The Journal of Symbolic Logic
Volume 80, Number 1, March 2015

COMPUTABILITY AND UNCOUNTABLE LINEAR ORDERS II: DEGREE SPECTRA

NOAM GREENBERG, ASHER M. KACH, STEFFEN LEMPP, AND DANIEL D. TURETSKY

Abstract. We study the computable structure theory of linear orders of size $\aleph_1$ within the framework of admissible computability theory. In particular, we study degree spectra and the successor relation.

§1. Introduction. This paper is the second part of [10], in which the study of the computable structure of uncountable linear orders was begun. This is part of a larger program of studying uncountable structures through admissible computability theory. We refer the reader to the previous paper for relevant background.

In Section 2, we study degree spectra of (order-types of) linear orderings of size $\aleph_1$. Jockusch and Soare [12] showed that there is a countable order-type having low presentations but no computable presentation. Various strengthenings of this result included the construction of R. Miller [14] of a countable linear ordering which has a copy in every nonzero Δ^0_2 Turing degree, but no computable copy; Downey later observed that in fact this ordering has a copy in every hyperimmune degree. In Theorem 2.7, we give an uncountable analogue of R. Miller's result.

In the countable context, Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon [9] showed that there are structures whose degree spectra consist of exactly the nonlow degrees; it is unknown if there is a countable linear ordering with this degree spectrum. In Theorem 2.18, we show that for any finite n, there is a linear ordering of size $\aleph_1$ whose degree spectrum is the collection of ω_1-nonlow$_n$ degrees. This again is a testament to the stronger (or at least easier) coding power vested in uncountable linear orderings.

In the same section, we also discuss finite jump degrees. As mentioned above, Richter [17] showed that the only degree of a countable order-type is $\mathbf{0}$. Knight [13] showed that the only jump degree of a countable order-type is $\mathbf{0}'$. However, Downey and Knight [4] (building on work of Ash, Jockusch, and Knight [1] and Ash and Knight [2]) showed that for all computable ordinals $\alpha \geq 2$, every degree $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$ is the proper α^{th} jump degree of a countable order-type. As mentioned above, Greenberg and Knight [11] showed that every ω_1-Turing degree is the degree of an

Received January 28, 2013.
2010 *Mathematics Subject Classification.* Primary: 03D60, secondary: 03C57.
Key words and phrases. uncountable linear orders, computability, computable categoricity.

0022-4812/15/8001-0007
DOI:10.1017/jsl.2014.69

order-type. We show in Theorem 2.21 that every ω_1-Turing degree $\mathbf{d} \geq \mathbf{0}^{(n)}$ is the proper n^{th} jump degree of an order-type. In Theorem 2.10, however, we show that the primary tool used by Downey and Knight for the countable case does not carry over to the ω_1-setting.

In Section 3 we study the complexity of the successor relation on a linear ordering. Recently, Downey, Lempp, and Wu [5] complemented work by Frolov [7] to show that for any ω-computable linear ordering $\mathcal{L}$, the collection of degrees of the successor relation in computable copies of $\mathcal{L}$ is upward closed in the c.e. degrees, as long as, of course, the order-type has infinitely many adjacent ordered pairs. For orderings of size $\aleph_1$, the situation is radically different. For example, in Example 3.2, we show that the successor relation can be intrinsically computable, that is, there is an ω_1-computable order-type λ such that the successor relation is computable in any computable presentation of λ. We identify a dichotomy between two kinds of linear orderings of size $\aleph_1$: Roughly speaking, between those which contain a copy of the rational numbers which demarcates the successivities of the linear ordering, and those which do not. The latter case behaves similarly to countable linear orderings in that the degrees of the successor relation in computable copies are upward closed in the c.e. degrees (Theorem 3.4). The other case is interesting; we identify an interval in the c.e. degrees which contains all the degrees of the successor relation in computable copies of the given linear ordering. The top and bottom degree in this interval are always realized as the degrees of the successor relation in some copy, but not all degrees in the interval need to be so realized (although they can be). As a corollary, we see that for any ω_1-c.e. degree $\mathbf{d}$, there is an ω_1-computable linear ordering $\mathcal{L}$ such that the degree of the successor relation in every ω_1-computable copy of $\mathcal{L}$ is $\mathbf{d}$.

1.1. Notation, Terminology, Background. Throughout this paper, we will always work under the assumption that all reals are constructible. We refer the reader to our previous paper for much of our notation. Here we mention only the new notions.

Definition 1.1. Let $\mathcal{A} = (A, <_{\mathcal{A}})$ be a linear ordering. If $B \subseteq A$, we let

$$\mathrm{dcl}(B) := \{b \in A : (\exists c \in B)[b \leq_{\mathcal{A}} c]\}$$

and

$$\mathrm{ucl}(B) := \{b \in A : (\exists c \in B)[b \geq_{\mathcal{A}} c]\}$$

be the downward closure and upward closure of B, respectively. When $\mathcal{A}$ is possibly ambiguous, we write $\mathrm{dcl}_{\mathcal{A}}(B)$ and $\mathrm{ucl}_{\mathcal{A}}(B)$, respectively.

We will make use of the linear orderings $\mathbb{Z}^\alpha$, where $\alpha \leq \omega_1$.

Definition 1.2. By recursion on ordinals α, we define a directed system of linear orderings and embeddings $\langle \mathbb{Z}^\alpha, \iota_{\beta,\alpha} \rangle$. We let $\mathbb{Z}^0 := 1$. Given $\mathbb{Z}^\alpha$, we let $\mathbb{Z}^{\alpha+1} := \mathbb{Z}^\alpha \cdot \mathbb{Z}$, and define $\iota_{\alpha,\alpha+1} : \mathbb{Z}^\alpha \to \mathbb{Z}^{\alpha+1}$ by letting $\iota_{\alpha,\alpha+1}(x) := (x, 0)$. In other words, $\mathbb{Z}^{\alpha+1}$ is obtained from $\mathbb{Z}^\alpha$ by adding ω many copies of $\mathbb{Z}^\alpha$ to the right, and ω^* many copies of $\mathbb{Z}^\alpha$ to the left. For $\beta < \alpha$, we let $\iota_{\beta,\alpha+1} := \iota_{\alpha,\alpha+1} \circ \iota_{\beta,\alpha}$. At limit stages δ, we let $\mathbb{Z}^\delta$ be the direct limit of the system $\langle \mathbb{Z}^\alpha, \iota_{\beta,\alpha} \rangle_{\beta<\alpha<\delta}$, and the maps $\iota_{\beta,\delta}$ be the limit of the maps $\langle \iota_{\beta,\alpha} \rangle_{\beta<\alpha<\delta}$.

By induction, it is easy to see that every map $\iota_{\beta,\alpha}$ is a *convex* embedding of $\mathbb{Z}^\beta$ into $\mathbb{Z}^\alpha$ (i.e., its image is convex), that each $\mathbb{Z}^\alpha$ is discrete, and that the maximal blocks in each $\mathbb{Z}^\alpha$ (for $\alpha > 0$) are all infinite.

LEMMA 1.3. *Let* $\alpha \le \omega_1$.

(1) $(\mathbb{Z}^\alpha)^* \cong \mathbb{Z}^\alpha$.
(2) *There is no embedding of* $\mathbb{Z}^\alpha$ *into a proper initial segment of itself; so, a fortiori, if* $\gamma < \beta \le \omega_1$, *then there is no embedding of* $\mathbb{Z}^\beta$ *into* $\mathbb{Z}^\gamma$.

PROOF. (1) is proved by induction on α, taking direct limits on both sides at limit stages.

(2) is proved by induction on α. Suppose this is known for α. Suppose that there is an embedding of $\mathbb{Z}^{\alpha+1}$ into a proper initial segment of itself. Then there is an embedding $f : \mathbb{Z}^{\alpha+1} \to \mathbb{Z}^\alpha \cdot \omega^*$. By taking a rightmost copy of $\mathbb{Z}^\alpha$ in $\mathbb{Z}^\alpha \cdot \omega^*$ intersecting the range of f, we get an embedding of $\mathbb{Z}^\alpha \cdot \omega$ into $\mathbb{Z}^\alpha$, contradicting the induction assumption for $\mathbb{Z}^\alpha$.

Let α be a limit ordinal, suppose that the lemma is verified for all $\beta < \alpha$, and suppose that f is an embedding of $\mathbb{Z}^\alpha$ into a proper initial segment of itself. Since $\mathbb{Z}^\alpha = \bigcup_{\beta<\alpha} \iota_{\beta,\alpha}\left[\mathbb{Z}^\beta\right]$, and since each embedding $\iota_{\beta,\alpha}$ is convex, there is a nonempty final segment of $\mathbb{Z}^\alpha$ whose image under f is contained in $\iota_{\beta,\alpha}[\mathbb{Z}^\beta]$ for some $\beta < \alpha$. This allows us to find an embedding of $\mathbb{Z}^{\beta+1}$ into $\mathbb{Z}^\beta$, again contradicting the induction assumption. ⊣

We also use shuffle sums of linear orders. We recall that in the countable setting, an η_0-shuffle sum of a countable collection of linear orders $\{\mathcal{L}_i\}_{i\in I}$ (denoted $\sigma_0(\{\mathcal{L}_i\}_{i\in I})$) is the linear order obtained by partitioning η_0 into $|I|$ many dense, codense sets and replacing each point in the ith set by a copy of $\mathcal{L}_i$.

DEFINITION 1.4. Let $\mathbb{Q}_1 \in \eta_1$, that is, let $\mathbb{Q}_1$ be a saturated linear ordering of size $\aleph_1$. A set $Z \subseteq \mathbb{Q}_1$ is *saturated in* $\mathbb{Q}_1$ if for all countable $A, B \subset \mathbb{Q}_1$, the interval $(A, B)_{\mathbb{Q}_1} \cap Z$ is nonempty. A standard construction shows that for any cardinal $\kappa \le \aleph_1$, there is a partition of $\mathbb{Q}_1$ into sets $\langle Z_\alpha \rangle_{\alpha<\kappa}$, each of which is saturated in $\mathbb{Q}_1$.

Let $\kappa \le \aleph_1$ be a cardinal and let $\langle \mathcal{L}_\alpha \rangle_{\alpha<\kappa}$ be a sequence of linear orderings. The η_1-*shuffle sum* of this sequence is obtained by replacing each point in Z_α by $\mathcal{L}_\alpha$. A back-and-forth argument shows that the order-type of the shuffle sum does not depend on the choice of the sets Z_α, nor does it depend on the ordering of the sequence $\langle \mathcal{L}_\alpha \rangle_{\alpha<\kappa}$. We can thus unambiguously define, for a *set* Λ of order-types such that $|\Lambda| \le \aleph_1$, the order-type $\sigma_1(\Lambda)$ of the shuffle sum of the order-types in Λ.

Finally, we list results of ω-computability theory and ω-computable structure theory (stated in the ω_1-framework) which also hold in the ω_1-framework, with similar or easier proofs.

FACT 1.5.

(1) *There is an* ω_1*-computable bijection between* ω_1 *and the universe* H_{ω_1}. *This bijection induces an* ω_1*-computable ordering of* H_{ω_1} *of order-type* ω_1, *denoted by* $<_{\omega_1}$.
(2) *There is a uniformly* ω_1*-computable list* $\langle \mathcal{L}_\beta \rangle_{\beta<\omega_1}$ *of* ω_1*-computable linear orderings such that for any* ω_1*-computable linear ordering* $\mathcal{A}$ *there is some* $\beta < \omega_1$ *such that* $\mathcal{A} \cong \mathcal{L}_\beta$.

(3) *For any ω_1-degrees $\mathbf{b}' \leq \mathbf{d}$, there is an ω_1-degree $\mathbf{a} > \mathbf{b}$ such that $\mathbf{a}' = \mathbf{d}$. In fact, there are incomparable ω_1-degrees $\mathbf{a}_1$ and $\mathbf{a}_2$ such that $\mathbf{a}_1' = \mathbf{d} = \mathbf{a}_2'$. Hence, there are non-$\omega_1$-computable low degrees.*

(4) *For any $n < \omega$ and any ω_1-degree $\mathbf{d} \geq \mathbf{0}^{(n)}$, there is an ω_1-degree $\mathbf{a}$ such that $\mathbf{a}^{(n)} = \mathbf{d}$. Moreover, provided $\mathbf{d} > \mathbf{0}^{(n)}$, for every ω_1-degree $\mathbf{a}_1$ with $\mathbf{d} = \mathbf{a}_1^{(n)}$, there is an ω_1-degree $\mathbf{a}_2$ with $\mathbf{d} = \mathbf{a}_2^{(n)}$ and $\mathbf{a}_1^{(m)} \mid \mathbf{a}_2^{(m)}$ for any $m < n$.*

§2. Degree Spectra of Linear Orderings. In this section, we exhibit an order-type whose degree spectrum includes all hyperimmune ω_1-degrees but omits $\mathbf{0}$ (Subsection 2.1); a transfer theorem for all order-types (Subsection 2.2); for every finite n, an order-type whose degree spectrum is precisely the collection of nonlow$_n$ ω_1-degrees (Subsection 2.3); and for each degree $\mathbf{d} \geq \mathbf{0}'$, an order-type of proper jump degree $\mathbf{d}$ (Subsection 2.4).

We recall the definition of the degree spectrum of an order-type.

Definition 2.1. For an order-type λ of size at most $\aleph_1$, we let $\mathrm{DegSpec}(\lambda)$, the *degree spectrum* of λ, be the collection of ω_1-Turing degrees of presentations of λ.

In this paper we assume that the universe of any linear ordering is a subset of H_{ω_1}, and so every linear ordering indeed has a Turing degree.

We abuse notation slightly by writing $\mathrm{DegSpec}(\mathcal{L})$ for $\mathrm{DegSpec}(\mathrm{otp}(\mathcal{L}))$ for a linear ordering $\mathcal{L}$ of size at most $\aleph_1$.

A theorem of Knight [13] generalizes to the ω_1-context; for any order-type λ of size $\aleph_1$, an ω_1-Turing degree $\mathbf{d}$ is in the degree spectrum of λ if and only if it computes a presentation of λ.

2.1. A Hyperimmune Spectrum. As mentioned above, R. Miller [14] demonstrated the existence of a countable, non-ω-computable order-type that has a presentation in every nonzero Δ^0_2 ω-degree. Miller built an order-type λ of the form $\sum_{i\in\omega}(\sigma_i + \kappa_i)$, where $\sigma_i = 1 + \eta + i + \eta + 1$ and κ_i was either ω or $c_i + \zeta$ for some $c_i < \omega$.

The purpose of the *separators* σ_i (the idea of which originates in [12]) was to divide λ into countably many intervals; the purpose of the *diagonalizers* κ_i was to diagonalize against the i^{th} computable linear order.

An inspection of Miller's proof shows that the linear ordering he constructed has a copy in every hyperimmune ω-degree. Recall that Rice [16] and Uspenskii [20] showed an ω-Turing degree is *hyperimmune* if and only if it computes a total function $f : \omega \to \omega$ such that for any total ω-computable function $g : \omega \to \omega$ there are infinitely many numbers n such that $f(n) > g(n)$.

Beyond ω, Chong and Wang [3] studied hyperimmune and hyperimmune-free α-degrees for various admissible ordinals α. Under our assumption that all reals are constructible, every subset of ω_1 is amenable and admissible (we refer the reader to [18] for these terms). Under these conditions, Chong and Wang give a straightforward generalization of the countable concept: an ω_1-Turing degree $\mathbf{a}$ is *hyperimmune* if and only if it contains a set A such that for every computable list $\langle F_\alpha \rangle$ of pairwise disjoint countable subsets of ω_1, there is some $\alpha < \omega_1$ such that $F_\alpha \cap A \neq \emptyset$. Chong and Wang show that an ω_1-Turing degree is hyperimmune if and only if it computes a total function $f : \omega_1 \to \omega_1$ such that for any total

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

ω_1-computable function $g : \omega_1 \to \omega_1$ there are uncountably many ordinals β such that $f(\beta) > g(\beta)$.

We build a linear order which has no ω_1-computable copy, but whose degree spectrum contains every hyperimmune ω_1-degree. This ordering $\mathcal{L}$ will be of the form $\sum_{\beta \in \omega_1} (\mathcal{S}_\beta + \mathcal{K}_\beta)$. The orderings $\mathcal{S}_\beta$ will serve as *separators*, denoting the location of the *diagonalizers* $\mathcal{K}_\beta$. We first discuss these building blocks of $\mathcal{L}$, and then give the construction defining $\mathcal{L}$.

Definition 2.2. Fix an enumeration $\langle q_i \rangle_{i<\omega}$ of the rational numbers $\mathbb{Q}$ and a computable enumeration $\langle r_\alpha \rangle_{\alpha<\omega_1}$ of the irrational numbers $\mathbb{I}$. We let $\mathcal{S}_\beta$ be obtained from $\mathbb{R}$ by omitting all irrational numbers but r_β, and by replacing the rational number q_i by $i + 2$ many points.

Formally, for $r \in \mathbb{R}$, we define

$$\mathcal{C}_{r,\beta} := \begin{cases} 1 & \text{if } r = r_\beta, \\ i+2 & \text{if } r = q_i, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\mathcal{S}_\beta := \sum_{r \in \mathbb{R}} \mathcal{C}_{r,\beta}$.

Each linear ordering $\mathcal{S}_\beta$ is countable, and the map $\beta \mapsto \mathcal{S}_\beta$ is computable.

