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# SOME NEW RESULTS ON STOCHASTIC COMPARISONS OF COHERENT SYSTEMS USING SIGNATURES

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**Abstract**

We consider coherent systems with independent and identically distributed components. While it is clear that the system's life will be stochastically larger when the components are replaced with stochastically better components, we show that, in general, similar results may not hold for hazard rate, reverse hazard rate, and likelihood ratio orderings. We find sufficient conditions on the signature vector for these results to hold. These results are combined with other well-known results in the literature to get more general results for comparing two systems of the same size with different signature vectors and possibly with different independent and identically distributed component lifetimes. Some numerical examples are also provided to illustrate the theoretical results.

*Keywords:* Likelihood ratio order; hazard rate order; reversed hazard rate order; coherent systems

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## 1. Introduction

Consider a coherent system consisting of $n$ components with lifetimes $X_1, \ldots, X_n$, which are assumed to be independent and identically distributed (i.i.d.) continuous random variables. It follows from the coherent property of the system that the lifetime of the system $T_{\boldsymbol{X}}$ corresponds to exactly one of the order statistics, $X_{i:n}$, $i = 1, \ldots, n$. Samaniego (1985) introduced the concept of the "signature" of a system which depends on the design of the system. Let

$$p_i = \mathrm{P}[T_{\boldsymbol{X}} = X_{i:n}], \quad i = 1, \ldots, n, \qquad \textstyle\sum_{i=1}^{n} p_i = 1,$$

be the probability that the system fails upon the occurrence of the $i$th component failure. The vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ is called the signature of the system. The survival function of the lifetime of the underlying coherent system can be expressed as

$$\mathrm{P}(T_{\boldsymbol{X}}(\boldsymbol{p}) > t) = \sum_{i=1}^{n} p_i \mathrm{P}(X_{i:n} > t). \tag{1}$$

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Because of the fundamental property of a system's signature $\boldsymbol{p}$, namely that the distribution of the system lifetime $T$, given i.i.d. component lifetimes with cumulative distribution function $F$, can be expressed as a function of $\boldsymbol{p}$ and $F$ alone, we use the notation $T_{\boldsymbol{X}}(\boldsymbol{p})$ to denote the lifetime of the system.

This representation was used by Kochar et al. (1999) to stochastically compare two systems, each with the same i.i.d. components. These ordering properties are distribution free in the sense that they do not depend on the common distribution of the components. As mentioned in Kochar et al. (1999) and also explained in Navarro et al. (2008), the above results continue to hold when the vector $(X_1, \dots, X_n)$ has an exchangeable distribution. Moreover, they obtain distribution-free ordering properties used to compare systems having different numbers of exchangeable components. For further references on this topic, see Samaniego (2007) and Navarro et al. (2010).

First, we review some notions of stochastic orderings. Throughout this paper, *increasing* and *decreasing* stand for *non-decreasing* and *non-increasing*, respectively.

Let $X$ and $Y$ be two non-negative continuous random variables with density functions $f$ and $g$, distribution functions $F$ and $G$, survival functions $\overline{F} = 1 - F$ and $\overline{G}$, hazard rate functions $h_X = f/\overline{F}$ and $h_Y$, and reversed hazard rate functions $\tilde{h}_X = f/F$ and $\tilde{h}_Y$, respectively.

(a) $X$ is said to be larger than $Y$ in the *likelihood ratio order* (denoted by $X \geq_{\text{lr}} Y$) if $\frac{f(t)}{g(t)}$ is increasing in $t$.

(b) $X$ is said to be larger than $Y$ in the *hazard rate order* (denoted by $X \geq_{\text{hr}} Y$) if $\frac{\overline{F}(t)}{\overline{G}(t)}$ is increasing in $t$, or, equivalently, $h_X(t) \leq h_Y(t)$ for all $t$.

(c) $X$ is said to be larger than $Y$ in the *reversed hazard rate order* (denoted by $X \geq_{\text{rh}} Y$) if $\frac{F(t)}{G(t)}$ is increasing in $t$, or, equivalently, $\tilde{h}_X(t) \geq \tilde{h}_Y(t)$ for all $t$.

(d) $X$ is said to be larger than $Y$ in the usual stochastic order (denoted by $X \geq_{\text{st}} Y$) if $\overline{F}(t) \geq \overline{G}(t)$ for all $t$.

The above stochastic orders can be defined on the same lines to compare two discrete random variables with sample space $\{1, \dots, n\}$. For an $n$-dimensional probability vector $\boldsymbol{p} = (p_1, \dots, p_n)$, we denote by $h_{\boldsymbol{p}}(j) = \frac{p_j}{\sum_{i=j}^{n} p_i}$ the hazard rate of $\boldsymbol{p}$, and $\tilde{h}_{\boldsymbol{p}}(j) = \frac{p_j}{\sum_{i=1}^{j} p_i}$ the reverse hazard rate of $\boldsymbol{p}$. For two discrete distributions $\boldsymbol{p}$ and $\mathbb{Q}$ on the integers $\{1, \dots, n\}$, we write

(a) $\boldsymbol{p} \geq_{\text{st}} \boldsymbol{q}$ if and only if $\sum_{i=j}^{n} p_i \geq \sum_{i=j}^{n} q_i$ for $j = 1, \dots, n-1$.

(b) $\boldsymbol{p} \geq_{\text{hr}} \boldsymbol{q}$ if and only if $h_{\boldsymbol{p}}(i) \leq h_{\boldsymbol{q}}(i)$ for $i = 1, \dots, n$.

(c) $\boldsymbol{p} \geq_{\text{rhr}} \boldsymbol{q}$ if and only if $\tilde{h}_{\boldsymbol{p}}(i) \geq \tilde{h}_{\boldsymbol{q}}(i)$ for $i = 1, \dots, n$.

(d) $\boldsymbol{p} \geq_{\text{lr}} \boldsymbol{q}$ if and only if $\frac{p_i}{q_i}$ is increasing in $i$ for $i = 1, \dots, n$.

It is well known that

$$X \geq_{\text{lr}} Y \Longrightarrow X \geq_{\text{hr[rh]}} Y \Longrightarrow X \geq_{\text{st}} Y,$$

but neither reversed hazard rate nor hazard rate orders imply each other. One may refer to Shaked and Shanthikumar (2007) and Müller and Stoyan (2002) for more details.

Let $T_{\boldsymbol{Y}}(\boldsymbol{q})$ be the lifetime of another coherent system with signature vector $\boldsymbol{q} = (q_1, \dots, q_n)$ and with component lifetimes $Y_1, \dots, Y_n$. It is of interest to compare the two systems $T_{\boldsymbol{X}}(\boldsymbol{p})$

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and $T_{\boldsymbol{Y}}(\boldsymbol{q})$ according to various stochastic orders. Kochar et al. (1999) proved that if random lifetimes $X_1, \ldots, X_n$ are i.i.d., then

$$\boldsymbol{p} \geq_* \boldsymbol{q} \Longrightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_* T_{\boldsymbol{X}}(\boldsymbol{q}), \tag{2}$$

where $*$ stands for lr, hr, rh, and st orders. In fact, they pointed out that the above results hold when $X_1, \ldots, X_n$ are exchangeable random variables and the corresponding consecutive order statistics are ordered according to $*$ ordering, i.e. if $X_{i+1:n} \geq_* X_{i:n}$, $i = 1, 2, \ldots, n-1$. Rychlik et al. (2018) have shown that, in general, the converse of (2) may not hold for stochastic ordering. That is, there exist some coherent systems with i.i.d. components for which $T_{\mathbf{X}}(\boldsymbol{p}) \geq_{\text{st}} T_{\boldsymbol{X}}(\boldsymbol{q})$, but the signature vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ may not be stochastically ordered. Navarro (2016) proved that in the case of coherent systems composed of items with i.i.d. lifetimes, there exist systems with hr-ordered (lr-ordered) lifetimes whose signatures are not hr ordered (lr ordered) for any component lifetime distribution. These problems have been studied by many other researchers. For instance, the reader may refer to Belzunce et al. (2001), Khaledi and Shaked (2007), Nanda et al. (1998), Navarro et al. (2008), Zhang (2010), and Zhang and Meeker (2013), among others.

