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A Gap Principle for Subvarieties with Finitely Many Periodic Points

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Abstract. Let $f: X \to X$ be a quasi-finite endomorphism of an algebraic variety X defined over a number field K and fix an initial point $a \in X$. We consider a special case of the Dynamical Mordell–Lang Conjecture, where the subvariety V contains only finitely many periodic points and does not contain any positive-dimensional periodic subvariety. We show that the set $\{n \in \mathbb{Z}_{\geqslant 0} \mid f^n(a) \in V\}$ satisfies a strong gap principle.

1 Introduction

The Dynamical Mordell–Lang Conjecture predicts that given an endomorphism $f: X \to X$ of a complex quasi-projective variety X, for any point $a \in X$ and any subvariety $V \subseteq X$, the set $S_V := \{n \in \mathbb{Z}_{\geq 0} \mid f^n(a) \in V\}$ is a finite union of arithmetic progressions (sets of the form $\{a, a+d, a+2d, \dots\}$ with $a, d \in \mathbb{Z}_{\geq 0}$). The Dynamical Mordell–Lang Conjecture was proposed in [16]. See also [1] and [10] for earlier works. In the case of étale maps we know that the Dynamical Mordell–Lang Conjecture is true (see [5] and [8]). Xie proved in [22] the Dynamical Mordell–Lang Conjecture for polynomial endomorphisms of the affine plane.

The p-adic interpolation method is one of the most important tools for tackling the Dynamical Mordell–Lang Conjecture. The basic idea is to construct a p-adic analytic function $G: \mathbb{Z}_p \to X$ such that $G(n) = f^n(a)$ (replacing f by an iterate if necessary). This allows one to use the tools from p-analytic functions to calculate the set S_V.

However, in general it is unknown whether such an interpolation exists for any endomorphism f or any initial point a. For some evidence of its nonexistence see [2]. Then we might not be able to prove the Dynamical Mordell–Lang Conjecture using the p-adic interpolation in those cases. However, by a discovery of Bell, Ghioca and Tucker (see [8, Theorem 11.11.3.1] or Theorem 4.1 below), we might expect an approximating function G such that $G(n)$ approximates $f^n(a)$ very closely. Suppose Q is a polynomial vanishing on V. Then the roots of $Q \circ G$ give much restriction to those n such that $f^n(a) \in V$. This allows us to prove a weaker version of the Dynamical Mordell–Lang Conjecture in certain cases.

To apply the approximation result, we need to exclude a case when the orbit converges p-adically to a periodic point on V. One way to guarantee this is to ensure that the residue class of the orbit does not meet the periodic point mod p after a certain number of iterates. Theorem 1 of [2] gives this kind of avoidance result for $\mathbb{P}^1$. In this

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paper, we will generalize this result to quasi-finite endomorphisms of quasi-projective varieties. This enables us to prove a strong gap principle under certain conditions.

The main result of this paper is the following theorem. For the definition of nonsingular variety, see [19, Section 1.5].

Theorem 1.1 *Suppose $f: X \to X$ is a quasi-finite endomorphism of a nonsingular variety X. Let $V \subseteq X$ be a subvariety. Assume that X, V, and f are all defined over a number field K. Assume that there are only finitely many periodic points in $V(\overline{K})$ and that V does not contain a positive-dimensional periodic subvariety. Assume that $a \in X(K)$ is not preperiodic under f. Then the set $S_V := \{n \in \mathbb{Z}_{\geq 0} \mid f^n(a) \in V(K)\}$ has the property that*

$$\#\{i \leq n \mid i \in S_V\} = o\big(\log^{(m)}(n)\big)$$

for any fixed $m \in \mathbb{Z}_{>0}$ (where $\log^{(m)}$ is the m-th iterated logarithmic function).

This theorem is an analog of Theorem 1.4 of [3]. See also [7, Theorem 1.4] for another result along this line. In a broader sense, all these theorems can be thought of as dynamical analogs of the gap principle in Diophantine geometry. In [11] the authors showed that good approximations of algebraic numbers have heights growing rapidly. Let C be a curve of genus at least 2. Mumford proved in [21] that if we order the rational points of C according to Weil height, then their Weil height grows exponentially. Faltings later proved in [13] that the number of rational points in C is actually finite. In this article, by sparsity of S_V we mean a large gap between consecutive numbers, ordered increasingly. One difference is that in the results in dynamics the "gaps" are much larger than in the Diophantine geometry. But our ultimate goal is also to show that S_V is finite in these cases.

We say that a morphism $f: X \to X$ of a normal variety X is *polarizable* if there is an ample divisor L of X such that $f^*L \cong L^{\otimes d}$ for some $d > 1$. Our result is related to the following.

Conjecture 1.2 (Dynamical Manin–Mumford, [23]) *Suppose $f: X \to X$ is a polarizable endomorphism of a variety X. If a subvariety $V \subseteq X$ contains a Zariski dense set of preperiodic points, then V is itself preperiodic.*

In the case when f is a split rational map on $(\mathbb{P}^1)^n$, the Dynamical Manin–Mumford Conjecture is proved in [14] (see also [15]). Unfortunately, there is a counterexample in [18] to Conjecture 1.2. In the same paper, the authors also proposed a refined conjecture in Conjecture 2.4 there.