Lemma 2.3. *Let $\beta, \gamma < \omega_1$ be distinct.*

(1) *The linear order $\mathcal{S}_\beta$ is not isomorphic to any proper convex subset of itself.*
(2) *The linear order $\mathcal{S}_\beta$ is not isomorphic to any convex subset of $\mathcal{S}_\gamma$.*

Proof. The point is that for all $i < \omega$ and all $\beta < \omega_1$, the suborder $\mathcal{C}_{q_i,\beta}$ is the unique maximal block of $\mathcal{S}_\beta$ of size $i + 2$. Hence if $f : \mathcal{S}_\beta \to \mathcal{S}_\gamma$ is a convex embedding, then for all $i < \omega$ it must be that $f[\mathcal{C}_{q_i,\beta}]$ equals $\mathcal{C}_{q_i,\gamma}$. This implies that the range of f is $\mathcal{S}_\gamma$, and so also that $\beta = \gamma$. ⊣

The diagonalizers $\mathcal{K}_\beta$ are built as sums of the linear orders $\mathbb{Z}^\alpha$ for $\alpha < \omega_1$. For $\beta \leq \omega_1$, we let $\mathcal{A}_\beta := \sum_{\alpha<\beta} \mathbb{Z}^\alpha$ and $\mathcal{B}_\beta := (\mathcal{A}_\beta)^*$; the latter is isomorphic to $\sum_{\alpha \in \beta^*} \mathbb{Z}^\alpha$ (with an abuse of notation). For $\beta < \gamma \leq \omega_1$, let $j_{\beta,\gamma}$ be the canonical initial segment embedding of $\mathcal{A}_\beta$ into $\mathcal{A}_\gamma$.

Lemma 2.4. *Let $\beta \leq \omega_1$.*

(1) *There is no embedding of $\mathcal{A}_\beta$ into a proper initial segment of itself.*
(2) *If β is a limit ordinal, then there is no proper initial segment of $\mathcal{A}_\beta$ into which there is an embedding of $\mathcal{A}_\gamma$ for all $\gamma < \beta$.*

Proof. Both parts follow from Lemma 1.3(2). ⊣

It follows that if a linear order $\mathcal{L}$ is isomorphic to the sum $\mathcal{A}_\alpha + \mathcal{B}_\beta$ for some ordinals α and β, then there is a unique decomposition of $\mathcal{L}$ as a sum of linear orderings $\mathcal{L}_1 + \mathcal{L}_2$ such that $\mathcal{L}_1 \cong \mathcal{A}_\alpha$ and $\mathcal{L}_2 \cong \mathcal{B}_\beta$.

Lemma 2.5. *Let $\beta < \omega_1$.*

(1) *For any limit ordinal $\delta \leq \omega_1$, the order $\mathcal{A}_\delta$ is isomorphic to the direct limit of the directed system $\langle \mathcal{A}_\beta, j_{\beta,\gamma} \rangle_{\beta<\gamma<\delta}$.*
(2) *For any nonempty initial segment $\mathcal{C}$ of $\mathcal{B}_{\omega_1}$, there is an embedding of $\mathcal{A}_{\beta+1}$ into $\mathcal{A}_\beta + \mathcal{C}$.*
(3) *There is an embedding of $\mathcal{A}_\beta + \mathcal{B}_\beta$ into $\mathcal{A}_{\beta+1}$ extending $j_{\beta,\beta+1}$.*

PROOF. (1) is immediate. For (2), it suffices to show that for all β, the order $\mathbb{Z}^\beta$ is embeddable in $\mathcal{C}$, which is immediate.

For (3), it suffices to show that there is an embedding of $\mathcal{B}_\beta$ into $\mathbb{Z}^\beta$. This is proved by induction. Suppose that f_β is an embedding of $\mathcal{B}_\beta$ into $\mathbb{Z}^\beta$. As $\mathcal{B}_{\beta+1} \cong \mathbb{Z}^\beta + \mathcal{B}_\beta$, we can extend f_β to an embedding of $\mathcal{B}_{\beta+1}$ into $\mathbb{Z}^\beta \cdot 2$, and hence into $\mathbb{Z}^{\beta+1}$. For a limit ordinal β, let $\langle \beta_n \rangle_{n<\omega}$ be an increasing and cofinal sequence in β; for $n < \omega$, let f_{β_n} be an embedding of $\mathcal{B}_{\beta_n}$ into $\mathbb{Z}^{\beta_n}$. If $j^*_{\beta_n,\beta_{n+1}}$ is the canonical final segment embedding of $\mathcal{B}_{\beta_n}$ into $\mathcal{B}_{\beta_{n+1}}$ (the analogue of $j_{\beta_n,\beta_{n+1}}$), we can inductively construct embeddings $g_{\beta_n} : \mathcal{B}_{\beta_n} \to \mathbb{Z}^{\beta_n+1}$ so that $g_{\beta_{n+1}} \circ j_{\beta_n,\beta_{n+1}}$ agrees with g_{β_n}. The limit of these maps is then an embedding of $\mathcal{B}_\beta$ into $\mathbb{Z}^\beta$. ⊣

Our separators and building blocks do not interact:

LEMMA 2.6. *For all $\alpha, \beta < \omega_1$, no nonempty initial or final segment of $\mathcal{S}_\beta$ is isomorphic to any convex subset of $\mathcal{A}_\alpha$ or $\mathcal{B}_\alpha$.*

PROOF. For any $\alpha > 0$, every maximal block in $\mathbb{Z}^\alpha$ is infinite, whereas $\mathcal{S}_\beta$ contains no infinite blocks. Hence for any α, no nonempty initial or final segment of $\mathcal{S}_\beta$ is isomorphic to any convex subset of $\mathbb{Z}^\alpha$. The lemma follows. ⊣

We are now ready to prove the main result of this subsection. We note that the construction below only relies on the properties of the orderings $\mathcal{S}_\beta$, $\mathcal{A}_\beta$, and $\mathcal{B}_\beta$ detailed in Lemma 2.3, Lemma 2.4, Lemma 2.5, and Lemma 2.6. In a sense, this is a modular approach to the construction, which we believe sheds light on Miller's construction as well.

THEOREM 2.7. *There is a linear ordering $\mathcal{L}$ of size $\aleph_1$ such that* DegSpec($\mathcal{L}$) *contains every hyperimmune ω_1-degree, but does not contain* **0**.

PROOF. The linear order $\mathcal{L}$ we construct will be $\sum_{\beta\in\omega_1} \left(\mathcal{S}_\beta + \mathcal{K}_\beta\right)$, where $\mathcal{K}_\beta$ is either $\mathcal{A}_{\omega_1}$ or $\mathcal{A}_\alpha + \mathcal{B}_{\omega_1}$ for some countable ordinal α. By Fact 1.5(2), we fix a sequence $\{\mathcal{L}_\beta\}_{\beta\in\omega_1}$ of all computable linear orderings. The purpose of $\mathcal{K}_\beta$ is to diagonalize against $\mathcal{L}_\beta$.

Lemma 2.6 implies that for all $\beta < \omega_1$ and $\gamma < \omega_1$, no nonempty initial or final segment of $\mathcal{S}_\beta$ is isomorphic to a convex subset of $\mathcal{K}_\gamma$. Lemma 2.3 and Lemma 2.6 now guarantee that if built according to our plan, for all $\beta < \omega_1$, there is a unique convex subset of $\mathcal{L}$ isomorphic to $\mathcal{S}_\beta$. We identify $\mathcal{S}_\beta$ with that convex subset of $\mathcal{L}$.

Construction: For each $\beta < \omega_1$, we need to determine the largest ordinal $\alpha = \alpha(\beta) \leq \omega_1$ such that $\mathcal{A}_\alpha$ should be an initial segment of $\mathcal{K}_\beta$. If $\alpha = \omega_1$ then $\mathcal{K}_\beta := \mathcal{A}_{\omega_1}$, and if $\alpha < \omega_1$ then $\mathcal{K}_\beta := \mathcal{A}_\alpha + \mathcal{B}_{\omega_1}$. The choice of $\alpha(\beta)$, of course, will not be done effectively since we want to ensure that otp($\mathcal{L}$) is not computable. However, we need to make this choice "as computably as possible" so that any sufficiently fast-growing function does have the ability to compute, uniformly in β, a copy of $\mathcal{K}_\beta$.

The choice of each $\alpha(\beta)$ is made independently, based only on $\mathcal{L}_\beta$. If $\mathcal{L}_\beta$ were to be isomorphic to $\mathcal{L}$, then $\mathcal{L}_\beta$ would have a unique convex subset $S = S(\beta)$ isomorphic to $\mathcal{S}_\beta$, a unique convex subset $T = T(\beta)$ isomorphic to $\mathcal{S}_{\beta+1}$, and would have $S(\beta) <_{\mathcal{L}_\beta} T(\beta)$. Furthermore, any isomorphism between $\mathcal{L}_\beta$ and $\mathcal{L}$ would have to extend the isomorphisms between S and $\mathcal{S}_\beta$, and T and $\mathcal{S}_{\beta+1}$; so the isomorphism would map $(S, T)_{\mathcal{L}_\beta}$ onto $\mathcal{K}_\beta$. Since S and T are countable, both subsets would be enumerated into $\mathcal{L}_\beta$ in their entirety by some countable stage.

Thus, at each stage $s < \omega_1$, we let $(S_s(\beta), T_s(\beta))$ be the $<_{\omega_1}$-least pair of convex subsets of $\mathcal{L}_\beta \upharpoonright s$ such that $S_s(\beta)$ is seen (at stage s) to be isomorphic to $\mathcal{S}_\beta$, $T_s(\beta)$ is seen (at stage s) to be isomorphic to $\mathcal{S}_{\beta+1}$, and $S_s(\beta) <_{\mathcal{L}_\beta} T_s(\beta)$, if such a pair exists. We then let $\mathcal{I}_s(\beta) = (S_s(\beta), T_s(\beta))_{\mathcal{L}_\beta}$ be the $\mathcal{L}_\beta$-interval (*not* the $(\mathcal{L}_\beta \upharpoonright s)$-interval) determined by these subsets. The plan is to ensure that if $\mathcal{I}_s(\beta)$ stabilizes, then it is not isomorphic to $\mathcal{K}_\beta$. If such subsets $S_s(\beta)$ and $T_s(\beta)$ are not found, then $\mathcal{I}_s(\beta)$ is undefined.

We describe how to define $\alpha_s = \alpha_s(\beta)$, our stage s approximation for the ordinal $\alpha(\beta)$. This approximation will be nondecreasing and continuous. The sequences $\langle \mathcal{I}_s(\beta) \rangle_{s<\omega_1}$ and $\langle \alpha_s(\beta) \rangle_{s<\omega_1}$ will be ω_1-computable, uniformly in β.

We try to pick a point $x_s = x_s(\beta) \in \mathcal{I}_s(\beta)$ which will aid our diagonalization efforts. Once picked, we only change our choice of point if the ambient interval $\mathcal{I}_s(\beta)$ changes. That is:

- If $s = t + 1$ is a successor stage, $\mathcal{I}_s(\beta) = \mathcal{I}_t(\beta)$ are both defined, and x_t is defined, then we let $x_s := x_t$;
- If s is a limit stage, there is some $t < s$ such that for all stages $r \in [t, s)$, $\mathcal{I}_r(\beta) = \mathcal{I}_s(\beta)$ are all defined, and x_t is defined, then $x_s := x_t$.

If $\mathcal{I}_s(\beta)$ is defined, but x_s is not yet defined by the previous clause, and there is some $x \in \mathcal{I}_s(\beta) \upharpoonright s$ such that $\mathcal{A}_{\alpha_s}$ is seen, at stage s, to be embeddable into $(-\infty, x)_{\mathcal{I}_s(\beta)}$, then we let x_s be the $<_{\omega_1}$-least such x; if there is no such x, then we leave x_s undefined.

If x_s is undefined, then we let $\alpha_{s+1} := \alpha_s$. If x_s is defined, then we let α_{s+1} be the supremum of the ordinals $\alpha < \omega_1$ such that at stage s, $\mathcal{A}_\alpha$ is seen to be embeddable into $(-\infty, x_s)_{\mathcal{I}_s(\beta)}$. By induction on s, we can easily see that if x_s is defined, then $\mathcal{A}_{\alpha+1}$ is embeddable into $(-\infty, x_s)_{\mathcal{I}_s(\beta)}$ for all $\alpha < \alpha_s$, and so $\alpha_{s+1} \geq \alpha_s$.

We let $\alpha(\beta) = \alpha_{\omega_1}(\beta) := \sup_{s<\omega_1} \alpha_s(\beta)$. This determines $\mathcal{K}_\beta$, and so completes the definition of the linear ordering $\mathcal{L}$.

Verification: Before we formally show that $\mathcal{L}$ is not isomorphic to $\mathcal{L}_\beta$ for any $\beta < \omega_1$, and so that $\mathbf{0} \notin \mathrm{DegSpec}(\mathcal{L})$, we explain what goes wrong if we follow a naive strategy for computing a copy of $\mathcal{L}$. For $s < \omega_1$, we let $\mathcal{K}_{\beta,s} = \mathcal{A}_{\alpha_s(\beta)} + \mathcal{B}_s$. Suppose that, uniformly in β, we want to enumerate a direct system of embeddings $f_{s,t} \colon \mathcal{K}_{\beta,s} \to \mathcal{K}_{\beta,t}$, whose direct limit will be $\mathcal{K}_\beta$. If $\alpha_{s+1}(\beta) = \alpha_s(\beta)$, then we add a copy of $\mathbb{Z}^s$ between $\mathcal{A}_{\alpha_s(\beta)}$ to $\mathcal{B}_s$ to get a copy of $\mathcal{K}_{\beta,s+1}$; in other words, $f_{s,s+1}$ is the "disjoint union" of $j_{\alpha_s,\alpha_{s+1}}$ and $j^*_{s,s+1}$. If $\alpha_{s+1}(\beta) > \alpha_s(\beta)$, then we want to "swallow" $\mathcal{K}_{\beta,s}$ in $\mathcal{A}_{\alpha_{s+1}(\beta)}$, and then add a copy of $\mathcal{B}_s$ to the right; in other words, we want $f_{s,s+1}$ to be an embedding of $\mathcal{K}_{\beta,s}$ in $\mathcal{A}_{\alpha_{s+1}}$ extending $j_{\alpha_s,\alpha_{s+1}}$. The swallowing is necessary so that if $\alpha(\beta) = \omega_1$, then all copies of $\mathcal{B}_s$ disappear into copies of greater $\mathcal{A}_\alpha$'s and at the end we would get $\mathcal{K}_\beta = \mathcal{A}_{\omega_1}$. The problem is that Lemma 2.5 (3) only ensures that $\mathcal{K}_{\beta,s}$ is embeddable in a copy of $\mathcal{A}_{s+1}$, and it may be that $\alpha_{s+1}(\beta)$, while greater than $\alpha_s(\beta)$, is still smaller that $s + 1$, and so $\mathcal{A}_{\alpha_{s+1}}$ is not large enough to swallow $\mathcal{K}_{\beta,s}$. This failure can be translated into a proof that $\mathcal{L}$ has no computable copy, and modified (by looking sufficiently far into the future) into a construction showing that any hyperimmune degree can compute a copy of $\mathcal{L}$.

Noncomputability: We now show that for each $\beta \in \omega_1$, we have $\mathcal{L} \ncong \mathcal{L}_\beta$, and so $\mathbf{0} \notin \mathrm{DegSpec}(\mathcal{L})$. Let $\beta < \omega_1$, and for a contradiction suppose that $f : \mathcal{L}_\beta \to \mathcal{L}$ is an isomorphism.

Let $S := S(\beta) := f^{-1}\mathcal{S}_\beta$ and $T := T(\beta) := f^{-1}\mathcal{S}_{\beta+1}$. As already noted, this implies $S <_{\mathcal{L}_\beta} T$, the set S is the unique convex subset of $\mathcal{L}_\beta$ isomorphic to $\mathcal{S}_\beta$, and T is the unique convex subset of $\mathcal{L}_\beta$ isomorphic to $\mathcal{S}_{\beta+1}$. Hence, for every pair (S', T') of subsets of $\mathcal{L}_\beta$ which precede (S, T) in the canonical ordering $<_{\omega_1}$ of H_{ω_1} such that $S' <_{\mathcal{L}_\beta} T'$, $S' \cong \mathcal{S}_\beta$ and $T' \cong \mathcal{S}_{\beta+1}$, either S' is not a convex subset of $\mathcal{L}_\beta$, or T' is not a convex subset of $\mathcal{L}_\beta$. It follows that for each pair $(S', T') <_{\omega_1} (S, T)$ there is some stage $s < \omega_1$ such that for all $t \geq s$, $(S', T') \neq (S_t(\beta), T_t(\beta))$. Since ω_1 is regular, for all but countably many stages s, we have $S_s(\beta) = S$ and $T_s(\beta) = T$. Let s_0 be the least stage such that for all $s \geq s_0$, $(S_s(\beta), T_s(\beta)) = (S, T)$. Let $\mathcal{I} = (S, T)_{\mathcal{L}_\beta} = f^{-1}\mathcal{K}_\beta$; then for all $s \geq s_0$, $\mathcal{I}_s(\beta) = \mathcal{I}$. We show that there is some stage $s \geq s_0$ at which $x_s(\beta)$ is defined. For the sake of a contradiction, suppose that for no $s \geq s_0$ is $x_s(\beta)$ defined. Then for all $s \geq s_0$, $\alpha_s(\beta) = \alpha_{s_0}(\beta)$, and so $\alpha(\beta) = \alpha_{s_0}(\beta)$, and $\mathcal{K}_\beta = \mathcal{A}_{\alpha_{s_0}(\beta)} + \mathcal{B}_{\omega_1}$. But then $f^{-1} \restriction \mathcal{A}_{\alpha(\beta)}$ is an embedding of $\mathcal{A}_{\alpha(\beta)}$ into a proper initial segment of $\mathcal{I}$. This embedding is discovered at some countable stage, at which we would define $x_s(\beta)$.

