MOMENTUM MAPS AND THE KÄHLER PROPERTY FOR BASE SPACES OF REDUCTIVE PRINCIPAL BUNDLES

DANIEL GREB[1] AND CHRISTIAN MIEBACH[2]

[1]*Essener Seminar für Algebraische Geometrie und Arithmetik, Fakultät für Mathematik, Universität Duisburg–Essen, Essen 45117, Germany* (daniel.greb@uni-due.de)
[2]*Université du Littoral Côte d'Opale, UR 2597 – LMPA – Laboratoire de mathématiques pures et appliquées Joseph Liouville, Calais F-62100, France* (christian.miebach@univ-littoral.fr)

Abstract We investigate the complex geometry of total spaces of reductive principal bundles over compact base spaces and establish a close relation between the Kähler property of the base, momentum maps for the action of a maximal compact subgroup on the total space, and the Kähler property of special equivariant compactifications. We provide many examples illustrating that the main result is optimal.

Keywords: complex reductive group; Kähler manifold; symplectic reduction; principal bundle; equivariant compactification

1. Introduction

Complex-reductive Lie groups $G = K^{\mathbb{C}}$ acting holomorphically on Kählerian manifolds X appear naturally in many questions of complex geometry. Classical examples, one of which found by Lescure that we recall in § 5 below, show that even if the action is proper and free with compact quotient manifold B, so that X becomes a G-principal bundle over B, the Kähler property does in general not descend to B.

In these examples, the induced action of the maximal compact subgroup K of G is not Hamiltonian with respect to any Kähler form on X, that is, there is no equivariant momentum map $\mu : X \to \mathfrak{k}^* = \mathrm{Lie}(K)^*$. So, a natural guess might be that the Kähler property descends for Hamiltonian G-actions. We will see that it follows from the theory of Kählerian Reduction (see [12, 15, 20] that this is true once, in addition, the zero fibre $\mu^{-1}(0)$ of the momentum map is compact, and conversely the principal G-bundles over

218

compact Kähler manifolds always admit Kähler structures such that the K-action admits a momentum map with compact zero fibre.

The group G is an affine algebraic group in a unique and canonical way. As such, it admits a (non-unique) equivariant smooth projective completion; that is, there exists a smooth projective variety $\overline{G}$ admitting an effective algebraic action of G and containing a point $e \in \overline{G}$ that has trivial stabilizer in G and open G-orbit $G \cong G \bullet e \subset \overline{G}$. Any principal G-bundle $\pi \colon X \to B$ admits a fibrewise partial compactification to a $\overline{G}$-fibre bundle $\overline{\pi} \colon \overline{X} \to B$ having the same transition functions as π; using the associated bundle construction, this can be written as $\overline{X} = X \times_G \overline{G} \to X/G$. With this notation, our main result can now be formulated as follows:

Theorem 1.1. *Let $G = K^{\mathbb{C}}$ be a complex-reductive Lie group acting holomorphically, properly and freely on the connected complex manifold X such that $Q = X/G$ is compact. Then, the following are equivalent:*

(a) *The quotient Q is Kähler.*
(b) *There exists a K-invariant Kähler form on X with respect to which there is a momentum map $\mu \colon X \to \mathrm{Lie}(K)^*$ with compact zero fibre $\mu^{-1}(0) \neq \emptyset$.*
(c) *The natural compactification of the principal G-bundle $\pi \colon X \to Q$ to a $\overline{G}$-fibre bundle $\overline{\pi} \colon \overline{X} \to Q$ is a Kähler manifold.*

The implication '(a) $\Rightarrow$ (b)' can be seen as a first instance of a Hamiltonian version of Mumford's famous GIT-statement [18, Converse 1.12], which has been vastly generalized to show that any open subset of a smooth quasi-projective variety admitting a projective good quotient is actually the set of semistable points with respect to some linearized line bundle (see [13]).

In § 5, we give several examples showing that Theorem 1.1 is optimal. In particular, we construct an example of a proper, free action of a complex-reductive group $G = K^{\mathbb{C}}$ on a Kähler manifold X such that the K-action is Hamiltonian (and in fact X admits an equivariant Kähler compactification on which the K-action is still Hamiltonian) with compact, non-Kähler quotient B; in particular, the K-action is Hamiltonian with respect to *some* Kähler form on X, but there is *no* Kähler form on X with K-momentum map having non-empty compact zero fibre.

Notation

Throughout, we will work over the field $\mathbb{C}$ of complex numbers. A *variety* is a reduced scheme of finite type over $\mathbb{C}$, that is, not necessarily irreducible. All complex spaces are assumed to be Hausdorff and second countable, so that smooth complex spaces are complex manifolds. If G acts holomorphically on a complex manifold X, for any $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$, we will denote the induced (real holomorphic) *fundamental vector field* on X by ξ_X; that is, for any $f \in \mathcal{C}^{\infty}(X)$, we have

$$\xi_X(f)(p) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f(\exp(t\xi) \bullet p)).$$

If a Lie group K with Lie algebra $\mathfrak{k} := \mathrm{Lie}(K)$ acts on a Kähler manifold (X, ω) preserving the Kähler form, a *momentum map* is a K-equivariant map $\mu : X \to \mathfrak{k}^*$ whose components $\mu^\xi(\cdot) = \mu(\cdot)(\xi) \in \mathcal{C}^\infty(X)$ satisfy the Hamiltonian equations

$$d\mu^\xi = \imath_{\xi_X} \omega.$$

If the complex structure of X is denoted by $J \in \mathrm{End}(TX)$, slightly nonstandard, but useful for our purposes, for any $f \in \mathcal{C}^\infty(X)$, we set $d^c(f) := df \circ J$, so that $2i\partial\bar{\partial} = -dd^c$.

