context in $\mathcal{C}$, we have by [3, Prop. 5.5] that $\mathrm{id}_{\mathcal{GP}_{(\mathcal{X},\omega)}}(\mathcal{Y}) = 0$. This fact, along with the assumption (a), implies that $\mathcal{X} \subseteq \mathcal{GP}_{(\mathcal{X},\omega)} \subseteq {}^{\perp_1}\mathcal{Y} \subseteq \mathcal{X}$, and thus $\mathcal{X} = \mathcal{GP}_{(\mathcal{X},\omega)}$, which is left thick since $(\mathcal{X}, \omega)$ is a GP-admissible pair (see Proposition 1.9). Then, $\mathcal{Y} \subseteq \mathcal{GP}_{(\mathcal{X},\omega)}^\wedge = \mathcal{X}^\wedge$ by (lABc3).

The implication (b) $\Rightarrow$ (c) is clear. Finally, let us show (c) $\Rightarrow$ (a). So suppose that $(\mathcal{X}, \mathcal{Y})$ is a left weak AB-context. Thus, we have that $\mathcal{Y}$ is right thick and that $\mathcal{Y} \subseteq \mathcal{X}^\wedge \subseteq \mathcal{GP}_{(\mathcal{X},\omega)}^\wedge$. It is only left to show that $(\mathcal{GP}_{(\mathcal{X},\omega)}, \mathcal{GP}_{(\mathcal{X},\omega)} \cap \mathcal{Y})$ is a left Frobenius pair in $\mathcal{C}$. We know that $\mathcal{GP}_{(\mathcal{X},\omega)}$ is left thick by Proposition 1.9. For conditions (lFp2) and (lFp3), it suffices to show that $\mathcal{GP}_{(\mathcal{X},\omega)} \cap \mathcal{Y} = \omega$. Indeed, using Propositions 1.5 and 1.9, the equality $\mathrm{Ext}_\mathcal{C}^1(\mathcal{GP}_{(\mathcal{X},\omega)}, \omega^\wedge) = 0$ and the fact that $\mathcal{X}$ is closed under direct summands, we can note that $\mathcal{GP}_{(\mathcal{X},\omega)} \cap \mathcal{X}^\wedge = \mathcal{X}$. This, along with the assumption $\mathcal{Y} \subseteq \mathcal{X}^\wedge$, yields $\omega \subseteq \mathcal{GP}_{(\mathcal{X},\omega)} \cap \mathcal{Y} \subseteq \mathcal{X} \cap \mathcal{Y} = \omega$. $\qquad\square$

The previous theorem basically asserts that $(\mathcal{X}, \mathcal{X} \cap \mathcal{Y})$-Gorenstein projective objects in the sense of Xu [46] are trivial in the case they are part of a left weak AB-context. In other words, given a hereditary complete cotorsion pair $(\mathcal{X}, \mathcal{Y})$, it is not useful to apply the theory of *absolute* AB-contexts [3] to the objects in $\mathcal{GP}_{(\mathcal{X},\mathcal{X}\cap\mathcal{Y})}$, in the sense that any result obtained this way is simply a property for the subcategories $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X} \cap \mathcal{Y}$. This leads to the need of a relativization for the notion of AB context to subcategories of $\mathcal{C}$. One interesting aspect about the relativization proposed in Definition 3.14 is that the correspondence proved in [3] for the absolute case, between AB contexts, Frobenius pairs and relative cotorsion pairs, is still going to be valid. We shall prove this assertion in a series of results which are part of Section 4. For the moment, we give a small preamble to this by constructing below in Proposition 3.20 three examples of complete cut cotorsion pairs involving the subcategories $\mathcal{X}$, $\mathcal{GP}_{(\mathcal{X},\mathcal{Y})}$ and $\mathcal{GP}_{(\mathcal{X},\omega)}$, which were considered in Example 3.16 in the construction of relative weak AB-contexts.

The following establishes results similar to those in Proposition 1.5 (7), in the setting of complete cut cotorsion pairs.

Proposition 3.19. *Let $\mathcal{X}$ and ω be subcategories of $\mathcal{C}$. The following holds:*

(1) If ω is closed under extensions and direct summands, and $\mathrm{id}_\omega(\omega) = 0$, then $\omega^\wedge$ is also closed under extensions and direct summands.

If $\mathcal{X}$ is closed under extensions and direct summands, then the following also hold:

(2) If ω is an $\mathcal{X}$-injective relative cogenerator in $\mathcal{X}$, then $\mathcal{X}^\wedge \in \mathrm{lCuts}(\mathcal{X}, \omega^\wedge)$. If in addition, ω is closed under extensions and direct summands, then $\mathcal{X}^\wedge \in \mathrm{rCuts}(\mathcal{X}, \omega^\wedge)$ (and so $\mathcal{X}^\wedge \in \mathrm{Cuts}(\mathcal{X}, \omega^\wedge)$).

(3) If ω is an $\mathcal{X}$-projective relative generator in $\mathcal{X}$, then $\mathcal{X}^\wedge$ is closed under extensions and direct summands. If in addition ω is closed under direct summands, then $\mathcal{X}^\wedge \in \mathrm{Cuts}(\omega, \mathcal{X}^\wedge)$.

Proof.

(1) Follows by taking $\mathcal{S} := \mathcal{C}$ in Lemma 3.5 (2) and (3).

(2) From Proposition 1.5, we know that for any $C \in \mathcal{X}^\wedge$ there are exact sequences $\eta \colon Y \rightarrowtail X \twoheadrightarrow C$ and $\epsilon \colon C \rightarrowtail Y' \twoheadrightarrow X'$ with $X, X' \in \mathcal{X}$ and $Y, Y' \in \omega^\wedge$. We show that $\mathcal{X} = {}^{\perp_1}(\omega^\wedge) \cap \mathcal{X}^\wedge$. Since ω is $\mathcal{X}$-injective, we have from [32, dual of Lem. 2.13 (a)] that $\mathrm{Ext}^1_\mathcal{C}(\mathcal{X}, \omega^\wedge) = 0$ and so the containment $\subseteq$ holds true. Now, let $C \in {}^{\perp_1}(\omega^\wedge) \cap \mathcal{X}^\wedge$ and consider η as above. Since $C \in {}^{\perp_1}(\omega^\wedge)$ we get that η splits and so C is a direct summand of $X \in \mathcal{X}$. Hence, $C \in \mathcal{X}$ and the equality holds true. We thus have $\mathcal{X}^\wedge \in \mathrm{lCuts}(\mathcal{X}, \omega^\wedge)$. For the assertion $\mathcal{X}^\wedge \in \mathrm{rCuts}(\mathcal{X}, \omega^\wedge)$, suppose that ω is closed under extensions and direct summands. From part (1), $\omega^\wedge$ is closed under direct summands. Then, it remains to show the equality $\omega^\wedge = \mathcal{X}^{\perp_1} \cap \mathcal{X}^\wedge$. The containment $\subseteq$ follows from [32, dual of Lem. 2.13] again. Now, let $C \in \mathcal{X}^{\perp_1} \cap \mathcal{X}^\wedge$ and consider ϵ as above. Notice that ϵ splits since $C \in \mathcal{X}^{\perp_1}$. Then, C is a direct summand of an object in $\omega^\wedge$. Therefore, $C \in \omega^\wedge$.

(3) The first part follows by Lemma 3.5 (2) and (3). For the second part, we have now that $\mathcal{X}^\wedge$ and ω are closed under direct summands. Now let us show the equality $\omega = {}^{\perp_1}(\mathcal{X}^\wedge) \cap \mathcal{X}^\wedge$. The inclusion ($\subseteq$) is clear since ω is $\mathcal{X}$-projective. Now let $C \in {}^{\perp_1}(\mathcal{X}^\wedge) \cap \mathcal{X}^\wedge$. Then there is a short exact sequence $\xi \colon K \rightarrowtail X \twoheadrightarrow C$ with $X \in \mathcal{X}$ and $K \in \mathcal{X}^\wedge$. Since $C \in {}^{\perp_1}(\mathcal{X}^\wedge)$, we have that ξ splits and so C is a direct summand of X. It follows that $C \in \mathcal{X}$. Then, since ω is a generator in $\mathcal{X}$, there exists a short exact sequence $\eta \colon X' \rightarrowtail W \twoheadrightarrow C$ with $W \in \omega$ and $X' \in \mathcal{X}$. Again, using the fact that $C \in {}^{\perp_1}(\mathcal{X}^\wedge)$, we have that η splits, and so C is a direct summand of W. Therefore, $C \in \omega$. The other equality $\mathcal{X}^\wedge = \omega^{\perp_1} \cap \mathcal{X}^\wedge$ follows from [32, dual of Lem. 2.13 (a)]. It is clear that $(\omega, \mathcal{X}^\wedge)$ satisfies (rccp3). On the other hand, to show (lccp3) let $C \in \mathcal{X}^\wedge$ and set $n := \mathrm{resdim}_\mathcal{X}(C)$. Then, we have a short exact sequence $K \xrightarrow{\alpha} X \twoheadrightarrow C$ where $X \in \mathcal{X}$ and $\mathrm{resdim}_\mathcal{X}(K) \leq n - 1$. Since ω is a relative generator in $\mathcal{X}$, there exists also a short exact sequence $X' \rightarrowtail W \xrightarrow{p} X$ with $W \in \omega$ and $X' \in \mathcal{X}$. Taking the pullback of α and p gives rise to the following solid diagram:

$$
\begin{array}{ccc}
X' & =\!=\!= & X' \\
\downarrow & & \downarrow \\
W' \rightarrowtail & W & \twoheadrightarrow C \\
\downarrow \quad \mathrm{pb} & \downarrow & \| \\
K \rightarrowtail & X & \twoheadrightarrow C
\end{array}
$$

Using the fact that $\mathcal{X}^\wedge$ is closed under extensions, the central row is the desired short exact sequence. $\qquad\square$

Proposition 3.20. *Let $(\mathcal{X}, \mathcal{Y})$ be a GP-admissible pair in $\mathcal{C}$ such that $\mathcal{X}$ and $\omega := \mathcal{X} \cap \mathcal{Y}$ are closed under direct summands. Then, the following assertions hold true:*

(1) $(\mathcal{X}, \omega^\wedge)$ is a complete cotorsion pair cut along $\mathcal{X}^\wedge$.
(2) $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \omega^\wedge)$ is a complete cotorsion pair cut along $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}^\wedge$.
(3) $(\mathcal{GP}_{(\mathcal{X}, \omega)}, \omega^\wedge)$ is a complete cotorsion pair cut along $\mathcal{GP}_{(\mathcal{X}, \omega)}^\wedge$.

Proof. For this proof, keep in mind the properties from Proposition 1.9. We first note that ω is closed under extensions. Then part (1) follows by Proposition 3.19 (2). For part (2), we have that $(\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}, \mathcal{Y})$ is a GP-admissible pair with $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})} \cap \mathcal{Y} = \omega$. Moreover, we also know that $\mathcal{GP}_{(\mathcal{X}, \mathcal{Y})}$ is closed under direct summands. Thus, the results follows as an application of part (1). Part (3) is in turn an application of part (2) by considering the GP-admissible pair $(\mathcal{X}, \omega)$. $\qquad\square$

4. Correspondences between complete cut cotorsion pairs, cut Frobenius pairs, and cut Auslander–Buchweitz contexts

We study the interplay between cut Frobenius pairs and cut AB contexts. This interaction will depend on two equivalence relations: one defined on $\mathfrak{F}_\mathcal{S}$, the class of left Frobenius pairs cut along $\mathcal{S}$, and the other one defined on the class $\mathfrak{C}_\mathcal{S}$ of left weak AB-contexts cut along $\mathcal{S}$. Using these relations, we shall prove that there exists a one-to-one correspondence between the corresponding quotient classes. We shall also show that there exists a one-to-one correspondence between cut AB contexts and complete cut cotorsion pairs. For this, we consider an equivalence relation defined on the class $\mathfrak{P}_\mathcal{S}$ of complete cotorsion pairs $(\mathcal{F}, \mathcal{G})$ cut along the smallest thick subcategory containing $\mathcal{F}$ and which satisfy a certain relation with $\mathcal{S}$. Let us commence proving the following lemma, which is a relativization of Proposition 1.5 (2).

Lemma 4.1. *Let $\mathcal{X}, \omega, \mathcal{S} \subseteq \mathcal{C}$ satisfying the following conditions:*

(1) ω is closed under extensions and isomorphisms;
(2) $\omega \cap \mathcal{S}$ is closed under direct summands and a relative generator in ω;
(3) $\omega \cap \mathcal{S} \subseteq \mathcal{X} \cap \mathcal{S}$;
(4) $\mathcal{X} \cap \mathcal{S}$ is closed under epi-kernels; and
(5) $\mathrm{id}_{\mathcal{X} \cap \mathcal{S}}(\omega \cap \mathcal{S}) = 0$.