So let $s_1 \geq s_0$ be the least stage $s \geq s_0$ at which $x_s(\beta)$ is defined. Let $x = x_{s_1}(\beta)$; then for all $s \geq s_1$, we have $x_s(\beta) = x$. The definition of $\alpha(\beta)$ implies that $\alpha(\beta)$ is the supremum of the ordinals α such that $\mathcal{A}_\alpha$ is embeddable into $\mathcal{I}(< x)$.

Now either $f(x) \in \mathcal{A}_{\alpha(\beta)}$ or $f(x) \in \mathcal{B}_{\omega_1}$; in either case, we reach a contradiction. If $f(x) \in \mathcal{B}_{\omega_1}$, then $\alpha(\beta) < \omega_1$; but by Lemma 2.5 (2), there is an embedding of $\mathcal{A}_{\alpha(\beta)+1}$ into $\mathcal{A}_{\alpha(\beta)} + \mathcal{B}_{\omega_1}(< f(x))$, and so into $\mathcal{I}(< x)$, contradicting the definition of $\alpha(\beta)$.

On the other hand, suppose that $f(x) \in \mathcal{A}_{\alpha(\beta)}$. If $\alpha(\beta)$ is a successor ordinal, then by definition of $\alpha(\beta)$, there is an embedding g of $\mathcal{A}_{\alpha(\beta)}$ into $\mathcal{I}(< x)$. Composing g with f gives an embedding of $\mathcal{A}_{\alpha(\beta)}$ into a proper initial segment of $\mathcal{A}_{\alpha(\beta)}$, which is impossible by Lemma 2.4 (1). If $\alpha(\beta)$ is a limit ordinal, then the same argument shows that for all $\gamma < \alpha(\beta)$, there is an embedding of $\mathcal{A}_\gamma$ into the proper initial segment $\mathcal{A}_{\alpha(\beta)}(< f(x))$, which is impossible by Lemma 2.4 (2).

Hyperimmune Degrees: Let $g : \omega_1 \to \omega_1$ be a function such that for any computable function $f : \omega_1 \to \omega_1$, there are uncountably many ordinals $\beta < \omega_1$ such that $g(\beta) > f(\beta)$. We show that g can compute, uniformly in $\beta < \omega_1$, a copy of $\mathcal{K}_\beta$. Hence $\mathrm{DegSpec}(\mathcal{L})$ contains every hyperimmune degree.

Fix $\beta < \omega_1$; we omit the argument β and so write α_s for $\alpha_s(\beta)$, etc. We may assume that for all s, $g(s) > s$.

We define a g-computable closed unbounded subset I of ω_1. For $s \in I$, we let $\mathcal{K}_{\beta,s} = \mathcal{A}_{\alpha_s} + \mathcal{B}_s$. We define a g-computable system of embeddings $f_{t,s} : \mathcal{K}_{\beta,t} \to \mathcal{K}_{\beta,s}$ for $t < s$ in I, where, of course, if $t < r < s$ are in I then $f_{t,s} = f_{t,r} \circ f_{r,s}$. We ensure that for $t < s$ in I, $f_{t,s} \restriction \mathcal{A}_{\alpha_t} = j_{\alpha_t,\alpha_s}$. If $\mathcal{K}_\beta = \mathcal{A}_\gamma + \mathcal{B}_{\omega_1}$ for some γ, then we will also ensure that $f_{t,s} \restriction \mathcal{B}_{\alpha_t} = j^*_{s,s+1}$ for all $t \geq t_0$, for some t_0.

Let $s < \omega_1$, and suppose that we have already determined that $s \in I$, and that we have defined $f_{t,r}$ for $t < r \leq s$ in I. Now there are two possibilities:

- If $\alpha_{g(s)} > s$, then as $\alpha_s \leq s$, Lemma 2.5 (3) ensures that there is an embedding $f_{s,g(s)}$ of $\mathcal{K}_{\beta,s}$ into $\mathcal{A}_{\alpha_{g(s)}}$ extending $j_{\alpha_s,\alpha_{g(s)}}$. We let the next element of I after s be $g(s)$.
- If $\alpha_{g(s)} \leq s$, we let $s+1$ be in I. We let $f_{s,s+1} = j_{\alpha_s,\alpha_{s+1}} + j^*_{s,s+1}$. That is, $f_{s,s+1}$ embeds $\mathcal{A}_{\alpha_s}$ into $\mathcal{A}_{\alpha_{s+1}}$ and $\mathcal{B}_s$ into $\mathcal{B}_{s+1}$ canonically; and so $\mathcal{K}_{\beta,s+1} \setminus f[\mathcal{K}_{\beta,s}] = (f[\mathcal{A}_{\alpha_s}], f[\mathcal{B}_s])_{\mathcal{K}_{\beta,s+1}}$.

For bookkeeping, we let $J = \left\{ s \in I \,:\, \alpha_{g(s)} > s \right\}$.

Suppose that $s \leq \omega_1$ is a limit point of I (and so $s \in I$). Let $\mathcal{K}_{\beta,<s}$ be the direct limit of the system $\left\langle \mathcal{K}_{\beta,t}, f_{t,r} \right\rangle_{r,t \in I, r<t<s}$, and for $t < s$ in I, let $f_{t,<s}$ be the limit of the maps $\langle f_{t,r} \rangle_{r \in I, t<r<s}$. As each map $f_{t,r}$ extends j_{α_t,α_r}, and as $\alpha_s = \sup_{t<s} \alpha_t$, we see that for all $t < s$ in I, $f_{\alpha,<s} \restriction \mathcal{A}_{\alpha_t} = j_{\alpha_t,\alpha_s}$. As each j_{α_t,α_r} is an initial segment embedding of $\mathcal{A}_{\alpha_r}$ into $\mathcal{K}_{\beta,r}$, we see that $\mathcal{A}_{\alpha_s}$ is an initial segment of $\mathcal{K}_{\beta,<s}$.

There are two possibilities:

- If $J \cap s$ is unbounded in s, then for all $t < s$ in I, there is some $r \in I$ such that $t < r < s$ and such that $f_{t,r}[\mathcal{B}_t] \subseteq \mathcal{A}_{\alpha_r}$. This implies that $\mathcal{K}_{\beta,<s}$ is the direct limit of the maps j_{α_t,α_r} for $t < r < s$ in I, that is, $\mathcal{K}_{\beta,<s} = \mathcal{A}_{\alpha_s}$.
- If $J \cap s$ is bounded in s, let $t_0 = \sup(J) \cap s$. In this case, for all $t, r \in I$ such that $t_0 \leq t < r < s$, we have $f_{t,r} = j_{\alpha_t,\alpha_r} + j^*_{t,r}$, and so $\mathcal{K}_{\beta,<s}$, being the direct limit of these maps, is $\mathcal{A}_{\alpha_s} + \mathcal{B}_s = \mathcal{K}_{\beta,s}$.

In either case, we can let, for $t < s$ in I, $f_{t,s} = f_{t,<s}$, where in the first case, the maps are composed with the identity inclusion of $\mathcal{K}_{\beta,<s}$ into $\mathcal{K}_{\beta,s} = \mathcal{K}_{\beta,<s} + \mathcal{B}_s$.

Now we argue that $\mathcal{K}_{\beta,<\omega_1}$, which is computable in g, uniformly in β, is isomorphic to $\mathcal{K}_\beta$. We have verified that if J is bounded below ω_1, then $\mathcal{K}_{\beta,<\omega_1} \cong \mathcal{A}_{\alpha(\beta)} + \mathcal{B}_{\omega_1}$, and that if J is cofinal in ω_1, then $\mathcal{K}_{\beta,<\omega_1} \cong \mathcal{A}_{\alpha(\beta)}$. Certainly if J is unbounded in ω_1 then $\alpha(\beta) = \omega_1$. We thus only need to show that if $\alpha(\beta) = \omega_1$, then J is cofinal in ω_1.

Assume that $\alpha(\beta) = \omega_1$, and suppose, for contradiction, that J is bounded below ω_1. Let $s_0 = \sup(J)$. Then $(s_0, \omega_1) \subseteq I$. Define a computable function $h \colon \omega_1 \to \omega_1$ by letting $h(\gamma)$ be the least stage $s < \omega_1$ such that $\alpha_s > \gamma$. By our assumption, there is some $s > s_0$ such that $g(s) > h(s)$, so $\alpha_{g(s)} \geq \alpha_{h(s)} > s$. As $s \in I$, it follows that $s \in J$, contradicting $s > s_0$.

This completes the proof. ⊣

Remark 2.8. The construction is flexible in that it is not important that $\mathcal{L}$ be an ω_1-sum of separators and diagonalizers. For example, we can obtain $\mathcal{L}$ from $\mathbb{R}$ by replacing the i^{th} rational number q_i by $\mathcal{S}_i$, and the α^{th} irrational number r_α by $\mathcal{K}_\alpha$. We just need the location of $\mathcal{A}_\alpha$ to be determined by the location of a countable uniformly computable set of $\mathcal{S}_\beta$'s.

2.2. Transfer Theorems. Within the ω-setting, there are several well-known and widely used theorems stating that if an order-type λ is **a**-ω-computable (for some fixed theorem-dependent degree **a**), then $\kappa \cdot \lambda$ is ω-computable (for some fixed theorem-dependent order-type κ). For example, the following theorem has been used to exhibit linear orders having spectra exactly the nonlow$_n$ degrees for $n \geq 2$ (see [8]) and to exhibit linear orders having arbitrary α^{th} jump degree (see [4]).

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

THEOREM 2.9 (Downey and Knight [4]). *If λ is $\mathbf{0}'$-ω-computable, then $(\eta_0 + 2 + \eta_0) \cdot \lambda$ is ω-computable.*

Here, we show that there are no such simple transfer theorems of the above type (involving only multiplication of linear orders) in the ω_1-setting. The following theorem is an extension of Theorem 6.5 of Greenberg and Knight [11].

THEOREM 2.10. *For any degree $\mathbf{a} > \mathbf{0}$, there is an $\mathbf{a}$-ω_1-computable order-type λ such that $\kappa \cdot \lambda$ is not ω_1-computable for any (nonempty) order-type κ.*

Moreover, the order-type λ can be chosen so that, for any nonempty order-type κ, the degree spectrum of $\kappa \cdot \lambda$ is the intersection of DegSpec(κ) *with the cone of degrees above* $\mathbf{a}$.

PROOF OF THEOREM 2.10. Given an ω_1-degree $\mathbf{a}$, we fix a set $A \in \mathbf{a}$. Then the set $S := A \oplus (\omega_1 \setminus A)$ has the property that S is ω_1-c.e. in an ω_1-degree $\mathbf{b}$ if and only if $\mathbf{b} \geq \mathbf{a}$.

Let $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ be the collection of irrational real numbers. This is an uncountable computable set, and so is isomorphic to ω_1 by a computable bijection $h : \omega_1 \to \mathbb{I}$. Let $\mathcal{L}_S := \mathbb{Q} \cup h[S]$, with the ordering inherited from $\mathbb{R}$. We argue that $\lambda := \mathrm{otp}(\mathcal{L}_S)$ has the desired properties.

Let κ be any nonempty order-type. If $\mathbf{b} \in \mathrm{DegSpec}(\lambda) \cap \mathrm{DegSpec}(\kappa)$, then it is immediate that $\mathbf{b} \in \mathrm{DegSpec}(\kappa \cdot \lambda)$. For the reverse direction, we show that any linear order $\mathcal{B}$ in $\kappa \cdot \lambda$ computes both $\mathbf{a}$ and a presentation of κ.

Fix a presentation $\mathcal{B} \in \kappa \cdot \lambda$. Fix an order-preserving embedding $g : \mathbb{Q} \to \mathcal{B}$ by picking, for each rational $q \in \mathbb{Q}$, a point $g(q)$ in the qth copy of κ. Using g as a countable parameter, we show that $\mathcal{B}$ can enumerate the set S.

Indeed, for $x \in \omega_1$, let (L_x, R_x) be the cut of $\mathbb{Q}$ such that $(L_x, R_x)_{\mathbb{R}} = \{h(x)\}$. Then $x \in S$ if and only if $(g[L_x], g[R_x])_{\mathcal{B}}$ is nonempty. Since the cut (L_x, R_x) can be effectively obtained from x, this gives a $\Sigma^0_1(\mathcal{B})$ definition of S. By our choice of S, this implies $\mathcal{B} \geq_T \mathbf{a}$.

As $\mathbf{a} > \mathbf{0}$, it must be the case that S is nonempty. We fix $z \in S$ and consider the interval $(g[L_z], g[R_z])_{\mathcal{B}}$. It has order-type κ. As $g[L_z]$ and $g[R_z]$ are countable, it follows that $\mathcal{B} \upharpoonright (g[L_z], g[R_z])_{\mathcal{B}}$ is a $\mathcal{B}$-computable presentation of κ.

Thus, an arbitrary presentation $\mathcal{B}$ of $\kappa \cdot \lambda$ computes both $\mathbf{a}$ and a presentation of κ. ⊣

The proof of Theorem 2.10, or simply using the theorem with any computable order-type κ, yields the Greenberg–Knight result:

THEOREM 2.11 (Greenberg and Knight [11]). *For any ω_1-degree $\mathbf{a}$, there is a linear ordering whose degree spectrum is the cone of degrees above $\mathbf{a}$ (including $\mathbf{a}$).*

Although multiplication does not work, transfer theorems do exist.

DEFINITION 2.12. For a linear order $\mathcal{L}$, define an equivalence relation $\sim$ on subsets of $\mathcal{L}$ by

$$A_0 \sim A_1 \qquad \text{if and only if} \qquad \mathrm{dcl}_{A_0 \cup A_1}(A_0) = \mathrm{dcl}_{A_0 \cup A_1}(A_1).$$

It is easily checked that $\sim$ is an equivalence relation.

Define $\mathcal{L}^{\mathrm{c}}$ to be the smallest extension of $\mathcal{L}$ satisfying

$$|(\mathrm{dcl}_{\mathcal{L}}(A), \mathcal{L} - \mathrm{dcl}_{\mathcal{L}}(A))_{\mathcal{L}^{\mathrm{c}}}| = 1$$

for every at most countable $A \subseteq \mathcal{L}$. In other words, the linear ordering $\mathcal{L}^{\mathrm{c}}$ is the linear ordering formed from $\mathcal{L}$ by filling with one point every cut such that the set of points to the left of the cut has at most countable cofinality.

Define $\mathcal{L}^{\mathrm{t}}$ (termed the *transfer of* $\mathcal{L}$) to be the linear ordering

$$\mathcal{L}^{\mathrm{t}} := \sum_{x \in \mathcal{L}^{\mathrm{c}}} A_x,$$

where $A_x := 2$ if $x \in \mathcal{L}$ and $A_x := \eta_1$ if $x \in \mathcal{L}^{\mathrm{c}} - \mathcal{L}$.

Note that if $\mathcal{L}$ is computable, the linear orderings $\mathcal{L}^{\mathrm{c}}$ and $\mathcal{L}^{\mathrm{t}}$ are computable.

LEMMA 2.13. *Fix an* ω_1*-degree* $\mathbf{a}$. *A linear ordering* $\mathcal{L}$ *is* $\mathbf{a}'$*-computable if and only if* $\mathcal{L}^{\mathrm{t}}$ *is* $\mathbf{a}$*-computable. Further, the transition between* $\mathcal{L}$ *and* $\mathcal{L}^{\mathrm{t}}$ *is uniform in the indices in both directions.*

PROOF. ($\Longleftarrow$) Given an $\mathbf{a}$-computable presentation of $\mathcal{L}^{\mathrm{t}}$, let $\mathcal{K} := \mathrm{Succ}(\mathcal{L}^{\mathrm{t}})$, the set of adjacencies of $\mathcal{L}^{\mathrm{t}}$ with the natural ordering. Then $\mathcal{K}$ is $\mathbf{a}'$-computable and has the appropriate order-type when given the induced order from $\mathcal{L}^{\mathrm{t}}$.

($\Longrightarrow$) By the universal property of η_1, we may assume that $\mathcal{L}$ is an $\mathbf{a}'$-computable subset of a computable presentation of η_1. We will, of course, approximate $\mathcal{L}$ in an $\mathbf{a}$-computable manner, building a linear ordering $\mathcal{K} \in \mathrm{otp}(\mathcal{L}^{\mathrm{t}})$ from this approximation.

When we see an element enter $\mathcal{L}$, we add an appropriate pair of elements into $\mathcal{K}$. When we see an element leave $\mathcal{L}$, since we cannot remove the corresponding pair from $\mathcal{K}$, we instead incorporate it into the copy of η_1 immediately to its left. Since the approximation at every stage is at most countable, there are at most countably many points in the current approximation to $\mathcal{L}$ which are to the left of the removed point—call this set A. So there is always a copy of η_1 to the immediate left of the removed pair—the copy of η_1 corresponding to the unique element of $\mathcal{L}^{\mathrm{c}}$ in $(\mathrm{dcl}_{\mathcal{L}}(A), \mathcal{L} - \mathrm{dcl}_{\mathcal{L}}(A))_{\mathcal{L}^{\mathrm{c}}}$.

Of course, we must also build the copies of η_1. Naively, one might hope to consider every countable subset of the current approximation to $\mathcal{L}$ and build a corresponding copy of η_1. Unfortunately, there may be uncountably many such subsets, so we cannot do this in a single stage. Instead, at every stage we consider a single countable subset of η_1. If this set is a subset of the current approximation to $\mathcal{L}$, then we build a copy of η_1 for it. Every countable subset of $\mathcal{L}$ will eventually be a subset of the approximation, so as long as we arrange to consider every subset at uncountably many stages, every countable set of $\mathcal{L}$ will eventually be handled. We must also build a copy of η_1 if it does not already exist when we seek to incorporate a pair into it as described above.