A $k$-out-of-$n$ system consisting of $n$ components with lifetimes $X_1, \ldots, X_n$ functions if and only if at least $k$ out of the $n$ components function. That is, the lifetime of the system corresponds to the $(n-k+1)$th order statistic, $X_{(n-k+1:n)}$. Therefore, stochastically comparing two $k$-out-of-$n$ systems is equivalent to comparing the corresponding order statistics. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of i.i.d. random variables. Then, it is known that

$$X_1 \geq_* Y_1 \Longrightarrow X_{i:n} \geq_* Y_{i:n}, \tag{3}$$

where $*$ stands for lr, hr, rh, and st orders. Note that $X_{i:n}$ and $Y_{i:n}$ are the lifetimes of two coherent systems with common signature vector $\boldsymbol{p} = \boldsymbol{q} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1 being at the $i$th position), but with different lifetime distributions. It is of interest to investigate whether results like (3) hold for coherent systems with more general signature vectors.

It follows from the fact that

$$X_1 \geq_{\text{st}} Y_1 \Longrightarrow X_{i:n} \geq_{\text{st}} Y_{i:n}, \qquad \text{for } i = 1, \ldots, \text{n},$$

and equation (1) that when $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ are independent and identically distributed, then

$$X_1 \geq_{\text{st}} Y_1 \Rightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_{\text{st}} T_{\boldsymbol{Y}}(\boldsymbol{p}). \tag{4}$$

However, such a result may not hold for other stochastic orders like likelihood ratio, hazard rate, and reverse hazard rate, as shown in the following counterexample.

**Example 1.1.** Consider a coherent system of order four with lifetime

$$T_{\boldsymbol{X}}(\boldsymbol{p}) = \max\left(X_1, \min\left(X_2, X_3, X_4\right)\right)$$

and signature vector $\boldsymbol{p} = \left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, where $X_i$, $i = 1, \ldots, 4$, are independent exponential random variables with common hazard rate 0.7. Let $Y_i$, $i = 1, \ldots, 4$, be another set of independent exponential random variables with common hazard rate 1. Figure 1 shows the hazard rate functions of $T_{\boldsymbol{X}}(\boldsymbol{p})$ and $T_{\boldsymbol{Y}}(\boldsymbol{p})$, and it can be seen that the hazard rate functions cross at $t = 5.5$ even though $X_1 \geq_{\text{hr}} Y_1$. Therefore, a result similar to (4) does not hold, in general, for hazard rate ordering.

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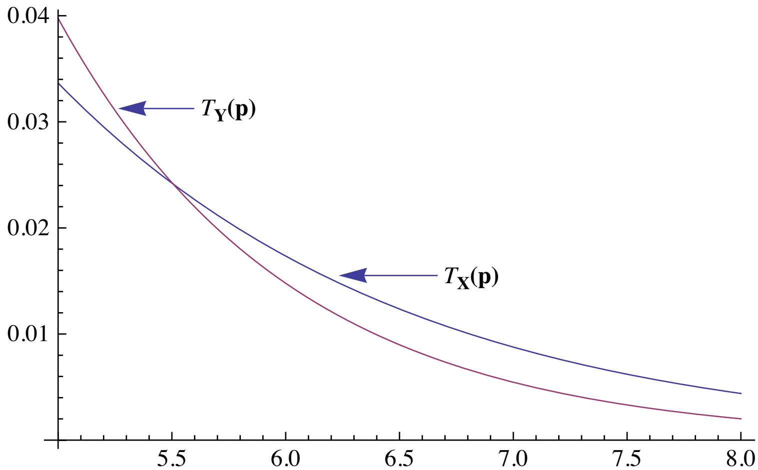


FIGURE 1: Plot of the hazard rate functions.

In the next section we find sufficient conditions on the signature vector $\boldsymbol{p}$ under which

$$X_1 \geq_* Y_1 \Rightarrow T_X(\boldsymbol{p}) \geq_* T_Y(\boldsymbol{p})$$

holds for hazard rate, reverse hazard, and likelihood ratio orderings. This problem has also been studied by Navarro et al. (2013), but as we will see later, their results seem to differ from ours.

The rest of the paper is organized as follows. The main results are given in the next section. These new results are combined with the other well-known results in the literature to get more general results when two systems of the same size with different signature vectors have differently distributed sets of component lifetimes. In Section 3 we illustrate our results with some numerical examples and list, in Tables 1 and 2, all coherent systems of size 3, 4, and 5 for which the conditions of Theorem 2.1 are satisfied. The results of the paper are summarized in the last section.

## 2. Main results

From (1), the survival function of $T_{\mathbf{X}}(\boldsymbol{p})$ can be written as

$$\begin{aligned}\overline{F}_{T_X(\boldsymbol{p})}(t) &= \sum_{i=1}^{n} p_i \sum_{j=0}^{i-1} \binom{n}{j} F^j(t)\overline{F}^{n-j}(t) \\ &= \sum_{j=0}^{n-1} \left( \sum_{i=j+1}^{n} p_i \right) \binom{n}{j} F^j(t)\overline{F}^{n-j}(t)\end{aligned}$$

by changing the order of summation.

From this, we find that the density function of $T_X(\boldsymbol{p})$ is

$$f_{T_X(\boldsymbol{p})}(t) = \sum_{i=1}^{n} i p_i \binom{n}{i} F^{i-1}(t)\overline{F}^{n-i}(t) f(t),$$

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TABLE 1: Coherent systems of sizes 3 and 4.