2. Momentum maps for K-representations and their projective compactifications

Let V be a finite-dimensional complex G-representation. Assume that we are given a K-invariant Hermitian product $\langle \cdot, \cdot \rangle$ on V, making the induced K-representation unitary. Let $\chi_V : V \to \mathbb{R}^{\geqslant 0}$ be defined by $v \mapsto \langle v, v \rangle$ and consider the Kähler form $-dd^c\chi_V$ on V. Note that the good quotient $\pi_V : V \to V /\!\!/ G = \mathrm{Spec}\left(\mathbb{C}[V]^G\right)$ exists; it is endowed with a Kähler structure by Kählerian reduction with respect to the momentum map

$$\mu_V : V \to \mathfrak{k}^*, \ v \mapsto (\xi \mapsto 2\langle i\xi \cdot v, v \rangle) = d^c\chi_V(\xi_V)(v).$$

These, however, are not the right Kähler form and momentum map to consider in our situation, since they do not globalize to the situation of a G-vector bundle over a non-trivial base manifold with typical fibre V; see Example 3.4.

Instead, we are going to compactify the situation. For this, we consider the G-representation space $W := V \oplus \mathbb{C}$, where G acts trivially on $\mathbb{C}$, and embed V regularly and equivariantly into $\mathbb{P}(W) = \mathbb{P}(V \oplus \mathbb{C})$; explicitly, we consider

$$\theta : V \hookrightarrow \mathbb{P}(V \oplus \mathbb{C}), \ v \mapsto [v : 1].$$

We denote by $\mathbb{P}(W)^{ss}$ the Zariski-open subset of points that are semistable with respect to the linearization of the G-action on $\mathbb{P}(W)$ in W or equivalently the corresponding linearization in the line bundle $\mathcal{O}_{\mathbb{P}(W)}(1)$; the associated good quotient will be called $\pi_{\mathbb{P}(W)} \colon \mathbb{P}(W)^{ss} \to \mathbb{P}(W)^{ss} /\!\!/ G$. While not strictly needed for our subsequent arguments, the following observation regarding GIT is at least philosophically crucial.

Lemma 2.1. *The representation space V is mapped isomorphically to a $\pi_{\mathbb{P}(W)}$-saturated subset of $\mathbb{P}(W)^{ss}$ via the G-equivariant map θ.*

Proof. This is almost tautological: if we choose linear coordinates (i.e., linear forms) $z_1, \ldots, z_{\dim V}, z_0$ on $W = V \oplus \mathbb{C}$, then z_0 is G-invariant, as the action of G on $\mathbb{C}$ is trivial. This implies that the associated section $\sigma = \sigma_{z_0} \in H^0\left(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)\right)$ is G-invariant. Since obviously

$$\theta(V) = \{[w] \in \mathbb{P}(W) \mid \sigma(w) \neq 0\} = \mathbb{P}(W)_\sigma,$$

every point in $\theta(V)$ is semistable. Moreover, for any $\tau \in H^0\big(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)\big)^G$, the corresponding open subset $\mathbb{P}(W)_\tau$ is $\pi_{\mathbb{P}(W)}$-saturated; see the proof of [18, Thm. 1.10]. □

Consequently, we will consider $V \subset \mathbb{P}(V \oplus \mathbb{C})$, suppressing θ. Using the linear coordinate z_0 on $\mathbb{C}$ introduced in the previous proof, we endow the G-representation $W = V \oplus \mathbb{C}$ with the Hermitian form $\langle \cdot, \cdot \rangle$ corresponding to

$$\chi_W(v, z_0) := \chi_V(v) + |z_0|^2.$$

This in turn yields a Fubini-Study form on $\mathbb{P}(W)$, which is K-invariant and makes the K-action Hamiltonian with momentum map given by

$$\mu^\xi([w]) = \frac{2\langle i\xi \cdot w, w \rangle}{\langle w, w \rangle}.$$

The next observation is a momentum geometry counterpart of Lemma 2.1 and will be applied fibrewise in the bundle situation considered in § 3 below.

Lemma 2.2. *A potential for the restriction of the Fubini-Study Kähler form $\omega = \omega_{\mathbb{P}(W)}$ to $V \subset \mathbb{P}(W)$ is given by*

$$\omega|_V = -dd^c \log(\chi_V + 1) = 2i\partial\bar\partial \log(\chi_V + 1).$$

The function $\rho := \log(\chi_V + 1) \in \mathcal{C}^\infty(V)$ is an exhaustion of V, and if $\xi_{\mathbb{P}(W)}$ is the vector field on $\mathbb{P}(W)$ induced by the K-action, then

$$V \ni v \mapsto d^c\rho(\xi_{\mathbb{P}(W)})([v:1]) = \mu^\xi([v:1])$$

defines a momentum map for the K-action on V with respect to $\omega|_V$.