If the Dynamical Manin–Mumford Conjecture is true for $f: X \to X$ and $V \subseteq X$, then the condition that V does not contain a positive-dimensional subvariety would imply that the set of preperiodic points on V is not Zariski dense. In particular, if V is a curve, then the condition that V is not periodic would imply that there are finitely many periodic points on V. For more about the Dynamical Manin–Mumford Conjecture, see [24]. On the other hand, [12] shows that if f is polarizable, then the subset of $X(\overline{K})$ consisting of all periodic points of f is Zariski dense in X.

By Theorem 1.1 and the above discussion, we are able to prove the following theorem, which is also a consequence of [3, Theorem 1.4].

Theorem 1.3 *Let $X = (\mathbb{P}^1)^n$. Suppose $f\colon X \to X$ is given by $(x_1, \ldots, x_n) \mapsto \big(f_1(x_1), \ldots, f_n(x_n)\big)$ where each f_i is one-variable rational function of degree at least 2. Let $V \subseteq X$ be a non-periodic curve. Suppose X, V, and f are all defined over a number field K. Then the set $S_V := \{n \in \mathbb{Z}_{\geq 0} \mid f^n(a) \in V(K)\}$ has the property that*

$$\#\{i \leq n \mid i \in S_V\} = o\big(\log^{(m)}(n)\big)$$

for any fixed $m \in \mathbb{Z}_{>0}$ (where $\log^{(m)}$ is the m-th iterated logarithmic function).

The avoidance result below is a key step towards the proof of Theorem 1.1. It is useful in many other contexts. It is a generalization of Theorem 5 of [2] and its proof goes parallel with that of the latter theorem, with simplifications in certain steps.

Theorem 1.4 *Suppose $f\colon X \to X$ is a quasi-finite endomorphism of a nonsingular variety X. Let $V \subseteq X$ be a subvariety. Assume that X, V, and f are all defined over a number field K. Assume that there are only finitely many periodic points $\gamma_1, \ldots, \gamma_r$ in $V(\overline{K})$ and that V does not contain a positive-dimensional periodic subvariety. Then there are a finite set of places S of K, a positive integer M, and a set $\mathcal{P}$ of primes of K with positive density such that for all $\mathfrak{p}' \in \mathcal{P}$, for all $m \geq M$, for all $1 \leq i \leq r$, and for all $a \in X(\mathcal{O}_{S,K})$ not preperiodic, we have $f^m_{\mathfrak{p}'}(a_{\mathfrak{p}'}) \neq (\gamma_i)_{\mathfrak{p}'}$.*

A precursor of this kind of avoidance result can be found in [4, Section 4]. We follow the idea of the proof in [2] to prove Theorem 1.4. First, we find a single prime $\mathfrak{p}$ at which the reduction of a does not hit γ_i after the M-th iteration. Note that $\mathfrak{p}$ might ramify, but in contrast with the treatment in [2] we use Corollary 3.6 to obtain a unified way to find a Frobenius "coset" in both ramified and unramified case, and then we use the Chebotarev density theorem to find a family of primes with the same Frobenius conjugacy class, and hence prove the avoidance theorem.

For the proof of Theorem 1.1, after some reductions we use Theorem 4.1 to find an approximate interpolating function G. This allows us to use the zeros of G to obtain information about the set S_V. We need Theorem 1.4 to avoid one case in which we cannot say much about S_V. A key technical result is Proposition 4.2, which is a modification of Theorem 11.11.3.1 of [8].

The organization of this paper is as follows. Section 2 gives the basic notation and definitions, as well as some simple reductions. In Section 3, we prove the avoidance result. In Section 4, we apply the approximate p-adic interpolation to demonstrate Theorem 1.1.

2 Definitions and Preliminaries

Definition 2.1 Suppose $\phi\colon X \to Y$ is a morphism of algebraic varieties. We say that ϕ is *quasi-finite* if each point $y \in Y$ has finitely many preimages.

We need the following simple observation.

Lemma 2.2 *If Theorem 1.1 holds for f^k with initial points $a, f(a), f^2(a), \ldots, f^{k-1}(a)$, then Theorem 1.1 holds for f with initial point a.*

Proof The number k is independent of m and n. Then the result is clear as a sum of k numbers of size $o\big(\log^{(m)}(n)\big)$ is still of size $o\big(\log^{(m)}(n)\big)$. ∎

Suppose $f\colon X \to X$ is a quasi-finite endomorphism of a variety X. Let $V \subseteq X$ be a subvariety. Assume that X, V, and f are all defined over a number field K. Suppose $\gamma_1, \ldots, \gamma_r$ are all the periodic points of f on V. Replacing f by an iterate we may assume that all γ_i are fixed points. Note that by Lemma 2.2 this does not affect the gap property in Theorem 1.1. Extending K if necessary we may assume that all γ_i are defined over K. Suppose S is a finite set containing all the infinity places of K and all the finite places where the reduction of X is not a nonsingular variety or the reduction of f is not well-defined. Choose an S-integral model $\mathcal{X}$ of X, that is, a scheme $\mathcal{X}$ flat over the ring $\mathcal{O}_{S,K}$ of S-integers in K, such that the generic fiber $\mathcal{X} \times_{\mathcal{O}_{S,K}} K$ is isomorphic to X.