| $\boldsymbol{p}$ | $(n-j)h_{\boldsymbol{p}}(j+1)$ | $j\tilde{h}_{\boldsymbol{p}}(j)$ |
|---|---|---|
| $(\frac{1}{3}, \frac{2}{3}, 0)$ | $\uparrow j$ | |
| $(0, \frac{2}{3}, \frac{1}{3})$ | | $\downarrow j$ |
| $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{4}, \frac{3}{4}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ | $\uparrow j$ | |
| $(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$ | $\uparrow j$ | |
| $(0, \frac{5}{6}, \frac{1}{6}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{2}{3}, \frac{1}{3}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{2}{3}, \frac{1}{3}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{1}{2}, \frac{1}{2}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{1}{3}, \frac{2}{3}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{1}{3}, \frac{2}{3}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{1}{6}, \frac{5}{6}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, 0, \frac{1}{2}, \frac{1}{2})$ | | $\downarrow j$ |
| $(0, 0, \frac{3}{4}, \frac{1}{4})$ | | $\downarrow j$ |
| $(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ | | $\downarrow j$ |
| $(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$ | | $\downarrow j$ |

and the hazard rate function of $T_{\boldsymbol{X}}(\boldsymbol{p})$ is

$$\begin{aligned} h_{T_X(\boldsymbol{p})}(t) &= \frac{\sum_{i=1}^{n} ip_i\binom{n}{i}F^{i-1}(t)\overline{F}^{n-i+1}(t)}{\sum_{i=1}^{n} p_i \sum_{j=1}^{i-1}\binom{n}{j}F^j(t)\overline{F}^{n-j}(t)} h_X(t) \\ &= \frac{\sum_{i=0}^{n-1}(n-i)p_{i+1}\binom{n}{i}F^i(t)\overline{F}^{n-i}(t)}{\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} p_j\right)\binom{n}{i}F^i(t)\overline{F}^{n-i}(t)} h_X(t) \\ &= \Psi_1\left(\frac{F(t)}{\overline{F}(t)}\right) h_X(t), \end{aligned} \tag{5}$$

where

$$\Psi_1(x) = \frac{\sum_{i=0}^{n-1}(n-i)\binom{n}{i}p_{i+1}x^i}{\sum_{i=0}^{n-1}\left(\sum_{j=i+1}^{n} p_j\right)\binom{n}{i}x^i}.$$

Likewise, the reverse hazard rate function of $T_{\boldsymbol{X}}(\boldsymbol{p})$ is

$$\begin{aligned} \tilde{h}_{T_X(\boldsymbol{p})}(t) &= \frac{\sum_{i=1}^{n} ip_i\binom{n}{i}F^i(t)\overline{F}^{n-i}(t)}{\sum_{i=1}^{n}\left(\sum_{j=1}^{i} p_j\right)\binom{n}{i}F^i(t)\overline{F}^{n-i}(t)} \tilde{h}_{\boldsymbol{X}}(t) \\ &= \Psi_2\left(\frac{F(t)}{\overline{F}(t)}\right) \tilde{h}_{\boldsymbol{X}}(t), \end{aligned}$$

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TABLE 2: Coherent systems of size 5.

| $\boldsymbol{p}$ | $(n-j)h_{\boldsymbol{p}}(j+1)$ | $j\tilde{h}_{\boldsymbol{p}}(j)$ |
|---|---|---|
| $(\frac{3}{5}, \frac{2}{5}, 0, 0, 0)$ | $\uparrow j$ | |
| $(\frac{2}{5}, \frac{3}{5}, 0, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{4}{5}, 0, 0, 0)$ | $\uparrow j$ | |
| $(\frac{2}{5}, \frac{1}{2}, \frac{1}{10}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{7}{10}, \frac{1}{10}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{2}{5}, \frac{3}{10}, \frac{3}{10}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{1}{2}, \frac{3}{10}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{3}{10}, \frac{1}{2}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 0, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{10}, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{1}{5}, \frac{1}{2}, \frac{1}{10}, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0)$ | $\uparrow j$ | |
| $(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, 0)$ | $\uparrow j$ | |
| $(0, 0, 0, \frac{3}{5}, \frac{2}{5})$ | | $\downarrow j$ |
| $(0, 0, 0\frac{2}{5}, \frac{3}{5})$ | | $\downarrow j$ |
| $(0, 0, 0, \frac{1}{5}, \frac{4}{5})$ | | $\downarrow j$ |
| $(0, 0, \frac{2}{5}, \frac{1}{2}, \frac{1}{10})$ | | $\downarrow j$ |
| $(0, 0, \frac{1}{5}, \frac{7}{10}, \frac{1}{10})$ | | $\downarrow j$ |
| $(0, 0, \frac{2}{5}, \frac{3}{10}, \frac{3}{10})$ | | $\downarrow j$ |
| $(0, 0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5})$ | | $\downarrow j$ |
| $(0, 0, \frac{1}{5}, \frac{1}{2}, \frac{3}{10})$ | | $\downarrow j$ |
| $(0, 0, \frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ | | $\downarrow j$ |
| $(0, 0, \frac{1}{5}, \frac{3}{10}, \frac{1}{2})$ | | $\downarrow j$ |
| $(0, 0, \frac{1}{5}, \frac{1}{5}, \frac{3}{5})$ | | $\downarrow j$ |
| $(0, \frac{1}{5}, \frac{1}{2}, \frac{1}{5}, \frac{1}{10})$ | | $\downarrow j$ |
| $(0, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{10})$ | | $\downarrow j$ |
| $(0, \frac{1}{5}, \frac{1}{5}, \frac{1}{2}, \frac{1}{10})$ | | $\downarrow j$ |
| $(0, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5})$ | | $\downarrow j$ |
| $(0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ | | $\downarrow j$ |
| $(0, \frac{9}{10}, \frac{1}{10}, 0, 0)$ | $\uparrow j$ | $\downarrow j$ |

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TABLE 2: Continued.

| | | |
|---|---|---|
| $(0, \frac{4}{5}, \frac{1}{5}, 0, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{7}{10}, \frac{3}{10}, 0, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{3}{5}, \frac{2}{5}, 0, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{7}{10}, \frac{1}{5}, \frac{1}{10}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{3}{5}, \frac{3}{10}, \frac{1}{10}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{1}{2}, \frac{3}{10}, \frac{1}{5}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, \frac{1}{2}, \frac{3}{10}, \frac{1}{5}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, 0, \frac{9}{10}, \frac{1}{10}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, 0, \frac{4}{5}, \frac{1}{5}, 0)$ | $\uparrow j$ | $\downarrow j$ |
| $(0, 0, \frac{3}{5}, \frac{2}{5}, 0)$ | $\uparrow j$ | $\downarrow j$ |

where

$$\Psi_2(x) = \frac{\sum_{i=1}^{n} ip_i\binom{n}{i}x^i}{\sum_{i=1}^{n}\left(\sum_{j=1}^{i} p_j\right)\binom{n}{i}x^i}.$$

First, we prove the following lemma which gives simple sufficient conditions for the functions $\Psi_1$ and $\Psi_2$ to be monotone.

**Lemma 2.1.** *For $i = n_1 \ldots, n_2$, let $c_i$ and $d_i$ be non-negative constants. If $\ell_i = \frac{c_i}{d_i}$ is increasing (decreasing) in $i$, then so is*

$$\phi(x) = \frac{\sum_{i=n_1}^{n_2} c_i x^i}{\sum_{i=n_1}^{n_2} d_i x^i}$$

*on the set of positive numbers $x$.*

*Proof.* We only give the proof when $\ell_i$ is increasing in $i$.