Proof. The first part is well known, the exhaustion property is clear and the last part follows by direct computation that uses the obvious equality

$$\xi_W(v, 1) = (\xi_V(v), 0)$$

together with the fact that (being a G-equivariant isomorphism) θ^{-1} transforms the fundamental vector field $\xi_{\mathbb{P}(W)}$ into ξ_V. □

Remark 2.3. In the following, we will only need that ρ is an exhaustive K-invariant potential for a Kähler form on V. The fact that $d^c\rho$ then yields a momentum map for the K-action can be seen as follows: since ρ and hence $d^c\rho$ is K-invariant, we have $\mathcal{L}_{\xi_X}(d^c\rho) = 0$, and therefore Cartan's formula yields
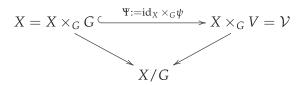
$$d\big(\imath_{\xi_X}(d^c\rho)\big) = -\imath_{\xi_X}\big(dd^c(\rho)\big) = \imath_{\xi_X}\omega,$$

so that indeed $\imath_{\xi_X}(d^c\rho)$ is the ξ-component of a momentum map with respect to ω, whose equivariance is easy to check as well.

3. Kähler forms on holomorphic principal bundles and vector bundles

Given a G-principal bundle $\pi\colon X \to Q$ over a compact Kählerian manifold $Q = X/G$, we will be looking for special Kähler structures on X. For this, the following construction will be useful.

Remark 3.1. As G is an affine algebraic G-variety (with algebraic action given by left-multiplication), it admits a closed (algebraic) embedding $\psi : G \hookrightarrow V$ into the vector space associated with a certain finite-dimensional G-representation $\varphi : G \to GL_{\mathbb{C}}(V)$, see for example [4, Prop. 2.2.5]. That is, there is a closed G-orbit $G \bullet v$ in V with $G_v = \{e\}$, and in particular, φ is injective. We may assume that V has a K-invariant Hermitian form h, so that $\varphi|_K : K \hookrightarrow SU(h)$. The associated vector bundle $\mathcal{V} := X \times_G V$ has a natural holomorphic G-action (preserving each fibre), as well as a Hermitian metric that is induced by h and therefore invariant by the associated K-action. Slightly abusing notation, we will call the bundle metric h as well. Writing $X = X \times_G G$, we may produce a closed G-equivariant holomorphic embedding

$$X = X \times_G G \overset{\Psi:=\mathrm{id}_X \times_G \psi}{\lhook\joinrel\longrightarrow} X \times_G V = \mathcal{V}$$

$$X/G$$

of X into $\mathcal{V}$ over $Q = X/G$.

Let now (B, ω) be a compact Kähler manifold, and let $\pi\colon \mathcal{V} \to B$ be a holomorphic vector bundle.

Remark 3.2. Setting $\mathcal{W} := \mathcal{V} \oplus \mathbb{C}$, it can be deduced from [3, Théorème Principal II] that the total space of $\mathbb{P}(\mathcal{W}) \to B$ is Kähler. In fact, there is a rather explicit way of constructing Kähler forms on projectivizations of vector bundles, for example, see [21, Prop. 3.18]. As we will see in more detail below, one uses the fact that there exists a $(1,1)$-form $\omega_{\mathcal{W}}$ on $\mathbb{P}(\mathcal{W})$ whose restriction to every fibre $\mathbb{P}(\mathcal{W}_x)$ is the Fubini-Study form of this fibre associated with a bundle metric on $\mathcal{W}$, namely (up to some positive constant) the Chern form of $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(1)$. Then one obtains a Kähler form on $\mathbb{P}(\mathcal{W})$ by adding a sufficiently large multiple of the pull-back of ω to $\mathbb{P}(\mathcal{W})$. Since the holomorphic vector bundle $\mathcal{V}$ embeds as an open subset into $\mathbb{P}(\mathcal{V} \oplus \mathbb{C})$, we see that $\mathcal{V}$ and hence $X \subset \mathcal{V}$ both inherit corresponding Kähler forms by restriction.

We will now analyse the construction of the Kähler form on $\mathcal{V} \subset \mathbb{P}(\mathcal{V} \oplus \mathbb{C})$ in more detail, with the aim of pointing out the specific features relevant to momentum map geometry.

Lemma 3.3. *Let $\pi\colon \mathcal{V} \to B$ be a holomorphic vector bundle of rank n over a compact Kähler manifold (B, ω) on which the compact Lie group K acts holomorphically via fibre-preserving vector bundle automorphisms. Set $\mathfrak{k} := \mathrm{Lie}(K)$. Let h be a K-invariant*

Hermitian metric on $\mathcal{V}$, with associated length function

$$\chi_h : \mathcal{V} \to \mathbb{R}^{\geqslant 0}, \ v \mapsto h(v, v).$$

Then, for all positive real numbers $c \gg 0$, the $(1,1)$-form

$$\omega_{\mathcal{V}} = 2i\partial\bar{\partial} \log(\chi_h + 1) + c \cdot \pi^*(\omega)$$

has the following properties:

(1) *$\omega_{\mathcal{V}}$ is Kähler,*
(2) *the K-action on $\mathcal{V}$ admits a momentum map $\mu \colon \mathcal{V} \to \mathfrak{k}^*$ with respect to $\omega_{\mathcal{V}}$,*
(3) *every point $x \in B$ has an open neighbourhood U such that on $\pi^{-1}(U)$ we may write $\omega = 2i\partial\bar{\partial}\rho$ for some function $\rho \in \mathcal{C}^\infty(\pi^{-1}(U))^K$ that has the following properties:*
    (a) *ρ is an exhaustion when restricted to $\mathcal{V}_y$ for all $y \in U$,*
    (b) *for all $\xi \in \mathfrak{k}$ and for all $v \in \pi^{-1}(U)$, we have*

$$\mu^\xi(v) = d^c\rho(\xi_{\mathcal{V}}(v)) = d\rho(J\xi_{\mathcal{V}}(v)), \tag{3.1}$$

where as usual $J \in \mathrm{End}(T\mathcal{V})$ denotes the (almost) complex structure of the complex manifold $\mathcal{V}$ and $\xi_{\mathcal{V}}$ is the vector field on $\mathcal{V}$ associated with ξ via the K-action.

Proof. We consider $\mathcal{W} = \mathcal{V} \oplus \mathbb{C}$ the direct sum of $\mathcal{V}$ with the trivial line bundle, and its projectivization $\mathbb{P}(\mathcal{W}) = (\mathcal{W} \setminus z_{\mathcal{W}})/\mathbb{C}^*$, which contains $\mathbb{P}(\mathcal{V})$ as a co-dimension one sub-bundle with (open) complement isomorphic to $\mathcal{V}$. Endow $\mathcal{W}$ with the Hermitian metric

$$h_{\mathcal{W}}(v \oplus z) := \chi_h(v) + |z|^2.$$

Using the section of $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(1)$ corresponding to the divisor $\mathbb{P}(\mathcal{V}) \subset \mathbb{P}(\mathcal{W})$, one computes that the *relative Fubini-Study form* $\omega_{FS}^{\mathcal{W}}$, that is, the curvature form of the natural Hermitian metric in $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(1)$ induced by h, when restricted to $\mathcal{V} = \mathbb{P}(\mathcal{W}) \setminus \mathbb{P}(\mathcal{V})$, is given by

$$c \cdot \omega_{FS}^{\mathcal{W}}|_{\mathcal{V}} = 2i\partial\bar{\partial} \log(\chi_h + 1) = -dd^c \log(\chi_h + 1) \qquad \text{for some } c \in \mathbb{R}^{>0};$$

see for example [6, Chap. V, § 15.C; p. 282], but notice the different convention regarding projectivizations of vector bundles. Restricting to any fibre $\mathcal{V}_y$, $y \in B$, we recover the Kähler form associated with the Hermitian scalar product h_y discussed in Lemma 2.2. Using compactness of B, Part (1) is now proven with the argument of [21, p. 78].

The statements made in Parts (2) and (3) will be proven simultaneously. Let $\{U_k\}_{k \in I}$ be a finite open covering of B so that $\omega|_{U_k}$ is given by a potential φ_k; that is, $\omega|_{U_k} = -dd^c\varphi_k$.

It is clear that $\omega_{\mathcal{V}}|_{\pi^{-1}(U_k)}$ has K-invariant potential

$$\rho_k := \log(\chi_h + 1)|_{\pi^{-1}(U_k)} + c \cdot \pi^*(\varphi_k). \tag{3.2}$$

On any of the open sets $\pi^{-1}(U_k)$, a corresponding momentum map for the K-action is given by $\mu_k^\xi := \iota_{\xi_{\mathcal{V}}} d^c \rho_k$, cf. Remark 2.3. Setting $\eta := \log(\chi_h + 1)$ in order to shorten notation, still on $\pi^{-1}(U_k)$, we compute

$$\begin{aligned}
\iota_{\xi_{\mathcal{V}}} d^c \big(\eta + c \cdot \pi^* \varphi_k\big) &= \iota_{\xi_{\mathcal{V}}} d^c \eta + c \cdot \iota_{\xi_{\mathcal{V}}} d^c \pi^* \varphi_k \\
&= \iota_{\xi_{\mathcal{V}}} d^c \eta + c \cdot \iota_{\xi_{\mathcal{V}}} \pi^* d^c \varphi_k \\
&= \iota_{\xi_{\mathcal{V}}} d^c \eta.
\end{aligned} \tag{3.3}$$

Here, we used the fact that $\xi_{\mathcal{V}}$ is tangential to the fibres of π, whereas the forms $\pi^* d^c \varphi_k$ vanish in fibre direction. Consequently, the maps μ_k on $\pi^{-1}(U_k)$ glue together to a well-defined momentum map $\mu_{\mathcal{V}} : \mathcal{V} \to \mathfrak{k}^*$ with respect to $\omega_{\mathcal{V}}$; as the computation Equation (3.3) shows, it is given by the map formally associated with the (in general only fibrewise strictly plurisubharmonic) function $\eta = \log(\chi_h + 1)$ defined on the whole of $\mathcal{V}$. This shows the statements made in Part (2). The properties listed in Part (3) then follow from the definition of ρ_k, see Equation (3.2), and the fibrewise properties established in Lemma 2.2. □

The following example explains the choice of the potential $\log(\chi_h + 1)$ and the remark at the end of the first paragraph of § 2.