Then, $\omega \cap \mathcal{S} = \mathcal{X} \cap \omega^\wedge \cap \mathcal{S}$. In particular, this equality holds if $(\mathcal{X}, \omega)$ is a left Frobenius pair cut along $\mathcal{S}$.

Proof. The containment $\omega \cap \mathcal{S} \subseteq \mathcal{X} \cap \omega^\wedge \cap \mathcal{S}$ is clear. For the converse, let $M \in \mathcal{X} \cap \omega^\wedge \cap \mathcal{S}$ and $W_n \rightarrowtail W_{n-1} \to \cdots \to W_1 \to W_0 \twoheadrightarrow M$ be an ω-resolution of M. If $n = 0$, then $M \in \omega \cap \mathcal{S}$ since ω is closed under isomorphisms. So we can assume that $n \geq 1$. Now, since $\mathcal{X} \cap \mathcal{S}$ is closed under epi-kernels, we have by Lemma 3.3 and its notation therein that there are exact sequences $\eta_j : X_{j+1} \rightarrowtail F_j \twoheadrightarrow X_j$ where $X_0 = M$ and $X_j \in \mathcal{X} \cap \mathcal{S}$ for all $1 \leq j \leq n-1$. Notice that $X_n \in \omega \cap \mathcal{S}$ since $G_{n-1}, E_n := W_n \in \omega$. Thus, η_{n-1} splits and $X_{n-1} \in \omega \cap \mathcal{S}$ since $\mathrm{id}_{\mathcal{X} \cap \mathcal{S}}(\omega \cap \mathcal{S}) = 0$ and $\omega \cap \mathcal{S}$ is closed under direct summands. Using again the preceding argument, we get that $X_j \in \omega \cap \mathcal{S}$ for all $1 \leq j \leq n$. Therefore, η_0 splits and so $M \in \omega \cap \mathcal{S}$. $\qquad\square$

Example 4.2. *We know from* [3, Prop. 6.1] *that $(\mathcal{X}, \omega) := (\mathsf{Mod}(R), \mathcal{P}(R))$ is a left Frobenius pair cut along $\mathcal{GP}(R)$, for any ring R. Notice that in this case the equality $\mathcal{X} \cap \omega^\wedge = \omega$ does not necessarily hold true, while $\mathcal{X} \cap \omega^\wedge \cap \mathcal{S} = \omega \cap \mathcal{S}$ does by the previous lemma.*

Cut Frobenius pairs versus cut AB-contexts. We give the precise definition for the equivalence relations mentioned at the beginning of this section.

Definition 4.3. *Let $\mathcal{S} \subseteq \mathcal{C}$. For $(\mathcal{X}, \omega), (\mathcal{X}', \omega') \in \mathfrak{F}_{\mathcal{S}}$ and $(\mathcal{A}, \mathcal{B}), (\mathcal{A}', \mathcal{B}') \in \mathfrak{C}_{\mathcal{S}}$, we shall say that:*

(1) *$(\mathcal{X}, \omega)$ is **related** to $(\mathcal{X}', \omega')$ in $\mathfrak{F}_{\mathcal{S}}$, denoted $(\mathcal{X}, \omega) \sim (\mathcal{X}', \omega')$, if $\mathcal{X} \cap \mathcal{S} = \mathcal{X}' \cap \mathcal{S}$ and $\omega \cap \mathcal{S} = \omega' \cap \mathcal{S}$;*

(2) *$(\mathcal{A}, \mathcal{B})$ is **related** to $(\mathcal{A}', \mathcal{B}')$ in $\mathfrak{C}_{\mathcal{S}}$, denoted $(\mathcal{A}, \mathcal{B}) \sim (\mathcal{A}', \mathcal{B}')$, if $\mathcal{A} \cap \mathcal{S} = \mathcal{A}' \cap \mathcal{S}$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{S} = \mathcal{A}' \cap \mathcal{B}' \cap \mathcal{S}$.*

Notice that (1) and (2) in the previous definition are equivalence relations. We denote by $[\mathcal{X}, \omega]_{\mathfrak{F}_{\mathcal{S}}}$ the equivalence class of $(\mathcal{X}, \omega)$ in $\mathfrak{F}_{\mathcal{S}}/\sim$. Similarly, $[\mathcal{A}, \mathcal{B}]_{\mathfrak{C}_{\mathcal{S}}}$ denotes the equivalence class of $(\mathcal{A}, \mathcal{B})$ in $\mathfrak{C}_{\mathcal{S}}/\sim$.

Example 4.4. *Let $(\mathcal{X}, \mathcal{Y})$ be a GP-admissible pair with $\mathcal{X}$ closed under epi-kernels and direct summands, and such that $\mathcal{Y}$ is right thick. We know from Example 3.16 that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{GP}_{(\mathcal{X}, \omega)}, \mathcal{Y})$ are left weak AB-contexts cut along $\mathcal{X}^\wedge$, that is, $(\mathcal{X}, \mathcal{Y}), (\mathcal{GP}_{(\mathcal{X}, \omega)}, \mathcal{Y}) \in \mathfrak{C}_{\mathcal{X}^\wedge}$. Also, one can verify using Proposition 1.5 that $\mathcal{X} = \mathcal{GP}_{(\mathcal{X}, \omega)} \cap \mathcal{X}^\wedge$ and $\mathcal{GP}_{(\mathcal{X}, \omega)} \cap \mathcal{Y} \cap \mathcal{X}^\wedge = \omega$. It follows that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{GP}_{(\mathcal{X}, \omega)}, \mathcal{Y})$ are related in $\mathfrak{C}_{\mathcal{X}^\wedge}$.*

Theorem 4.6 below is a generalization of the following result.

Theorem 4.5. *([3, part 1. of Thm. 5.4]) There is a one-to-one correspondence between the class $\mathfrak{F}$ of left Frobenius pairs $(\mathcal{X}, \omega)$ in $\mathcal{C}$, and the class $\mathfrak{C}$ of left weak AB contexts $(\mathcal{A}, \mathcal{B})$ in $\mathcal{C}$, given by $(\mathcal{X}, \omega) \mapsto (\mathcal{X}, \omega^\wedge)$ with inverse $(\mathcal{A}, \mathcal{B}) \mapsto (\mathcal{A}, \mathcal{A} \cap \mathcal{B})$.*

Theorem 4.6. *(**first correspondence theorem**) Let $\mathcal{S} \subseteq \mathcal{C}$, which is closed under epi-kernels and mono-cokernels. Then, there is a one-to-one correspondence*

$$\Phi_{\mathcal{S}} \colon \mathfrak{F}_{\mathcal{S}}/\sim \; \to \mathfrak{C}_{\mathcal{S}}/\sim \qquad \text{given by} \qquad [\mathcal{X}, \omega]_{\mathfrak{F}_{\mathcal{S}}} \mapsto [\mathcal{X}, \omega^\wedge]_{\mathfrak{C}_{\mathcal{S}}},$$

with inverse

$$\Psi_{\mathcal{S}} \colon \mathfrak{C}_{\mathcal{S}}/\sim \; \to \mathfrak{F}_{\mathcal{S}}/\sim \qquad \text{given by} \qquad [\mathcal{A}, \mathcal{B}]_{\mathfrak{C}_{\mathcal{S}}} \mapsto [\mathcal{A}, \mathcal{A} \cap \mathcal{B}]_{\mathfrak{F}_{\mathcal{S}}}.$$

Proof. First, we show that the mappings $\Phi_{\mathcal{S}}$ and $\Psi_{\mathcal{S}}$ are well defined. On the one hand, for $\Psi_{\mathcal{S}}$, we have that $(\mathcal{A}, \mathcal{A} \cap \mathcal{B}) \in \mathfrak{F}_{\mathcal{S}}$ for every $(\mathcal{A}, \mathcal{B}) \in \mathfrak{C}_{\mathcal{S}}$, by definition of cut left weak AB-context. Also, it is clear that $\Psi_{\mathcal{S}}([\mathcal{A}, \mathcal{B}]_{\mathfrak{C}_{\mathcal{S}}})$ does not depend on the chosen representative $(\mathcal{A}, \mathcal{B}) \in \mathfrak{C}_{\mathcal{S}}$. On the other hand, $\Phi_{\mathcal{S}}$ does not depend on representatives either by Lemma 4.1. Now we prove that if $(\mathcal{X}, \omega) \in \mathfrak{F}_{\mathcal{S}}$ then $(\mathcal{X}, \omega^\wedge) \in \mathfrak{C}_{\mathcal{S}}$ by checking (lcABc1), (lcABc2) and (lcABc3) in Definition 3.14:

- $(\mathcal{X}, \mathcal{X} \cap \omega^\wedge)$ is a left Frobenius pair cut along $\mathcal{S}$: Clearly, $\mathcal{X}$ is left thick by (lcFp1) in Definition 3.6. Now by Lemma 4.1, we have $(\mathcal{X} \cap \mathcal{S}, \mathcal{X} \cap \omega^\wedge \cap \mathcal{S}) = (\mathcal{X} \cap \mathcal{S}, \omega \cap \mathcal{S})$, which is a left Frobenius pair in $\mathcal{C}$. In order to show that $\omega \cap \mathcal{S}$ is an $(\mathcal{X} \cap \omega^\wedge)$-projective relative generator in $\mathcal{X} \cap \omega^\wedge$, note first that $\mathrm{pd}_{\omega^\wedge}(\omega \cap \mathcal{S}) = 0$ by Lemma 3.5 (1). Now let $M \in \mathcal{X} \cap \omega^\wedge$. Using again Lemma 3.5 (1), there exists a short exact sequence $M' \rightarrowtail P \twoheadrightarrow M$ with $P \in \omega \cap \mathcal{S}$ and $M' \in \omega^\wedge$. Since $\mathcal{X}$ is left thick and $\omega \cap \mathcal{S} \subseteq \mathcal{X} \cap \mathcal{S}$, we get that $M' \in \mathcal{X} \cap \omega^\wedge$. Finally, we note that $\mathcal{X} \cap \omega^\wedge$ is closed under extensions and direct summands. Indeed, since $(\mathcal{X}, \omega) \in \mathfrak{F}_{\mathcal{S}}$, we have by Remark 3.7 (3) that the statements of Lemma 3.5 hold true. In particular, $\omega^\wedge$ is closed under extensions and direct summands, and so the same holds for $\mathcal{X} \cap \omega^\wedge$ since $\mathcal{X}$ is left thick.
- $\omega^\wedge \cap \mathcal{S}$ is right thick and $\omega^\wedge \cap \mathcal{S} \subseteq (\mathcal{X} \cap \mathcal{S})^\wedge$: First, since $(\mathcal{X} \cap \mathcal{S}, \omega \cap \mathcal{S})$ is a left Frobenius pair in $\mathcal{C}$, we have by Theorem 4.5 that $(\omega \cap \mathcal{S})^\wedge$ is right thick and that $\omega \cap \mathcal{S} \subseteq \mathcal{X} \cap \mathcal{S}$. The rest follows by Lemma 3.5 (4).

Finally, one can check that the mappings $\Psi_{\mathcal{S}}$ and $\Phi_{\mathcal{S}}$ are inverse to each other by using Lemma 4.1. $\square$

Cut AB-contexts versus complete cut cotorsion pairs. In order to present the second correspondence mentioned earlier, let us point out the following facts about complete cut cotorsion pairs.

Lemma 4.7. *Let $(\mathcal{F}, \mathcal{G})$ be a complete cotorsion pair cut along $\mathsf{Thick}(\mathcal{F})$ with $\mathrm{id}_{\mathcal{F}}(\mathcal{G}) = 0$. Then, the following statements hold true:*

(1) $(\mathcal{F}, \mathcal{F} \cap \mathcal{G})$ is a left Frobenius pair in $\mathcal{C}$.
(2) $\mathcal{F} \cap \mathcal{G} = \mathcal{F} \cap \mathcal{F}^{\perp_1} = \mathcal{F} \cap (\mathcal{F} \cap \mathcal{G})^{\wedge}$.
(3) $(\mathcal{F} \cap \mathcal{G})^{\wedge} = \mathcal{F}^{\perp} \cap \mathcal{F}^{\wedge}$.
(4) $\mathcal{F}^{\wedge} = \mathsf{Thick}(\mathcal{F})$.