Of course, since η_1 is an uncountable object, we cannot actually build an entire copy of it at a single stage. Instead, we declare what we call a *saturating interval*. At uncountably many later stages, we will add points to this saturating interval, causing it to grow into a copy of η_1.

If a point x leaves the approximation to $\mathcal{L}$, we must consider the effect on the saturating intervals we have built so far. If I is a saturating interval built on behalf of the countable set X, and x is not the largest element of X, then we do not need to adjust I; since $X \sim X - \{x\}$, I can continue to be the saturating interval which

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

we build on behalf of $X - \{x\}$. If x is the largest point in X, however, then there is no longer a need for I. In this case, I must be the interval immediately to the right of the pair which corresponds to x. This pair will be merged with the saturating interval to its left, and we can merge I with the same interval.

Finally, we must concern ourselves with what happens at limit stages. We assume that the approximation to $\mathcal{L}$ at a limit stage is the limit infimum of the approximations at previous stages. Thus, the only points in the approximation at a limit stage are the points which were in for a terminal segment of previous stages. Hence, for pairs, there is nothing to do. For saturating intervals, however, we may need to cause more mergers.

For example, consider the following situation: The approximation to $\mathcal{L}$ at stage ω has order type ω^2. At stage ω, we have saturating intervals in order type $\omega + 1$, built on behalf of the "sets" $\emptyset$, ω, $\omega \cdot 2$, $\omega \cdot 3, \dots,$ and ω^2. Suppose that at every stage $\omega + \langle m, k\rangle$, the point corresponding to $\omega \cdot m + k$ leaves the approximation, but otherwise there is no change.

Then at every stage $\omega + n$, every pair of the original $\omega + 1$ saturating intervals is separated by countably many elements, and so will not merge. However, at stage $\omega + \omega$, the approximation is empty and so there are no elements separating any of the saturating intervals. As η_1 and $\eta_1 \cdot \omega$ are not isomorphic, we will need to merge these saturating intervals.

In general, at a limit stage we will merge all saturating intervals which are not separated by a pair.

We will also define a sequence of functions F_s and G_s, which will assist us in tracking the relationship between $\mathcal{L}$ and $\mathcal{K}$. The function F_s will map the elements of $\mathcal{L}_s$ to their corresponding pair in $\mathcal{K}_s$. The function G_s will map a saturating interval in $\mathcal{K}_s$ to its corresponding at most countable subset in $\mathcal{L}_s$. It will be convenient to assume that these subsets are downward closed. So even if we make no changes to a saturating interval I between stages s and $s+1$, we will redefine $G_{s+1}(I)$ to be the downward closure (in $\mathcal{L}_{s+1}$) of $G_s(I)$. It will be the case that $G_s(I)$ is downward closed automatically at limit stages.

Preliminaries: Let $(\mathcal{L}_s)_{s<\omega_1}$ be an **a**-computable sequence of countable subsets of η_1 satisfying:

- $\mathcal{L}_0 = \emptyset$;
- $\mathcal{L}_s \triangle \mathcal{L}_{s+1} = \{z_s\}$ for some z_s;
- for s a limit ordinal, $\mathcal{L}_s = \liminf_{t<s} \mathcal{L}_t$; and
- $\mathcal{L} = \lim_s \mathcal{L}_s$.

We construct $\mathcal{K}$ as the union of countable linear orders $(\mathcal{K}_s)_{s<\omega_1}$. Each $\mathcal{K}_s$ will be partitioned into *saturating intervals* and *pairs*.

As discussed earlier, we also build sequences of functions $(F_s)_{s<\omega_1}$ and $(G_s)_{s<\omega_1}$. The sequence $(F_s)_{s<\omega_1}$ will be continuous, and each F_s will be order-preserving. The map G_s will also be order-preserving, in that if $I <_{\mathcal{K}_s} J$, then $G_s(I) \subset G_s(J)$. For $x \in \mathcal{L}_s$, we let $(F_s(x))_1$ and $(F_s(x))_2$ denote the left and right elements of the pair $F_s(x)$, respectively.

We fix a computable enumeration $(A_s, B_s)_{s<\omega_1}$ of pairs from H_{ω_1} such that every pair occurs uncountably many times in the enumeration, and fix a computable enumeration $(Y_s)_{s<\omega_1}$ of H_{ω_1} such that every element occurs uncountably many times. These will be used in the creation of the saturating intervals.

Construction: At stage $s = 0$, we define $\mathcal{K}_0$, F_0, and G_0 to be empty.

At a successor stage $s + 1$, we work in three steps, building intermediate orders $\mathcal{K}^1_{s+1}$ and $\mathcal{K}^2_{s+1}$ and intermediate functions G^1_{s+1} and G^2_{s+1}: First, we adjust F_s and the pairs in $\mathcal{K}_s$ for the change from $\mathcal{L}_s$ to $\mathcal{L}_{s+1}$; second, we create new saturating intervals as necessary; third, we work to build the saturating intervals into η_1.

(1) If $\mathcal{L}_{s+1} = \mathcal{L}_s \cup \{z_s\}$, then we add a new pair to be the image of z_s. More precisely, let

$$R := \operatorname{ucl}_{\mathcal{K}_s} \{(F_s(y))_1 : y \in \mathcal{L}_s \text{ and } z_s <_{\mathcal{L}_{s+1}} y\}$$

and let $Q := \mathcal{K}_s - R$. We choose two new elements a and b and define $\mathcal{K}^1_{s+1} := \mathcal{K}_s \cup \{a, b\}$ with

$$Q <_{\mathcal{K}^1_{s+1}} a <_{\mathcal{K}^1_{s+1}} b <_{\mathcal{K}^1_{s+1}} R.$$

We make (a, b) a pair in $\mathcal{K}^1_{s+1}$ and define $F_{s+1} := F_s \cup \{(z_s, (a, b))\}$. For every saturating interval $I \subseteq \mathcal{K}_s$, we define $G^1_{s+1}(I) := \operatorname{dcl}_{\mathcal{L}_{s+1}}(G_s(I))$.

If instead $\mathcal{L}_s = \mathcal{L}_{s+1} \cup \{z_s\}$, then we merge the pair $F_s(z_s)$ with the saturating interval to its left. More precisely, let $(a, b) := F_s(z_s)$ and let

$$Q := \{y : y \in \mathcal{L}_s \text{ and } y <_{\mathcal{L}_s} z_s\}.$$

There may already exist saturating intervals $I, J \subseteq \mathcal{K}_s$ with $G_s(I) = Q$ and $G_s(J) = Q \cup \{z_s\}$. Let $L = I \cup \{a, b\} \cup J$, omitting I, J, or both when those intervals do not exist. We make L a saturating interval of MK^1_{s+1} with $G^1_{s+1}(L) = Q$.

We define $F_{s+1} := F_s \restriction \mathcal{L}_{s+1}$. We do not make (a, b) a pair in $\mathcal{K}^1_{s+1}$. All other pairs and saturating intervals of $\mathcal{K}_s$ other than I and J remain pairs and saturating intervals of $\mathcal{K}^1_{s+1}$, respectively. For any saturating interval $H \subseteq \mathcal{K}_s$ other than I and J, we define $G^1_{s+1}(H) := G_s(H) - \{z_s\}$.

(2) If there is no saturating interval $I \subseteq \mathcal{K}^1_{s+1}$ with $G^1_{s+1}(I) = \operatorname{dcl}_{\mathcal{L}^1_{s+1}}(Y_s)$, let

$$Q := \{(F_{s+1}(y))_2 : y \in Y_s\},$$
$$R := \{(F_{s+1}(y))_1 : y \in \mathcal{L}_{s+1} \text{ and } Y_s <_{\mathcal{L}_{s+1}} y\}.$$

We choose a new element c and define $\mathcal{K}^2_{s+1} := \mathcal{K}^1_{s+1} \cup \{c\}$ with

$$Q <_{\mathcal{K}^2_{s+1}} c <_{\mathcal{K}^2_{s+1}} R.$$

We make $\{c\}$ a saturating interval in $\mathcal{K}^2_{s+1}$ with $G^2_{s+1}(\{c\}) = \operatorname{dcl}_{\mathcal{L}^1_{s+1}}(Y_s)$.

Otherwise, we define $\mathcal{K}^2_{s+1} := \mathcal{K}^1_{s+1}$.

For every saturating interval $I \subseteq \mathcal{K}^1_{s+1}$, we define $G^2_{s+1}(I) := G^1_{s+1}(I)$, noting these are downward closed subsets.

(3) If there is some saturating interval $I \subseteq \mathcal{K}^2_{s+1}$ with $A_s, B_s \subseteq I$ and $A_s <_{\mathcal{K}^2_{s+1}} B_s$ and $(A_s, B_s)_{\mathcal{K}^2_{s+1}} = \emptyset$, we choose a new element d and define $\mathcal{K}_{s+1} := \mathcal{K}^2_{s+1} \cup \{d\}$. We define $<_{\mathcal{K}_{s+1}}$ by extending $<_{\mathcal{K}^2_{s+1}}$ with

$$A_s <_{\mathcal{K}_{s+1}} d <_{\mathcal{K}_{s+1}} B_s.$$

We make $I \cup \{d\}$ a saturating interval in $\mathcal{K}_{s+1}$ with $G_{s+1}(I \cup \{d\}) := G^2_{s+1}(I)$. For every other saturating interval $J \subseteq \mathcal{K}^2_{s+1}$, we define $G_{s+1}(J) := G^2_{s+1}(J)$.

At a limit stage s, we work in two steps, building an intermediate function G'_s: First we define the pairs and saturating intervals as the limits of the previous stages. Then we merge saturating intervals where necessary.

Before doing so, we define $\mathcal{K}_s := \bigcup_{t<s} \mathcal{K}_t$ and $F_s := \lim_{t<s} F_t$, noting the limit exists because $\mathcal{L}_s = \liminf_{t<s} \mathcal{L}_t$.

(1) We make (a, b) a pair in $\mathcal{K}_s$ if there is a stage $s_0 < s$ such that (a, b) is a pair in $\mathcal{K}_t$ for every t with $s_0 < t < s$.

By Claim 2.13.1, for every t with $s_0 < t < s$ and every saturating interval $I \subseteq \mathcal{K}_{s_0}$, there is a unique saturating interval $I_t \subseteq \mathcal{K}_t$ with $I \cap I_t \neq \emptyset$, and further this unique saturating interval satisfies $I \subseteq I_t$.

Thus for every $s_0 < s$ and every saturating interval $I \subseteq \mathcal{K}_{s_0}$, the set $I'_s := \bigcup_{s_0<t<s} I_t$ is convex. We let $G'_s(I'_s) := \liminf_{t<s} G_t(I_t)$, observing this is downward closed.

To see that this is well-defined, suppose $I'_s = J'_s$. Then there is some stage $r > s_0$ and some saturating interval $L \subseteq K_r$ with $I \cup J \subseteq L$. Then for all $t > r$, $I_t = J_t = L_t$, and so

$$\liminf_{t<s} G_t(I_t) = \liminf_{t<s} G_t(L_t) = \liminf_{t<s} G_t(J_t),$$

Thus, the choice of the stage s_0 and starting interval I is unimportant.

(2) As discussed above, there may be I and J such that $I'_s \neq J'_s$ but $G'_s(I'_s) = G'_s(J'_s)$. Note that in this case, there can be no $y \in \mathcal{L}_s$ with $F(y) = (a, b)$ and $I'_s <_{\mathcal{L}_s} a <_{\mathcal{L}_s} b <_{\mathcal{L}_s} J'_s$, because then y would be in $G'_s(J'_s) \setminus G'_s(I'_s)$. Also the converse holds, so if there is no such y, then $G'_s(I'_s) = G'_s(J'_s)$.

For every saturating interval $I \subset \mathcal{K}_t$ for some $t < s$, we make

$$I_s = \bigcup_{G'_s(J'_s)=G'_s(I'_s)} J'_s$$

a saturating interval in $\mathcal{K}_s$. We define $G_s(I_s) := G'_s(I'_s)$.

This completes the construction.

We let $\mathcal{K} := \mathcal{K}_{\omega_1}$, $F := F_{\omega_1}$ and $G := G'_{\omega_1}$. We note that sets in the range of G may be uncountable, unlike sets in the range of G_s for $s < \omega_1$; also we do not perform the final step of combining saturating intervals at stage ω_1 (we argue in Claim 2.13.4 that it is unnecessary).

Verification: Clearly $\mathcal{K}$ is **a**-computable, F is an order-preserving bijection from $\mathcal{L}$ to the pairs in $\mathcal{K}$, and G is an order-preserving map from the saturating intervals to the downward closed subsets of $\mathcal{L}$. Also, by the action of Step 3 at successor stages, every saturating interval in $\mathcal{K}$ has order type η_1.

CLAIM 2.13.1. *For every $t \leq s$ and every saturating interval $I \subseteq \mathcal{K}_t$, there is a unique saturating interval $I_s \subseteq \mathcal{K}_s$ with $I \cap I_s \neq \emptyset$. Furthermore, $I \subseteq I_s$ and $G_s(I_s)$ is contained in the downward closure of $G_t(I)$ in η_1 (recalling that $\mathcal{L} \subseteq \eta_1$).*

PROOF. Immediate by construction and induction on s. ⊣

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

Claim 2.13.2. *If $I \subseteq \mathcal{K}$ is a saturating interval, then there is an at most countable $Y \subseteq \mathcal{L}$ with $Y \sim G(I)$.*

Proof. Fix a saturating interval $J \subseteq \mathcal{K}_s$ such that $J \subseteq I$. By regularity, there is a stage $t > s$ such that $\mathcal{L}_{t'}$ extends $\mathcal{L}_t$ for all $t' > t$. Let J_t be the saturating interval of $\mathcal{K}_t$ containing J. Then $G_t(J_t) \subseteq G(I)$ by construction, and $G(I)$ is contained in the downward closure of $G_t(J_t)$. Hence, the set $G_t(J_t)$ suffices as a choice for Y. ⊣

Claim 2.13.3. *At every stage s, the map G_s is injective.*

Proof. This follows by induction on s: At limit stages, this is by explicit construction. At successor stages, this is by construction and the inductive hypothesis. ⊣

Claim 2.13.4. *For every $Y \in [\mathcal{L}]^{<\omega_1}$, there is precisely one saturating interval $I \subseteq \mathcal{K}$ with $G(I) \sim Y$.*

Proof. Let s_0 be a stage such that $Y \subseteq \mathcal{L}_s$ for all $s \geq s_0$, and let $s_1 > s_0$ be a stage such that $Y = Y_{s_1}$. Then there is a saturating interval $J \subseteq \mathcal{K}_{s_1+1}$ with $G_{s_1+1}(J) \sim Y$ (which is created if it did not already exist). For every $t > s_1$, let J_t be the unique saturating interval in $\mathcal{K}_t$ with $J \subseteq J_t$. By Claim 2.13.1, $G_t(J_t) \sim Y$ for all t. Thus the saturating interval $J'_{\omega_1} \subseteq \mathcal{K}$ has $G(J'_{\omega_1}) \sim Y$.

Towards uniqueness, assume there were two such intervals I_0 and I_1. Let s be a stage such that there are saturating intervals $J_0, J_1 \subseteq \mathcal{K}_s$ with $J_0 \subseteq I_0$ and $J_1 \subseteq I_1$, and such that $\mathcal{L}_t$ extends $\mathcal{L}_s$ for all $t > s$. Then by the argument in Claim 2.13.2, $G_s(J_0) \sim G_s(J_1)$. But since these sets are downward closed in $\mathcal{L}_s$, we would have $G_s(J_0) = G_s(J_1)$, contrary to Claim 2.13.3. ⊣

Claim 2.13.5. *If $I, J \subseteq \mathcal{K}$ are saturating intervals with $G(I) \subset G(J)$, then $I <_{\mathcal{K}} J$. Furthermore, if $y \in \mathcal{L}$ with $G(I) <_{\mathcal{L}} y$ and $y \in G(J)$, then $I <_{\mathcal{K}} (F(y))_1 <_{\mathcal{K}} (F(y))_2 <_{\mathcal{K}} J$.*

Proof. Fix s. By construction, this is true for any saturating intervals $I', J' \subseteq \mathcal{K}_s$ with $I' \subset I$ and $J' \subset J$. Thus it is true for $I, J \subseteq \mathcal{K} = \bigcup_{s<\omega_1} \mathcal{K}_s$. ⊣

Thus we can map $x \in \mathcal{L}^c$ to $A_x \subseteq \mathcal{K}$ by sending $x \in \mathcal{L}$ to $F(x)$ and $x \in \mathcal{L}^c - \mathcal{L}$ to $G^{-1}\{y \in \mathcal{L} \mid y < x\}$, and this map is order-preserving and its image covers $\mathcal{K}$.

This completes the proof of Lemma 2.13. ⊣

2.3. A Nonlow-n Spectrum. For $n \geq 2$, there are countable linear orderings whose degree spectrums consist of the nonlow$_n$ ω-degrees [8]. For $n = 1$, though, while it is known (see [9]) that the collection of nonlow ω-degrees is a degree spectrum, it is yet unknown if it is the degree spectrum of a linear order. We show that this problem has a solution in the ω_1-context: For every n, including $n = 1$, there is an order-type of size $\aleph_1$ whose degree spectrum consists of the nonlow$_n$ ω_1-degrees, that is, of the ω_1-Turing degrees $\mathbf{a}$ such that $\mathbf{a}^{(n)} > \mathbf{0}^{(n)}$.