Example 3.4. The total space of the line bundle $\mathcal{O}(1)$ over $\mathbb{P}_1$ can be explicitly realized as $\mathbb{P}_2 \setminus \{[0 : 0 : 1]\}$ with bundle projection $\pi([z_0 : z_1 : z_2]) = [z_0 : z_1]$. Let us consider the bundle metric on $\mathcal{O}(1)$ with associated length function

$$\chi_h([z_0 : z_1 : z_2]) = \frac{|z_2|^2}{|z_0|^2 + |z_1|^2}.$$

An explicit computation shows that there does not exist any $c > 0$ such that $i\partial\overline{\partial}\chi_h + c \cdot \pi^*\omega_{FS}$ is positive. Namely, in the affine coordinates (z, w) on $\mathbb{P}_2 \setminus \{[0 : 0 : 1]\} \cap \{z_0 \neq 0\} \cong \mathbb{C}^2 \setminus \{0\}$, the Hermitian form of $i\partial\overline{\partial}\chi_h + c \cdot \pi^*\omega_{FS}$ is given by

$$\frac{1}{(|z|^2 + |w|^2)^3} \begin{pmatrix} |z|^2 \left(1 + c|w|^2\right) - |w|^2 + c|w|^4 & \left(2 - c\left(|z|^2 + |w|^2\right)\right) z\overline{w} \\ \left(2 - c\left(|z|^2 + |w|^2\right)\right) \overline{z}w & \left(1 + c|z|^2\right)|w|^2 - |z|^2 + c|z|^4 \end{pmatrix},$$

which is never positive definite.

4. Proof of Theorem 1.1

Before we start the proof proper, we observe that in case that G is not connected with identity component G°, the quotient $Q^\circ := X/G^\circ$ is a finite topological covering of X/G

and therefore also compact. Moreover, for $l \in \{a, b, c\}$, statement (l) of Theorem 1.1 holds if and only if the corresponding statement (l°) holds for the action of G°, the maximal compact group $K^\circ := K \cap G^\circ$ of G°, the equivariant compactification $\overline{G^\circ} \subset \overline{G}$, and the quotient Q°. For this observation, the main point to notice is that $\overline{G} = G \times_{G^\circ} \overline{G^\circ}$, and hence

$$\overline{X} = X \times_G \overline{G} = X \times_G G \times_{G^\circ} \overline{G^\circ} \cong X \times_{G^\circ} \overline{G^\circ}.$$

We therefore assume for the remainder of the proof that $G = G^\circ$ is connected.

4.1. Implication '(b) ⇒ (a)'

We quickly summarize the fundamental results in the Kählerian version of the theory of Symplectic Reduction: Let $G = K^{\mathbb{C}}$ be a complex-reductive group and let X be a holomorphic G-manifold. Let ω be a K-invariant Kähler form on X such that the K-action on X is Hamiltonian with equivariant momentum map $\mu \colon X \to \mathfrak{k}^*$. If the zero fibre $\mu^{-1}(0)$ is non-empty, then the set

$$X^{ss}(\mu) := \left\{ x \in X \mid \overline{G \bullet x} \cap \mu^{-1}(0) \neq \emptyset \right\}$$

of semistable points is a non-empty open G-invariant of X admitting an *analytic Hilbert quotient* $q \colon X^{ss}(\mu) \to X^{ss}(\mu) /\!\!/ G$; that is, q is a surjective G-invariant, (locally) Stein map to a complex space $X^{ss}(\mu) /\!\!/ G$ fulfilling

$$q_*(\mathcal{O}_{X^{ss}(\mu)})^G = \mathcal{O}_{X^{ss}(\mu) /\!\!/ G}.$$

In particular, q is universal with respect to G-invariant holomorpic maps from $X^{ss}(\mu)$ to complex spaces. Moreover, every fibre of q contains a unique G-orbit that is closed in $X^{ss}(\mu)$; this orbit is the only one in $\pi^{-1}(0)$ that intersects $\mu^{-1}(0)$, and the intersection is a uniquely determined K-orbit. With more work, one can show that $X^{ss}(\mu) /\!\!/ G$ is homeomorphic to the symplectic reduction $\mu^{-1}(0)/K$ and via this homeomorphism inherits a Kähler structure induced by ω, see [15, 16, 20] and the detailed survey [14].

Suppose from now on that in addition to the assumptions made above the G-action on X is free and proper. In this situation, since all G-orbits are closed in X, we have

$$X^{ss}(\mu) = G \cdot \mu^{-1}(0),$$

and hence $X^{ss}(\mu) /\!\!/ G = X^{ss}(\mu)/G = q(X^{ss}(\mu))$ is an open subset of X/G that is homeomorphic to $\mu^{-1}(0)/K$ and possesses a Kähler form, whose construction is much easier in this simple case; for example, see [17, Thm. 3.1] and cf. [12, Thm. 3.5] or [14, Prop. 2.4.6]. Under the assumption made in (b), the non-empty subset $q(X^{ss}(\mu)) \simeq \mu^{-1}(0)/K$ is not only open but also compact. It therefore coincides with $X/G = Q$, which is hence Kähler.