Proof. Assertion (1) and the equality $\mathcal{F} \cap \mathcal{G} = \mathcal{F} \cap \mathcal{F}^{\perp_1}$ are easy to check from the definitions of complete cut cotorsion pairs and of left Frobenius pairs. The rest of the equalities appearing in (2), (3) and (4) follow from part (1) and Proposition 1.5. □

Lemma 4.8. *Let $\mathcal{S}$ be a thick subcategory of $\mathcal{C}$ and $(\mathcal{F}, \mathcal{G})$ be a complete cotorsion pair cut along $\mathsf{Thick}(\mathcal{F})$, with $\mathrm{id}_{\mathcal{F}}(\mathcal{G}) = 0$. If $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S}$ is both a relative generator and cogenerator in $\mathcal{F} \cap \mathcal{G}$, then $(\mathcal{F}, \mathcal{F} \cap \mathcal{G})$ is a left Frobenius pair cut along $\mathcal{S}$.*

Proof. We need to verify conditions (lcFp1), (lcFp2), (lcFp3), and (lcFp4) in Definition 3.6 for the subcategories $\mathcal{F}$, $\mathcal{F} \cap \mathcal{G}$, and $\mathcal{S}$. One can note that (lcFp1), (lcFp3) and (lcFp4) hold by Lemma 4.7. It remains to check that $(\mathcal{F} \cap \mathcal{S}, \mathcal{F} \cap \mathcal{G} \cap \mathcal{S})$ is a left Frobenius pair in $\mathcal{C}$. One can use again Lemma 4.7 to note that $\mathcal{F} \cap \mathcal{S}$ is left thick and that $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S} = \mathcal{F} \cap \mathcal{F}^{\perp_1} \cap \mathcal{S}$. Then, $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S}$ is closed under direct summands. Finally, since $\mathrm{id}_{\mathcal{F} \cap \mathcal{G} \cap \mathcal{S}}(\mathcal{F} \cap \mathcal{G}) = 0$, it is only left to show that $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S}$ is a relative cogenerator in $\mathcal{F} \cap \mathcal{S}$. So for any $F \in \mathcal{F} \cap \mathcal{S}$, by the relative right completeness of $(\mathcal{F}, \mathcal{G})$ there is a short exact sequence $F \rightarrowtail G \twoheadrightarrow F'$ with $G \in \mathcal{F} \cap \mathcal{G}$ and $F' \in \mathcal{F}$. Now by using that $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S}$ is a relative cogenerator in $\mathcal{F} \cap \mathcal{G}$, we can find a short exact sequence $G \rightarrowtail L \twoheadrightarrow G'$ with $L \in \mathcal{F} \cap \mathcal{G} \cap \mathcal{S}$ and $G' \in \mathcal{F} \cap \mathcal{G}$. The result follows after taking the pushout of $L \leftarrow G \rightarrow F'$. □

For a fixed subcategory $\mathcal{S} \subseteq \mathcal{C}$, let $\mathfrak{P}_{\mathcal{S}}$ denote the class of pairs $(\mathcal{F}, \mathcal{G})$ of subcategories of $\mathcal{C}$ such that $\mathsf{Thick}(\mathcal{F}) \in \mathrm{Cuts}(\mathcal{F}, \mathcal{G})$, $\mathrm{id}_{\mathcal{F}}(\mathcal{G}) = 0$ and $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S}$ is both a relative generator and cogenerator in $\mathcal{F} \cap \mathcal{G}$.

Definition 4.9. *Let $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}') \in \mathfrak{P}_{\mathcal{S}}$. We shall say that $(\mathcal{F}, \mathcal{G})$ is **related** to $(\mathcal{F}', \mathcal{G}')$ in $\mathfrak{P}_{\mathcal{S}}$, denoted $(\mathcal{F}, \mathcal{G}) \sim (\mathcal{F}', \mathcal{G}')$, if $\mathcal{F} \cap \mathcal{S} = \mathcal{F}' \cap \mathcal{S}$ and $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S} = \mathcal{F}' \cap \mathcal{G}' \cap \mathcal{S}$.*

Note that $\sim$ is an equivalence relation in $\mathfrak{P}_{\mathcal{S}}$. In what follows, let us denote by $[\mathcal{F}, \mathcal{G}]_{\mathfrak{P}_{\mathcal{S}}}$ the equivalence class of the representative $(\mathcal{F}, \mathcal{G}) \in \mathfrak{P}_{\mathcal{S}}$.

Example 4.10. *Let $(\mathcal{X}, \omega)$ be a left Frobenius pair in $\mathcal{C}$. We know by Proposition 2.3 that $(\mathcal{X}, \omega^{\wedge})$ and $(\omega, \mathcal{X}^{\perp_1})$ are complete cotorsion pairs cut along $\omega^{\wedge}$, which are related in $\mathfrak{P}_{\omega^{\wedge}}$ by Proposition 1.5. In particular, $(\mathcal{GP}(R), \mathcal{P}(R)^{\wedge}) \sim (\mathcal{P}(R), \mathcal{GP}(R)^{\perp_1})$ in $\mathfrak{P}_{\mathcal{P}(R)^{\wedge}}$.*

We are now ready to show the correspondence between the quotient classes $\mathfrak{P}_{\mathcal{S}}/\sim$ and $\mathfrak{C}_{\mathcal{S}}/\sim$, which generalizes the following previous result from [3].

Theorem 4.11. *([3, part 2. of Thm. 5.4]) The class $\mathfrak{C}$ from Theorem 4.5 equals the class $\mathfrak{P}$ of complete cotorsion pairs $(\mathcal{F}, \mathcal{G})$ in the exact subcategory $\mathsf{Thick}(\mathcal{F})$ with $\mathrm{id}_{\mathcal{F}}(\mathcal{G}) = 0$.*

Theorem 4.12. *(second correspondence theorem) Let $\mathcal{S} \subseteq \mathcal{C}$ be a thick subcategory. Then, there is a one-to-one correspondence*

$$\Lambda_{\mathcal{S}} : \mathfrak{P}_{\mathcal{S}}/\sim \; \to \; \mathfrak{C}_{\mathcal{S}}/\sim \qquad \text{given by} \qquad [\mathcal{F}, \mathcal{G}]_{\mathfrak{P}_{\mathcal{S}}} \mapsto [\mathcal{F}, (\mathcal{F} \cap \mathcal{G})^{\wedge}]_{\mathfrak{C}_{\mathcal{S}}},$$

with inverse

$$\Upsilon_{\mathcal{S}} : \mathfrak{C}_{\mathcal{S}}/\sim \; \to \mathfrak{P}_{\mathcal{S}}/\sim \qquad \text{given by} \qquad [\mathcal{A}, \mathcal{B}]_{\mathfrak{C}_{\mathcal{S}}} \mapsto [\mathcal{A} \cap \mathcal{S}, \mathcal{B} \cap \mathcal{S}]_{\mathfrak{P}_{\mathcal{S}}}.$$

Proof. First, we check that the mappings $\Lambda_{\mathcal{S}}$ and $\Upsilon_{\mathcal{S}}$ are well defined. On the one hand, let $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}') \in \mathfrak{P}_{\mathcal{S}}$ such that $(\mathcal{F}, \mathcal{G}) \sim (\mathcal{F}', \mathcal{G}')$. By Theorem 4.6 and Lemma 4.8, we get that $(\mathcal{F}, (\mathcal{F} \cap \mathcal{G})^{\wedge})$ is a left weak AB-context cut along $\mathcal{S}$. Moreover, by Lemma 4.7 we have that $\mathcal{F} \cap \mathcal{G} \cap \mathcal{S} = \mathcal{F} \cap (\mathcal{F} \cap \mathcal{G})^{\wedge} \cap \mathcal{S}$ and $\mathcal{F}' \cap \mathcal{G}' \cap \mathcal{S} = \mathcal{F}' \cap (\mathcal{F}' \cap \mathcal{G}')^{\wedge} \cap \mathcal{S}$. Thus, $(\mathcal{F}, (\mathcal{F} \cap \mathcal{G})^{\wedge}) \sim (\mathcal{F}', (\mathcal{F}' \cap \mathcal{G}')^{\wedge})$, and hence $\Lambda_{\mathcal{S}}$ does not depend on representatives. On the other hand, consider now $(\mathcal{A}, \mathcal{B}) \in \mathfrak{C}_{\mathcal{S}}$. By Remark 3.15 we have that $(\mathcal{A} \cap \mathcal{S}, \mathcal{B} \cap \mathcal{S})$ is a complete cotorsion pair cut along $\mathsf{Thick}(\mathcal{A} \cap \mathcal{S})$ with $\mathrm{id}_{\mathcal{A} \cap \mathcal{S}}(\mathcal{B} \cap \mathcal{S}) = 0$. The fact that $\Upsilon_{\mathcal{S}}$ does not depend on representatives is clear. Finally, the fact that $\Lambda_{\mathcal{S}}$ and $\Upsilon_{\mathcal{S}}$ are inverse to each other follows by Lemma 4.7 and Remark 3.15. $\square$

5. Applications and more examples

In this last section, we present more detailed examples and applications of the theory of complete cut cotorsion pairs. Our examples will be presented in settings more general than $\mathsf{Mod}(R)$, namely, in categories of chain complexes, quasi-coherent sheaves, and modules over extriangulated categories. We also explore some relations with the Finitistic Dimension Conjecture and Serre subcategories.

Induced cotorsion cuts in chain complexes. In the category $\mathsf{Ch}(\mathcal{C})$ of chain complexes in an abelian category $\mathcal{C}$, we shall write the extension bifunctors $\mathrm{Ext}^{i}_{\mathsf{Ch}(\mathcal{C})}$ as $\mathrm{Ext}^{i}_{\mathsf{Ch}}$, for simplicity.

In this section, we induce complete cut cotorsion pairs in chain complexes in the following sense: we consider complete cotorsion pairs $(\mathcal{A}, \mathcal{B})$ cut along $\mathcal{S} \subseteq \mathcal{C}$, and study under which conditions it is possible to get complete cut cotorsion pairs in $\mathsf{Ch}(\mathcal{C})$ involving the subcategories $\widetilde{\mathcal{A}}$ of $\mathcal{A}$-complexes and $\widetilde{\mathcal{B}}$ of $\mathcal{B}$-complexes. Our motivation comes from Gillespie's result [17, Prop. 3.6], which asserts that if $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in $\mathcal{C}$ with enough $\mathcal{A}$-objects and enough $\mathcal{B}$-objects, then $(\widetilde{\mathcal{A}}, \mathrm{dg}\widetilde{\mathcal{B}})$ and $(\mathrm{dg}\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ are complete cotorsion pairs in $\mathsf{Ch}(\mathcal{C})$.

In the case where $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair cut along $\mathcal{S} \subseteq \mathcal{C}$, we shall determine a subcategory of $\mathsf{Ch}(\mathcal{C})$ along which $\widetilde{\mathcal{A}}$, $\mathsf{Ch}_{\mathrm{acy}}(\widetilde{\mathcal{A}};\mathcal{B})$, $\widetilde{\mathcal{B}}$ and $\mathsf{Ch}_{\mathrm{acy}}(\mathcal{A}; \widetilde{\mathcal{B}})$ (see [27, Def. 5.34]) form a complete cut cotorsion pair. It is known from Yang and Liu's [47, Lem. 3.4] that if $(\mathcal{A}, \mathcal{B})$ is a hereditary complete cotorsion pair in $\mathsf{Ch}(R)$ then every exact complex admits a special $\widetilde{\mathcal{A}}$-precover and a special $\widetilde{\mathcal{B}}$-preenvelope. This suggests that the subcategory $\widetilde{\mathcal{S}}$ should be considered as a possible cotorsion cut. Below we impose some conditions on $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$ so that $(\widetilde{\mathcal{A}}, \mathsf{Ch}_{\mathrm{acy}}(\widetilde{\mathcal{A}}; \mathcal{B}))$ is a complete cotorsion pair cut along $\widetilde{\mathcal{S}}$. Let us recall and specify some of the notation previously displayed. Let $\mathcal{X}, \mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$:

- $\mathsf{Ch}(\mathcal{X})$ is the subcategory of complexes $X_{\bullet}$ with $X_m \in \mathcal{X}$ for every $m \in \mathbb{Z}$.
- $\widetilde{\mathcal{X}}$ is the subcategory of exact complexes $X_{\bullet}$ with $Z_m(X_{\bullet}) \in \mathcal{X}$ for every $m \in \mathbb{Z}$.
- $\mathsf{Ch}_{\mathrm{acy}}(\mathcal{A}; \widetilde{\mathcal{B}})$ denotes the subcategory of complexes $X_{\bullet} \in \mathsf{Ch}(\mathcal{A})$ such that the internal hom $\mathcal{H}om(X_{\bullet}, B_{\bullet})$[2] is an exact complex of abelian groups whenever $B_{\bullet} \in \widetilde{\mathcal{B}}$. Dually, $\mathsf{Ch}_{\mathrm{acy}}(\widetilde{\mathcal{A}}; \mathcal{B})$ denotes the subcategory of complexes $Y_{\bullet} \in \mathsf{Ch}(\mathcal{B})$ such that $\mathcal{H}om(A_{\bullet}, Y_{\bullet})$ is exact for every $A_{\bullet} \in \widetilde{\mathcal{A}}$.

Proposition 5.1. *Let* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ *closed under extensions and such that* $\mathrm{Ext}^{i}_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = 0$ *for every* $i = 1, 2$. *If* $\mathcal{A}$ *is closed under direct summands and* $\mathcal{S} \in \mathrm{lCuts}(\mathcal{A}, \mathcal{B})$, *then* $\widetilde{\mathcal{S}} \in \mathrm{lCuts}(\widetilde{\mathcal{A}}, \mathsf{Ch}_{\mathrm{acy}}(\widetilde{\mathcal{A}}; \mathcal{B}))$.