At the expense of expanding the size of S by a finite number, we may do several simplifications. We may assume that f can be extended to a morphism of schemes $f\colon \mathcal{X} \to \mathcal{X}$ defined over $\mathcal{O}_{S,K}$. We may also assume that the points $\gamma_1, \ldots, \gamma_r$ are defined over $\mathcal{O}_{S,K}$. Furthermore assume that no element in $f^{-1}(\gamma_i) \smallsetminus \{\gamma_i\}$ hits γ_i modulo $\mathfrak{p}$ for any $\mathfrak{p} \notin S$ and for any $1 \le i \le r$. For a prime $\mathfrak{p}$ of K outside S, denote by $(\gamma_1)_{\mathfrak{p}}, \ldots, (\gamma_r)_{\mathfrak{p}}$, the extended points of $\gamma_1, \ldots, \gamma_r$ on the special fiber $X_{\mathfrak{p}} := \kappa(\mathfrak{p}) \times_{\mathcal{O}_{S,K}} \mathcal{X}$, and denote by $f_{\mathfrak{p}}$ the reduction of f at $\mathfrak{p}$.

For any prime $\mathfrak{p}$ of K such that the reduction $X_{\mathfrak{p}}$ is nonsingular, we use the notation as below. Let p be the prime number such that $p\mathbb{Z}$ is the contraction of $\mathfrak{p}$ in $\mathbb{Z}$. Let $\mathfrak{o}_v$ be the completion of the local ring $(O_K)_{\mathfrak{p}}$ and suppose π is a uniformization element of $\mathfrak{o}_v$. For $f \in \mathfrak{o}_v[x_1, \ldots, x_m]$ we let $\|f\|$ be the supremum of the absolute values of the coefficients of f. We let $\mathfrak{o}_v\langle x_1, \ldots, x_m\rangle$ be the completion of $\mathfrak{o}_v[x_1, \ldots, x_m]$ with respect to $\|\cdot\|$; it consists of all power series in $x_1, \ldots, x_m$ with the property that the absolute values of its coefficients tend to 0.

3 The Proof of Theorem 1.4

We follow the idea of the proof in [2]. We need some lemmas.

Lemma 3.1 *With $f, X, \mathcal{X}, S$ as in Section 2, choose a prime $\overline{\mathfrak{p}}$ of $\mathcal{O}_{\overline{K}}$ such that $\overline{\mathfrak{p}} \cap \mathcal{O}_K \notin S$. Assume that $\gamma_{\mathfrak{p}}$ is not a periodic point modulo $\mathfrak{p}$. Then for each finite extension L/K, there is an integer M such that for all $m \ge M$ and for all $\beta \in X(\overline{K})$ with $f^m(\beta) = \gamma$, we have $[\mathcal{O}_{L(\beta)}/\tau : \mathcal{O}_L/\mathfrak{q}] > 1$ where $\tau = \overline{\mathfrak{p}} \cap \mathcal{O}_{L(\beta)}$ and $\mathfrak{q} = \overline{\mathfrak{p}} \cap \mathcal{O}_L$.*

Proof Since γ is not periodic modulo $\mathfrak{p}$ and the set $X_{\mathfrak{p}}(\mathcal{O}_L/\mathfrak{q})$ is finite, we know that there exists an M such that for all $m \ge M$, the set $\{x \in X_{\mathfrak{p}}(\mathcal{O}_L/\mathfrak{q}) \mid f_{\mathfrak{p}}^m(x) = \gamma_{\mathfrak{p}}\}$ is empty. It follows that $[\mathcal{O}_{L(\beta)}/\tau : \mathcal{O}_L/\mathfrak{q}] > 1$. ∎

Assume that γ is a fixed point modulo $\mathfrak{p}$. Suppose $\eta_j \in f^{-1}(\gamma) \smallsetminus \{\gamma\}$ are all the preimages of γ other than γ itself. Let E be the compositum of K with all the defining fields of η_j.