4.2. Implication '(a) ⇒ (b)'

Choose a G-equivariant closed holomorphic embedding $\Psi \colon X \hookrightarrow \mathcal{V}$ into the total space of a holomorphic G-vector bundle over Q, as constructed in Remark 3.1, and let h be a

corresponding K-invariant Hermitian metric on $\mathcal{V}$. Next, we apply Lemma 3.3 to obtain a constant $c \gg 0$ such that

$$\omega = 2i\partial\bar{\partial}\log(\chi_h + 1) + c \cdot \pi^*(\omega)$$

is a Kähler form with properties (2) and (3) stated in the Lemma. We will show that the restriction of ω to the closed submanifold $X \subset \mathcal{V}$, which we continue to denote by ω, has the properties claimed in Theorem 1.1(b).

First, we notice that the K-action on X is Hamiltonian with respect to ω, with momentum map being given by the restriction of the momentum map from $\mathcal{V}$ to X, which we continue to denote by μ. Now, let $q \in Q$ be any point, let $x \in X$ a point in the fibre over q, and let U and ρ be as in part (3) of Lemma 3.3. Then, by (3.a) the restriction of ρ to $X_q = G \bullet x \subset \mathcal{V}_q$ continues to be a strictly plurisubharmonic exhaustion function. Therefore, $\rho|_{X_q}$ has a minimum and in particular a critical value, say at $x_0 \in G \bullet x$. As $J\xi_{\mathcal{V}}(x_0) = J\xi_{X_q}(x_0) = J\xi_{G \bullet x_0}(x_0)$ is obviously tangent to the (complex) orbit $G \bullet x_0 = G \bullet x$, Formula (3.1) together with the fact that x_0 is critical implies that

$$\mu^\xi(x_0) = d\rho(J\xi_{\mathcal{V}}(x_0)) = d\rho(J\xi_{X_q}(x_0)) = 0 \qquad \text{for all } \xi \in \mathfrak{k},$$
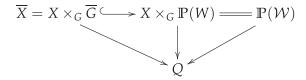
so that $x_0 \in \mu^{-1}(0)$. In other words, every fibre of $\pi\colon X \to Q$ intersects $\mu^{-1}(0)$, so that

$$\pi|_{\mu^{-1}(0)}\colon \mu^{-1}(0) \to Q$$

is a K-principal bundle over the compact manifold Q. In particular, $\mu^{-1}(0) \subset X$ is compact.

4.3. Implication '(a) $\Rightarrow$ (c)'

As a normal projective G-variety for the connected group G, the completion $\overline{G}$ admits a G-equivariant embedding into the projectivization $\mathbb{P}(W)$ of a finite-dimensional complex G-representation W; see [4, Prop. 5.2.1, Prop. 3.2.6]. Using the associated bundle construction, this leads to a closed holomorphic embedding

into the projectivization of the holomorphic vector bundle $\mathcal{W} = X \times_G W$ over Q. By assumption, Q is compact Kähler, so Remark 3.2 yields the claim.

4.4. Implication '(c) $\Rightarrow$ (a)'

Since $\overline{\pi}\colon \overline{X} \to Q$ is a fibre bundle with compact Kählerian total space (and connected fibres), the claim follows from [3, Prop. II.2].

This concludes the proof of Theorem 1.1.

5. Examples

The following example shows that even the 'Kähler' statement of the implication $(a) \implies$ (b) of Theorem 1.1 does not hold for *non-compact Q*.

Example 5.1. The homogeneous fibration

$$\mathrm{SL}(2,\mathbb{C})/\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right) \to \mathrm{SL}(2,\mathbb{C})/\left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$$

is a $\mathbb{C}^*$-principal bundle over the non-compact Kähler base $\mathbb{C}^2 \setminus \{0\}$. Owing to [2, Theorem 3.1], however, the total space $\mathrm{SL}(2,\mathbb{C})/\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$ is *not* Kähler.

A slight modification of Example 5.1 yields an example showing that an analogue of Lemma 3.3 does not hold if we replace the vector bundle $\pi \colon \mathcal{V} \to X$ by an arbitrary holomorphic fibre bundle having Stein fibres, even though these kind of fibres also admit strictly plurisubharmonic exhaustion functions similar to (the logarithm of) the norm square function used in the proof of Lemma 3.3.

Example 5.2. Let B be the Borel subgroup of $G = \mathrm{SL}(2,\mathbb{C})$ consisting of upper triangular matrices and let $\Gamma = \left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$. One verifies directly that B/Γ is biholomorphic to $(\mathbb{C}^*)^2$ and therefore a Stein manifold. Consequently, the homogeneous fibration $G/\Gamma \to G/B$ is a holomorphic fibre bundle over the compact Kähler manifold $\mathbb{P}_1$ with typical fibre $(\mathbb{C}^*)^2$ such that the total space G/Γ is *not* Kähler, again by [2, Theorem 3.1]. Note that this bundle is *not* principal, as Γ is not normal in B.

Next, we give an example (originally due to Lescure) which explains that even for $K^\mathbb{C}$-actions on *Kählerian* manifolds X, further conditions are needed to ensure that the quotient X/G is Kähler.