Proof. It is clear that $\widetilde{\mathcal{A}}$ is closed under diret summands. Moreover, following the proof given in [27, Rmk. 3.10], we obtain for every complex $S_{\bullet} \in \widetilde{\mathcal{S}}$ an exact sequence in $\mathsf{Ch}(\mathcal{C})$ of the form $\eta : B_{\bullet} \rightarrowtail A_{\bullet} \twoheadrightarrow S_{\bullet}$ where $A_{\bullet} \in \widetilde{\mathcal{A}}$ and $B_{\bullet} \in \widetilde{\mathcal{B}} \subseteq \mathsf{Ch}_{\mathrm{acy}}(\widetilde{\mathcal{A}}; \mathcal{B})$ (this inclusion follows by [27, Lem. 5.37]).

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[2]The reader can recall the definition of $\mathcal{H}om(X_{\bullet}, B_{\bullet})$ from [16, §2.1].

So, it suffices to show that $\tilde{\mathcal{A}} \cap \tilde{\mathcal{S}} = {}^{\perp_1}(\mathsf{Ch}_{\mathrm{acy}}(\tilde{\mathcal{A}}; \mathcal{B})) \cap \tilde{\mathcal{S}}$. On the one hand, considering η as above, with $S_\bullet \in {}^{\perp_1}(\mathsf{Ch}_{\mathrm{acy}}(\tilde{\mathcal{A}}; \mathcal{B})) \cap \tilde{\mathcal{S}}$, we have that η splits and then $S_\bullet \in \tilde{\mathcal{A}} \cap \tilde{\mathcal{S}}$. Thus, the containment $\supseteq$ holds true. For the remaining containment $\subseteq$, since $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = 0$, it follows from [27, Lem. 5.38 (1)] that $\mathrm{Ext}^1_{\mathsf{Ch}}(\tilde{\mathcal{A}}, \mathsf{Ch}_{\mathrm{acy}}(\tilde{\mathcal{A}}; \mathcal{B})) = 0$. Therefore, $\tilde{\mathcal{A}} \cap \tilde{\mathcal{S}} \subseteq {}^{\perp_1}(\mathsf{Ch}_{\mathrm{acy}}(\tilde{\mathcal{A}}; \mathcal{B})) \cap \tilde{\mathcal{S}}$. $\qquad\square$

In the next result, we give some conditions so that the converse of Proposition 5.1 holds. In what follows, given an object $C \in \mathcal{C}$, we denote by $D^1(C)$ the chain complex with C in the first and zeroth positions, and with 0 elsewhere, where the only nonzero differential is the identity on C. The complex $D^0(C)$ is defined similarly.

Proposition 5.2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{S} \subseteq \mathcal{C}$ such that $\mathcal{A}$ is closed under extensions, $\tilde{\mathcal{A}}$ is closed under direct summands, $0 \in \mathcal{S}$ and $\tilde{\mathcal{S}} \in \mathrm{lCuts}(\tilde{\mathcal{A}}, \mathsf{Ch}_{\mathrm{acy}}(\tilde{\mathcal{A}}; \mathcal{B}))$. Then, $\mathcal{S} \in \mathrm{lCuts}(\mathcal{A}, \mathcal{B})$ provided that $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = 0$ or $D^1(B) \in \mathsf{Ch}_{\mathrm{acy}}(\tilde{\mathcal{A}}; \mathcal{B})$ for every $B \in \mathcal{B}$.*

Proof. The result is easy to show assuming $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{A}, \mathcal{B}) = 0$. Assuming $D^1(B) \in \mathsf{Ch}_{\mathrm{acy}}(\tilde{\mathcal{A}}; \mathcal{B})$ for every $B \in \mathcal{B}$, the closure under direct summands for $\mathcal{A}$, condition (lccp3) and the containment $\mathcal{A} \cap \mathcal{S} \supseteq {}^{\perp_1}\mathcal{B} \cap \mathcal{S}$ are also easy to show, while $\mathcal{A} \cap \mathcal{S} \subseteq {}^{\perp_1}\mathcal{B} \cap \mathcal{S}$ follows by [17, Lem. 3.1] and Proposition 2.12. $\qquad\square$

Cut cotorsion pairs in the category of quasi-coherent sheaves. For the notions about sheaves and schemes appearing in this section, we recommend the reader to check Hartshorne's [25]. In what follows, X will be a scheme with structure sheaf $\mathcal{O}_X$, and $\mathfrak{Qcoh}(X)$ will denote the category of quasi-coherent sheaves on X. For simplicity, we shall refer to the objects in $\mathfrak{Qcoh}(X)$ simply as "sheaves". It is a well-known fact that $\mathfrak{Qcoh}(X)$ is a Grothendieck category which in general does not have enough projective objects. The latter makes us think that it is not likely to obtain complete cut cotorsion pairs in $\mathfrak{Qcoh}(X)$ involving the subcategory of Gorenstein projective sheaves. This suggests to consider the subcategory $\mathfrak{GF}(X)$ of Gorenstein flat sheaves on X as a more reliable source to obtain complete cut cotorsion pairs. Indeed, in [9, Thm. 2.2] Christensen, Estrada and Thompson proved that $(\mathfrak{GF}(X), (\mathfrak{GF}(X))^{\perp_1})$ is a hereditary complete cotorsion pair in $\mathfrak{Qcoh}(X)$, provided that X is a semi-separated noetherian scheme. In general, the orthogonal subcategory $(\mathfrak{GF}(X))^{\perp_1}$ does not always have an explicit description in terms of simpler sheaves, but studying the cotorsion of $\mathcal{A} := \mathfrak{GF}(X)$ along certain subcategories $\mathcal{S} \subseteq \mathfrak{Qcoh}(X)$ could overcome this limitation. The two questions that arise at this point are: (1) what can we expect for a suitable "local" orthogonal complement $\mathcal{B}$ of $\mathfrak{GF}(X)$?, and (2) which cotorsion cut $\mathcal{S}$ do we need to choose for $(\mathcal{A}, \mathcal{B})$?. First, we known from [3, Prop. 6.17] and [45, Coroll. 4.12] that $(\mathcal{GF}(R), \mathcal{F}(R) \cap (\mathcal{F}(R))^{\perp_1})$ is a left Frobenius pair in $\mathsf{Mod}(R)$, for any ring R, where $\mathcal{GF}(R)$ denotes the subcategory of Gorenstein flat R-modules. It follows by Theorem 4.11 that $(\mathcal{GF}(R), (\mathcal{F}(R) \cap (\mathcal{F}(R))^{\perp_1})^\wedge)$ is a $\mathsf{Thick}(\mathcal{GF}(R))$-cotorsion pair in $\mathsf{Mod}(R)$. On the other hand, in case R is commutative, we can regard $\mathsf{Mod}(R)$ as the category $\mathfrak{Qcoh}(\mathrm{Spec}(R))$. The previous, along with the correspondences in Theorems 4.6 and 4.12, suggests that we should take $\mathcal{B} := (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge$ (where $\mathfrak{F}(X)$ and $\mathfrak{C}(X) = \mathfrak{F}(X)^{\perp_1}$ denote the subcategories of flat and cotorsion sheaves on X, respectively) and $\mathcal{S}$ as a subcategory of $\mathfrak{Qcoh}(X)$ equivalent to $\mathsf{Mod}(\mathcal{O}_X(U))$, for some affine open set $U \subseteq X$. One can for instance determine an equivalence between $\mathcal{S}$ and $\mathsf{Mod}(\mathcal{O}_X(U))$ by using the inverse and direct image functors $i^* \colon \mathfrak{Qcoh}(X) \longrightarrow \mathfrak{Qcoh}(U)$ and $i_* \colon \mathfrak{Qcoh}(U) \longrightarrow \mathfrak{Qcoh}(X)$ induced by the inclusion $i \colon U \to X$.[3] Thus, we shall be working then with the following subcategories of sheaves:

- $\mathcal{A} := \mathfrak{GF}(X)$ for Gorenstein flat sheaves.[4]
- $\mathcal{B} := (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge$.
- $\mathcal{S} := i_*(\mathfrak{Qcoh}(U))$, where U is an affine open subset of X. That is, $\mathcal{S}$ is the subcategory of sheaves on X isomorphic to sheaves of the form $i_*(\mathcal{N})$, where $\mathcal{N}$ is a sheaf on U.

---

[3] Notice that these functors are well defined as they preserve quasi-coherence. See [25, Prop. II 5.8] and [29, Thm. 1.17].
[4] Recall from [9, Def. 1.2] that a sheaf $\mathcal{M}$ is *Gorenstein flat* if $\mathcal{M} = Z_0(\mathscr{F}_\bullet)$, where $\mathscr{F}_\bullet$ is an exact complex of flat sheaves on X such that $\mathscr{F}_\bullet \otimes \mathscr{I}$ is an exact complex of $\mathcal{O}_X$-modules for every injective sheaf $\mathscr{I}$.

We shall give sufficient conditions on X so that $(\mathcal{A}, \mathcal{B})$ is a complete right cotorsion pair cut along $\mathcal{S}$. More specifically, the goal of this section is to show the following result.

Theorem 5.3. *Let X be a semi-separated noetherian scheme, and $U \subseteq X$ be an affine open subset such that every $\mathcal{O}_X(U)$-module has finite Gorenstein flat dimension. Then, $(\mathfrak{GF}(X), (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge)$ is a complete right cotorsion pair cut along $i_*(\mathfrak{Qcoh}(U))$.*

In order to prove this theorem, according to the dual of Proposition 2.12, we need to show the following:

(1) $(\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge$ is closed under direct summands.
(2) $\text{Ext}^1_X(\mathfrak{GF}(X), (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge \cap i_*(\mathfrak{Qcoh}(U))) = 0$.[5]
(3) For every $\mathcal{G} \in i_*(\mathfrak{Qcoh}(U))$ there exists an exact sequence $\mathcal{G} \rightarrowtail \mathcal{B} \twoheadrightarrow \mathcal{A}$ with $\mathcal{A} \in \mathfrak{GF}(X)$ and $\mathcal{B} \in (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge$.

Property (1) is easy to note on any scheme X from our results. First, it is clear that $\mathfrak{F}(X) \cap \mathfrak{C}(X)$ is self-orthogonal and closed under extensions and direct summands. Then by Lemma 3.5 (3) we have that $(\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge$ is closed under direct summands. For (2), we need X to be a semi-separated scheme.[6]

Proposition 5.4. *Let X be a quasi-compact[7] and semi-separated scheme, and $U \subseteq X$ an affine open. Then, $\text{Ext}^i_X(\mathcal{A}, \mathcal{B}) = 0$ for every $\mathcal{A} \in \mathfrak{GF}(X)$, $\mathcal{B} \in (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge \cap i_*(\mathfrak{Qcoh}(U))$ and $i \geq 1$.*

Proof. First, let us write $\mathcal{B} \simeq i_*(\mathcal{N})$ where $\mathcal{N} \in \mathfrak{Qcoh}(U)$. Since X is semi-separated, it is known by Gillespie's [18, Lem. 6.5] that there is a natural adjunction $\text{Ext}^i_X(\mathcal{A}, \mathcal{B}) = \text{Ext}^i_X(\mathcal{A}, i_*(\mathcal{N})) \cong \text{Ext}^i_U(i^*(\mathcal{A}), \mathcal{N})$. On the other hand, since U is affine, by a well-known result of Grothendieck (see for instance [25, Coroll. II 5.5]) the categories $\mathfrak{Qcoh}(U)$ and $\text{Mod}(\mathcal{O}_X(U))$ are equivalent via the mapping $\mathcal{N} \mapsto \mathcal{M}(U)$, and so $\text{Ext}^i_X(\mathcal{A}, \mathcal{B}) \cong \text{Ext}^i_{\mathcal{O}_X(U)}(i^*(\mathcal{A})(U), \mathcal{M}(U))$. The result will follow after showing that $i^*(\mathcal{A})(U)$ is a Gorenstein flat $\mathcal{O}_X(U)$-module and that $\mathcal{M}(U) \in (\mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U)))^\wedge$.