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Lemma 3.2 *Let $f, X, \mathcal{X}, S$ be the same as in Section 2 and in the above paragraph. Choose a prime $\overline{\mathfrak{p}}$ of $\mathcal{O}_{\bar{K}}$ such that $\overline{\mathfrak{p}} \cap \mathcal{O}_{S,K} \notin S$. Then for each finite extension L/E, there is an integer M such that for all $m \geq M$ and for all $\beta \in X(\overline{K})$ with $f^m(\beta) = \gamma$ and $f^t(\beta) \neq \gamma$ for $t < m$, we have $[\mathcal{O}_{L(\beta)}/\tau : \mathcal{O}_L/\mathfrak{q}] > 1$ where $\tau = \overline{\mathfrak{p}} \cap \mathcal{O}_{L(\beta)}$ and $\mathfrak{q} = \overline{\mathfrak{p}} \cap \mathcal{O}_L$.*

Proof This is a slight generalization of the proof of Proposition 1 of [2]. We apply Lemma 3.1 to each η_j. The assumption about S implies that $\eta_j \not\equiv \gamma \pmod{\overline{\mathfrak{p}}}$. Therefore none of $(\eta_j)_{\mathfrak{p}}$ is periodic under $f_{\mathfrak{p}}$. We can find M_j such that for all $m \geq M_j$, if $\beta \in X(K)$ satisfies $f^m(\beta) = \eta_j$ and $f^t(\beta) \neq \eta_j$ for $t < m$, then we have $[\mathcal{O}_{L(\beta)}/\tau : \mathcal{O}_L/\mathfrak{q}] > 1$. Since there are finitely many η_j, we can set $M := \max M_j + 1$. Now for all $m \geq M$, if $\beta \in X(K)$ satisfies $f^m(\beta) = \gamma$ and $f^t(\beta) \neq \gamma$ for $t < m$, then $f^{m-1}(\beta) = \eta_j$ for some j. Hence we have $[\mathcal{O}_{L(\beta)}/\tau : \mathcal{O}_L/\mathfrak{q}] > 1$. ■

The lemma below shows that passing to integral closures affects only finitely many primes.

Lemma 3.3 *Suppose $B' \subseteq B''$ are both integral subrings with field of fraction E, a number field. Assume that B'' is integral over B'. Then there is a prime $\mathfrak{q}$ of B'' lying above $\mathfrak{p}$, and furthermore for all but finitely many prime $\mathfrak{p} \in B'$, the prime $\mathfrak{q}$ is unique and we have $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$.*

Proof The first conclusion holds by the going-up theorem. The ring $B'_{\mathfrak{p}}$ is a discrete valuation ring for all but finitely many $\mathfrak{p}$. For these $\mathfrak{p}$ the integral closure of $B'_{\mathfrak{p}}$ in its field of fraction is still $B'_{\mathfrak{p}}$. Hence $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$. ■

The advantage of passing to the integral closures is that we have the same automorphism group at the level of fields and at the level of rings. Clearly R is normal over R_i and $\mathcal{O}_{S,K}$. Let $G = \mathrm{Gal}(F/K)$ and let $H_i = \mathrm{Gal}(F/\mathrm{Frac}(R_i))$. Then $R^G = \mathcal{O}_{S,K}$ and $R^{H_i} = R_i$ as R and R_i are integrally closed. Let $\mathcal{T}_i$ be the set of left cosets of H_i in G. For each prime $\mathfrak{q}'$ of F, suppose $\mathfrak{p}' = \mathfrak{q}' \cap K$. Let $D_{\mathfrak{q}'} = D(\mathfrak{q}'/\mathfrak{p}')$ and $I_{\mathfrak{q}'} = I(\mathfrak{q}'/\mathfrak{p}')$ denote the decomposition and inertia groups of $\mathfrak{q}'$. We may view G as a group of permutations of the set $\mathcal{T}_i$. The following lemma is part of Lemmas 3.1 and 3.2 of [17]. This lemma gives a unified approach of treating ramified and unramified primes.

Lemma 3.4 *There is a group isomorphism $D_{\mathfrak{q}'}/I_{\mathfrak{q}'} \cong \mathrm{Gal}(\kappa(\mathfrak{q}')/\kappa(\mathfrak{p}'))$, and the number of primes $\mathfrak{q}'_i \subset R_i$ lying over $\mathfrak{p}'$ such that $\kappa(\mathfrak{q}'_i) = \kappa(\mathfrak{p}')$ is the number of common orbits of $D_{\mathfrak{q}'}$ and $I_{\mathfrak{q}'}$ on $\mathcal{T}_i$.* ■

We then have the following.

Lemma 3.5 *If there is no $\mathfrak{q}'_i$ lying over $\mathfrak{p}'$ with residue degree 1, then for all $\bar{x} \in X_{\mathfrak{p}}(\mathcal{O}/\mathfrak{p})$, the orbit under $f_{\mathfrak{p}}$ does not hit $\gamma_{\mathfrak{p}}$ for the first time at the M-th iterate.* ■

The above corollary applies to any $\mathfrak{p}'$, but we will apply it to $\mathfrak{p}$.

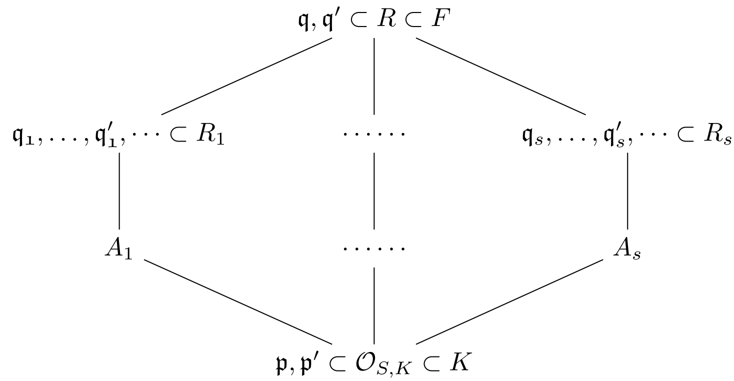

Figure 1: The Diagram.