Differentiating $\phi(x)$ with respect to $x$, we obtain a fraction with positive denominator and numerator equal to

$$\begin{aligned}
\phi'(x) &\overset{\text{sgn}}{=} \left(\sum_{i=n_1}^{n_2} ic_i x^{i-1}\right)\left(\sum_{j=n_1}^{n_2} d_j x^j\right) - \left(\sum_{i=n_1}^{n_2} id_i x^{i-1}\right)\left(\sum_{j=n_1}^{n_2} c_j x^j\right) \\
&= \sum_{i=n_1}^{n_2}\sum_{j=n_1}^{n_2}\left[ic_i d_j - id_i c_j\right] x^{i-1}x^j \\
&= \sum_{k=2n_1-1}^{2n_2-1}\left[\sum_{(i,j):n_1\le i,j\le n_2,\ i+j-1=k}(ic_i d_j - id_i c_j)\right]x^k \qquad (6) \\
&= \sum_{k=2n_1-1}^{2n_2-1}\left[\sum_{(i,j):n_1\le i,j\le n_2,\ i+j-1=k,\ i>j}(ic_i d_j - id_i c_j)\right]x^k \qquad (7)
\end{aligned}$$

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$$+\sum_{k=2n_1-1}^{2n_2-1}\left[\sum_{(i,j):n_1\le i,j\le n_2,\, i+j-1=k,\, j>i}(ic_id_j-id_ic_j)\right]x^k \tag{8}$$

since, in the inner sum in (6), for $i=j$, $ic_id_j-id_ic_j=0$. Now letting $j=i'$ and $i=j'$ in (8), it becomes

$$\sum_{k=2n_1-1}^{2n_2-1}\left[\sum_{(i',j'):n_1\le i',j'\le n_2,\, i'+j'-1=k,\, i'>j'}(j'c_{j'}d_{i'}-j'd_{j'}c_{j'})\right]x^k,$$

which can be rewritten as

$$\sum_{k=2n_1-1}^{2n_2-1}\left[\sum_{(i,j):n_1\le i,j\le n_2,\, i+j-1=k,\, i>j}(jc_jd_i-jc_id_j)\right]x^k.$$

Combining this with (7), (6) becomes

$$\sum_{k=2n_1-1}^{2n_2-1}\left[\sum_{(i,j):n_1\le i,j\le n_2,i+j-1=k,i>j}(i-j)(c_id_j-d_ic_j)\right]x^k.$$

If $\ell$ is increasing, then all the coefficients of $x^k$ are non-negative, thus proving our lemma. □

Using the above lemma, we prove the following result.

**Lemma 2.2.** *Let* $\mathbf{p}=(p_1,\ldots,p_n)$ *be a probability vector.*

(a) *If* $(n-i)h_{\mathbf{p}}(i+1)$ *is increasing in i for* $i\in\{0,\ldots,n-1\}$*, then*

$$\Psi_1(x)=\frac{\sum_{i=0}^{n-1}(n-i)p_{i+1}\binom{n}{i}x^i}{\sum_{i=0}^{n-1}\binom{n}{i}\sum_{j=i+1}^{n}p_jx^i}$$

*is increasing in* $x\ge 0$.

(b) *If* $i\tilde{h}_{\mathbf{p}}(i)$ *is decreasing in i for* $i\in\{1,\ldots,n\}$*, then*

$$\Psi_2(x)=\frac{\sum_{i=1}^{n}i\binom{n}{i}p_ix^i}{\sum_{i=1}^{n}\binom{n}{i}\sum_{j=1}^{i}p_jx^i}$$

*is decreasing in* $x\ge 0$.

*Proof.* (a) Letting $c_i=(n-i)\binom{n}{i}p_{i+1}$ and $d_i=\binom{n}{i}\sum_{j=i+1}^{n}p_j$, we find that $\ell_i=\frac{c_i}{d_i}=(n-i)h_{\mathbf{p}}(i+1)$ is increasing in $i$ under the given condition. By taking $n_1=0$, $n_2=n-1$, it follows from Lemma 2.1 that $\Psi_1(x)$ is increasing in $x$.

(b) Letting $c_i=i\binom{n}{i}p_i$ and $d_i=\binom{n}{i}\sum_{j=1}^{i}p_j$, we find that $\ell_i=\frac{c_i}{d_i}=\ell_i=i\tilde{h}_{\mathbf{p}}(i)$ is decreasing in $i\in\{1,\ldots,n\}$ under the given condition. By taking $n_1=1$, $n_2=n$, it follows from Lemma 2.1 that $\Psi_2(x)$ is decreasing in $x$. □

The rational function $\Psi_1(x)$ seems to appear at several places in the literature. For example, Samaniego (1985) proved that a coherent system with $n$ i.i.d. IFR (increasing failure rate) components is IFR if and only if the function $\Psi_1(x)$ is increasing in $x\in(0,\infty)$. A similar

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result holds for a coherent system with DFR components. The above lemma gives simple conditions in terms of the hazard rate and the reverse hazard rates of the signature vector for the monotonicity of the functions $\Psi_1$ and $\Psi_2$.

Now we state the main result of this paper.

**Theorem 2.1.** *Let $T_{\boldsymbol{X}}(\boldsymbol{p})$ be the lifetime of a coherent system with signature vector $\boldsymbol{p}$ and with the component lifetimes $X_1, \ldots, X_n$, which are independent with a common continuous distribution $F$ and density function $f$. Let $Y_1, \ldots, Y_n$ be i.i.d. with common continuous distribution $G$ and density function $g$. Then:*

(a) *if*

$$(n-j)h_{\boldsymbol{p}}(j+1) \text{ is increasing in } j \text{ for any } j=0, \ldots, n-1 \tag{9}$$

*then*

$$X_1 \geq_{\text{hr}} Y_1 \Longrightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_{\text{hr}} T_{\boldsymbol{Y}}(\boldsymbol{p});$$

(b) *if*

$$j\tilde{h}_{\boldsymbol{p}}(j) \text{ is decreasing in } j \text{ for any } j=1, \ldots, n \tag{10}$$

*then*

$$X_1 \geq_{\text{rh}} Y_1 \Longrightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_{\text{rh}} T_{\boldsymbol{Y}}(\boldsymbol{p}).$$

(c) *Let $\Phi(u) = \sum_{j=0}^{n} \xi(j)u^{n-j}$ for all $u \in (0, 1)$, where*

$$\xi(j) = \sum_{i=j}^{n} p_{i+1}(-1)^{i-j}\binom{n-j-1}{n-i-1}\binom{n}{j}(n-j), \qquad j=0, \ldots, n.$$

*If*

$$\frac{h_{\boldsymbol{X}}(t)}{h_{\boldsymbol{Y}}(t)} \text{ is increasing in } t$$

*and, for some point $v \in (0, 1)$,*

(i) $\frac{u\Phi'(u)}{\Phi(u)}$ *is decreasing and positive for all $u \in (0, v)$,*

(ii) $\frac{(1-u)\Phi'(u)}{\Phi(u)}$ *is decreasing and negative for all $u \in (v, 1)$,*

*then*

$$X_1 \geq_{\text{lr}} Y_1 \Longrightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_{\text{lr}} T_{\boldsymbol{Y}}(\boldsymbol{p}).$$

*Proof.* (a) From (5),

$$\begin{aligned} h_{T_{\boldsymbol{X}(\boldsymbol{p})}}(t) &= \Psi_1\left(\frac{F(t)}{\overline{F}(t)}\right) h_X(t) \\ &\leq \Psi_1\left(\frac{G(t)}{\overline{G}(t)}\right) h_Y(t) \\ &= h_{T_{Y(\boldsymbol{p})}}(t), \end{aligned}$$

as $\Psi_1(x)$ is increasing in $x > 0$ and $h_X(t) \le h_Y(t)$ for all $t$; also, $h_X(t) \le h_Y(t) \Rightarrow F(t) \le G(t)$ for all $t$, which in turn implies that $F(t)/\overline{F}(t) \le G(t)/\overline{G}(t)$ for all $t$.

(b) The proof is similar to that of part (a), and hence is omitted.