- $i^*(\mathcal{A})(U) \in \mathcal{GF}(\mathcal{O}_X(U))$: First, we know that $\mathcal{A} = Z_0(\mathscr{F}_\bullet)$ for an exact complex $\mathscr{F}_\bullet$ of flat sheaves on X such that $\mathscr{F}_\bullet \otimes \mathscr{I}$ is exact, for every injective sheaf $\mathscr{I} \in \mathfrak{Qcoh}(X)$. By the implication (i) $\Rightarrow$ (ii) in [8, Prop. 2.10] we have that $\mathscr{F}_\bullet(U)$ is an exact complex of flat $\mathcal{O}_X(U)$-modules, such that $\mathscr{F}_\bullet(U) \otimes_{\mathcal{O}_X(U)} I$ is an exact complex of abelian groups for every injective $\mathcal{O}_X(U)$-module I, that is, $Z_0(\mathscr{F}_\bullet(U))$ is a Gorenstein flat $\mathcal{O}_X(U)$-module. On the other hand, the functor i^* is the restriction on U, and so $i^*(\mathcal{A})(U) = Z_0(\mathscr{F}_\bullet)|_U(U) = Z_0(\mathscr{F}_\bullet)(U) = Z_0(\mathscr{F}_\bullet(U)) \in \mathcal{GF}(\mathcal{O}_X(U))$.
- $\mathcal{M}(U) \in (\mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U)))^\wedge$: We proof this claim by induction on the flat-cotorsion resolution dimension of $\mathcal{B}$. So suppose first that $\mathcal{B} = i_*(\mathcal{N}) \in \mathfrak{F}(X) \cap \mathfrak{C}(X)$. To see that $\mathcal{M}(U)$ is a flat $\mathcal{O}_X(U)$-module, we verify that the functor $\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} -$ is exact. Since $i_*(\mathcal{N}) \otimes -$ is exact, U is affine and $i_*(\mathcal{N})$ is quasi-coherent, we have that $(i_*(\mathcal{N}) \otimes -)(U) = i_*(\mathcal{N})(U) \otimes_{\mathcal{O}_X(U)} - = \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} -$ is exact (see [13, Proof of Prop. 3.3]). We now show that $\mathcal{M}(U)$ is also a cotorsion $\mathcal{O}_X(U)$-module. For let $F \in \mathcal{F}(\mathcal{O}_X(U))$ and consider the sheaf on U, $\tilde{F} \in \mathfrak{Qcoh}(U)$, associated to F (see [25, II 5.]). Note that $\tilde{F} \simeq i^*(i_*(\tilde{F}))$, and so we obtain $\text{Ext}^1_{\mathcal{O}_X(U)}(F, \mathcal{M}(U)) \cong \text{Ext}^1_U(\tilde{F}, \mathcal{N}) \cong \text{Ext}^1_X(i_*(\tilde{F}), i_*(\mathcal{N}))$. Now since F is a flat $\mathcal{O}_X(U)$-module, one can note that $i_*(\tilde{F})$ is a flat sheaf on X. Also, $i_*(\mathcal{N})$ is a cotorsion sheaf on X by assumption. Hence, we obtain $\text{Ext}^1_X(i_*(\tilde{F}), i_*(\mathcal{N})) = 0$ and so $\mathcal{M}(U)$ is a cotorsion $\mathcal{O}_X(U)$-module.

---

[5]By $\text{Ext}^1_X(-, -)$, we mean the extension bifunctor $\text{Ext}^1_{\mathfrak{Qcoh}(X)}(-, -)$.
[6]Recall that X is *semi-separated* if it has an open cover by affine open sets with affine intersections. See for instance Neeman's [39].
[7]Recall from [22, Def. 3.16 (b)] that a scheme $(X, \mathcal{O}_X)$ is quasi-compact if the underlying topological space X is quasi-compact, that is, if any open covering of X has a finite subcovering.

So far we have shown that $\mathcal{M}(U) \in \mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U))$ if $i_*(\mathcal{N}) \in \mathfrak{F}(X) \cap \mathfrak{C}(X)$. The more general case where $i_*(\mathcal{N})$ has positive flat-cotorsion dimension will follow by using Auslander–Buchweitz Approximation Theory. Specifically, we shall use the equalities

$$(\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge = \mathfrak{F}(X)^\perp \cap \mathfrak{F}(X)^\wedge, \tag{i}$$

$$(\mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U)))^\wedge = \mathcal{F}(\mathcal{O}_X(U))^\perp \cap \mathcal{F}(\mathcal{O}_X(U))^\wedge. \tag{ii}$$

In order to prove (i), note first that since X is a quasi-compact and semi-separated scheme, we have by [12, Coroll. 4.2] that $(\mathfrak{F}(X), \mathfrak{C}(X))$ is a complete cotorsion pair in $\mathfrak{Qcoh}(X)$. On the other hand, by [11, Lem. A.1] we know that the category $\mathfrak{Qcoh}(X)$ has a flat generator. Then, by [42, Lem. 4.25] we finally have that $(\mathfrak{F}(X), \mathfrak{C}(X))$ is also a hereditary cotorsion pair in $\mathfrak{Qcoh}(X)$. It follows that $\mathfrak{F}(X)$ and $\mathfrak{F}(X) \cap \mathfrak{C}(X)$ satisfy the conditions of [3, Prop. 2.13], and so (i) holds. The equality (ii) is simply the affine case of (i).

Let us now show that $\mathcal{M}(U) \in \mathcal{F}(\mathcal{O}_X(U))^\perp \cap \mathcal{F}(\mathcal{O}_X(U))^\wedge$. We already know from previous arguments that $\mathcal{M}(U) \in \mathcal{F}(\mathcal{O}_X(U))^\perp$. Now let us check $\mathcal{M}(U) \in \mathcal{F}(\mathcal{O}_X(U))^\wedge$. Since $i_*(\mathcal{N}) \in \mathfrak{F}(X)^\wedge$, there is an exact sequence $\mathscr{F}_m \rightarrowtail \mathscr{F}_{m-1} \to \cdots \to \mathscr{F}_1 \to \mathscr{F}_0 \twoheadrightarrow i_*(\mathcal{N})$, for some $m > 0$, where $\mathscr{F}_k$ is a flat sheaf on X for every $0 \leq k \leq m$. Apply now the exact functor i^* (see [18, 6.3]) to obtain the following exact sequence in $\mathfrak{Qcoh}(U)$:

$$
\begin{array}{ccccccccc}
i^*(\mathscr{F}_m) & \rightarrowtail & i^*(\mathscr{F}_{m-1}) & \longrightarrow & \cdots & \longrightarrow & i^*(\mathscr{F}_1) & \longrightarrow & i^*(\mathscr{F}_0) & \longrightarrow\!\!\!\!\!\rightarrow & i^*(i_*(\mathcal{N})) \\
\| & & \| & & & & \| & & \| & & \| \\
\mathscr{F}_m|_U & \rightarrowtail & \mathscr{F}_{m-1}|_U & \longrightarrow & \cdots & \longrightarrow & \mathscr{F}_1|_U & \longrightarrow & \mathscr{F}_0|_U & \longrightarrow\!\!\!\!\!\rightarrow & \mathcal{N}
\end{array}
$$

Since the previous sequence is formed by quasi-coherent sheaves on U, it remains exact after applying the functor of global sections $\Gamma(U, -)$ (see [25, Prop. II 5.6]):

$$
\begin{array}{ccccccccc}
\Gamma(U, \mathscr{F}_m|_U) & \rightarrowtail & \Gamma(U, \mathscr{F}_{m-1}|_U) & \longrightarrow & \cdots & \longrightarrow & \Gamma(U, \mathscr{F}_1|_U) & \longrightarrow & \Gamma(U, \mathscr{F}_0|_U) & \longrightarrow\!\!\!\!\!\rightarrow & \Gamma(U, \mathcal{N}) \\
\| & & \| & & & & \| & & \| & & \| \\
\mathscr{F}_m(U) & \rightarrowtail & \mathscr{F}_{m-1}(U) & \longrightarrow & \cdots & \longrightarrow & \mathscr{F}_1(U) & \longrightarrow & \mathscr{F}_0(U) & \longrightarrow\!\!\!\!\!\rightarrow & \mathcal{N}(U)
\end{array}
$$

Here, each $\mathscr{F}_k(U)$ is a flat $\mathcal{O}_X(U)$-module by the case $m = 0$ settled previously. Then, $\mathcal{M}(U) \in \mathcal{F}(\mathcal{O}_X(U))^\wedge$. Hence, from (ii) we can conclude that $\mathcal{M}(U) \in (\mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U)))^\wedge$.

Therefore, since $i^*(\mathscr{A})(U) \in \mathcal{GF}(\mathcal{O}_X(U))$ and $\mathcal{M}(U) \in (\mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U)))^\wedge$, we conclude that $\mathrm{Ext}^i_X(\mathscr{A}, \mathscr{B}) \cong \mathrm{Ext}^i_{\mathcal{O}_X(U)}(i^*(\mathscr{A})(U), \mathcal{M}(U)) = 0$. □

Proposition 5.5. *Let X be a noetherian[8] semi-separated scheme, and $U \subseteq X$ be an open affine subset such that every $\mathcal{O}_X(U)$-module has finite Gorenstein flat dimension. Then, for every $\mathscr{S} \in i_*(\mathfrak{Qcoh}(U))$ there exists a short exact sequence $\mathscr{S} \rightarrowtail \mathscr{B} \twoheadrightarrow \mathscr{A}$ with $\mathscr{A} \in \mathfrak{GF}(X)$ and $\mathscr{B} \in (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge$.*

Proof. Let us write $\mathscr{S} \simeq i_*(\mathcal{N})$ with $\mathcal{N} \in \mathfrak{Qcoh}(U)$. First, note by [25, Prop. II 5.4] that we can write $\mathcal{N} \simeq \tilde{N}$ for some $\mathcal{O}_X(U)$-module $N \in \mathsf{Mod}(\mathcal{O}_X(U))$. Since N has finite Gorenstein flat dimension, and $(\mathcal{GF}(\mathcal{O}_X(U)), \mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U)))$ is a left Frobenius pair in, there exists a short exact sequence $N \rightarrowtail B \twoheadrightarrow A$ with $A \in \mathcal{GF}(\mathcal{O}_X(U))$ and $B \in (\mathcal{F}(\mathcal{O}_X(U)) \cap \mathcal{C}(\mathcal{O}_X(U)))^\wedge$. The previous induces by [25, Prop. II 5.2] an exact sequence $\tilde{N} \rightarrowtail \tilde{B} \twoheadrightarrow \tilde{A}$ of associated sheaves on U. Now since the functor i_* is exact by [18, Lem. 6.5], the previous sequence induces in turn a short exact sequence $\mathscr{S} \rightarrowtail i_*(\tilde{B}) \twoheadrightarrow i_*(\tilde{A})$ of sheaves on X. The result will follow after showing that $i_*(\tilde{A}) \in \mathfrak{GF}(X)$ and $i_*(\tilde{B}) \in (\mathfrak{F}(X) \cap \mathfrak{C}(X))^\wedge$:

---

[8]Recall that X is noetherian if it has a finite covering by affine open sets $\mathrm{Spec}(A_i)$, where each A_i is a (commutative) noetherian ring.

- $i_*(\tilde{A})$ is a Gorenstein flat sheaf on X: Since A is a Gorenstein flat $\mathcal{O}_X(U)$-module, we have that $A = Z_0(F_\bullet)$ for some exact complex $F_\bullet$ of flat $\mathcal{O}_X(U)$-modules such that $F_\bullet \otimes_{\mathcal{O}_X(U)} I$ is exact for every injective $I \in \mathsf{Mod}(\mathcal{O}_X(U))$. Using the assumption that X is a noetherian semi-separated scheme, we can apply [35, Lem. 4.8] to deduce that $i_*(\tilde{F}_\bullet)$ is an exact complex of flat sheaves on X such that $i_*(\tilde{F}_\bullet) \otimes \mathscr{I}$ is exact for every injective sheaf $\mathscr{I} \in \mathfrak{Qcoh}(X)$. Hence, $i_*(\tilde{A}) = Z_0(i_*(\tilde{F}_\bullet))$ is a Gorenstein flat sheaf on X.
- $i_*(\tilde{B})$ has finite flat-cotorsion dimension: For B there is a flat-cotorsion resolution of B, say $F_m \rightarrowtail F_{m-1} \to \cdots \to F_1 \to F_0 \twoheadrightarrow B$, which induces an exact sequence $i_*(\tilde{F}_m) \rightarrowtail i_*(\tilde{F}_{m-1}) \to \cdots \to i_*(\tilde{F}_1) \to i_*(\tilde{F}_0) \twoheadrightarrow i_*(\tilde{B})$ where each $i_*(\tilde{F}_k)$ is a flat sheaf on X by our previous comments in the proof of Proposition 5.4. Moreover, since U is affine, for every flat sheaf $\mathscr{F}$ and $0 \le k \le m$ we have that $\mathrm{Ext}^i_X(\mathscr{F}, i_*(\tilde{F}_k)) \cong \mathrm{Ext}^i_U(i^*(\mathscr{F}), F_k) \cong \mathrm{Ext}^1_{\mathcal{O}_X(U)}(\mathscr{F}(U), F_k) = 0$. Hence, the previous sequence is a flat-cotorsion resolution of $i_*(\tilde{B})$. $\qquad\square$

Cut cotorsion pairs and the Finitistic Dimension Conjecture. Among the homological conjectures studied nowadays, the *Finitistic Dimension Conjecture* has a remarkable importance in representation theory of algebras, as it implies the validity of other well-known conjectures, such as the *Nunke Condition* and the (*generalized*) *Nakayama Conjecture*. The Finitistic Dimension Conjecture was stated by H. Bass in 1960 [2], and it says that the *small finitistic dimension* of an Artin algebra is always finite. This problem still remains open, but has been proved in several cases (see for instance [23, 24, 28]).