Corollary 3.6 *Use notation as in the paragraph before Lemma 3.4. Suppose there is no $1 \le i \le s$ such that $\kappa(\mathfrak{q}_i') = \kappa(\mathfrak{p}')$. Then there is an element $\sigma \in G$ such that σ has no fixed point on any of $\mathcal{T}_i$.*

Proof By Lemma 3.4, $D_{\mathfrak{q}'}/I_{\mathfrak{q}'} \cong \mathrm{Gal}(\kappa(\mathfrak{q}')/\kappa(\mathfrak{p}'))$ is cyclic. Choose $\sigma \in G$ such that $\bar{\sigma} \in D_{\mathfrak{q}'}/I_{\mathfrak{q}'}$ is a generator of $D_{\mathfrak{q}'}/I_{\mathfrak{q}'}$. Assume by contradiction that some $t \in \mathcal{T}_i$ is fixed by σ and write $D_{\mathfrak{q}'} = \bigcup_{j=1}^m I_{\mathfrak{q}'}\sigma^j$ where $m = |D_{\mathfrak{q}'}/I_{\mathfrak{q}'}|$. Now $D_{\mathfrak{q}'}(t) = \bigcup_{j=1}^m I_{\mathfrak{q}'}\sigma^j(t) = I_{\mathfrak{q}'}(t)$. Hence by Lemma 3.4 this is a contradiction. ∎

Example 3.7 The converse of Corollary 3.6 is not true. Let $K = \mathbb{Q}$, $s = 1$, $\mathfrak{p}' = (2)$ and let $\mathrm{Frac}(R_1) = F = \mathbb{Q}(\sqrt{-1})$. Then the generator σ of G lies in $I_{\mathfrak{q}'}$, has no fixed point on $\mathcal{T}_1$, and yet $\kappa(\mathfrak{q}_1') = \kappa(\mathfrak{p}')$.

From now on we replace K by E as in the paragraph before Lemma 3.2. Our goal is to apply the Chebotarev density theorem to obtain infinitely many primes $\mathfrak{p}'$ satisfying Lemma 3.2. However, the prime $\mathfrak{p}$ we found might be ramified. So, *a priori*, we might need a different way of finding a conjugacy class of a certain Galois group to apply the Chebotarev density theorem. By Corollary 3.6, there is a unified way to treat the cases when the inertia group is identity or not.

Proof of Theorem 1.4 Use S as in Section 2. Fix a prime $\overline{\mathfrak{p}}$ of $\mathcal{O}_{\bar{K}}$ as in Lemma 3.2. Let F/K be obtained by composing the defining field of all $\beta \in X(K)$ such that the orbit of β hits the set $\{\gamma_1, \ldots, \gamma_r\}$ for the first time at the M-th iterate. Denote by B the set of all such β. Suppose $B = \{\beta_1, \ldots, \beta_s\}$. Suppose A_i and A are the smallest $\mathcal{O}_{S,K}$-algebras over which β_i and B are defined. By our assumption in Section 2, A_i and A are all finitely generated modules over $\mathcal{O}_{S,K}$. Then we have inclusions of rings $\mathcal{O}_{S,K} \subseteq A_i \subseteq A$. Let R be the integral closure of A in its field of fraction and let R_i be the integral closure of A_i in its field of fraction. Then the primes of R_i can be identified with the primes of $\mathrm{Frac}(R_i)$ whose intersection with K is not in S. See Figure 1 for the relationship among these rings and fields.

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By Lemma 3.2, there is no primes $\mathfrak{q}_i \in R_i$ lying over $\mathfrak{p}$ such that $\kappa(\mathfrak{q}_i) = \kappa(\mathfrak{p})$. Then we find a $\sigma \in G$ as in the proof of Corollary 3.6. Look at the nonzero prime ideals of K. By the Chebotarev density theorem, there exists a set of primes $\mathcal{P} \subseteq M_K$ of positive density, which consists of unramified primes $\mathfrak{p}'$ not in the finite set given by Lemma 3.3 such that the Frobenius conjugacy class $\mathrm{Fr}_{\mathfrak{p}'} \in \mathrm{Gal}(F/K)$ coincides with the conjugacy class of σ. In this case we have $I_{\mathfrak{p}'} = 1$. By the above paragraph and Lemma 3.4, for all such $\mathfrak{p}'$ there is no prime $\mathfrak{q}'_i \subseteq R_i$ lying over $\mathfrak{p}'$ such that $\kappa(\mathfrak{q}'_i) = \kappa(\mathfrak{p}')$. It follows that none of $\mathfrak{q}'_i$ has residue degree 1 over $\kappa(\mathfrak{p}')$. By Lemma 3.5, there is no $x_{\mathfrak{p}} \in X_{\mathfrak{p}}(\mathcal{O}/\mathfrak{p})$ whose orbit under $f_{\mathfrak{p}}$ hits $\gamma_{\mathfrak{p}}$ for the first time at the M-th iteration. On the other hand, if $x_{\mathfrak{p}}$ hits $(\gamma_i)_{\mathfrak{p}}$ for the first time at the m-th iteration with some $m > M$, then the forward image $f_{\mathfrak{p}}^{m-M}(x_{\mathfrak{p}})$ hits $(\gamma_i)_{\mathfrak{p}}$ for the first time at the M-th iteration. This is a contradiction as $\mathfrak{p} \in \mathcal{P}$.