(c) It is known that the survival function of $X_{i:n}$ can be written as (see Gupta (2002, p. 839)

$$\mathrm{P}(X_{i:n} > t) = \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{j} \binom{n-j-1}{n-i} \overline{F}^{n-j}(t),$$

from which its density function is

$$f_{i:n}(t) = h_X(t) \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{n}{j} \binom{n-j-1}{n-i} (n-j) \overline{F}^{n-j}(t),$$

where $h_{\boldsymbol{X}}(t)$ is the hazard rate function of $\boldsymbol{X}$. Hence, it follows that

$$\begin{aligned} f_{T_{X(\boldsymbol{p})}}(t) &= h_X(t) \sum_{i=1}^{n} \sum_{j=0}^{i-1} p_i (-1)^{i-j-1} \binom{n}{j} \binom{n-j-1}{n-i} (n-j) \overline{F}^{n-j}(t) \\ &= h_X(t) \sum_{j=0}^{n} \sum_{i=j}^{n} p_{i+1} (-1)^{i-j} \binom{n-j-1}{n-i-1} \binom{n}{j} (n-j) \overline{F}^{n-j}(t) \\ &= h_X(t) \sum_{j=0}^{n} \xi(j) \overline{F}^{n-j}(t), \end{aligned}$$

where $\xi(j) = \sum_{i=j}^{n} p_{i+1} (-1)^{i-j} \binom{n-j-1}{n-i-1} \binom{n}{j} (n-j), j = 0, \ldots, n$.

The required result is equivalent to proving that

$$\frac{f_{T_{X(\boldsymbol{p})}}(t)}{f_{T_{Y(\boldsymbol{p})}}(t)} = \frac{h_X(t)}{h_Y(t)} \cdot \frac{\sum_{j=0}^{n} \xi(j) \overline{F}^{n-j}(t)}{\sum_{j=0}^{n} \xi(j) \overline{G}^{n-j}(t)}$$

is increasing in $t$. From the assumption that $\frac{h_X(t)}{h_Y(t)}$ is increasing in $t$, we only need to show that

$$\Delta(t) = \frac{\Phi(\overline{F}(t))}{\Phi(\overline{G}(t))}$$

is increasing in $t$, where $\Phi(u) = \sum_{j=0}^{n} \xi(j) u^{n-j}(t)$.

Suppose that for some $x_v$, $\overline{F}(x_v) = v$.

*Case* 1: For all $t \in (x_v, \infty)$, i.e. $\overline{F}(t) \in (0, v)$, the sign of the derivative $\Delta(t)$ is

$$\begin{aligned} \Delta'(t) &=^{\mathrm{sgn}} g(t) \frac{\Phi'(\overline{G}(t))}{\Phi(\overline{G}(t))} - f(t) \frac{\Phi'(\overline{F}(t))}{\Phi(\overline{F}(t))} \\ &= h_Y(t) \frac{\overline{G}(t) \Phi'(\overline{G}(t))}{\Phi(\overline{G}(t))} - h_X(t) \frac{\overline{F}(t) \Phi'(\overline{F}(t))}{\Phi(\overline{F}(t))} \end{aligned}$$

$$\geq h_Y(t)\left[\frac{\overline{G}(t)\Phi'(\overline{G}(t))}{\Phi(\overline{G}(t))}-\frac{\overline{F}(t)\Phi'(\overline{F}(t))}{\Phi(\overline{F}(t))}\right]$$
$$= h_Y(t)\left[\frac{u_1\Phi'(u_1)}{\Phi(u_1)}-\frac{u_2\Phi'(u_2)}{\Phi(u_2)}\right]$$
$$\geq 0,$$

where $\overline{G}(t)=u_1$ and $\overline{F}(t)=u_2$. The first inequality follows from the assumption that $\frac{u\Phi'(u)}{\Phi(u)}$ is positive and the fact that $X\geq_{\mathrm{lr}} Y \Longrightarrow X\geq_{\mathrm{hr}} Y$. It follows from $X\geq_{\mathrm{lr}} Y \Longrightarrow X\geq_{\mathrm{st}} Y$ that $u_1\leq u_2$. Using this observation, the second inequality follows from the assumption that $\frac{u\Phi'(u)}{\Phi(u)}$ is decreasing and positive for $u\in(0, v)$.

*Case* 2: For all $t\in(0, x_v)$, i.e. $\overline{F}(t)\in(v, 1)$, the sign of the derivative $\Delta(t)$ with respect to $t$ is

$$\Delta'(t) \overset{\mathrm{sgn}}{=} g(t)\frac{\Phi'(\overline{G}(t))}{\Phi(\overline{G}(t))}-f(t)\frac{\Phi'(\overline{F}(t))}{\Phi(\overline{F}(t))}$$
$$=\tilde{h}_{\mathbf{Y}}(t)\frac{(1-\overline{G}(t))\Phi'(\overline{G}(t))}{\Phi(\overline{G}(t))}-\tilde{h}_{\mathbf{X}}(t)\frac{(1-\overline{F}(t))\Phi'(\overline{F}(t))}{\Phi(\overline{F}(t))}$$
$$=\tilde{h}_{\mathbf{X}}(t)\left[-\frac{(1-\overline{F}(t))\Phi'(\overline{F}(t))}{\Phi(\overline{F}(t))}\right]-\tilde{h}_{\mathbf{Y}}(t)\left[-\frac{(1-\overline{G}(t))\Phi'(\overline{G}(t))}{\Phi(\overline{G}(t))}\right]$$
$$\geq\tilde{h}_{\mathbf{Y}}(t)\left[-\frac{(1-u_2)\Phi'(u_2)}{\Phi(u_2)}\right]-\tilde{h}_{\mathbf{Y}}(t)\left[-\frac{(1-u_1)\Phi'(u_1)}{\Phi(u_1)}\right]$$
$$\geq 0,$$

where $\overline{G}(t)=u_1$ and $\overline{F}(t)=u_2$. The first inequality follows from the assumption that $-\frac{(1-u)\Phi'(u)}{\Phi(u)}$ is positive for all $u\in(v, 1)$ and the fact that $X\geq_{\mathrm{lr}} Y \Longrightarrow X\geq_{\mathrm{rh}} Y$. It follows from $X\geq_{\mathrm{lr}} Y \Longrightarrow X\geq_{\mathrm{st}} Y$ that $u_1\leq u_2$. Using this observation, the second inequality follows from the assumption that $-\frac{(1-u)\Phi'(u)}{\Phi(u)}$ is increasing and positive in $u\in(v, 1)$.

Equations (9) and (10) give simple sufficient conditions on the signature vector for two systems to be ordered according to hazard rate and reverse hazard rate orderings. This problem has also been studied by Navarro et al. (2013). However, as the examples in the next section show, their results are different from the results of this paper. Kochar et al. (1999) established the following results for comparing two coherent systems with signature vectors $\mathbf{p}$ and $\mathbf{q}$, but with i.i.d. component lifetimes $X_1, \ldots X_n$.