Example 5.3. Let us consider the Hopf surface $Y = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ with respect to the $\mathbb{Z}$-action given by $m \bullet v = 2^m v$. Let $p \colon \mathbb{C}^2 \setminus \{0\} \to Y$ be the universal covering. The group $G = \mathrm{GL}(2,\mathbb{C})$ acts transitively on Y, and a direct calculation shows that the isotropy group of $y_0 := p(1,0) \in Y$ is

$$G_{y_0} = \left\{ \begin{pmatrix} 2^m & b \\ 0 & a \end{pmatrix}; \ m \in \mathbb{Z}, b \in \mathbb{C}, a \in \mathbb{C}^* \right\}.$$

Let us consider the closed subgroups

$$H := \left\{ \begin{pmatrix} 2^m & b \\ 0 & 1 \end{pmatrix}; \ m \in \mathbb{Z}, b \in \mathbb{C} \right\} \text{ and } T := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}; \ a \in \mathbb{C}^* \right\}$$

of G_{y_0}. One verifies directly that $T \cong \mathbb{C}^*$ normalizes H and thus that $G_{y_0} = TH \cong T \ltimes H$. Hence, we have the holomorphic $\mathbb{C}^*$-principal bundle $X := G/H \to Y = G/(HT)$.

We claim that X is a Kähler manifold. For this, let $S := \mathrm{SL}(2,\mathbb{C})$ and note that

(1) $S \cap H = \left(\begin{smallmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{smallmatrix}\right)$ is an algebraic subgroup of S, as well as that

(2) $SH = \left\{ 2^{m/2}g; \ m \in \mathbb{Z}, g \in S \right\}$ is a closed subgroup of G. (In fact, $G/(SH)$ is an elliptic curve.)

Now we can apply [9, Theorem 5.1] to deduce that X is Kähler.

However, there is no momentum map for the S^1-action on $X = G/H$ with respect to any S^1-invariant Kähler form: Indeed, since the subgroup H has infinitely many connected components and is therefore not algebraic, the action of $\mathrm{U}(2)$ on X is not Hamiltonian, see [9, Theorem 4.11]. Since $\mathrm{U}(2) = \mathrm{SU}(2)S^1$, this implies that X is not a Hamiltonian S^1-manifold, either.

In Example 5.3 above, while X is Kähler, the action of a maximal compact group K is not Hamiltonian with respect to any Kähler form. Therefore, one might wonder whether the existence of a momentum map for the K-action is actually sufficient to conclude that the base of a Kählerian principal bundle is Kähler. Candidates of counterexamples to this statement can be obtained by the Cox construction, which realizes toric varieties associated with simplicial fans and without torus factors as geometric quotients of certain domains in $\mathbb{C}^N$ by a linear action of a complex torus T, see [5, Theorem 5.1.11]. The following concrete example, which is in some sense minimal, consists of a smooth complete non-projective toric threefold for which Cox' geometric quotient is a T-principal bundle.

Example 5.4. Let Σ be the simplicial fan from [8, Example 2]. It is shown that the associated toric threefold X_Σ is smooth, complete, and non-projective, hence non-Kähler by Moishezon's Theorem, [10, Chap. VII, Thm. 6.23]. Since $\left| \Sigma(1) \right| = 8$ and X_Σ is smooth, we have

$$\mathrm{Cl}(X_\Sigma) = \mathrm{Pic}(X_\Sigma) \cong \mathbb{Z}^{|\Sigma(1)| - \dim X(\Sigma)} = \mathbb{Z}^5,$$

see [5, Theorem 4.1.3 and Proposition 4.2.6]. Moreover, since X_Σ has no torus factors, the relevant group is $T = \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^5, \mathbb{C}^*) \cong (\mathbb{C}^*)^5$, see [5, Lemma 5.1.1]. Consequently, an application of [5, Theorem 5.1.11] implies that there exists a geometric T-quotient $\pi \colon \mathbb{C}^8 \setminus Z(\Sigma) \to X_\Sigma$ where T acts linearly on $\mathbb{C}^8$. In particular, there is a momentum map for the action of any compact real form of T on $\mathbb{C}^8 \setminus Z(\Sigma)$, described explicitly in § 2 above.

Using again the fact that X_Σ is smooth, we may apply [1, Proposition 2.1.4.6] to see that T acts *freely* on $\mathbb{C}^8 \setminus Z(\Sigma)$. Since the orbit space $\left(\mathbb{C}^8 \setminus Z(\Sigma) \right)/T \cong X_\Sigma$ is Hausdorff and since the holomorphic slice theorem for Hamiltonian actions, for example see [14, Thm. 4.1.], yields local slices for the T-action, it follows that T acts *properly* on $\mathbb{C}^8 \setminus Z(\Sigma)$, see [19, Theorem 1.2.9] ; that is, π is indeed a T-principal bundle.

Remark 5.5. As in § 2 above, Example 5.4 can be equivariantly compactified $\mathbb{C}^8 \hookrightarrow \mathbb{P}^8 = \mathbb{P}(\mathbb{C}^8 \oplus \mathbb{C})$ to a Hamiltonian action of $K = (S^1)^5$ on $\mathbb{P}^8$, say with momentum map μ. Now, given any point p in the open subset $\mathbb{C}^8 \setminus Z(\Sigma)$, we may actually shift the momentum map by a constant $\xi \in \mathfrak{k}^*$ to define a new momentum map $\hat{\mu}$ with $p \in \hat{\mu}^{-1}(0)$. The corresponding set of semistable points is Zariski-open in $\mathbb{P}^8$; hence, its intersection with $\mathbb{C}^8 \setminus Z(\Sigma)$ yields a Zariski-open subset $U \subset \left(\mathbb{C}^8 \setminus Z(\Sigma) \right)/T \cong X_\Sigma$ admitting a Kähler form ω_U. As all quotient maps are in fact meromorphic, X_Σ is bimeromorphic to the compact Kähler space $\hat{\mu}^{-1}(0)/K$ on which ω_U extends to an honest Kähler structure

by Kählerian reduction; cf. [7]. That is, the Kählerian reduction theory explains in a precise way how X_Σ is in Fujiki's class $\mathcal{C}$ (and not Kähler).