We begin with the case $n = 1$. The order-type whose degree spectrum is the nonlow degrees will be the η_1-shuffle sum of linear orders coding a family $\mathcal{F}$ of sets which is Σ^0_2 in every nonlow ω_1-degree, but not Σ^0_2.

As in Section 2.1, let $\mathcal{A}_\beta := \sum_{\alpha<\beta} \mathbb{Z}^\alpha$ and $\mathcal{B}_\beta := \mathcal{A}^*_\beta$.

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

LEMMA 2.14. *Let $S \subseteq \omega_1$ and $\mathbf{a}$ be an ω_1-Turing degree. There is a sequence of uniformly $\mathbf{a}$-computable linear orders $\langle \mathcal{L}_i \rangle_{i<\omega_1}$ such that*

$$\mathcal{L}_i \cong \begin{cases} \mathcal{A}_{\omega_1} & \text{if } i \in S, \\ \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1} & \text{otherwise,} \end{cases}$$

if and only if the set S is $\Pi^0_2(\mathbf{a})$.

Moreover, the passage between an $\mathbf{a}$-computable index for the sequence of ω_1-computable linear orders and a $\Pi^0_2(\mathbf{a})$-index for S is effective.

PROOF. ($\Longrightarrow$) Let $\langle \mathcal{L}_i \rangle_{i<\omega_1}$ be a uniformly $\mathbf{a}$-computable sequence of linear orders. Then the collection of $i < \omega_1$ such that $\mathrm{cf}(\mathcal{L}_i) = \omega_1$ is $\Pi^0_2(\mathbf{a})$, as $\mathrm{cf}(\mathcal{L}_i) = \omega_1$ if and only if every countable subset of L_i is strictly bounded in $\mathcal{L}_i$. It is easy to see that $\mathrm{cf}(\mathcal{A}_{\omega_1}) = \omega_1$ and that $\mathrm{cf}(\mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1}) = 1$.

($\Longleftarrow$) Fix a $\Pi^0_2(\mathbf{a})$ set S. We can, uniformly in $\mathbf{a}$, enumerate sets U_i such that for all i, the set U_i is uncountable if and only if $i \in S$. Fixing i, at stage s we define $\mathcal{C}_s := \mathcal{A}_s + \mathcal{B}_s$ and an embedding $f^i_{s,s+1}$ of $\mathcal{C}_s$ into $\mathcal{C}_{s+1}$ extending the initial segment embedding $j_{s,s+1}$ of $\mathcal{A}_s$ into $\mathcal{A}_{s+1}$. If a new number is enumerated into U_i (i.e., we see new evidence that $i \in S$), then we let $f_{s,s+1}$ embed $\mathcal{C}_s$ into $\mathcal{A}_{s+1}$ (i.e., we move past work built for $\mathcal{B}$ into $\mathcal{A}$); otherwise, we let $f_{s,s+1} = j_{s,s+1} + j^*_{s,s+1}$ (i.e., we continue building $\mathcal{A}$ and $\mathcal{B}$ separately). We let $\mathcal{L}_i$ be the direct limit of the system $\langle \mathcal{C}_s, f^i_{s,t} \rangle_{s\leq t<\omega_1}$. The arguments of the previous section show that if U_i is uncountable, then all copies of $\mathcal{B}_s$ are “swallowed” and we get $\mathcal{L}_i \cong \mathcal{A}_{\omega_1}$; otherwise, we get $\mathcal{L}_i \cong \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1}$. ⊣

As is done in the countable framework, we say that a set $\mathcal{F}$ of subsets of ω_1 is ω_1-c.e. in some degree $\mathbf{a}$ if there is a uniformly $\mathbf{a}$-c.e. sequence of sets $\langle F_i \rangle_{i<\omega_1}$ such that $\mathcal{F} = \{F_i : i < \omega_1\}$. Similarly, a set $\mathcal{F}$ of subsets of ω_1 is ω_1-Σ^0_2 in $\mathbf{a}$ if there is a uniformly $\Sigma^0_2(\mathbf{a})$ sequence of sets $\langle F_i \rangle_{i<\omega_1}$ such that $\mathcal{F} = \{F_i : i < \omega_1\}$.

LEMMA 2.15. *There is a family $\mathcal{F}$ of sets which is Σ^0_2 in a degree $\mathbf{a}$ if and only if $\mathbf{a}$ is nonlow. In fact, fixing a degree $\mathbf{c}$, there is a family $\mathcal{F}$ of sets which is Σ^0_2 in a degree $\mathbf{a}$ if and only if $\mathbf{a}$ is nonlow over $\mathbf{c}$.*

PROOF. As in the countable framework, for any ω_1-degree $\mathbf{d}$, a set is $\Sigma^0_2(\mathbf{d})$ if and only if it is ω_1-c.e. in $\mathbf{d}'$. Hence, we are looking for a family $\mathcal{F}$ of sets which is ω_1-c.e. in $\mathbf{a}'$ for every $\mathbf{a}$ with $\mathbf{a}' > \mathbf{0}'$ but is not ω_1-c.e. in $\mathbf{0}'$.

The construction of $\mathcal{F}$ is the relativization to $\emptyset'$ of Wehner [21] of a family of sets which is c.e. in every nonzero ω-Turing degree but is not c.e. The change of setting to ω_1 does not change any of the details. Namely, we let

$$\mathcal{F} := \left\{ \{\alpha\} \oplus A \ : \ A \text{ is countable, and } A \neq W^{\emptyset'}_\alpha \right\}.$$

The Recursion Theorem shows that $\mathcal{F}$ is not ω_1-c.e. in $\mathbf{0}'$; but $\mathcal{F}$ is ω_1-c.e. in every degree $\mathbf{a} > \mathbf{0}'$, because $\mathbf{a}$ can code, element by element, a set $W \in \mathbf{a}$ which is not Σ^0_2, to escape equality with a given $W^{\emptyset'}_\alpha$. ⊣

We introduce the order-types that will be used to code the sets in $\mathcal{F}$.

DEFINITION 2.16. Again fix an enumeration $\langle q_i \rangle_{i<\omega}$ of the set of rational numbers $\mathbb{Q}$. Let $\mathbb{I}$ be the set of irrationals. For $q = q_i$, let $\mathcal{P}_q = i + 2$.

For $X \subseteq \mathbb{I}$ and $r \in \mathbb{R}$, define

$$\mathcal{Q}_{X,r} := \begin{cases} \mathcal{P}_r & \text{if } r \in \mathbb{Q}, \\ \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1} & \text{if } r \in X, \\ \mathcal{A}_{\omega_1} & \text{if } r \in \mathbb{I} \setminus X, \end{cases}$$

and let $\mathcal{Q}_X := \sum_{r \in \mathbb{R}} \mathcal{Q}_{X,r}$.

Let $\mathcal{P} := \sum_{q \in \mathbb{Q}} \mathcal{P}_q$. For $X \subseteq \mathbb{I}$, let f_X be the natural embedding of $\mathcal{P}$ into $\mathcal{Q}_X$; for $q \in \mathbb{Q}$, f_X maps the copy of $\mathcal{P}_q$ in $\mathcal{P}$ to $\mathcal{Q}_{X,q}$. The range of f_X consists of those points in $\mathcal{Q}_X$ which are contained in finite maximal blocks of size larger than one.

Furthermore, the argument of Lemma 2.3 shows that if $X, Y \subseteq \mathbb{I}$ and $X \neq Y$, then $\mathcal{Q}_X$ is not isomorphic to any convex subset of $\mathcal{Q}_Y$.

We see that the linear ordering $\mathcal{Q}_X$ indeed "jump-codes" the set X.

Lemma 2.17. *For any $X \subseteq \mathbb{I}$ and ω_1-degree $\mathbf{a}$, the set X is $\Sigma^0_2(\mathbf{a})$ if and only if $\mathbf{a} \in \mathrm{DegSpec}(\mathcal{Q}_X)$. Furthermore, the equivalence is uniform: From a $\Sigma^0_2(\mathbf{a})$-index for X we can effectively pass to an $\mathbf{a}$-computable index for a linear ordering isomorphic to $\mathcal{Q}_X$, and vice versa.*

Proof. Suppose first that X is $\Sigma^0_2(\mathbf{a})$. Taking an effective bijection between ω_1 and $\mathbb{I}$, by Lemma 2.14, there is a uniformly $\mathbf{a}$-computable sequence $\langle \mathcal{L}_r \rangle_{r \in \mathbb{I}}$ of linear orderings such that if $r \in X$ then $\mathcal{L}_r \cong \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1}$, and if $r \notin X$ then $\mathcal{L}_r \cong \mathcal{A}_{\omega_1}$. We then see that $\sum_{r \in \mathbb{R}} \mathcal{D}_r$, where

$$\mathcal{D}_r := \begin{cases} \mathcal{P}_r & \text{if } r \in \mathbb{Q}, \\ \mathcal{L}_r & \text{if } r \in \mathbb{I}, \end{cases}$$

is $\mathbf{a}$-computable and is isomorphic to $\mathcal{Q}_X$.

For the other direction, suppose that $\mathcal{L}$ is $\mathbf{a}$-computable, and that $g \colon \mathcal{Q}_X \to \mathcal{L}$ is an isomorphism. We first note that if we did not insist on uniformity, then the conclusion that X is $\Sigma^0_2(\mathbf{a})$ follows from Lemma 2.14 as follows. Since $g \circ f_X$ and $\mathcal{P}$ are countable, we can fix them as parameters. For $r \in \mathbb{I}$, let $C_r := \bigcup_{q<r} \mathcal{P}_q$ and $D_r := \bigcup_{q>r} \mathcal{P}_q$ be the indicated subsets of $\mathcal{P}$, noting that the pair (C_r, D_r) can be obtained effectively from r. Let $\mathcal{L}_r := ((g \circ f_X)[C_r], (g \circ f_X)[D_r])_{\mathcal{L}}$. Then $\mathcal{L}_r = g[\mathcal{Q}_{X,r}]$ and so $\langle \mathcal{L}_r \rangle_{r \in \mathbb{I}}$ is a sequence which witnesses, by Lemma 2.14, that X is $\Sigma^0_2(\mathbf{a})$.

However, this argument is nonuniform, as it required fixing the parameter $g \circ f_X$. To obtain uniformity, we will prove that $\mathbf{a}'$ can find this parameter. The argument of the previous paragraph and of the easy direction of Lemma 2.14 then shows that, given this parameter, the ω_1-degree $\mathbf{a}'$ can enumerate X: For each r, the ω_1-degree $\mathbf{a}'$ can obtain an $\mathbf{a}$-computable index for $\mathcal{L}_r$ and can then enumerate those r for which it discovers a maximal element in $\mathcal{L}_r$.

To show that $g \circ f_X$ can be uniformly obtained from $\mathcal{L}$ in a $\Delta^0_2(\mathbf{a})$-fashion, we unfortunately cannot use the characterization of $g \circ f_X$ as the unique isomorphism between $\mathcal{P}$ and the set of points in $\mathcal{L}$ contained in maximal finite blocks of size greater than one. This is because, in general, the computation of the maximal block containing an element takes two jumps rather than one jump. However, there are $\mathbf{a}'$-computable properties whose conjunction is satisfied only by $g \circ f_X$.

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

For $q \in \mathbb{Q}$, let A_q and B_q be the subsets of $\mathcal{P}$ (for the copy we fixed above) such that $\mathcal{P} = A_q + \mathcal{P}_q + B_q$. Since the copy of $\mathcal{P}$ is fixed, this decomposition (note that $\mathcal{P}_q$ is a subset of $\mathcal{P}$, not an order-type, so it is unique within $\mathcal{P}$) is effective in q. We claim that $g \circ f_X$ is the unique embedding h of $\mathcal{P}$ into $\mathcal{L}$ such that for all $q \in \mathbb{Q}$,

(1) $h[\mathcal{P}_q]$ is a convex subset of $\mathcal{L}$; and
(2) $(h[A_q], h[\mathcal{P}_q])_{\mathcal{L}}$ and $(h[\mathcal{P}_q], h[B_q])_{\mathcal{L}}$ are both empty.

Both conditions are $\Pi^0_1(\mathbf{a})$, since it is $\Pi^0_1(\mathbf{a})$ to tell, given countable $C, D \subset \mathcal{L}$, whether $(C, D)_{\mathcal{L}}$ is empty or not. Certainly $g \circ f_X$ satisfies both conditions for all $q \in \mathbb{Q}$. To show that this is the only embedding of $\mathcal{P}$ into $\mathcal{L}$ which satisfies both conditions for all $q \in \mathbb{Q}$, we show that f_X is the only embedding of $\mathcal{P}$ into $\mathcal{Q}_X$ which satisfies the corresponding conditions for all $q \in \mathbb{Q}$.

Suppose that $h \colon \mathcal{P} \to \mathcal{Q}_X$ is an embedding, that for all $q \in \mathbb{Q}$, the set $h[\mathcal{P}_q]$ is a convex subset of $\mathcal{Q}_X$, and that for all $q \in \mathbb{Q}$, both $(h[A_q], h[\mathcal{P}_q])_{\mathcal{Q}_X}$ and $(h[\mathcal{P}_q], h[B_q])_{\mathcal{Q}_X}$ are empty. We first show that $h[\mathcal{P}] \subseteq f_X[\mathcal{P}]$. In other words, we show if $r \in \mathbb{I}$ and $q \in \mathbb{Q}$ then $h[\mathcal{P}_q] \cap \mathcal{Q}_{X,r}$ is empty. If not, then as $h[\mathcal{P}_q]$ is a finite convex subset of $\mathcal{Q}_X$ and the maximal blocks of $\mathcal{Q}_{X,r}$ are of size one or infinite, we must have $h[\mathcal{P}_q] \subset \mathcal{Q}_{X,r}$, and the initial segment of $\mathcal{Q}_X$ consisting of the points to the left of $h[\mathcal{P}_q]$ contains a greatest element x. Now A_q does not contain a greatest element, so $h[A_q]$ cannot contain x; so $h[A_q] <_{\mathcal{Q}_X} x <_{\mathcal{Q}_X} h[\mathcal{P}_q]$, contradicting the assumption on h. A similar argument shows that if $i < j$ then $h[\mathcal{P}_{q_i}]$ cannot intersect $\mathcal{Q}_{X,q_j}$

Finally, if $q, r \in \mathbb{Q}$ and $h[\mathcal{P}_q] \cap \mathcal{Q}_{X,r}$ is nonempty, then as $\mathcal{Q}_{X,r}$ is a maximal block of $\mathcal{Q}_X$ and $h[\mathcal{P}_q]$ is convex in $\mathcal{Q}_X$, we must have $h[\mathcal{P}_q] \subseteq \mathcal{Q}_{X,r}$. This shows that if $i > j$, then $h[\mathcal{P}_{q_i}]$ does not intersect $\mathcal{Q}_{X,q_j}$. Hence for all $q \in \mathbb{Q}$, $h[\mathcal{P}_q] = \mathcal{Q}_{X,q}$, which shows that $h = f_X$. ⊣

THEOREM 2.18. *There is an order-type whose degree spectrum consists of the nonlow ω_1-degrees. In fact, fixing a degree* $\mathbf{c}$, *there is an order-type whose degree spectrum consists of the ω_1-degrees nonlow over* $\mathbf{c}$.

PROOF. Fix a family $\mathcal{F}$ as in Lemma 2.15; by fixing an effective bijection between H_{ω_1} and $\mathbb{I}$, we may assume that every element of $\mathcal{F}$ is a subset of $\mathbb{I}$. We show that the η_1-shuffle sum

$$\lambda := \sigma_1 \left(\{ \mathcal{Q}_X : X \in \mathcal{F} \} \right)$$

(recall Definition 1.4) has presentations in exactly the nonlow ω_1-degrees. By Lemma 2.15, it is sufficient to show that a degree $\mathbf{a}$ computes a presentation of λ if and only if $\mathcal{F}$ is Σ^0_2 in $\mathbf{a}$.

Let $\mathbf{a}$ be an ω_1-Turing degree. Suppose first that $\mathcal{F}$ is Σ^0_2 in $\mathbf{a}$. Then the uniformity guaranteed by Lemma 2.17 shows that there is a sequence $\langle \mathcal{L}_\alpha \rangle_{\alpha < \omega_1}$ of uniformly $\mathbf{a}$-computable linear orders such that

$$\{ \mathrm{otp}(\mathcal{L}_\alpha) \, : \, \alpha < \omega_1 \} = \{ \mathrm{otp}(\mathcal{Q}_X) \, : \, X \in \mathcal{F} \} .$$

From the sequence $\langle \mathcal{L}_\alpha \rangle$ we can easily build a presentation of λ, noting that a computable presentation $\mathbb{Q}_1$ of η_1 can be split into a partition of ω_1-many uniformly computable subsets, each saturated in $\mathbb{Q}_1$.

For the converse, suppose that $\mathcal{L}$ is an $\mathbf{a}$-computable presentation of λ. With oracle $\mathbf{a}'$, we enumerate the sets in $\mathcal{F}$. To do so, with this oracle, we enumerate all the countable functions $g \circ f_X$, where $X \in \mathcal{F}$ and g is a convex embedding of $\mathcal{Q}_X$ into $\mathcal{L}$. The $\Delta^0_2(\mathbf{a})$-conditions on an embedding $h \colon \mathcal{P} \to \mathcal{L}$ to be one of these functions are the conditions (1) and (2) of the proof of Lemma 2.17, together the following condition:

(3) For all $r \in \mathbb{I}$, the interval $(h[C_r], h[D_r])_{\mathcal{L}}$ is *scattered* (i.e., does not contain a copy of $\mathbb{Q}$). Here, again, $C_r := \bigcup_{q<r} \mathcal{P}_q$ and $D_r := \bigcup_{q>r} \mathcal{P}_q$.