**Theorem 2.2.** *(Kochar et al. (1999)) Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be the signature vectors of two coherent systems with the same number of components, and let $T_{\mathbf{X}}(\boldsymbol{p})$ and $T_{\mathbf{X}}(\boldsymbol{q})$ be their lifetimes, where the elements of the vector $\mathbf{X}$ are i.i.d. Then:*

(a) $\boldsymbol{p}\geq_{\mathrm{st}}\boldsymbol{q}\Rightarrow T_{\mathbf{X}}(\boldsymbol{p})\geq_{\mathrm{st}} T_{\mathbf{X}}(\boldsymbol{q})$,

(b) $\boldsymbol{p}\geq_{\mathrm{hr}}\boldsymbol{q}\Rightarrow T_{\mathbf{X}}(\boldsymbol{p})\geq_{\mathrm{hr}} T_{\mathbf{X}}(\boldsymbol{q})$,

(c) $\boldsymbol{p}\geq_{\mathrm{lr}}\boldsymbol{q}\Rightarrow T_{\mathbf{X}}(\boldsymbol{p})\geq_{\mathrm{lr}} T_{\mathbf{X}}(\boldsymbol{q})$.

A similar result holds for reverse hazard rate ordering. Combining the results of the above theorem with the previous theorem, we obtain the following general results which show how the hazard rate, the reversed hazard rate, and the likelihood ratio orders between $\boldsymbol{X}$ and $\boldsymbol{Y}$ and signature vectors **p** and **q** are preserved by the lifetimes of the corresponding coherent systems.

**Theorem 2.3.** *Under the assumptions of Theorem 2.1,*

(a) *if either* $(n-j)h_{\boldsymbol{p}}(j+1)$ *or* $(n-j)h_{\boldsymbol{q}}(j+1)$ *is increasing in* $j \in \{j=0, \ldots, n-1\}$*, then* $\boldsymbol{p} \geq_{\text{hr}} \boldsymbol{q} \Rightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_{\text{hr}} T_{\boldsymbol{Y}}(\boldsymbol{q})$*;*

(b) *if either* $j\tilde{h}_{\boldsymbol{p}}(j)$ *or* $j\tilde{h}_{\boldsymbol{q}}(j)$ *is decreasing in* $j$ *for* $j=1, \ldots, n$*, then* $\boldsymbol{p} \geq_{\text{rh}} \boldsymbol{q} \Rightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_{\text{rh}} T_{\boldsymbol{Y}}(\boldsymbol{q})$*;*

(c) *if the conditions of Theorem 2.3 (c) are satisfied by either* $\boldsymbol{p}$ *or* $\boldsymbol{q}$*, then* $\boldsymbol{p} \geq_{\text{lr}} \boldsymbol{q} \Rightarrow T_{\boldsymbol{X}}(\boldsymbol{p}) \geq_{\text{lr}} T_{\boldsymbol{Y}}(\boldsymbol{q})$*.*

## 3. Some numerical examples

In this section we present some coherent systems that satisfy the conditions of Theorem 2.1.

**Example 3.1.** Consider a coherent system of order four with lifetime

$$T_{\boldsymbol{X}}(\boldsymbol{p}) = \min(X_1, \max(X_2, X_3, X_4)).$$

It is easy to show that the signature vector of this system is $\boldsymbol{p} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$. Note that the condition "$(n-j)h_{\boldsymbol{p}}(j+1)$ is increasing in $j$" is equivalent to $\frac{(n-r)p_{r+1}}{(n-k)p_{k+1}} \geq \frac{\sum_{j=r+1}^{n} p_j}{\sum_{j=k+1}^{n} p_j}$ for all $r \geq k$, whenever $p_{k+1} > 0$. We check this condition for various cases of $k$ and $r$:

- If $r=3$ and $k=2$, then $\frac{(n-r)p_{r+1}}{(n-k)p_{k+1}} = 0 \geq \frac{\sum_{j=r+1}^{n} p_j}{\sum_{j=k+1}^{n} p_j} = 0.$
- If $r=3$ and $k=1$, then $\frac{(n-r)p_{r+1}}{(n-k)p_{k+1}} = 0 \geq \frac{\sum_{j=r+1}^{n} p_j}{\sum_{j=k+1}^{n} p_j} = 0.$
- If $r=2$ and $k=1$, then $\frac{(n-r)p_{r+1}}{(n-k)p_{k+1}} = \frac{8}{3} \geq \frac{\sum_{j=r+1}^{n} p_j}{\sum_{j=k+1}^{n} p_j} = \frac{4}{3}.$
- If $r=1$ and $k=0$, then $\frac{(n-r)p_{r+1}}{(n-k)p_{k+1}} = \frac{3}{4} \geq \frac{\sum_{j=r+1}^{n} p_j}{\sum_{j=k+1}^{n} p_j} = \frac{3}{4}.$

That is, this system satisfies condition (a) of Theorem 2.1.

**Example 3.2.** Consider a coherent system of order four with lifetime

$$T_{\boldsymbol{X}}(\boldsymbol{p}) = \max(X_1, \min(X_2, X_3, X_4)).$$

It is easy to show that the signature vector of this system is $\boldsymbol{p} = (0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Note that the condition "$j\tilde{h}_{\boldsymbol{p}}(j)$ is decreasing in $j$" is equivalent to $\frac{\sum_{i=1}^{r} p_i}{\sum_{i=1}^{k} p_i} \geq \frac{rp_r}{kp_k}$ for all $r \geq k$, whenever $p_k > 0$.

The above inequality is evaluated for various possible values of $k$ and $r$:

- If $r=3$ and $k=2$, then $\frac{\sum_{i=1}^{r} p_i}{\sum_{i=1}^{k} p_i} = \frac{3}{2} \geq \frac{rp_r}{kp_k} = \frac{3}{4}.$
- If $r=4$ and $k=3$, then $\frac{\sum_{i=1}^{r} p_i}{\sum_{i=1}^{k} p_i} = \frac{4}{3} \geq \frac{rp_r}{kp_k} = \frac{4}{3}.$

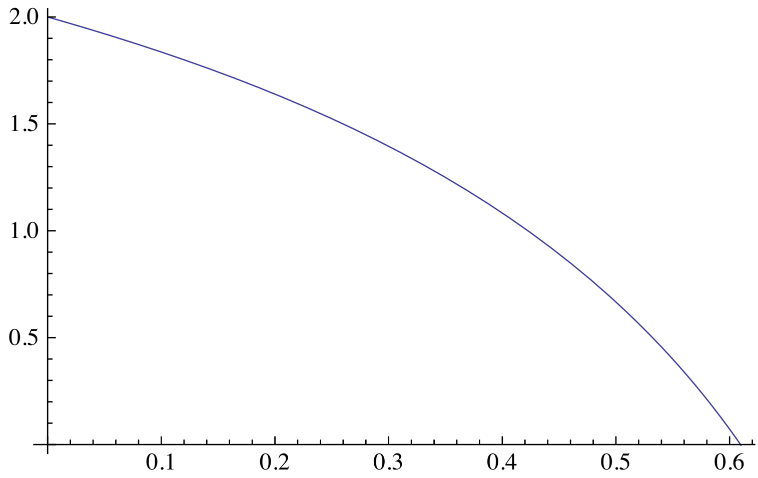


FIGURE 2: Plot of $\Delta_1(u)$ for all $u \in (0, 0.61)$.

- If $r = 4$ and $k = 2$, then $\frac{\sum_{i=1}^{r} p_i}{\sum_{i=1}^{k} p_i} = 2 \geq \frac{rp_r}{kp_k} = 1$.

That is, the condition in Theorem 2.1 (b) is satisfied by such a system.