Therefore, for all $m \geq M$ and for all $\mathfrak{p}' \in \mathcal{P}$ there is no $\eta_{\mathfrak{p}'} \in X_{\mathfrak{p}'}(\kappa(\mathfrak{p}'))$ such that $f_{\mathfrak{p}'}^m(\eta_{\mathfrak{p}'}) \in \{(\gamma_1)_{\mathfrak{p}'}, \dots, (\gamma_r)_{\mathfrak{p}'}\}$. ∎

4 Applying the Approximate p-adic Interpolation

We need the following theorem.

Theorem 4.1 ([8, Theorem 11.11.3.1]) *Let E be an N-by-N matrix with entries in $\mathfrak{o}_v$. Suppose $E^2 = E$. If $f \in \mathfrak{o}_v\langle x_1, \dots, x_N\rangle^N$ satisfies $f(x) \equiv Ex \pmod{p^c}$ for some $c > 1/(p-1)$, then there exists $g \in \mathfrak{o}_v\langle x_1, \dots, x_N, z\rangle^N$ such that $\|g(x,n) - f^n(x)\| \leq p^{-nc}$ for each $n \in \mathbb{Z}_{\geq 0}$.*

For each point $a \in \mathcal{X}(\mathfrak{o}_v)$, let $\mathcal{U}_{\bar{a}}$ be the set $\{\beta \in \mathcal{X}(\mathfrak{o}_v) \mid \beta_{\mathfrak{p}} = a_{\mathfrak{p}}\}$. By the argument in [6], the completed local ring $\hat{\mathcal{O}}_{\bar{a}}$ is isomorphic to a formal power series ring $\mathfrak{o}_v[[x_1, \dots, x_N]]$. For completeness, we include that proof here. Since $a \in \mathcal{X}$ is smooth, the quotient $\hat{\mathcal{O}}_{\bar{a}}/(\pi)$ is regular. By the Cohen structure theorem for regular local rings (see [9, Theorem 9] or [20, Theorem 29.7]), the quotient ring $\hat{\mathcal{O}}_{\bar{a}}/(\pi)$ is isomorphic to a formal power series ring of the form $\kappa_{\mathfrak{p}}[[y_1, \dots, y_N]]$. Choosing $x_i \in \hat{m}_v$ for $i = 1, \dots, N$ such that the residue class of each x_i is equal to y_i, we obtain a minimal basis $\pi, x_1, \dots, x_N$ for the maximal ideal $\hat{m}_v$ of $\hat{\mathcal{O}}_{\bar{a}}$ (see [9]).

By the basic theory of ideals, there is a one-to-one correspondence between the points in $\mathcal{X}(\mathfrak{o}_v)$ that reduces to $\bar{a}$ and the primes $\mathfrak{q}$ in $\mathcal{O}_{\bar{a}}$ such that $\mathcal{O}_{\bar{a}}/\mathfrak{q} \cong \mathfrak{o}_v$. Passing to completion, we have a one-to-one correspondence between the points in $\mathcal{X}(\mathfrak{o}_v)$ that reduces to $\bar{a}$ and the primes $\mathfrak{q}$ of $\hat{\mathcal{O}}_{\mathcal{X},\bar{a}}$ such that $\mathfrak{q}$ of $\hat{\mathcal{O}}_{\mathcal{X},\bar{a}}/\mathfrak{q} \cong \mathfrak{o}_v$.

Replacing f by an iterate and replacing a by a forward image under f, we may assume that f maps $\mathcal{U}_{\bar{a}}$ to itself. Then there is a $\mathfrak{p}$-adic analytic isomorphism $\iota\colon \mathcal{U}_{\bar{a}} \to \mathfrak{o}_v^N$, and the restriction of $f|_{\mathcal{U}_{\bar{a}}}$ can be conjugated to an analytic endomorphism defined over $\mathfrak{o}_v$. Denote by $F = (F_1, \dots, F_N)$ this function. Moreover, for each i we have

$$F_i(x_1, \dots, x_N) = \frac{1}{\pi} H_i(\pi x_1, \dots, \pi x_N)$$

with $H_i \in \mathfrak{o}_v[[x_1, \dots, x_N]]$. The advantage of conjugating by $(x_1, \dots, x_N) \mapsto (\pi x_1, \dots, \pi x_N)$ is that the coordinates of the initial point are mapped to the maximal ideal $\mathfrak{p}$. The reader might refer to [5, pp. 1658–1659] and [6, pp. 9–11] for more details.