Condition (3), together with the previous conditions, implies that $h[\mathcal{P}]$ must be contained in a single convex copy of some $\mathcal{Q}_X$ inside $\mathcal{L}$. Otherwise, fix some convex copy $\mathcal{K}$ of some $\mathcal{Q}_X$ in $\mathcal{L}$ which intersects $h[\mathcal{P}]$. Again, if $q \in \mathbb{Q}$ and $h[\mathcal{P}_q] \cap \mathcal{K} \neq \emptyset$ then $h[\mathcal{P}_q] \subset \mathcal{K}$. If it is not the case that $h[\mathcal{P}] \subset \mathcal{K}$, say, without loss of generality, that there are some $s, q \in \mathbb{Q}$ such that $s < q$, $h[\mathcal{P}_q] \subset \mathcal{K}$ and $h[\mathcal{P}_s] \cap \mathcal{K} = \emptyset$, then let r be the greatest lower bound of the rationals q such that $h[\mathcal{P}_q] \subset \mathcal{K}$. Now condition (2) implies that $r \in \mathbb{I}$; but the interval $(h[C_r], h[D_r])_{\mathcal{L}}$ must embed η_1, and so the rationals, contradicting (3). Then the argument proving Lemma 2.17 shows that $h = g \circ f_X$ where $g \colon \mathcal{Q}_X \to \mathcal{K}$ is an isomorphism.

Condition (3) is $\Pi^0_1(\mathbf{a})$, the universal quantification being over both irrational numbers and potential embeddings of $\mathbb{Q}$ into the intervals $(h[C_r], h[D_r])_{\mathcal{L}}$. Hence condition (3) can also be verified by $\mathbf{a}'$. The method, from the proof of Lemma 2.17, of enumerating X with oracle $\mathbf{a}'$ from $g \circ f_X$, is now applied to each of these maps, giving the desired $\mathbf{a}'$-computable enumeration of $\mathcal{F}$. ⊣

We can now use the result for $n = 1$ to extend it to all finite ordinals.

Theorem 2.19. *For any degree* $\mathbf{a}$ *and any nonzero* $n < \omega$, *there is an order-type whose degree spectrum is* $\{\mathbf{b} : \mathbf{b} > \mathbf{a}$ *and* $\mathbf{b}^{(n)} > \mathbf{a}^{(n)}\}$.

In particular, for any nonzero $n < \omega$, *there is an order-type whose degree spectrum consists of exactly the nonlow*$_n$ *degrees.*

Proof. We induct on n, simultaneously for all degrees $\mathbf{a}$, beginning with the case $n = 1$.

First, we relativize the proof of Theorem 2.18 to $\mathbf{a}$, obtaining a linear order $\mathcal{L}$ with presentations in every degree $\mathbf{b}$ with $\mathbf{b} > \mathbf{a}$ and $\mathbf{b}' > \mathbf{a}'$. Furthermore, the linear order $\mathcal{L}$ does not have a presentation in any degree $\mathbf{b}$ with $\mathbf{b} \geq \mathbf{a}$ and $\mathbf{b}' = \mathbf{a}'$.

Next, in order to handle degrees $\mathbf{b}$ with $\mathbf{b} \not\geq \mathbf{a}$, using Theorem 2.11, we fix a linear order $\mathcal{K}$ whose degree spectrum is the cone above $\mathbf{a}$. Then the degree spectrum of $\mathcal{L} + 1 + \mathcal{K}$ is the intersection of the degree spectra of $\mathcal{L}$ and $\mathcal{K}$, and so is as desired.

For $n > 1$, let $\mathcal{L}$ be a linear order whose degree spectrum consists of the degrees $\mathbf{b} > \mathbf{a}'$ such that $\mathbf{b}^{(n-1)} > \mathbf{a}^{(n)}$ (by the inductive hypothesis applied to $\mathbf{a}'$). Then the transfer $\mathcal{L}^{\mathrm{t}}$ has presentations in every degree $\mathbf{b}$ with $\mathbf{b} > \mathbf{a}$ and $\mathbf{b}^{(n)} > \mathbf{a}^{(n)}$. Furthermore, $\mathcal{L}^{\mathrm{t}}$ does not have a presentation in any degree $\mathbf{b}$ with $\mathbf{b} \geq \mathbf{a}$ and $\mathbf{b}^{(n)} = \mathbf{a}^{(n)}$. As in the case $n = 1$, the order $\mathcal{L}^{\mathrm{t}} + 1 + \mathcal{K}$ is as desired. ⊣

2.4. Arbitrary Finite Jump Degrees. The results of the previous section allow us to obtain results about the finite jump degrees of linear orders.

Definition 2.20. Fix a structure $\mathcal{A}$, a natural number $n < \omega$, and a degree $\mathbf{a}$. The structure $\mathcal{A}$ has n^{th} *jump degree* $\mathbf{a}$ if $\mathbf{a}$ is the least element of the set

$$\{\mathbf{d}^{(n)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}.$$

When $n = 0$, we say that $\mathcal{A}$ has *degree* $\mathbf{a}$.

For $n > 0$, the structure $\mathcal{A}$ has *proper n^{th} jump degree* $\mathbf{a}$ if $\mathcal{A}$ has n^{th} jump degree $\mathbf{a}$, but does not have any $(n-1)^{\text{st}}$ jump degree.

Thus, Theorem 2.11 can be restated as saying that every ω_1-degree is the degree of some linear ordering. Of course, as already noted, this contrasts rather sharply with the countable setting, where Richter [17] showed if a linear ordering has degree, that ω-degree must be $\mathbf{0}$. Furthermore, Knight [13] showed that if a countable linear ordering has a first jump degree, then this jump-degree must be $\mathbf{0}'$; whereas Downey and Knight [4] showed that for all $n \geq 2$, every degree $\mathbf{a} \geq \mathbf{0}^{(n)}$ is the proper n^{th} jump-degree of a countable linear ordering. In the uncountable setting, for every $n < \omega$, all possible (proper) jump degrees are realized.

Theorem 2.21. *Fix a finite ordinal $n < \omega$. For every ω_1-degree $\mathbf{b} \geq \mathbf{0}^{(n)}$, there is an order-type with proper n^{th} jump degree $\mathbf{b}$.*

Proof. For $n = 0$, this is Theorem 2.11.

For $n = 1$, from Fact 1.5, we obtain an ω_1-degree $\mathbf{a}$ with $\mathbf{a}' = \mathbf{b}$. We then relativize the proof of Theorem 2.7 to $\mathbf{a}$, obtaining a linear order $\mathcal{L}$. Then $\mathcal{L}$ has a presentation in every ω_1-degree $\mathbf{c}$ with $\mathbf{c} > \mathbf{a}$ and $\mathbf{c} \in \Delta^0_2(\mathbf{a})$. Notably, there are such ω_1-degrees $\mathbf{c}$ that are low over $\mathbf{a}$. Furthermore, the linear ordering $\mathcal{L}$ does not have a presentation in $\mathbf{a}$. As in the proof of Theorem 2.19, we take $\mathcal{L} + 1 + \mathcal{K}$, where $\mathcal{K}$ is a linear ordering such that $\text{DegSpec}(\mathcal{K})$ is the cone above $\mathbf{a}$.

For $n > 1$, from Fact 1.5 we obtain an ω_1-degree $\mathbf{a}$ with $\mathbf{a}^{(n)} = \mathbf{b}$. From Theorem 2.19 with $\mathbf{a}$ and $n-1$, we obtain a linear ordering $\mathcal{L}$ with degree spectrum $\{\mathbf{c} : \mathbf{c} > \mathbf{a} \text{ and } \mathbf{c}^{(n-1)} > \mathbf{a}^{(n-1)}\}$. By Fact 1.5 again, there is a $\mathbf{d} > \mathbf{a}^{(n-1)}$ with $\mathbf{d}' = \mathbf{b}$, and an $\mathbf{m}$ with $\mathbf{m}^{(n-1)} = \mathbf{d}$. Then $\mathbf{m} \in \text{DegSpec}(\mathcal{L})$, and $\mathbf{m}^{(n)} = \mathbf{b}$. Conversely, for every $\mathbf{c} \in \text{DegSpec}(\mathcal{L})$, since $\mathbf{c} > \mathbf{a}$, $\mathbf{c}^{(n)} \geq \mathbf{a}^{(n)} = \mathbf{b}$. ⊣

2.5. Open Questions on Degree Spectra. We close this section with some open questions on the degree spectra of linear orders.

Question 2.22. *Is there an order-type of size $\aleph_1$ whose degree spectrum consists of the nonzero ω_1-degrees?*

Question 2.23. *Is there, for each ordinal $\alpha < \omega_1$, an order-type with proper α^{th} jump degree $\mathbf{a}^{(\alpha)}$?*

§3. The Successor Relation. The *successor* (or *adjacency*) relation is central to understanding countable linear orders, both classically and effectively. For example, Hausdorff's analysis of universal (nonscattered) countable linear orders relies on his derivative operation of identifying adjacent points. Effectively, we mentioned the Remmel-Dzgoev characterization of computably categorical linear orderings in terms of their successor relation. Moses [15] showed that a computable linear ordering $\mathcal{L}$ is 1-decidable if and only if the successor relation on $\mathcal{L}$ is computable. This is one reason why the complexity of the successor relation on computable linear orderings was studied intensively, in particular in the theorem of Downey, Lempp and Wu mentioned above. Their result states that the Turing degrees of the successor relation of computable presentations of a computable order-type are closed upwards in the c.e. degrees, as long as, of course, the order-type has infinitely many adjacent ordered pairs. In this section, we show that the Downey–Lempp–Wu

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

theorem can fail for uncountable linear orderings and consider the consequences of this failure.

For a linear order $\mathcal{L}$, we denote the set of adjacent pairs in $\mathcal{L}$ by $\mathrm{Succ}(\mathcal{L})$.

DEFINITION 3.1. Let λ be an ω_1-computable order-type. Define

$$\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) := \{\deg_{\mathrm{T}}(\mathrm{Succ}(\mathcal{L})) : \mathcal{L} \text{ is a computable presentation of } \lambda\}.$$

Since the successor relation $\mathrm{Succ}(\mathcal{L})$ has a $\Pi^0_1(\mathcal{L})$-definition, for any ω_1-computable order-type λ, the set $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ consists only of ω_1-c.e. degrees. We start by demonstrating that the natural analogue of the Downey, Lempp, and Wu theorem (the assumption that λ contains uncountably many adjacent pairs) fails in the uncountable setting. We then provide a sufficient condition for upward closure.

EXAMPLE 3.2. The ω_1-computable order-type $2 \cdot \rho$ (where ρ is the order-type of $\mathbb{R}$) has uncountably many adjacent pairs and satisfies $\mathrm{DegSpec}_{\mathrm{Succ}}(2 \cdot \rho) = \{\mathbf{0}\}$. For let $\mathcal{L}$ be a computable presentation of $2 \cdot \rho$; let $f : 2 \cdot \mathbb{R} \to \mathcal{L}$ be an isomorphism, and let $Q := f[2 \cdot \mathbb{Q}]$. Then x, y in $\mathcal{L}$ are adjacent if and only if they lie in the same Q-interval. Since we can fix Q as a countable parameter, this gives an algorithm for computing $\mathrm{Succ}(\mathcal{L})$.

Our previous paper noted that $2 \cdot \rho$ is also ω_1-computably categorical, despite having uncountably many adjacent pairs. The sufficient condition we offer for upwards closure (Theorem 3.4) is also related to the condition for ω_1-computable categoricity. Here, the difference is that any level of density (rather than only $\aleph_1$-saturation) suffices as the successor relation is empty within any dense interval (regardless of whether or not it is saturated). Nonetheless, again the crucial hypothesis is the existence of something like a copy of the rational numbers, relative to which the intervals behave in a uniform way. The linear ordering $2 \cdot \mathbb{R}$ is "ρ-like": It contains a countable subset Q such that every Q-interval is finite.

DEFINITION 3.3. A linear order $\mathcal{L}$ is *weakly separable* if it contains a countable subset Q such that every Q-interval is either finite or dense.

THEOREM 3.4. *If λ is an ω_1-computable order-type which is not weakly separable, then the spectrum* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ *is closed upwards in the ω_1-c.e. degrees.*

PROOF. Let $\mathcal{L}$ be an ω_1-computable presentation of λ. Let W be an ω_1-c.e. set which computes $\mathrm{Succ}(\mathcal{L})$. Let $\langle W_s \rangle_{s<\omega_1}$ be a computable, increasing sequence of countable sets with $W = \bigcup_s W_s$. We build an ω_1-computable presentation $\mathcal{K} \in \lambda$ such that $\mathrm{Succ}(\mathcal{K}) \equiv_{\mathrm{T}} W$.

As in previous constructions, we let $\mathcal{L}_s := \mathcal{L} \upharpoonright s$ and build $\mathcal{K}$ as the union of an increasing, ω_1-computable sequence $\langle \mathcal{K}_s \rangle$ of countable linear orderings. To ensure that $\mathcal{K}$ is isomorphic to $\mathcal{L}$, we construct a Δ^0_2-isomorphism $F : \mathcal{K} \to \mathcal{L}$ as the limit of an ω_1-computable sequence of isomorphisms $F_s : \mathcal{K}_s \to \mathcal{L}_s$. Of course, we cannot make $\mathcal{K}$ and $\mathcal{L}$ computably isomorphic, else we would have $\mathrm{Succ}(\mathcal{K}) \equiv_{\mathrm{T}} \mathrm{Succ}(\mathcal{L})$.

To get W to compute $\mathrm{Succ}(\mathcal{K})$, we will ensure that W computes F. To get $\mathrm{Succ}(\mathcal{K})$ to compute W, we will ensure that the complement of W is ω_1-c.e. in $\mathrm{Succ}(\mathcal{K})$. We define an *enumeration functional* Φ; axioms enumerated into Φ at stage s will name countably many successor pairs in $\mathcal{K}_s$, and declare that if all of these pairs are indeed

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

successor pairs in $\mathcal{K}$, then some number x is enumerated into the Succ($\mathcal{K}$)-c.e. set $\Phi(\mathrm{Succ}(\mathcal{K}))$. At stage s, we let $\Phi(\mathrm{Succ}(\mathcal{K}))[s]$ be the result of applying Φ_s, the functional as enumerated up to stage s, on the collection of adjacencies in $\mathcal{K}_s$. For all $i < \omega_1$, requirement R_i states that $i \in \Phi(\mathrm{Succ}(\mathcal{K}))$ if and only if $i \notin W$.

Informally, we describe the strategy for meeting a requirement R_i. As long as $i \notin W_s$, we take the $<_{\omega_1}$-least available successor pair (a, b) in $\mathcal{K}_s$, and with the information that $(a, b) \in \mathrm{Succ}(\mathcal{K}_s)$ we enumerate i into $\Phi(\mathrm{Succ}(\mathcal{K}))[s]$. If later we see that i enters W_t, we want to enumerate a new element into $\mathcal{K}_t$ between a and b. We need to, in advance, pick the pair (a, b) so that adding such an element will still allow us to embed $\mathcal{K}_t$ into $\mathcal{L}_t$, possibly by changing F. Not surprisingly, the choice of (a, b) depends on whether $\mathcal{L}_s$ is scattered or nonscattered; in the scattered case, we will in fact need to use all the pairs in some infinite block.

Meanwhile, if i does not enter W, we need to maintain the adjacency of the pair (a, b). Of course $\mathcal{L}$ may force us to enumerate an element between a and b, by enumerating an element between $F_s(a)$ and $F_s(b)$. In this case, we just need to pick another pair; this will reach a limit. However, we need to actively prevent weaker requirements R_j for $j > i$ from enumerating elements between a and b. This is done by imposing restraint; weaker requirements are not allowed to change $F_s(a)$ and $F_s(b)$. This accumulated restraint gives a requirement R_i a countable set on which it is not allowed to change F; it needs to work in the intervals determined by this countable set, and find adjacencies in one of them. This is where the assumption on the structure of $\mathcal{L}$ comes into use.

Construction: For $j < \omega_1$, by recursion, we let $I_{j,s}$ be the set of stages less than s at which requirement R_j *requires attention* (as defined below). We define

$$r_{j,s} := \sup\left\{ t + 1 : t \in \bigcup_{i<j} I_{i,s} \right\}.$$

Let $s < \omega_1$, and suppose that $\mathcal{K}_s$ and F_s are recursively defined. A requirement R_j *requires attention* at stage s if $j < s$, $\Phi(\mathrm{Succ}(\mathcal{K}))(j) = W(j)[s]$ (i.e. $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s] \iff j \in W_s$), and there is some $\mathcal{K}_{r_{j,s}}$-interval of $\mathcal{K}_s$ (i.e. a maximal interval in $\mathcal{K}_s$ disjoint from $\mathcal{K}_{r_{j,s}}$) which is infinite and not dense. We act on behalf of the strongest requirement which requires attention, as described below. If no requirement requires attention at stage s, then we simply let $\mathcal{K}_{s+1}$ and F_{s+1} be extensions of $\mathcal{K}_s$ and F_s such that $F_{s+1} : \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$ is an isomorphism.