**Example 3.3.** Consider a coherent system of order four with signature vector $\boldsymbol{p} = (0, \frac{1}{3}, \frac{2}{3}, 0)$. It can be seen that

$$\begin{aligned}\Psi(u) &= \xi(0)u^4 + \xi(1)u^3 + \xi(2)u^2 \\ &= 4u^4 - 12u^3 + 8u^2, \\ \Delta_1(u) &= \frac{u\Psi'(u)}{\Psi(u)} = \frac{16u^4 - 36u^3 + 16u^2}{4u^4 - 12u^3 + 8u^2},\end{aligned}$$

and

$$\Delta_2(u) = \frac{(1-u)\Psi'(u)}{\Psi(u)} = \frac{(1-u)(16u^3 - 36u^2 + 16u)}{4u^4 - 12u^3 + 8u^2}.$$

The graphs of the functions $\Delta_1(u)$ and $\Delta_2(u)$ are given in Figures 2 and 3, respectively. It can be seen that $\Delta_1(u)$ is decreasing and is positive in $u \in (0, 0.61)$, and $\Delta_2(u)$ is decreasing and is negative in $u \in (0.61, 1)$, which shows that conditions (i) and (ii) of Theorem 2.1 (c) are satisfied.

**Example 3.4.** Consider a coherent system of order four with signature vector $\boldsymbol{p} = (0, \frac{1}{2}, \frac{1}{2}, 0)$. It can be seen that

$$\begin{aligned}\Psi(u) &= \xi(0)u^4 + \xi(1)u^3 + \xi(2)u^2 \\ &= -6u^3 + 6u^2, \\ \Delta_3(u) &= \frac{u\Psi'(u)}{\Psi(u)} = \frac{12u^2 - 18u^3}{6u^2 - 6u^3},\end{aligned}$$

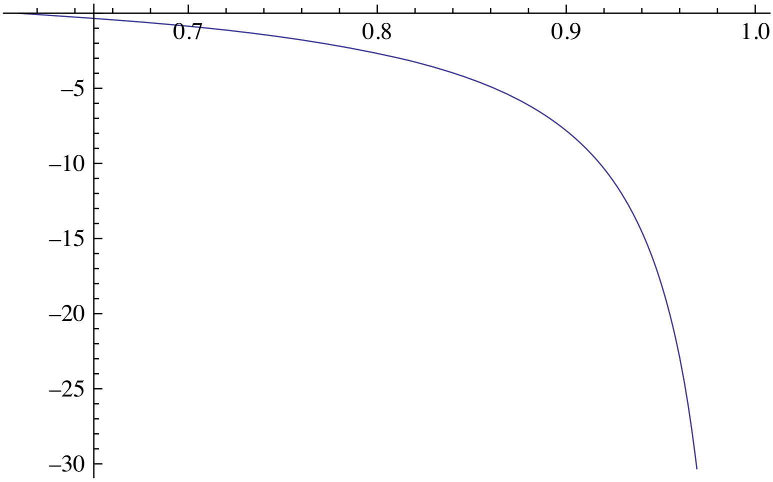


FIGURE 3: Plot of $\Delta_2(u)$ for all $u \in (0.61, 1)$.

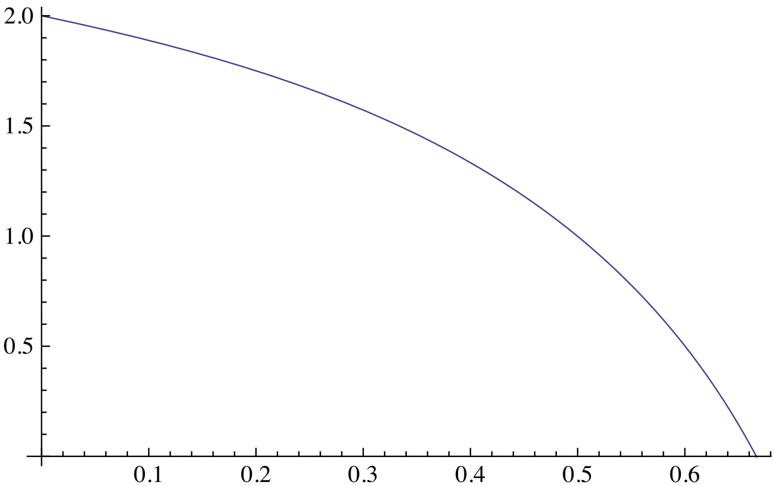


FIGURE 4: Plot of $\Delta_3(u)$ for all $u \in (0, 0.667)$.

and

$$\Delta_4(u) = \frac{(1-u)\Psi'(u)}{\Psi(u)} = \frac{(1-u)(12u-18u^2)}{6u^2-6u^3}.$$

The graphs of the functions $\Delta_3(u)$ and $\Delta_4(u)$ are shown in Figures 4 and 5, respectively. It can be seen that $\Delta_3(u)$ is decreasing and is positive in $u \in (0, 0.667)$, and $\Delta_2(u)$ is decreasing and is negative in $u \in (0.667, 1)$, which shows that conditions (i) and (ii) of Theorem 2.1 (c) are satisfied.

In Tables 1 and 2 we characterize all possible coherent systems of sizes 3, 4, and 5 for which the results of Theorem 2.1 (a) and (b) hold.

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TABLE 3: Coherent systems with sizes 4 and 5.

| $\boldsymbol{p}$ | $\frac{u\Psi'(u)}{\Psi(u)} > 0$ | $\frac{(1-u)\Psi'(u)}{\Psi(u)} < 0$ |
|---|---|---|
| $(0, \frac{1}{2}, \frac{1}{2}, 0)$ | $\downarrow u \in (0, 0.667)$ | $\downarrow u \in (0.667, 1)$ |
| $(0, \frac{2}{3}, \frac{1}{3}, 0)$ | $\downarrow u \in (0, 0.708)$ | $\downarrow u \in (0.708, 1)$ |
| $(0, \frac{1}{3}, \frac{2}{3}, 0)$ | $\downarrow u \in (0, 0.61)$ | $\downarrow u \in (0.61, 1)$ |
| $(0, \frac{7}{10}, \frac{3}{10}, 0, 0)$ | $\downarrow u \in (0, 0.767)$ | $\downarrow u \in (0.767, 1)$ |
| $(0, \frac{3}{5}, \frac{2}{5}, 0, 0)$ | $\downarrow u \in (0, 0.75)$ | $\downarrow u \in (0.75, 1)$ |
| $(0, \frac{2}{5}, \frac{3}{5}, 0, 0)$ | $\downarrow u \in (0, 0.702)$ | $\downarrow u \in (0.702, 1)$ |
| $(0, \frac{4}{5}, \frac{1}{5}, 0, 0)$ | $\downarrow u \in (0, 0.461)$ | $\downarrow u \in (0.461, 1)$ |
| $(0, 0, \frac{3}{5}, \frac{2}{5}, 0)$ | $\downarrow u \in (0, 0.546)$ | $\downarrow u \in (0.546, 1)$ |

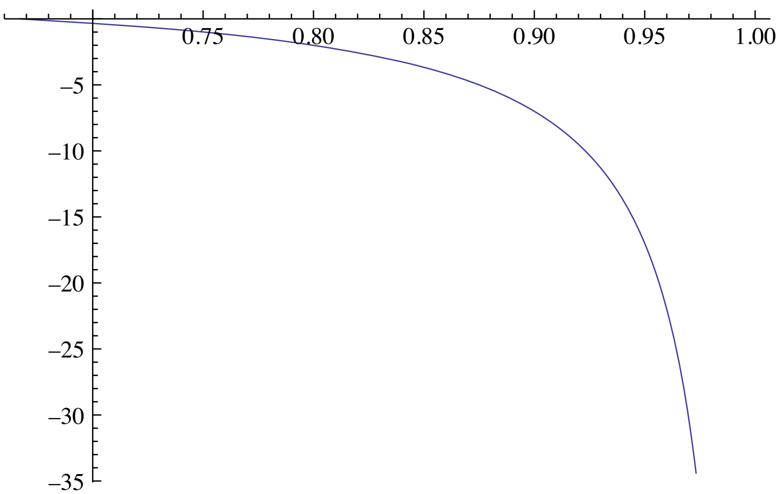


FIGURE 5: Plot of $\Delta_4(u)$ for all $u \in (0.667, 1)$.