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We need the following theorem, which is a modification to the analytic case of Theorem 4.1.

Proposition 4.2 *Let $p \geq 3$ be a prime number, and let $N \geq 2$ be an integer. Suppose Y be the N-dimensional unit disk $\mathfrak{o}_v^N$ over $K_\mathfrak{p}$ and suppose $F: Y \to Y$ is an algebraic endomorphism defined over $K_\mathfrak{p}$. Suppose $V' \subseteq Y$ is an analytic subvariety defined over $K_\mathfrak{p}$, and let $a \in Y(\mathfrak{o}_v)$ be a point. Assume the following conditions are verified:*

(i) *$a = (a_1, \dots, a_N)$ and each $a_i \in \mathfrak{p}$.*

(ii) *The endomorphism F is given by*

$$(x_1, \dots, x_N) \mapsto \big(F_1(x), \dots, F_N(x)\big)$$

for analytic functions $F_i \in \mathfrak{o}_v\langle x_1, \dots, x_n\rangle$, and furthermore for each $i = 1, \dots, N$ we have

$$F_i(x_1, \dots, x_n) \equiv \sum_{j=1}^{N} a_{ij} x_j \pmod{\mathfrak{p}}$$

with $a_{ij} \in \mathfrak{o}_v$, and the matrix $A := (a_{ij})$ satisfies that $A^2 = A$.

(iii) *Define $G: \mathfrak{o}_v \to \mathfrak{o}_v^N$ such that $G(n) = g(a, n)$ where g is the function defined in Theorem 4.1. There we have $\|G(n) - f^n(a)\| \leq p^{-nc}$. Assume that there is no $M \in \mathbb{Z}_{\geq 0}$ such that the function G satisfies that $G(n) \in V'$ for all $n \geq M$; note that such a G exists because of condition (ii).*

(iv) *For any $n \in \mathbb{Z}_{>0}$, the orbit $O_{F^n}(a)$ does not converge $\mathfrak{p}$-adically to a periodic point of F lying on V'.*

Then the set $S_{V'} := \{n \in \mathbb{Z}_{\geq 0} \mid F^n(a) \in V'(K_\mathfrak{p})\}$ has the property that

$$\#\{i \leq n \mid i \in S_{V'}\} = o\big(\log^{(m)}(n)\big)$$

for any fixed $m \in \mathbb{Z}_{>0}$ (where $\log^{(m)}$ is the m-th iterated logarithmic function).

Remark We need condition (iii) to apply Theorem 4.1. It can be satisfied under certain reductions. For more details, see the proof of Theorem 1.1.

Proof As in [8], we may assume that V' is irreducible, $S_{V'}$ is infinite, and the orbit $O_F(a)$ does not converge $\mathfrak{p}$-adically to a point in Y.

By Theorem 4.1 the function $G: \mathfrak{o}_v \to \mathfrak{o}_v^N$ satisfies that for each $n \in \mathbb{Z}_{\geq 0}$, we have

$$\|F^n(a) - G(n)\| \leq p^{-nc}$$

where $\|(x_1, \dots, x_n)\| = \max\{|x_1|_\mathfrak{p}, \dots, |x_n|_\mathfrak{p}\}$. We also have

$$F\big(G(n)\big) = G(n+1) \tag{4.1}$$

for all n. There are two cases.

Case 1. G is not constant. Consider the sets

$$S_{V',i} := \{n \in S_{V'} : n \equiv i \pmod{p^k}\}.$$

By condition (ii), there is no $M \in \mathbb{Z}_{\geq 0}$ such that for all $n \geq M$ we have $G(n) \in V(K_\mathfrak{p})$. It follows that there exists an Q vanishing on V such that $L := Q \circ G: \mathfrak{o}_v \to \mathfrak{o}_v$ is not identically zero. Since $Q \circ F^n(a) = 0$ for all $n \in S_{V,i}$, by equation (4.1) we then have

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$|L(n)|_{\mathfrak{p}} \leq p^{-nc}$. By the discreteness of the zeros of $\mathfrak{p}$-adic nonzero analytic functions we can cover $\mathfrak{o}_v$ by disks $\bar{D}(i, p^{-k})$ with finitely many $i \in \mathfrak{o}_v$ for some $k > 0$ such that there is at most one zero of L in each of these disks.