Otherwise, let R_j be the strongest requirement requiring attention at stage s. If $j \notin W_s$, we let (S_1, S_2) be the $<_{\omega_1}$-least cut of $\mathcal{K}_{r_{j,s}}$ such that $A_s := (S_1, S_2)_{\mathcal{K}_s}$ is infinite and not dense. If A_s is scattered, let T_s be the $<_{\omega_1}$-least infinite block of A_s. If A_s is nonscattered, let T_s be the $<_{\omega_1}$-least subset $\{a, b\}$ of A_s such that a and b are adjacent in A_s. In either case, enumerate a new axiom into Φ, enumerating j into $\Phi(\mathrm{Succ}(\mathcal{K}))[s + 1]$. The use of this computation is $\mathrm{Succ}(T_s) \cup \big(\mathrm{Succ}(\mathcal{K}_{r_{j,s}}) \cap \mathrm{Succ}(\mathcal{K}_s)\big)$, the collection of all successor pairs in T_s, along with all successor pairs in $\mathcal{K}_{r_{j,s}}$ that remain successor pairs in $\mathcal{K}_s$. We again let $\mathcal{K}_{s+1}$ and F_{s+1} be extensions so that $F_{s+1} : \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$ is an isomorphism.

If $j \in W_s$, we need to change $\mathcal{K}_{s+1}$ to extract j from $\Phi(\mathrm{Succ}(\mathcal{K}))$. Let $t < s$ be the stage at which the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined. (Note that at most one such computation can apply to the current oracle $\mathrm{Succ}(\mathcal{K})[s]$ at any stage.) Say

$A_t = (S_1, S_2)_{\mathcal{K}_t}$. Then T_t is still a convex subset of $A_s = (S_1, S_2)_{\mathcal{K}_s}$, as otherwise j would already be extracted from $\Phi(\mathrm{Succ}(\mathcal{K}))$. We can find a self-embedding f of A_s such that for some adjacent $a, b \in T_t$, $f(a)$ and $f(b)$ are not adjacent in A_s; this is either because T_t is an infinite block of A_s, or A_s is nonscattered. As A_s is a convex subset of $\mathcal{K}_s$, we extend f to a self-embedding of $\mathcal{K}_s$ by being the identity outside A_s. We then extend $\mathcal{K}_s$ and $F_s \circ f$ to $\mathcal{K}_{s+1}$ and an isomorphism $F_{s+1} \colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$. This definition ensures the enumeration of some point between some successor pair of T_t, and so $j \notin \Phi(\mathrm{Succ}(\mathcal{K}))[s+1]$.

At limit stages, we define $\mathcal{K}_{<s} := \bigcup_{t<s} \mathcal{K}_t$, and define $F_{<s} := \lim_{t \to s} F_t$ to be the limit embedding of $\mathcal{K}_{<s}$ into $\mathcal{L}_s$. We then let $\mathcal{K}_s$ and F_s be an extension of $\mathcal{K}_{<s}$ and $F_{<s}$ to an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$.

We argue now that $F_{<s}$ is well-defined, using Claim 3.4.2. Suppose that $\langle F_t \rangle_{t<s}$ is not increasing on some final segment of s. One of two cases must hold. Suppose first that there is some limit $j \leq s$ such that for all $i < j$, $r_{i,s} < s$ but $s = \sup_{i<j} r_{i,s}$. In this case, $F_{<s} = \bigcup_{i<j} F_{r_{i,s}}$.

Otherwise, there is some $j < s$ such that $r_{j,s} < s$ but $I_{j,s}$ is unbounded in s. In this case, consider $I_{j,s} \cap [r_{j,s}, s)$. If $r_{j,s} < q < s$ and t is the greatest element of $I_{j,s} \cap [r_{j,s}, s)$ with $t < q$, then $r_{j+1,q} = t + 1$. By Claim 3.4.2, F_q extends F_{t+1}. At the same time, since R_j requires attention at cofinally many stages before stage s, it must be that $j \notin W_s$, so at each $t \in I_{j,s} \cap [r_{j,s}, s)$, our action makes F_{t+1} an extension of F_t. Thus $(F_{t+1})_{t \in I_{j,s} \cap [r_{j,s}, s)}$ is an increasing sequence. Since F_q extends F_{t+1} for t the greatest element of $I_{j,s} \cap [r_{j,s}, s)$ with $t < q$, it follows that $F_{<s} = \bigcup_t F_{t+1}$ for $t \in I_{j,s} \cap [r_{j,s}, s)$.

Verification: First, we show that restraints are respected. The following two claims are proved by simultaneous induction on s, and verify the promises made during the construction.

Claim 3.4.1. *Fix $i, j, s < \omega_1$ with $i < j < s$, $i \notin W_s$, and $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$. Then $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$. Moreover, the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined before the stage at which the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined. It follows that every successor pair used in the computation of $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ is also used in the computation of $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$.*

Proof. Let $t < s$ be the stage at which the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined. Then there is some infinite, nondense $\mathcal{K}_{r_{j,t}}$-interval of $\mathcal{K}_t$. Since $r_{j,t} \geq r_{i,t}$, there is an infinite, nondense $\mathcal{K}_{r_{i,t}}$-interval of $\mathcal{K}_t$. Since $i \notin W_t$, we can conclude that $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$, as otherwise R_i would require attention at stage t. Let $u < t$ be the stage at which the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$ was defined by R_i.

By construction, the computation asserting that $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ uses every successor pair in $\mathcal{K}_{r_{j,t}}$. Since $u \in I_{i,t}$, we have $r_{j,t} > u$, and thus $r_{j,t} \geq u + 1$. By construction, every successor pair used in the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$ is a successor pair of $\mathcal{K}_{u+1}$. Since $t \geq r_{j,t} \geq u + 1$, and the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$ persisted from stage $u + 1$ to stage t, it must be that every successor pair used in this second computation remains a successor pair at stage t, and so also at stage $r_{j,t}$. So the computation asserting that $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ uses every successor pair that was used in the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$. Since

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

$j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ holds, all of these pairs must be successor pairs of $\mathcal{K}_s$, so $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$. ⊣

CLAIM 3.4.2. *For all $j, s < \omega_1$ with $j < s$, the map F_s extends the map $F_{r_{j,s}}$.*

PROOF. We prove this by induction on s. As we take limits at limit stages, we need only consider a successor stage s. Suppose that $r_{j,s} < s$. By induction, the map F_{s-1} extends the map F_r. If F_s extends F_{s-1} then we are done. Suppose otherwise. Some requirement R_i receives attention at stage $s-1$, and extracts i from $\Phi(\mathrm{Succ}(\mathcal{K}))[s]$. Since $r_{j,s} < s$, we must have $i \geq j$ by definition of $r_{j,s}$. Let $t < s-1$ be the stage at which the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s-1]$ was defined. Note that we have $F_s(x) = F_{s-1}(x)$ for all $x \notin A_{s-1}$.

Since $i \geq j$, we have $r_{i,s-1} \geq r_{j,s-1} = r_{j,s}$. Suppose R_k is some stronger requirement. By Claim 3.4.1, the persistence of the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s-1]$ shows that there can be no stage $q \in [t, s-1)$ with $k \notin W_q$ and R_k requiring attention.

Suppose there is a stage $q \in [t, s-1)$ with $k \in W_q$ and we act for R_k at stage q. Let $u < q$ be the stage at which the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[q]$ was defined. Then at this stage R_k required attention and $k \notin W_u$, so $u < t$. But then $r_{j,t} \geq u+1$ by definition, so every successor pair used in the computation $k \in \Phi(\mathrm{Succ}(\mathcal{K}))[q]$ was also used in the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s-1]$. But our action at stage q enumerated a point between one of these pairs, contrary to the persistence of $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s-1]$. So there can be no such stage q.

The fact that we acted for R_i shows that no requirement R_k stronger than R_i required attention at stage $s-1$. So no stronger requirement required attention at any stage in $[t, s-1]$. Hence $r_{i,s-1} = r_{i,t}$. Since A_t is a $\mathcal{K}_{r_{i,t}}$-interval, it follows that A_{s-1} is an $\mathcal{K}_{r_{j,s}}$-interval. Hence A_{s-1} and $\mathcal{K}_{r_{j,s}}$ are disjoint; so F_s and F_{s-1} agree on $\mathcal{K}_{r_{j,s}}$ as required. ⊣

The argument defining $F_{<s}$ for a limit stage s shows that $F := F_{<\omega_1} = \lim_{s<\omega_1} F_s$ is well-defined, and is an isomorphism from $\mathcal{K} := \mathcal{K}_{<\omega_1}$ to $\mathcal{L}$.

CLAIM 3.4.3. *For all $j < \omega_1$, $r_{j,\omega_1} < \omega_1$, and requirement R_j is met.*

PROOF. To show $r_{j,\omega_1} < \omega_1$, for all j, it suffices to show that I_{j,ω_1} is bounded for all j. This is proved by induction.

If $r_{j,\omega_1} < \omega_1$, then we show that I_{j,ω_1} is bounded. If $j \in W$, then R_j requires attention at most once after a stage s at which $j \in W_s$; when we act for R_j then, we ensure $j \notin \Phi(\mathrm{Succ}(\mathcal{K}))$, and then by definition R_j never again requires attention.

Suppose that $j \notin W$. Let $S := \mathcal{K}_{r_{j,\omega_1}}$. Since $\mathcal{L}$ is not weakly separable, neither is $\mathcal{K}$. Hence there is some S-interval of $\mathcal{K}$ which is infinite and nondense. Since S is countable, there is a stage $t \geq r_{j,\omega_1}$ such that if $(a, b) \in \mathrm{Succ}(\mathcal{K}_t)$ and $a, b \in S$ then $(a, b) \in \mathrm{Succ}(\mathcal{K})$. Let (S_1, S_2) be the $<_{\omega_1}$-least cut of S such that $(S_1, S_2)_{\mathcal{K}}$ is infinite and nondense. If $(S_1, S_2)_{\mathcal{K}}$ is scattered, let T be the $<_{\omega_1}$-least infinite block of $(S_1, S_2)_{\mathcal{K}}$. If $(S_1, S_2)_{\mathcal{K}}$ is nonscattered, let T be the $<_{\omega_1}$-least adjacent pair of $(S_1, S_2)_{\mathcal{K}}$. Then if requirement R_j requires attention at cofinitely many stages, eventually a computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))$ is created at some stage $s > t$ where the use of this computation is $\mathrm{Succ}(T) \cup \big(\mathrm{Succ}(\mathcal{K}_{r_{j,s}}) \cap \mathrm{Succ}(\mathcal{K}_s)\big)$. Since $s > t$, $\mathrm{Succ}(\mathcal{K}_{r_{j,s}}) \cap \mathrm{Succ}(\mathcal{K}_s) \subseteq \mathrm{Succ}(\mathcal{K})$. By assumption, $\mathrm{Succ}(T) \subseteq \mathrm{Succ}(\mathcal{K})$. Thus this computation will persist at all later stages, implying both that R_j never again requires attention, and that R_j ensures its requirement. ⊣

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

Claim 3.4.4. *F is computable from W.*

Proof. Let $x \in \mathcal{K}$. To compute $F(x)$ with oracle W, find a stage $s < \omega_1$ and an index j such that $x \in \mathcal{K}_{r_{j,s}}$, $W \upharpoonright j = W_s \upharpoonright j$, and $\Phi(\text{Succ}(\mathcal{K}))(i) \neq W(i)\,[s]$ for all $i < j$. We claim that $F(x) = F_s(x) = F_{r_{j,s}}(x)$. This is because no requirement R_i, for $i < j$, will cause a redefinition of F_t after stage s, and so for all $t > s$, the map $F_{r_{j,t}}$ extends the map $F_{r_{j,s}}$, and so F_t extends $F_{r_{j,s}}$ (Claim 3.4.2). ⊣

Since W computes both F and $\text{Succ}(\mathcal{L})$, it also computes $\text{Succ}(\mathcal{K})$. Moreover, $\Phi(\text{Succ}(\mathcal{K}))$ and therefore the complement of W are ω_1-c.e. in $\text{Succ}(\mathcal{K})$, and so $\text{Succ}(\mathcal{K})$ computes W. This completes the proof. ⊣

We turn our attention now to weakly separable linear orders. Example 3.2 shows that upward closure can fail for such orders. It is natural to ask if this is the only way in which such failure can occur; if $\text{DegSpec}_{\text{Succ}}(\mathcal{L})$ is not upwards closed, must it be $\{\mathbf{0}\}$? We require the following definition.

Definition 3.5. Let A be an uncountable ω_1-c.e. set. If $f : \omega_1 \to A$ and $g : \omega_1 \to A$ are injective ω_1-computable enumerations of A, then for all $B \subseteq A$, the sets $f^{-1}B$ and $g^{-1}B$ are Turing equivalent (indeed they are 1-1 equivalent). We thus define, for all $B \subseteq A$, $\deg_{\text{T}}(B|A)$ to be the Turing degree of $f^{-1}B$, where f is any injective computable enumeration of A.

The point is that passing from B to $f^{-1}B$ erases the complexity of A. Certainly $\deg_{\text{T}}(A|A) = \mathbf{0}$. For all $B \subseteq A$, $\deg_{\text{T}}(B|A) \le \deg_{\text{T}}(B)$. If A is computable, then for all $B \subseteq A$, $\deg_{\text{T}}(B|A) = \deg_{\text{T}}(B)$.

The degree $\deg_{\text{T}}(B|A)$ is the amount of information coded in B once we know that it is a subset of A. This intuition is explained as follows. For all C, $\deg_{\text{T}}(C) \le \deg_{\text{T}}(B|A)$ if and only if there is a reduction of C to B which only queries the oracle on elements of A. Similarly, $\deg_{\text{T}}(B|A) \le \deg_{\text{T}}(C)$ if and only if there is a partial reduction Φ such that for all $x \in A$, $B(x) = \Phi(C, x)$; the reduction $\Phi(C)$ may not halt on inputs outside A. This is why we informally write, for example, $C \le_{\text{T}} (B|A)$, even though there is no fixed *set* $B|A$.

This definition also works for strong reducibilities. We say that $B \le_{\text{wtt}} C$ (where B and C are subsets of ω_1) if there is a Turing functional Φ and a computable function φ such that $\Phi(C) = B$ and such that for all $x < \omega_1$, $\Phi(C \upharpoonright \varphi(x))$ extends $B \upharpoonright x$. In other words, the use of the computation is bounded by φ. We say that $B \le_{\text{m}} C$ if there is a computable function g with $x \in B \iff g(x) \in C$. For $B \subseteq A$, we write $\deg_{\text{wtt}}(B|A)$ for $\deg_{\text{wtt}}(f^{-1}B)$, where f is any injective computable enumeration of A. Similarly, we write $\deg_{\text{m}}(B|A)$ for $\deg_{\text{m}}(f^{-1}B)$ for any such B. We note that neither of these depend on the choice of computable function f.

Definition 3.6. Let $\mathcal{L}$ be an ω_1-computable, weakly separable linear order, witnessed by a countable subset Q of $\mathcal{L}$.

For a set $\mathfrak{C}$ of cardinals, we let $I^Q_{\mathfrak{C}}(\mathcal{L})$ be the set of cuts (Q_1, Q_2) of Q such that the size of $(Q_1, Q_2)_{\mathcal{L}}$ is in $\mathfrak{C}$. We use obvious abbreviations: For example, we write $I^Q_\kappa(\mathcal{L})$ for $I^Q_{\{\kappa\}}(\mathcal{L})$, $I^Q_{>\kappa}(\mathcal{L})$ for $I^Q_{(\kappa,\aleph_1]}(\mathcal{L})$, and $I^Q_\infty(\mathcal{L})$ for $I^Q_{\{\aleph_0,\aleph_1\}}(\mathcal{L})$.

Observe that $I^Q_{>1}(\mathcal{L})$ is a c.e. set.

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

LEMMA 3.7. *Let λ be an ω_1-computable, weakly separable order-type. Let $\mathfrak{C}$ be a set of finite cardinals. Then* $\deg_{\mathrm{m}}(I^{Q}_{\mathfrak{C}}(\mathcal{L}))$ *does not depend on the choice of the computable presentation $\mathcal{L}$ of λ and the countable subset Q of $\mathcal{L}$ witnessing that $\mathcal{L}$ is weakly separable.*

PROOF. If $\mathcal{L}$ and $\mathcal{K}$ are ω_1-computable presentations of λ, $F\colon \mathcal{L} \to \mathcal{K}$ is an isomorphism, and Q witnesses that $\mathcal{L}$ is weakly separable, then $I^{Q}_{\mathfrak{C}}(\mathcal{L})$ and $I^{F[Q]}_{\mathfrak{C}}(\mathcal{K})$ are 1-1 equivalent. Here F need not be computable, since we only use the countable parameter $F \restriction Q$.

Hence it suffices to fix an ω_1-computable presentation $\mathcal{L}$ of λ and show that if S and Q both witness that $\mathcal{L}$ is weakly separable, then $I^{Q}_{\mathfrak{C}}(\mathcal{L}) \leq_{\mathrm{m}} I^{S}_{\mathfrak{C}}(\mathcal{L})$.

Fix such Q and S. If $I^{Q}_{\mathfrak{C}}(\mathcal{L})$ is computable, then there is nothing to show. Thus we may assume that $\mathcal{L}$ contains uncountably many maximal finite blocks. Since Q and S are countable, they intersect only countably many maximal (finite) blocks of $\mathcal{L}$. Since infinite S- and Q-intervals of $\mathcal{L}$ are dense, it follows that for all but countably many cuts (Q_1, Q_2) of Q, if the interval $(Q_1, Q_2)_{\mathcal{L}}$ is finite, then it is a maximal block of $\mathcal{L}$, and must be an S-interval as well.