In the next lines, we give some examples of cut cotorsion pairs and complete cut cotorsion pairs that arise when studying the finiteness of the big and small finitistic dimensions of a ring. Moreover, in the last part of this section, we shall provide a characterization of the Finitistic Dimension Conjecture in terms of complete cut cotorsion pairs.

In what follows, we let $\mathcal{C}$ be an abelian category with enough projective and injective objects. The *finitistic dimension of* $\mathcal{C}$ is defined as

$$\mathrm{Findim}(\mathcal{C}) := \mathrm{pd}(\mathcal{P}^\wedge).$$

Note that $\mathcal{P}^\wedge$ is a resolving subcategory, and since $\mathcal{C}$ has enough projectives, one has that $(\mathcal{P}^\wedge)^{\perp_1} = (\mathcal{P}^\wedge)^\perp$. Hence, by setting the subcategories

$$\mathcal{G} := (\mathcal{P}^\wedge)^{\perp_1} \text{ and } \mathcal{F} := {}^{\perp_1}\mathcal{G} = {}^{\perp_1}((\mathcal{P}^\wedge)^{\perp_1})$$

one forms a hereditary cotorsion pair $(\mathcal{F}, \mathcal{G})$, which turns out to be useful for computing the finitistic dimension of $\mathcal{C}$, as we show in the following result.

Proposition 5.6. $\mathrm{Findim}(\mathcal{C}) = \mathrm{coresdim}_\mathcal{G}(\mathcal{C}) = \mathrm{pd}(\mathcal{F})$.

Proof. For any $M \in \mathcal{C}$ we have that $\mathrm{coresdim}_\mathcal{G}(M) = \mathrm{coresdim}_{(\mathcal{P}^\wedge)^\perp}(M)$. Now by the dual of [4, Lem. 2.11], we get that $\mathrm{coresdim}_{(\mathcal{P}^\wedge)^\perp}(M) = \mathrm{id}_{\mathcal{P}^\wedge}(M)$, which yields $\mathrm{coresdim}_{(\mathcal{P}^\wedge)^\perp}(\mathcal{C}) = \mathrm{id}_{\mathcal{P}^\wedge}(\mathcal{C}) = \mathrm{pd}(\mathcal{P}^\wedge) =: \mathrm{Findim}(\mathcal{C})$. On the other hand, using again [4, Lem. 2.11], we get that $\mathrm{pd}(\mathcal{F}) = \mathrm{id}_{{}^{\perp_1}\mathcal{G}}(\mathcal{C}) = \mathrm{coresdim}_{({}^{\perp_1}\mathcal{G})^\perp}(\mathcal{C}) = \mathrm{coresdim}_\mathcal{G}(\mathcal{C})$. Hence, the result follows. $\qquad\square$

In the next result, we aim to characterize the finiteness of $\mathrm{Findim}(\mathcal{C})$ by means of the existence of a certain cut cotorsion pair.

Theorem 5.7. *The following conditions are equivalent for any $n \ge 0$ and $\mathcal{G} := (\mathcal{P}^\wedge)^{\perp_1}$:*

(a) $\mathrm{Findim}(\mathcal{C}) \le n$.
(b) $(\mathcal{P}_n^\wedge, \mathcal{G})$ *is a left cotorsion pair cut along* $\mathcal{P}_n^\wedge \cup {}^{\perp_1}\mathcal{G}$.
(c) $\mathcal{P}_n^\wedge = {}^{\perp_1}\mathcal{G}$.

Moreover, $\mathrm{Findim}(\mathcal{C}) = \mathrm{coresdim}_\mathcal{G}(\mathcal{C}) = \mathrm{pd}({}^{\perp_1}\mathcal{G})$.

Proof. The implication (c) $\Rightarrow$ (b) is immediate, while for (a) $\Rightarrow$ (c) we have by Proposition 5.6 that $\mathrm{pd}(^{\perp_1}\mathcal{G}) = \mathrm{Findim}(\mathcal{C}) \leq n$, and so $^{\perp_1}\mathcal{G} \subseteq \mathcal{P}_n^\wedge \subseteq \mathcal{P}^\wedge \subseteq {}^{\perp_1}\mathcal{G}$. Finally, for (b) $\Rightarrow$ (a), we can note that $\mathcal{P}_n^\wedge = {}^{\perp_1}\mathcal{G}$. We then have by Proposition 5.6 that $\mathrm{Findim}(\mathcal{C}) = \mathrm{pd}(^{\perp_1}\mathcal{G}) = \mathrm{pd}(\mathcal{P}_n^\wedge) \leq n$. $\qquad\square$

In the particular case where $\mathcal{C}$ is the category $\mathsf{Mod}(R)$ of modules over a ring R, let $\mathrm{Findim}(R)$ denote the finitistic dimension of $\mathsf{Mod}(R)$. Using [10, Thms. 2.2 3.2], we can add to the equivalence in Theorem 5.7 an additional condition, and also improve condition (b).

Note that in Theorem 5.7, we only need conditions (lccp1) and (lccp2) to characterize the finiteness of $\mathrm{Findim}(\mathcal{C})$, that is, in some cases left completeness is not required for objects along the cut. This is the case shown in the following example.

Example 5.8. *In [6], the authors introduced the notion of objects of finite type in Grothen-dieck categories, as a generalization for finitely n-presented modules in the sense of [7, §1]. An object F in a Grothendieck category $\mathcal{C}$ is said to be **of type FP_n** if the functor $\mathrm{Ext}_\mathcal{C}^k(F, -)$ preserves direct limits for every $0 \leq k \leq n - 1$ (see [6, Def. 2.1]). Recall also that $\mathcal{C}$ is **locally finitely presented** if it has a generating family of finitely presented objects.*

*Let $\mathcal{FP}_n$ denote the subcategory of objects of type FP_n in a locally finitely presented Grothen-dieck category $\mathcal{C}$, with $n \geq 2$. Consider also the subcategory $\mathcal{FP}_n^{\perp_1}$ of **FP_n-injective objects**. By [6, Part 4 of Prop. 2.8], we know that $\mathcal{FP}_n$ is closed under direct summands. Moreover, by [6, Prop. 3.8] the equality $\mathcal{FP}_n \cap \mathcal{FP}_{n-1} = {}^{\perp_1}(\mathcal{FP}_n^{\perp_1}) \cap \mathcal{FP}_{n-1}$ holds true. Hence, $(\mathcal{FP}_n, \mathcal{FP}_n^{\perp_1})$ and $\mathcal{FP}_{n-1}$ satisfy (lccp1) and (lccp2). Also, it is clear that (rccp1) and (rccp2) hold for $(\mathcal{FP}_n, \mathcal{FP}_n^{\perp_1})$ and $\mathcal{FP}_{n-1}$.*

Moreover, in $\mathsf{Mod}(R)$, $\mathsf{Ch}(R)$ or $\mathfrak{Qcoh}(X)$ (with X semi-separated), $(\mathcal{FP}_1, \mathcal{FP}_1^{\perp_1})$ is a cotorsion pair cut along the subcategory of finitely generated objects. See [6, Rmk. 3.9 Prop. B.2] for details.

Proposition 5.9. *Let R be an arbitrary ring. The following are equivalent for the subcategory $\mathcal{G} := (\mathcal{P}(R)^\wedge)^{\perp_1}$ and any integer $n \geq 0$:*

(a) $\mathrm{Findim}(R) \leq n$.
(b) $\mathcal{P}(R)_n^\wedge \cup {}^{\perp_1}\mathcal{G} \in \mathrm{lCuts}(\mathcal{P}(R)_n^\wedge, \mathcal{G})$.
(c) $\mathcal{P}(R)_n^\wedge = {}^{\perp_1}\mathcal{G}$.
(d) There exists $\mathcal{S} \in \mathrm{rCuts}(\mathcal{P}(R), \mathcal{G}_n^\vee)$ such that $R^{(R)} \in \mathcal{S}$.

Moreover, $\mathrm{Findim}(R) = \mathrm{coresdim}_\mathcal{G}(R^{(R)})$.

Proof. We already have from Theorem 5.7 the implications (a) $\Leftrightarrow$ (c) and (b) $\Rightarrow$ (a) and (c).

- (c) $\Rightarrow$ (b): From (c) we have that $(\mathcal{P}(R)_n^\wedge, \mathcal{G}) = ({}^{\perp_1}\mathcal{G}, \mathcal{G})$ is a hereditary cotorsion pair in $\mathsf{Mod}(R)$, which is also complete by [14, Thm. 7.4.6]. In particular, $(\mathcal{P}(R)_n^\wedge, \mathcal{G})$ is a complete cotorsion pair cut along any subcategory of $\mathsf{Mod}(R)$.
- (c) $\Rightarrow$ (a): It follows from [10, Thm. 3.2].
- (a) $\Rightarrow$ (d): From [10, Thm. 3.2] and Proposition 5.6, we get that $\mathcal{G}_n^\vee = \mathsf{Mod}(R)$. Then, $(\mathcal{P}(R), \mathcal{G}_n^\vee) = (\mathcal{P}(R), \mathsf{Mod}(R))$ is clearly a complete right cotorsion pair cut along $\mathcal{S} := \mathsf{Mod}(R)$.
- (d) $\Rightarrow$ (a): Condition (d) yields $\mathcal{G}_n^\vee \cap \mathcal{S} = \mathcal{P}(R)^{\perp_1} \cap \mathcal{S} = \mathcal{S}$. Thus, $R^{(R)} \in \mathcal{S} \subseteq \mathcal{G}_n^\vee$. By [10, Dual of Prop. 1.11 & Thm. 3.2], the latter is equivalent to saying that $\mathrm{Findim}(R) \leq n$. $\qquad\square$

Condition (d) in the previous theorem can be simplified for certain rings. Specifically, using the proof of (a) $\Leftrightarrow$ (d), along with [10, Coroll. 3.3], we have the following result.

Corollary 5.10. *Let R be a left perfect and right coherent ring and $\mathcal{G} := (\mathcal{P}(R)^\wedge)^{\perp_1}$. Then, $\mathrm{Findim}(R) \leq n$ if, and only if, there exists a subcategory $\mathcal{S} \subseteq \mathsf{Mod}(R)$ such that $\mathcal{S} \in \mathrm{rCuts}(\mathcal{P}(R), \mathcal{G}_n^\vee)$ with $R \in \mathcal{S}$.*

In the rest of this section, we apply Theorem 5.7 to establish a relation between cotorsion cuts and the small finitistic dimension of a ring. We shall work with a slight generalization of this dimension.

Let $\mathcal{FP}_\infty(R) = \bigcup_{n \geq 0} \mathcal{FP}_n(R)$, where $\mathcal{FP}_n(R)$ denotes the subcategory of R-modules of type FP_n (see Example 5.8 above). The R-modules in $\mathcal{FP}_\infty(R)$ are known as *modules of type* FP_∞.

Definition 5.11. *Let R be a ring. The **FP-finitistic dimension** of R is*

$$\mathrm{FP\text{-}findim}(R) := \mathrm{pd}(\mathcal{P}(R)^\wedge \cap \mathcal{FP}_\infty(R)).$$

*Let $\mathsf{mod}(R)$ be the subcategory of finitely generated R-modules[9]. The **small finitistic dimension** of R is*

$$\mathrm{findim}(R) := \mathrm{pd}(\mathcal{P}(R)^\wedge \cap \mathsf{mod}(R)).$$

Remark 5.12. *It is known that in the case where R is a left noetherian ring, $\mathcal{FP}_\infty(R) = \mathsf{mod}(R)$, and so $\mathrm{FP\text{-}findim}(R) = \mathrm{findim}(R)$.*

In what follows, let us consider the subcategory

$$\mathcal{G}^{<\infty} := (\mathcal{P}(R)^\wedge \cap \mathcal{FP}_\infty(R))^{\perp_1}.$$

The subcategory $\mathcal{FP}_\infty(R)$ is thick by [5, Prop. 2.3], and so we can note that $\mathcal{G}^{<\infty} = (\mathcal{P}(R)^\wedge \cap \mathcal{FP}_\infty(R))^\perp$. From this equality, and following the arguments in Proposition 5.6 and Theorem 5.7, one can show that

$$\mathrm{FP\text{-}findim}(R) = \mathrm{coresdim}_{\mathcal{G}^{<\infty}}(\mathsf{Mod}(R)) = \mathrm{pd}(^{\perp_1}(\mathcal{G}^{<\infty})). \tag{iii}$$

The following result is an extension of [10, Thm. 3.4] in the setting of cotorsion cuts.

Proposition 5.13. *The following assertions are equivalent for any integer $n \geq 0$:*

(a) $\mathrm{FP\text{-}findim}(R) \leq n$.