If there is no $\eta \in \bar{D}(i, p^{-k})$ with $L(\eta) = 0$, then $S_{V',i}$ is a finite set, as (4.1) implies that an accumulation point of $S_{V',i}$ will result in a zero of L in $S_{V',i}$. If $S_{V',i}$ is an infinite set, we list the elements in $S_{V',i}$ in increasing order as $\{n_j\}_{j\geq 1}$. Suppose $L(\eta) = 0$ with $\eta \in S_{V',i}$. Let d be the order of vanishing of L at η. Suppose

$$L(n) = a_d(n-\eta)^d + a_{d+1}(n-\eta)^{d+1} + \cdots.$$

Then $|L(n_j)| \leq p^{-n_j}$ implies that

$$|n_j - \eta|_{\mathfrak{p}} \leq p^{-n_j/d + O(1)}.$$

Suppose $S_{V',i} = \{n_1, n_2, \dots\}$. Then we also have

$$|n_{j+1} - \eta|_{\mathfrak{p}} \leq p^{-n_{j+1}/d + O(1)}.$$

Combining them, and by the triangle inequality, we have

$$|n_j - n_{j+1}|_{\mathfrak{p}} \leq p^{-n_j/d + O(1)}.$$

Hence there exists $C > 1$ such that for all sufficiently large j, we have $n_{j+1} - n_j \geq C^{n_j}$.

Case 2. G is constant. Suppose $G(n) = \beta$ identically. By (4.1) we know that $F^n(a)$ converges to β, and by equation (4.1), β is fixed under F. This contradicts condition (iv) above. ∎

Proof of Theorem 1.1 As in Section 2, replacing f by an iterate we may assume that all the periodic points $\gamma_1, \dots, \gamma_r$ are fixed points. Let $\mathcal{P}$ be as in Theorem 1.4. Fix a prime $\mathfrak{p} \in \mathcal{P}$. We may assume further that the reduction $a_{\mathfrak{p}}$ is fixed under $f_{\mathfrak{p}}$. Fix an analytic isomorphism $\iota: \mathcal{U}_{\hat{a}} \to Y$ where $\mathcal{U}_{\bar{a}} = \{\beta \in \mathcal{X}(\mathfrak{o}_v) : \bar{\beta} = \bar{a}\}$ as under the statement of Theorem 4.1 and Y is the N-dimensional unit disk $\mathfrak{o}_v^N$. Then we obtain a map $F: Y \to Y$ as in Proposition 4.2.

By affine changes of coordinate and replacing f by an iterate, we can assume that conditions (i) and (ii) of Proposition 4.2 are verified. More precisely, suppose

$$F_i(x) = b_0 + \sum_{j=1}^{N} b_j x_j + \sum_{|n| := n_1 + \cdots + n_N \geq 2} b_{n_1,\dots,n_N} x_1^{n_1} \cdots x_N^{n_N}$$

where we omit the subindex i. Look at the orbit of any $\eta \in \mathfrak{o}_v^N$ under F modulo π^2: $\overline{\eta}, \overline{f(\eta)}, \overline{f^2(\eta)}, \dots$. Replacing $\overline{\eta}$ by a forward image we may assume that $\overline{\eta}$ is periodic. Replacing F by an iterate we may assume that $\overline{\eta}$ is fixed. Choose any η such that its residue class modulo π^2 is $\overline{\eta}$. Under the assumptions above, if we do the translation $\sigma_1: x \mapsto x - \eta$, the constant terms of $\sigma_1^{-1} \circ F \circ \sigma_1$ will be divisible by π^2, and all coefficients of $\sigma_1^{-1} \circ F \circ \sigma_1$ will still belong to $\mathfrak{o}_v$. Replacing F by $\sigma_1^{-1} \circ F \circ \sigma_1$, we may assume that the constant terms of F are divisible by π^2 and all the coefficients of F belong to $\mathfrak{o}_v$.

After that we make a change a variable of the form

$$\sigma : (a_1, \dots, a_N) \mapsto (\pi a_1, \dots, \pi a_N),$$

and then the conjugated map is

$$\sigma^{-1} \circ F_i \circ \sigma(x) = \frac{b_0}{\pi} + \sum_{j=1}^{N} b_j x_j + \sum_{|n|=n_1+\cdots+n_N \geq 2} \pi^{|n|-1} \cdot b_{n_1,\ldots,n_N} x_1^{n_1} \cdots x_N^{n_N}.$$

This conjugation makes the terms of F with degree at least 2 divisible by π, and yet the constant term is still divisible by π. Now, again replacing F by an iterate, we may assume that the matrix A of the linear part of F satisfies $A^2 = A$. This enables us to apply Proposition 4.2.

Furthermore as we have observed, after the transformation

$$(a_1, \ldots, a_N) \mapsto (\pi a_1, \ldots, \pi a_N),$$

our configuration will meet the requirement of condition (i).

By Theorem 1.4 we know that $\mathfrak{p}$-adically any subsequence of $\{f^n(a)\}_{n \geq 0}$ is not convergent to any of the periodic points on V. It is equivalent to the statement that $F^n\big(\iota(a)\big)$ does not converge to any of the periodic points on V'. In other words, condition (iv) of Proposition 4.2 is satisfied. Condition (iii) is satisfied, as otherwise the Zariski closure of $\left\{\iota^{-1}\big(G(n)\big) \mid n \geq M\right\}$ would be a positive-dimensional periodic subvariety of V. Then we can apply Proposition 4.2 to conclude the requested gap principle. ■

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