Ross et al. (1980) proved that the number $N$ of failed components at the time a system fails has the discrete increasing failure rate on average property. As elaborated in Navarro and Samaniego (2017), this fact implies that the signature of a system of arbitrary order $n$ cannot have internal zeros; that is, there exist no integers $i \in \{1, \ldots, n-2\}$ and $j \in \{2, \ldots, n-i\}$ for which $p_i > 0$ and $p_{i+j} > 0$ while $p_{i+1} = \cdots, p_{i+j-1} = 0$. They also give an elementary proof of this result. Looking at Tables 1 and 2, we may conjecture that every signature vector of the type $(p_1, p_2, \ldots, p_j, 0, 0, \ldots, 0)$ for some $j < n$ satisfies the condition (9), and that every signature vector of the type $(0, 0, \ldots, 0, p_j, p_{j+1}, \ldots, p_n)$ for some $j > 1$ satisfies the condition (10) in Theorem 2.1.

In the following, we characterize all possible coherent systems of sizes 4 and 5 for which the results of Theorem 2.1 (c) can be applied. Again, from Table 3 we conjecture that for a system with signature vector of the type $(0, 0, 0, \ldots, p_j, \ldots, p_k, 0, 0, \ldots, 0)$ for some $2 < j < k < n-1$, the conditions for the likelihood ratio order in Theorem 2.1 are satisfied.

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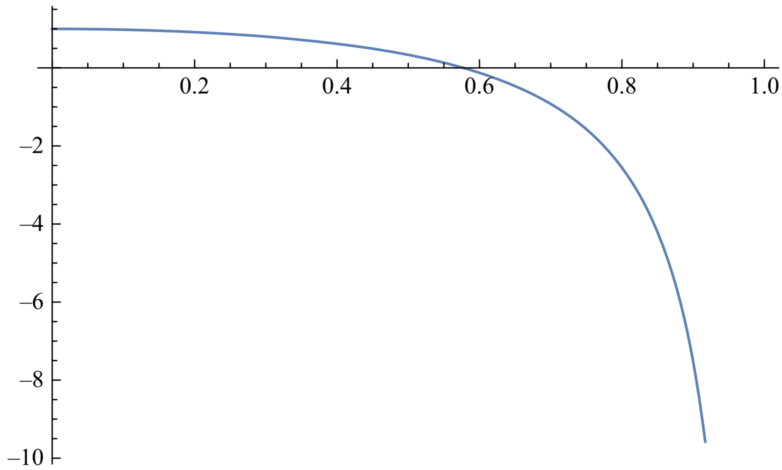


FIGURE 6: Plot of $\Delta_5(u)$ for all $u \in (0, 1)$.

Navarro et al. (2013) also obtained some stochastic comparison results for coherent systems with dependent but identically distributed components. Let $T_X$ and $T_Y$ be the lifetimes of two coherent systems with the same structure function, with i.i.d. component lifetimes, and with common absolutely continuous cumulative distribution functions $F$ and $G$, respectively. Let $h$ be the common domination function, and let us assume that it is twice differentiable. Among other results, Navarro et al. (2013) proved the following result in their Theorem 2.6 (iv):

If $\boldsymbol{X} \geq_{\text{lr}} \boldsymbol{Y}$ and $\frac{uh''(u)}{h'(u)}$ is non-negative and decreasing in (0, 1), then $T_X \geq_{\text{lr}} T_Y$.

We now provide a counterexample to illustrate that condition (iv) in Theorem 2.6 of Navarro et al. (2013) for establishing likelihood ratio ordering is not satisfied, but it satisfies the conditions of our main result on likelihood ratio ordering between systems.

**Example 3.5.** Consider a coherent system of order four with signature vector $\boldsymbol{p} = (0, \frac{2}{3}, \frac{1}{3}, 0)$ and lifetime $T_{\mathbf{X}}(\boldsymbol{p}) = \max(\min(X_1, X_2), \min(X_3, X_4))$, where $X_1, X_2, X_3$, and $X_4$ are independent and identically distributed. As shown in Table 1 of Navarro et al. (2013), the domination function of this system is

$$h(u) = 2u^2 - u^4,$$

and

$$\Delta_5(u) = \frac{uh''(u)}{h'(u)} = \frac{4u - 12u^3}{4(u - u^3)}.$$

The graph of the function $\Delta_5(u)$ is given in Figure 6. It can be seen that $\Delta_5(u)$ is not always non-negative. Therefore, condition (iv) of Theorem 2.6 in Navarro et al. (2013) for establishing likelihood ratio ordering is not satisfied, but conditions (i) and (ii) of Theorem 2.1 (c) are satisfied, as seen from Table 3. Hence, $\boldsymbol{X} \geq_{\text{lr}} \boldsymbol{Y} \Longrightarrow T_X \geq_{\text{lr}} T_Y$. These observations reveal that the conditions established in Theorem 2.1 are quite general.

## 4. Conclusions

In this paper we have considered the problem of stochastically comparing the lifetimes $T_X(\boldsymbol{p})$ and $T_Y(\boldsymbol{q})$ of two coherent systems with signature vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ of the same size and with i.i.d. component lifetimes distributed according to $X$ and $Y$, respectively. The results established in this paper generalize some of the known results in the literature. Most of the existing results in the literature deal with the case when the two coherent systems have component lifetimes which are identically distributed. In this paper, we first considered the case when a coherent system operates under two different sets of independent and identically distributed component lifetimes. We found simple sufficient conditions on the distribution of the signature vector under which the two systems are ordered according to hazard rate and reverse hazard rate orderings. In particular, we showed that if $(n-j)h_{\boldsymbol{p}}(j+1)$ $(j\tilde{h}_{\boldsymbol{p}}(j))$ is increasing in $j$ (decreasing in $j$) for any $j \geq 1$ and $X_1 \geq_{\mathrm{hr(rh)}} Y_1$, then a system with lifetime $T_{\mathbf{X}}(\boldsymbol{p})$ is larger than a system with lifetime $T_{\mathbf{Y}}(\boldsymbol{p})$ according to the hazard rate order (the reversed hazard rate order). We also gave sufficient conditions on the signature vectors for a similar result for likelihood ratio order. Then we combined these new results with (3) to compare two coherent systems consisting of components with different lifetime distributions and also with possible different signatures. We characterized possible coherent systems of size 3, 4, and 5 for which the above results can be applied and also the proposed two conjectures for coherent systems of size $n > 5$ are true. It will be interesting to examine whether our conjectures are true. We plan to pursue this problem in the near future.

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