(b) $(\mathcal{P}(R)_n^\wedge \cup {}^{\perp_1}(\mathcal{G}^{<\infty})) \cap \mathcal{FP}_\infty(R) \in \mathrm{lCuts}(\mathcal{P}(R)_n^\wedge, \mathcal{G}^{<\infty})$.

(c) $\mathcal{P}(R)_n^\wedge \cap \mathcal{FP}_\infty(R) = {}^{\perp_1}(\mathcal{G}^{<\infty}) \cap \mathcal{FP}_\infty(R)$.

(d) *There exists $\mathcal{S} \subseteq \mathsf{Mod}(R)$ such that $\mathcal{S} \in \mathrm{rCuts}(\mathcal{P}(R), (\mathcal{G}^{<\infty})_n^\vee)$ with $R^{(R)} \in \mathcal{S}$.*

Proof.

- (a) $\Rightarrow$ (c): The equality (iii) implies ${}^{\perp_1}(\mathcal{G}^{<\infty}) \cap \mathcal{FP}_\infty(R) \subseteq \mathcal{P}(R)_n^\wedge \cap \mathcal{FP}_\infty(R)$. The other containment is clear.
- (c) $\Rightarrow$ (b): It is easy to verify from (c) that $(\mathcal{P}(R)_n^\wedge, \mathcal{G}^{<\infty})$ is a left cotorsion pair cut along $(\mathcal{P}(R)_n^\wedge \cup {}^{\perp_1}(\mathcal{G}^{<\infty})) \cap \mathcal{FP}_\infty(R)$. Condition (lccp3) is also clear from the assumption.
- (b) $\Rightarrow$ (a): From (b) we can note $\mathcal{P}(R)_n^\wedge \cap \mathcal{FP}_\infty(R) = {}^{\perp_1}(\mathcal{G}^{<\infty}) \cap \mathcal{FP}_\infty(R)$. This implies that $\mathcal{P}(R)^\wedge \cap \mathcal{FP}_\infty(R) = \mathcal{P}(R)_n^\wedge \cap \mathcal{FP}_\infty(R)$, and so from (iii) we have that $\mathrm{FP\text{-}findim}(R) = \mathrm{pd}(\mathcal{P}(R)_n^\wedge \cap \mathcal{FP}_\infty(R)) \leq n$.
- (a) $\Rightarrow$ (d): Similar to the corresponding implication in Proposition 5.9 and follows by using [10, Thm. 3.4].
- (d) $\Rightarrow$ (a): Suppose there exists $\mathcal{S} \in \mathrm{rCuts}(\mathcal{P}(R), (\mathcal{G}^{<\infty})_n^\vee)$ with $R^{(R)} \in \mathcal{S}$. Then, it follows that $\mathcal{S} = (\mathcal{G}^{<\infty})_n^\vee \cap \mathcal{S}$, and so $R^{(R)} \in (\mathcal{G}^{<\infty})_n^\vee$. The latter along with [10, Thm. 3.4 dual of Prop. 1.11] implies that $\mathrm{FP\text{-}findim}(R) \leq n$. □

From the previous result, we can obtain the following characterization for the finiteness of $\mathrm{FP\text{-}findim}(R)$, provided that R is coherent, in terms of right cotorsion cuts. This way we extend [10, Coroll. 3.5].

---

[9]The reader should be warned that in [10] the notation $\mathsf{mod}(R)$ is used for the subcategory of all R-modules admitting a projective resolution consisting of finitely generated modules.

Corollary 5.14. *For any left coherent ring R, the following assertions are equivalent:*

  (a) FP-findim$(R) \leq n$.
  (b) *There exists* $\mathcal{S} \subseteq \mathcal{C}$ *such that* $\mathcal{S} \in \mathrm{rCuts}(\mathcal{P}(R), (\mathcal{G}^{<\infty})^{\vee}_n)$ *with* $R \in \mathcal{S}$.

Recall from Remark 3.12 that FP-findim$(R) =$ findim(R) provided that R is a left noetherian ring. Since any artinian ring is noetherian, one can deduce the following extension of [10, Coroll. 3.6] by using Propositions 5.9 and 5.13.

Corollary 5.15. *The following statements hold true for any two-sided artinian ring R:*

  (1) Findim$(R) = n$ *if, and only if, n is the smallest nonnegative integer such that there exists* $\mathcal{S} \subseteq$ Mod(R), *with* $R \in \mathcal{S}$ *and such that* $(\mathcal{P}(R), \mathcal{G}^{\vee}_n)$ *is a complete right cotorsion pair cut along* $\mathcal{S}$.
  (2) findim$(R) = n$ *if, and only if, n is the smallest nonnegative integer such that there exists* $\mathcal{S} \subseteq$ Mod(R), *with* $R \in \mathcal{S}$ *and such that* $(\mathcal{P}(R), (\mathcal{G}^{<\infty})^{\vee}_n)$ *is a complete right cotorsion pair cut along* $\mathcal{S}$.

Relations with Serre subcategories. Let $\mathcal{C}$ be a locally small abelian category. Recall that a subcategory $\mathcal{S} \subseteq \mathcal{C}$ is a *Serre subcategory* if for every short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{C}$, one has that $Y \in \mathcal{S}$ if, and only if, $X, Z \in \mathcal{S}$. In particular, Serre subcategories are clearly thick, and closed under subobjects and quotients.

If $\mathcal{S} \subseteq \mathcal{C}$ is a Serre subcategory, we can consider the *Serre quotient* $\mathcal{C}/\mathcal{S}$, which is abelian and whose objects are the same objects in $\mathcal{C}$. In Ogawa's [40], the author gives several outcomes from the existence of a right adjoint for the associated quotient functor $Q \colon \mathcal{C} \to \mathcal{C}/\mathcal{S}$. The purpose of this section is to characterize the latter via complete right cut cotorsion pairs. Let us begin proving the following consequence of having a right adjoint for Q.

Proposition 5.16. *Let* $\mathcal{S}$ *be a Serre subcategory of* $\mathcal{C}$. *If the Serre quotient functor* $Q \colon \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{S}$ *admits a right adjoint, then* $(\mathcal{S}, \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1})$ *is a complete right cotorsion pair cut along* $\mathcal{S}^{\perp_0}$.

Proof. It is clear that the dual of conditions (1) and (2) in Proposition 2.12 are satisfied. It is only left to show that for every object $M \in \mathcal{S}^{\perp_0}$ there exists a short exact sequence $M \rightarrowtail F \twoheadrightarrow K$ with $F \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$ and $K \in \mathcal{S}$. So let us take $M \in \mathcal{S}^{\perp_0}$. By [40, Props. 1.1 & 1.3], there exists an exact sequence $S \xrightarrow{f} M \xrightarrow{g} Y$ with $S \in \mathcal{S}$ and $Y \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$. Since $M \in \mathcal{S}^{\perp_0}$, we have that $f = 0$, and so g is a monomorphism. We can thus consider the short exact sequence $M \xrightarrow{g} Y \twoheadrightarrow \mathrm{CoKer}(g)$. Let us now apply again [40, Props. 1.1 & 1.3] to the object $\mathrm{CoKer}(g)$. We get an exact sequence $D \xrightarrow{h} \mathrm{CoKer}(g) \xrightarrow{i} E$ with $D \in \mathcal{S}$ and $E \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$. Let us factor h and i through their images, so that we get the following commutative diagram

$$
\begin{array}{ccccccc}
D & \longrightarrow & \mathrm{CoKer}(g) & \longrightarrow & E & \longrightarrow & \mathrm{CoKer}(i) \\
& \searrow & \nearrow & & \searrow & \nearrow & \\
& & K & & & C &
\end{array}
\qquad \text{(iv)}
$$

where $K := \mathrm{Im}(h) = \mathrm{Ker}(i)$ and $C := \mathrm{CoKer}(\mathrm{Ker}(i)) \simeq \mathrm{Ker}(\mathrm{CoKer}(i))$. Notice that $K \in \mathcal{S}$ since $D \in \mathcal{S}$ and $\mathcal{S}$ is closed under quotients. Taking the pullback of $K \to \mathrm{CoKer}(g) \leftarrow Y$ yields the following solid diagram:

$$
\begin{array}{ccc}
M & =\!=\!=\!=\!=\!= & M \\
\downarrow & & \downarrow \\
F & \rightarrowtail Y \twoheadrightarrow & C \\
\downarrow & \mathbf{pb} \quad \downarrow & \| \\
K & \rightarrowtail \mathrm{CoKer}(g) \twoheadrightarrow & C
\end{array}
\tag{v}
$$

We show that $F \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$. Let $S' \in \mathcal{S}$ and apply the functor $\mathrm{Hom}_{\mathcal{C}}(S', -)$ to the central row in (v). We get the following exact sequence:

$$
\mathrm{Hom}_{\mathcal{C}}(S', F) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(S', Y) \to \mathrm{Hom}_{\mathcal{C}}(S', C) \to \mathrm{Ext}^1_{\mathcal{C}}(S', F) \to \mathrm{Ext}^1_{\mathcal{C}}(S', Y),
$$

where $\mathrm{Hom}_{\mathcal{C}}(S', Y) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(S', Y)$ since $Y \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$. It follows that $\mathrm{Hom}_{\mathcal{C}}(S', F) = 0$ and $\mathrm{Hom}_{\mathcal{C}}(S', C) \cong \mathrm{Ext}^1_{\mathcal{C}}(S', F)$. Now consider the short exact sequence $C \rightarrowtail E \twoheadrightarrow \mathrm{CoKer}(i)$ in (iv). By applying the functor $\mathrm{Hom}_{\mathcal{C}}(S', -)$ to this sequence we obtain the monomorphism $\mathrm{Hom}_{\mathcal{C}}(S', C) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(S', E)$, where $\mathrm{Hom}_{\mathcal{C}}(S', E) = 0$ since $E \in \mathcal{S}^{\perp_0}$. Then, $\mathrm{Ext}^1_{\mathcal{C}}(S', F) \cong \mathrm{Hom}_{\mathcal{C}}(S', C) = 0$ for every $S' \in \mathcal{S}$. Therefore, $F \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$, and thus the left-hand column in (v) is the desired exact sequence. $\square$

In the next result, we prove the converse of the previous proposition with an additional condition: we shall need $\mathcal{C}$ to be cocomplete. We need to recall from [30, Def. 2.1] that two subcategories $\mathcal{T}, \mathcal{F} \subseteq \mathcal{C}$ form a *torsion pair* $(\mathcal{T}, \mathcal{F})$ in $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}}(\mathcal{T}, \mathcal{F}) = 0$ and if for every $C \in \mathcal{C}$ there is an exact sequence $T_M \rightarrowtail M \twoheadrightarrow F_M$ with $T_M \in \mathcal{T}$ and $F_M \in \mathcal{F}$. In this case, $\mathcal{T}$ and $\mathcal{F}$ are called the *torsion subcategory* and the *torsion-free subcategory*, respectively. If $\mathcal{C}$ is cocomplete, it is known that $(\mathcal{T}, \mathcal{F})$ is a torsion pair if, and only if, $\mathcal{T}$ is closed under extensions, quotients and coproducts (See for instance [44, Prop. VI. 2.1]). In particular, every Serre subcategory $\mathcal{S}$ of a cocomplete locally small abelian category $\mathcal{C}$, which is closed under coproducts, is torsion.

Theorem 5.17. *Let $\mathcal{S}$ be a Serre subcategory of a cocomplete abelian category $\mathcal{C}$. If $\mathcal{S}$ is closed under coproducts, then the following conditions are equivalent:*

(a) $Q \colon \mathcal{C} \to \mathcal{C}/\mathcal{S}$ admits a right adjoint.
(b) $(\mathcal{S}, \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1})$ is a complete right cotorsion pair cut along $\mathcal{S}^{\perp_0}$.

Proof. The implication (a) $\Rightarrow$ (b) is Proposition 5.16. For the implication (b) $\Rightarrow$ (a), by [40, Prop. 1.3] it suffices to show that for every $M \in \mathcal{C}$ there exists an exact sequence $S \xrightarrow{f} M \xrightarrow{g} Y$ where $S \in \mathcal{S}$ and $Y \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$. Indeed, for $M \in \mathcal{C}$, we have an exact sequence $S \xrightarrow{f} M \xrightarrow{g} S_0$ with $S \in \mathcal{S}$ and $S_0 \in \mathcal{S}^{\perp_0}$, since $\mathcal{S}$ is torsion. On the other hand, since $(\mathcal{S}, \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1})$ is a complete right cotorsion pair cut along $\mathcal{S}^{\perp_0}$, there exists a monomorphism $S_0 \xrightarrow{h} Y$ with $Y \in \mathcal{S}^{\perp_0} \cap \mathcal{S}^{\perp_1}$. Hence, we can form the exact sequence $S \xrightarrow{f} M \xrightarrow{h \circ g} Y$. $\square$

Cuts from extriangulated categories. We conclude this article with a final application of complete cut cotorsion pairs in the context of extriangulated categories. Such categories where introduced by H. Nakaoka and Y. Palu in [38] as a simultaneous generalization of triangulated categories and exact categories.