Remark 5.6. With respect to item (c) of Theorem 1.1, Example 5.4 shows that (for connected G) it is not enough to assume

(1) some compactification $\widehat{X}$ of X to be Kähler, and
(2) the action map $G \times X \to X$ to extend to a meromorphic map $\overline{G} \times \widehat{X} \dashrightarrow \widehat{X}$

in order for X/G to be Kähler. While under these two assumptions, there will always be a momentum map $\hat{\mu} \colon \widehat{X} \to \mathfrak{k}^*$ for the K-action, see for example, the discussion of classical results regarding this connection in [11, Rem. 2.2], as in the example the intersection of the compact subset $\widehat{\mu}^{-1}(0)$ with the open subset $X \subset \widehat{X}$ might always be non-compact (or empty).

Competing Interests. The authors declare none.

References

(1) I. Arzhantsev, U. Derenthal, J. Hausen and A. Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, Volume 144 (Cambridge University Press, Cambridge, 2015).
(2) F. Berteloot and K. Oeljeklaus, Invariant plurisubharmonic functions and hypersurfaces on semisimple complex Lie groups, *Math. Ann.* **281**(3) (1988), 513–530.
(3) A. Blanchard, Sur les variétés analytiques complexes, *Ann. Sci. Ec. Norm. Supér.(3)* **73** (1956), 157–202.
(4) M. Brion, *Linearization of algebraic group actions*, Handbook of Group actions. Vol. IV, Advanced Lectures in Mathematics (ALM), Volume 41, pp. 291–340 (International Press, Somerville, MA, 2018).
(5) D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, Volume 124 (American Mathematical Society, Providence, RI, 2011).
(6) J. -P. Demailly, Complex analytic and algebraic geometry, available at http://www-fourier.ujf-grenoble.fr/~demailly/books.html, 2012.
(7) A. Fujiki, *KäHler quotient and equivariant cohomology*, Moduli of Vector Bundles (Sanda, 1994; Kyoto, 1994), Lecture Notes in Pure and Applied Mathematics, Volume 179, pp. 39–53 (Dekker, New York, 1996).
(8) O. Fujino and S. Payne, Smooth complete toric threefolds with no nontrivial nef line bundles, *Proc. Japan Acad. Ser. A Math. Sci.* **81**(10) (2006), 174–179.

(9)   B. Gilligan, C. Miebach and K. Oeljeklaus, Homogeneous Kähler and Hamiltonian manifolds, *Math. Ann.* **349**(4) (2011), 889–901.

(10) H. Grauert, T. Peternell and R. Remmert, (eds.) *Several complex variables VII, sheaf-theoretical methods in complex analysis*, Encyclopaedia of Mathematical Sciences, Volume 74 (Springer, Berlin, 1994).

(11) D. Greb and C. Miebach, Hamiltonian actions of unipotent groups on compact Kähler manifolds, *ÉPijournal Géom. AlgéBrique* **2** (2018), 30, Art. 10.

(12) V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67**(3) (1982), 515–538.

(13) J. Hausen, Geometric invariant theory based on Weil divisors, *Compos. Math.* **140**(6) (2004), 1518–1536.

(14) P. Heinzner and A. Huckleberry, *Analytic Hilbert quotients.* Several complex variables (Berkeley, CA, 1995–1996), Mathematical Science Research Institute Publications, Volume 37, pp. 309–349 (Cambridge University Press, Cambridge, 1999).

(15) P. Heinzner and F. Loose, Reduction of complex Hamiltonian G-spaces, *Geom. Funct. Anal.* **4**(3): (1994), 288–297.

(16) P. Heinzner, A. Huckleberry and F. Loose, Kählerian extensions of the symplectic reduction, *J. Reine Angew. Math.* **455** (1994), 123–140.

(17) N. Hitchin, A. Karlhede, U. Lindström and M. Roček, Hyper-Kähler metrics and supersymmetry, *Comm. Math. Phys.* **108**(4) (1987), 535–589.

(18) D. Mumford, J. Fogarty and F. Clare Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathe-ma-tik und Ihrer Grenzgebiete, 2. Folge, Volume 34 (Springer, Berlin, 1994).

(19) R. Palais, On the existence of slices for actions of non-compact Lie groups, *Ann. Math. (2)* **73** (1961), 295–323.

(20) R. Sjamaar, Holomorphic slices, symplectic reduction and multiplicities of representations, *Ann. Math. (2)* **141**(1) (1995), 87–129.

(21) C. Voisin, *Hodge theory and complex algebraic geometry*, 1st, english ed., Cambridge Studies in Advanced Mathematics, Volume 76 (Cambridge University Press, Cambridge, 2007).