In what follows, we let $(\mathfrak{A}, \mathbb{E}, \mathfrak{s})$ denote an extriangulated category. Here, $\mathfrak{A}$ is a skeletally small additive category, $\mathbb{E} \colon \mathfrak{A}^{\mathrm{op}} \times \mathfrak{A} \longrightarrow \mathsf{Mod}(\mathbb{Z})$ is a biadditive functor with an additive realization $\mathfrak{s}$ satisfying a series of axioms (see [38, Def. 2.12] for details). We shall also consider the following categories constructed from $\mathfrak{A}$:

- $\mathsf{mod}(\mathfrak{A}^{\mathrm{op}})$ denotes the subcategory of the (Grothendieck) category $\mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})$ formed by the right $\mathfrak{A}$-modules (or $\mathfrak{A}^{\mathrm{op}}$-modules) $F\colon \mathfrak{A}^{\mathrm{op}} \longrightarrow \mathsf{Mod}(\mathbb{Z})$ which are *finitely presented*, that is, for which there exists an exact sequence $\mathrm{Hom}_{\mathfrak{A}}(-, A_1) \to \mathrm{Hom}_{\mathfrak{A}}(-, A_0) \twoheadrightarrow F$ in $\mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})$ with $A_0, A_1 \in \mathfrak{A}$.
- For any subcategory $\mathcal{X} \subseteq \mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})$ of $\mathfrak{A}^{\mathrm{op}}$-modules, $\overrightarrow{\mathcal{X}}$ is the subcategory of $\mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})$ of direct limits of objects in $\mathcal{X}$.
- $\mathrm{Lex}(\mathfrak{A}^{\mathrm{op}})$ and $\mathrm{lex}(\mathfrak{A}^{\mathrm{op}})$ denote the subcategories of $\mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})$ and $\mathsf{mod}(\mathfrak{A}^{\mathrm{op}})$, respectively, of all left exact $\mathfrak{A}^{\mathrm{op}}$-modules. Recall that an $\mathfrak{A}^{\mathrm{op}}$-module is *left exact* if it maps kernels in $\mathfrak{A}$ into cokernels in $\mathsf{Mod}(\mathbb{Z})$.
- $\mathrm{def}(\mathfrak{A}^{\mathrm{op}})$ denotes the subcategory of $\mathsf{mod}(\mathfrak{A}^{\mathrm{op}})$ consisting of all finitely presented $\mathfrak{A}^{\mathrm{op}}$-modules isomorphic to defects (see [40, Def. 2.4]).

For skeletally small extriangulated categories, we can obtain from the subcategories $\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})}$ and $\mathrm{Lex}(\mathfrak{A}^{\mathrm{op}})$ the following example of a complete cut cotorsion pair.

Proposition 5.18. *Let* $(\mathfrak{A}, \mathbb{E}, \mathfrak{s})$ *be a skeletally small extriangulated category with weak kernels. Then,*

$$(\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})}, \mathrm{Lex}(\mathfrak{A}^{\mathrm{op}}))$$

is a complete right cotorsion pair cut along $(\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})})^{\perp_0}$.

Proof. By [40, Prop. 2.5] we know that $\mathrm{def}(\mathfrak{A}^{\mathrm{op}})$ is a Serre subcategory of $\mathsf{mod}(\mathfrak{A}^{\mathrm{op}})$. On the other hand, by Krause's [31, Thm. 2.8] we have that $\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})}$ is a Serre subcategory of $\mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})$, and by [40, Thm. 3.1] the quotient functor $Q\colon \mathsf{Mod}(\mathfrak{A}^{\mathrm{op}}) \longrightarrow \mathsf{Mod}(\mathfrak{A}^{\mathrm{op}})/\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})}$ admits a right adjoint. It then follows by Proposition 5.16 that $(\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})}, (\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})})^{\perp_0} \cap (\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})})^{\perp_1})$ is a complete right cotorsion pair cut along $(\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})})^{\perp_0}$. Finally, $\mathrm{Lex}(\mathfrak{A}^{\mathrm{op}}) = (\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})})^{\perp_0} \cap (\overrightarrow{\mathrm{def}(\mathfrak{A}^{\mathrm{op}})})^{\perp_1}$ follows by [40, Lem. 3.3]. $\square$

The following is the finitely presented version of the previous proposition.

Proposition 5.19. *Let* $(\mathfrak{A}, \mathbb{E}, \mathfrak{s})$ *be a skeletally small extriangulated category with weak kernels. If the quotient functor* $Q\colon \mathsf{mod}(\mathfrak{A}^{\mathrm{op}}) \longrightarrow \mathsf{mod}(\mathfrak{A}^{\mathrm{op}})/\mathrm{def}(\mathfrak{A}^{\mathrm{op}})$ *admits a right adjoint, then*

$$(\mathrm{def}(\mathfrak{A}^{\mathrm{op}}), \mathrm{lex}(\mathfrak{A}^{\mathrm{op}}))$$

is a complete right cotorsion pair cut along $(\mathrm{def}(\mathfrak{A}^{\mathrm{op}}))^{\perp_0}$.

Proof. Follows as Proposition 5.18 by using [40, Props. 2.5, 2.8 (2), Thm. 2.9] and Proposition 5.16. $\square$

The existence of the previous cut cotorsion pair can be also guaranteed in the particular case where $\mathfrak{A}$ is an exact category, under some mild additional assumptions, as we specify below.

Corollary 5.20. *Let* $\mathfrak{A}$ *be a skeletally small exact category with weak kernels and enough projectives. Then,*

$$(\mathrm{def}(\mathfrak{A}^{\mathrm{op}}), \mathrm{lex}(\mathfrak{A}^{\mathrm{op}}))$$

is a complete right cotorsion pair cut along $(\mathrm{def}(\mathfrak{A}^{\mathrm{op}}))^{\perp_0}$.

Proof. By [40, Props. 2.16 (1) 2.17] we have that the quotient functor

$$Q\colon \mathsf{mod}(\mathfrak{A}^{\mathrm{op}}) \longrightarrow \mathsf{mod}(\mathfrak{A}^{\mathrm{op}})/\mathrm{def}(\mathfrak{A}^{\mathrm{op}})$$

admits a right adjoint. Hence, the result follows from Proposition 5.19. $\square$

We conclude this section also covering the other particular case where $\mathfrak{A}$ is a triangulated category. We show how to induce from a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathfrak{A}$, the complete right cotorsion pair $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{V}^{\mathrm{op}}))$ cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$ in $\mathrm{mod}(\mathfrak{A}^{\mathrm{op}})$. Moreover, we show that $(\mathcal{U}, \mathcal{V})$ is a co-t-structure if, and only if, $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{V}^{\mathrm{op}}))$ is also a complete left cotorsion pair cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$.

In what follows, let us fix a skeletally small triangulated category $\mathfrak{A}$ with translation automorphism $[1]\colon \mathfrak{A} \longrightarrow \mathfrak{A}$. Given (full) additive subcategories $\mathcal{U}, \mathcal{V} \subseteq \mathfrak{A}$, recall that $(\mathcal{U}, \mathcal{V})$ is a *cotorsion pair* in $\mathfrak{A}$ if the following two conditions are satisfied:

(1) $\mathrm{Hom}_{\mathfrak{A}}(U, V') = 0$ for every $U \in \mathcal{U}$ and $V' \in \mathcal{V}[1]$. Here, $\mathcal{V}[1]$ denotes the subcategory of objects in $\mathfrak{A}$ isomorphic to objects of the form $[1](V)$ with $V \in \mathcal{V}$.

(2) $\mathfrak{A} = \mathcal{U} * \mathcal{V}[1]$, that is, if every $C \in \mathfrak{A}$ admits a distinguished triangle $U \to C \to V' \to U[1]$ where $U \in \mathcal{U}$ and $V' \in \mathcal{V}[1]$.

Following [4, 40], if $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair in $\mathfrak{A}$, then $\mathcal{U}$ gives rise to an extriangulated category with weak kernels, translation automorphism $[1]|_{\mathcal{U}}$ and biadditive functor $\mathbb{E}(+, -) := \mathrm{Hom}_{\mathcal{U}}(+, -[1])\colon \mathcal{U}^{\mathrm{op}} \times \mathcal{U} \longrightarrow \mathsf{Mod}(\mathbb{Z})$. Here, $\mathfrak{A}^+ = \mathcal{W} * \mathcal{V}[1]$ and $\mathfrak{A}^- = \mathcal{U}[-1] * \mathcal{W}$, where $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$. Moreover, by [40, Prop. 4.2] the quotient functor $Q\colon \mathsf{mod}(\mathcal{U}^{\mathrm{op}}) \longrightarrow \mathsf{mod}(\mathcal{U}^{\mathrm{op}})/\mathrm{def}(\mathcal{U}^{\mathrm{op}})$ has a right adjoint, and so from Proposition 5.19 we deduce the following result.

Corollary 5.21. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in $\mathfrak{A}$. Then, $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{U}^{\mathrm{op}}))$ is a complete right cotorsion pair cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$.*

Recall that a *co-t-structure* is a pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathfrak{A}$ such that $(\mathcal{X}[1], \mathcal{Y})$ is a cotorsion pair in $\mathfrak{A}$ satisfying $\mathcal{X} \subseteq \mathcal{X}[1]$.

Proposition 5.22. *The following are equivalent for every cotorsion pair $(\mathcal{U}, \mathcal{V})$ in $\mathfrak{A}$:*

(a) $(\mathcal{U}, \mathcal{V})$ is a co-t-structure in $\mathfrak{A}$ (that is, $\mathcal{U}[-1] \subseteq \mathcal{U}$).
(b) $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{U}^{\mathrm{op}}))$ is a complete cotorsion pair cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$.

Proof. First, suppose condition (a) holds. By the previous corollary, we have that $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{U}^{\mathrm{op}}))$ is a complete right cotorsion pair cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$. So we focus on showing that $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{U}^{\mathrm{op}}))$ is a complete left cotorsion pair cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$. Consider the heart of the cotorsion pair $(\mathcal{U}, \mathcal{V})$ given by $\underline{\mathcal{H}} = (\mathfrak{A}^+ \cap \mathfrak{A}^-)/\mathcal{W}$. It is known by Nakaoka's [36, Thm. 6.4] that $\underline{\mathcal{H}}$ is an abelian category. Moreover, by [40, Thm. 4.7] it is also known that $\underline{\mathcal{H}}$ and $\mathrm{lex}(\mathcal{U}^{\mathrm{op}})$ are naturally equivalent. Using the assumption (a) that $(\mathcal{U}, \mathcal{V})$ is a co-t-structure in $\mathfrak{A}$, we can note that $\mathfrak{A}^+ \cap \mathfrak{A}^- \subseteq \mathcal{U} \cap \mathcal{V}$, and so $\underline{\mathcal{H}} = 0$, and hence $\mathrm{lex}(\mathcal{U}^{\mathrm{op}}) = 0$. On the other hand, by [40, Prop. 4.2] and [41, Thm. IV.4.5], we can note that for every $X \in \mathsf{mod}(\mathcal{U}^{\mathrm{op}})$ there exists an epimorphism $D \twoheadrightarrow X$ with $D \in \mathrm{def}(\mathcal{U}^{\mathrm{op}})$, and since $\mathrm{def}(\mathcal{U}^{\mathrm{op}})$ is a Serre subcategory, the previous implies that $\mathsf{mod}(\mathcal{U}^{\mathrm{op}}) = \mathrm{def}(\mathcal{U}^{\mathrm{op}})$. It then follows that $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{U}^{\mathrm{op}})) = (\mathsf{mod}(\mathcal{U}^{\mathrm{op}}), 0)$, which is clearly a complete left cotorsion pair cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$.

Now let us assume (b). We thus have that $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}), \mathrm{lex}(\mathcal{U}^{\mathrm{op}}))$ is a complete cotorsion pair cut along $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$, and so for every $X \in (\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0}$ there exists an epimorphism $D \twoheadrightarrow X$ with $D \in \mathrm{def}(\mathcal{U}^{\mathrm{op}})$. Again, since $\mathrm{def}(\mathcal{U}^{\mathrm{op}})$ is a Serre subcategory, we have that $X \in (\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0} \cap \mathrm{def}(\mathcal{U}^{\mathrm{op}}) = 0$. If follows that $(\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0} = 0$, which in turn and along with [40, Prop. 2.8 (2)] implies that $\mathrm{lex}(\mathcal{U}^{\mathrm{op}}) = (\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_0} \cap (\mathrm{def}(\mathcal{U}^{\mathrm{op}}))^{\perp_1} = 0$. Hence, the cotorsion pair $(\mathcal{U}, \mathcal{V})$ is a co-t-structure in $\mathfrak{A}$ by [37, Rmk. 2.6] and [40, Thm. 4.7]. □

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