So outside a countable set of cuts, given a cut (Q_1, Q_2) of Q, we search for either a cut (S_1, S_2) such that $(Q_1, Q_2)_{\mathcal{L}} = (S_1, S_2)_{\mathcal{L}}$, or a stage at which we see that $(Q_1, Q_2)_{\mathcal{L}}$ is infinite. In the former case, we, of course, know that $(Q_1, Q_2) \in I^{Q}_{\mathfrak{C}}(\mathcal{L})$ if and only if $(S_1, S_2) \in I^{S}_{\mathfrak{C}}(\mathcal{L})$. In the latter case, we know without consulting the oracle that $(Q_1, Q_2) \notin I^{Q}_{\mathfrak{C}}(\mathcal{L})$. ⊣

DEFINITION 3.8. Let λ be an ω_1-computable weakly separable order-type with uncountably many adjacencies. Fix any computable presentation $\mathcal{L}$ of λ and any set $Q \subseteq \mathcal{L}$ witnessing that $\mathcal{L}$ is weakly separable. We define the following degrees:

$$\mathbf{min}(\lambda) := \deg_{\mathrm{T}}\left(I^{Q}_{\infty}(\mathcal{L}) \mid I^{Q}_{>1}(\mathcal{L})\right),$$

$$\mathbf{min}_{\mathrm{wtt}}(\lambda) := \deg_{\mathrm{wtt}}\left(I^{Q}_{\infty}(\mathcal{L}) \mid I^{Q}_{>1}(\mathcal{L})\right),$$

$$\mathbf{max}(\lambda) := \bigvee_{n\geq 2} \deg_{\mathrm{T}}\left(I^{Q}_{n}(\mathcal{L}) \mid I^{Q}_{>1}(\mathcal{L})\right),$$

and

$$\mathbf{max}_{\mathrm{wtt}}(\lambda) := \bigvee_{n\geq 2} \deg_{\mathrm{wtt}}\left(I^{Q}_{n}(\mathcal{L}) \mid I^{Q}_{>1}(\mathcal{L})\right).$$

By Lemma 3.7, these do not depend on the choice of $\mathcal{L}$ and Q. The set $I^{Q}_{\infty}(\mathcal{L})$ is c.e., and so $\mathbf{min}(\lambda)$ is a c.e. degree. It is not immediately clear, but we will see that $\mathbf{max}(\lambda)$ is also a c.e. degree.

As we shall immediately see, the degrees $\mathbf{min}(\lambda)$ and $\mathbf{max}(\lambda)$ constrain the degree spectrum $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$. This explains why they are both defined inside $I^{Q}_{>1}(\mathcal{L})$: In measuring the complexity of $\mathrm{Succ}(\mathcal{L})$, we need to avoid the false complexity that can be added by the set of intervals containing fewer than two points. Of course, such intervals cannot add complexity to the successor relation.

THEOREM 3.9. *Let λ be an ω_1-computable, weakly separable order-type with uncountably many adjacencies. Then* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ *is contained in the interval of degrees* $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$.

In fact, for every computable presentation $\mathcal{L}$ of λ, $\mathrm{Succ}(\mathcal{L}) \leq_{\mathrm{wtt}} \mathbf{max}_{\mathrm{wtt}}(\lambda)$.

Proof. Let $\mathcal{L}$ be an ω_1-computable presentation of λ; let Q witness that $\mathcal{L}$ is weakly separable. We need to show that $\mathrm{Succ}(\mathcal{L}) \geq_{\mathrm{T}} \left(I^Q_\infty(\mathcal{L}) \mid I^Q_{>1}(\mathcal{L})\right)$ and that $\mathrm{Succ}(\mathcal{L}) \leq_{\mathrm{wtt}} \bigoplus_{n\geq 2} \left(I^Q_n(\mathcal{L}) \mid I^Q_{>1}(\mathcal{L})\right)$.

Since $I^Q_\infty(\mathcal{L})$ is c.e., to compute it from $\mathrm{Succ}(\mathcal{L})$ inside $I^Q_{>1}(\mathcal{L})$ it is sufficient to enumerate its complement inside $I^Q_{>1}(\mathcal{L})$, i.e., to enumerate the set $\bigcup_{n\geq 2} I^Q_n(\mathcal{L})$ with oracle $\mathrm{Succ}(\mathcal{L})$. To do so, given some cut (Q_1, Q_2) such that the interval $(Q_1, Q_2)_\mathcal{L}$ contains at least two points, we enumerate (Q_1, Q_2) if we find some pair (a, b) in $\mathrm{Succ}(\mathcal{L})$ with $a, b \in (Q_1, Q_2)_\mathcal{L}$; the point, of course, is that the interval is infinite if and only if it is dense. Note that the use of this enumeration may not be bounded by a computable function, as the $<_{\omega_1}$-least successor pair in a finite interval $(Q_1, Q_2)_\mathcal{L}$ may appear much later than the cut (Q_1, Q_2).

For the second reduction, we first note that

$$\left(I^Q_\infty(\mathcal{L})|I^Q_{>1}(\mathcal{L})\right) \leq_{\mathrm{wtt}} \bigoplus_{n\geq 2} \left(I^Q_n(\mathcal{K})|I^Q_{>1}(\mathcal{L})\right),$$

which is, of course, necessary for the theorem. This is because inside $I^Q_{>1}(\mathcal{L})$, $I^Q_\infty(\mathcal{L})$ and $\bigcup_{n\geq 2} I^Q_n(\mathcal{L})$ are complements. In other words, given $(Q_1, Q_2) \in I^Q_{>1}(\mathcal{L})$, we need only make the queries "$(Q_1, Q_2) \in I^Q_n(\mathcal{L})$?" for all $n \geq 2$. If any of these queries returns positively, then $(Q_1, Q_2) \notin I^Q_\infty(\mathcal{L})$, while if they all return negatively, then $(Q_1, Q_2) \in I^Q_\infty(\mathcal{L})$. Since this is only countably many queries, it describes a Turing reduction. Further, since we can precisely compute the set of queries we will need from the input, there is a computable bound on (the codes for) the queries.

We compute $\mathrm{Succ}(\mathcal{L})$ from $\bigoplus_{n\geq 2} \left(I^Q_n(\mathcal{L}) \mid I^Q_{>1}(\mathcal{L})\right)$. Let $a <_\mathcal{L} b$ be elements of $\mathcal{L}$; we want to decide if $(a, b) \in \mathrm{Succ}(\mathcal{L})$. We may assume that $a, b \notin Q$. This is because $\mathrm{Succ}(\mathcal{L}) \cap ((Q \times \mathcal{L}) \cup (\mathcal{L} \times Q))$ is countable, as Q is countable and every element of Q has at most one successor and one predecessor.

We first decide if a and b are in the same Q-interval; if not, then $(a, b) \notin \mathrm{Succ}(\mathcal{L})$. If so, let (Q_1, Q_2) be the cut of Q such that $a, b \in (Q_1, Q_2)_\mathcal{K}$. Then $(Q_1, Q_2) \in I^Q_{>1}(\mathcal{L})$. We may therefore ask the oracle if the interval $(Q_1, Q_2)_\mathcal{L}$ is finite, using the reduction just described above. If not, then it is dense, and so $(a, b) \notin \mathrm{Succ}(\mathcal{L})$. If so, the oracle gives us the size n of $(Q_1, Q_2)_\mathcal{K}$. We wait for a stage s such that $(Q_1, Q_2)_{\mathcal{K}\restriction s}$ already contains n points; then $(a, b) \in \mathrm{Succ}(\mathcal{L})$ if and only if a and b are adjacent in $\mathcal{L} \restriction s$.

The use of this computation is bounded by a computable function because the cut (Q_1, Q_2) is obtained effectively from a and b. ⊣

Having shown that the complexity of the successor relation is bounded within an interval, we turn to seeing which degrees in this interval belong to the spectrum of the successor relation. We first show that both endpoints always belong to the spectrum.

Theorem 3.10. *Let λ be an ω_1-computable, weakly separable order-type with uncountably many adjacencies. Then* $\mathbf{max}(\lambda) \in \mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$.

In particular, the degree $\mathbf{max}(\lambda)$ is c.e.

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

Proof. Let $\mathcal{L}$ be an ω_1-computable presentation of λ, and let Q witness that $\mathcal{L}$ is weakly separable. We build a computable copy $\mathcal{K}$ of $\mathcal{L}$ and an isomorphism $F: \mathcal{K} \to \mathcal{L}$ such that $F^{-1}Q = Q$, and such that for all $n \geq 2$, $(I_n^Q(\mathcal{K}) \mid I_{>1}^Q(\mathcal{K})) \leq_T \mathrm{Succ}(\mathcal{K})$. By Theorem 3.9 this is sufficient. Note that uniformity in n is free in ω_1-computability, but is anyway obvious from the proof.

By regularity of ω_1, let $\langle \mathcal{L}_s \rangle$ be a continuous, computable and increasing sequence of countable linear orderings such that $\mathcal{L} = \bigcup_s \mathcal{L}_s$, such that $\mathcal{L}_0 = Q$, and such that for all s, every Q-interval of $\mathcal{L}_s$ is either finite or dense. We define $\mathcal{K}$ as the union of a computable and increasing sequence $\langle \mathcal{K}_s \rangle$; for all s, we define an isomorphism $F_s : \mathcal{K}_s \to \mathcal{L}_s$. We start with $\mathcal{K}_0 = \mathcal{L}_0 = Q$ and $F_0 = \mathrm{id}_Q$. For all s, F_s will extend F_0, so to define $\mathcal{K}_s$ and F_s, it is sufficient, given a nonempty Q-interval $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$ of $\mathcal{L}_s$, to define $A_s = (Q_1, Q_2)_{\mathcal{K}_s}$ and the isomorphism $F_s \restriction A_s$ from A_s to B_s.

The idea for coding $I_n^Q(\mathcal{L})$ for each $n \geq 2$ into $\mathrm{Succ}(\mathcal{K})$ is by copying $\mathcal{L}$, but whenever we extend a finite Q-interval A_s to a larger A_{s+1}, we insert new points so that we destroy at least one adjacency in A_s. This way, $\mathrm{Succ}(\mathcal{K})$ can keep track of the size of $(Q_1, Q_2)_{\mathcal{L}}$.

So the instructions are simple. At stage s, given $\mathcal{K}_s$ and F_s, fix a cut (Q_1, Q_2) of Q such that $B_{s+1} = (Q_1, Q_2)_{\mathcal{L}_{s+1}}$ is nonempty. Suppose that $B_{s+1} \neq B_s$ (where, of course, $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$), that B_{s+1} is finite and that $A_s = (Q_1, Q_2)_{\mathcal{K}_s}$ contains at least two points. We then define A_{s+1} extending A_s which has the same size as B_{s+1}, but such that some $a, b \in A_s$ which are adjacent in A_s are no longer adjacent in A_{s+1}. We then let $F_{s+1} \restriction A_{s+1}$ be the unique isomorphism from A_{s+1} to B_{s+1}.

In all other cases (if $B_{s+1} = B_s$, or $|A_s| \leq 1$, or B_{s+1} is infinite), we let $F_{s+1} \restriction A_{s+1}$ be an extension of $F_s \restriction A_s$ to an isomorphism from A_{s+1} to B_{s+1}, and, of course, define A_{s+1} accordingly.

At a limit stage s, let $\mathcal{K}_{<s} = \bigcup_{t<s} \mathcal{K}_t$. Let $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$ be a nonempty Q-interval of $\mathcal{L}_s$; let $A_{<s} = (Q_1, Q_2)_{\mathcal{K}_{<s}} = \bigcup_{t<s} A_t$, where, of course, $A_t = (Q_1, Q_2)_{\mathcal{K}_t}$. We define an embedding $F_{<s} \restriction A_{<s}$ from $A_{<s}$ to B_s, and then extend it to an isomorphism $F_s \restriction A_s$ from A_s to B_s by adding points to $A_{<s}$. If $\langle F_t \restriction A_t \rangle_{t<s}$ is increasing on some final segment of s, then we let $F_{<s} \restriction A_{<s}$ be the limit of these maps. Otherwise, since $F_t \restriction A_t$ only changes when $B_{t+1} \neq B_t$, we see that B_s is infinite, and so dense, so we let $F_{<s}$ be any embedding of $A_{<s}$ into B_s.

This defines $\mathcal{K}$. We argue that $F = \lim_s F_s$ is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. This is because for every Q-interval A_{ω_1} of $\mathcal{K}$, the sequence $\langle F_s \restriction A_s \rangle$ is eventually increasing. For either A_{ω_1} is finite, in which case eventually the sequence stabilizes; or eventually A_s is infinite, after which the sequence is increasing.

Now let $n \geq 2$; and we will see how to compute $(I_n^Q(\mathcal{K}) | I_{>1}^Q(\mathcal{K}))$ from $\mathrm{Succ}(\mathcal{K})$. Let (Q_1, Q_2) be a cut of Q, and suppose that $A_{\omega_1} = (Q_1, Q_2)_{\mathcal{K}}$ contains at least two points. With oracle $\mathrm{Succ}(\mathcal{K})$ we can find a stage s such that either $A_s = (Q_1, Q_2)_{\mathcal{K}_s}$ is infinite, or A_s is finite, contains at least two points, and every adjacency in A_s is an adjacency in $\mathcal{K}$. The construction ensures that in the latter case we have $A_s = A_{\omega_1}$, so we can compute the size of A_{ω_1}. ⊣

Again we emphasize the need to work within $I_{>1}^Q(\mathcal{L})$. The procedure above will not halt if we start with a cut (Q_1, Q_2) such that $(Q_1, Q_2)_{\mathcal{L}}$ contains at most one point. This is why $\deg_T(\mathrm{Succ}(\mathcal{K}))$ lies above each $\deg_T(I_n^Q(\mathcal{K}) \mid I_{>1}^Q(\mathcal{K}))$, and not necessarily above $\deg_T(I_n^Q(\mathcal{K}))$.

Note that the use of the reduction of $(I_n^Q(\mathcal{K}) \mid I_{>1}^Q(\mathcal{K}))$ to $\mathrm{Succ}(\mathcal{K})$ is not necessarily computably bounded. We do not know if there is always a computable presentation $\mathcal{L}$ of λ such that $\mathrm{Succ}(\mathcal{L}) \in \mathbf{max}_{\mathrm{wtt}}(\lambda)$.

THEOREM 3.11. *Let λ be an ω_1-computable, weakly separable order-type with uncountably many adjacencies. Then* $\mathbf{min}(\lambda) \in \mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$.

In fact, we can build an ω_1-computable presentation $\mathcal{L}$ of λ such that $\mathrm{Succ}(\mathcal{L}) \in \mathbf{min}_{\mathrm{wtt}}(\lambda)$.

PROOF. The construction is the opposite of that of Theorem 3.10. We fix $\langle \mathcal{L}_s \rangle$ and build $\langle \mathcal{K}_s \rangle$ and $\langle F_s \rangle$ as before, but in this construction we preserve adjacencies in finite Q-intervals. So the construction is identical to that of the previous proposition, but when extending A_s to A_{s+1} in the case that A_s contains at least two points and B_{s+1} is finite, we make sure to define A_{s+1} so that every adjacency in A_s is still an adjacency in A_{s+1} (by, say, enumerating all new points in A_{s+1} to the right of A_s). This too may require changing the value of F on A_s, as some adjacencies in B_s may no longer be adjacencies in B_{s+1}.

Given $a <_{\mathcal{K}} b$, we want to decide, with oracle $(I_\infty^Q(\mathcal{K}) | I_{>1}^Q(\mathcal{K}))$, whether $(a, b) \in \mathrm{Succ}(\mathcal{K})$. As in the proof of Theorem 3.9, we may assume that $a, b \notin Q$, and that a and b lie in the same Q-interval $(Q_1, Q_2)_{\mathcal{K}}$. We know that this interval contains at least two points, so we can ask the oracle if this interval is infinite or not. If it is infinite, then it is dense, so $(a, b) \notin \mathrm{Succ}(\mathcal{K})$. If it is finite, then $(a, b) \in \mathrm{Succ}(\mathcal{K})$ if and only if $(a, b) \in \mathrm{Succ}(\mathcal{K}_s)$, where s is any stage such that $a, b \in \mathcal{K}_s$. This has bounded use since Q_1 and Q_2 can be effectively determined from a and b.

For the other direction, we modify slightly the algorithm given in the proof of Theorem 3.9. Given some cut (Q_1, Q_2), we wait until the first stage s such that $|(Q_1, Q_2)_{\mathcal{K}}| > 1$. If $(Q_1, Q_2)_{\mathcal{K}_s}$ is finite and for some $a, b \in (Q_1, Q_2)_{\mathcal{K}_s}$, $(a, b) \in \mathrm{Succ}(\mathcal{K})$, then we know $(Q_1, Q_2) \notin I_\infty^Q(\mathcal{K})$. Otherwise, we know $(Q_1, Q_2) \in I_\infty^Q(\mathcal{K})$.

Note that for any $(Q_1, Q_2) \in I_{>1}^Q(\mathcal{K})$, this algorithm will halt. For such (Q_1, Q_2), we can effectively compute the least s with $(Q_1, Q_2)_{\mathcal{K}_s}$ of size greater than one. If $(Q_1, Q_2)_{\mathcal{K}_s}$ is infinite, then the algorithm makes no queries of the oracle. Otherwise, the queries made are precisely those of the form "$(a, b) \in \mathrm{Succ}(\mathcal{K})$?", for $(a, b) \in ((Q_1, Q_2)_{\mathcal{K}_s})^2$. Thus we can compute the set of queries we will make, and since this set is finite, we can compute a bound on (the codes for) the queries. This establishes $(I_\infty^Q(\mathcal{K}) | I_{>1}^Q(\mathcal{K})) \leq_{\mathrm{wtt}} \mathrm{Succ}(\mathcal{K})$. ⊣

We note that for $\lambda = 2 \cdot \rho$ (see Example 3.2), $\mathbf{max}(\lambda) = \mathbf{min}(\lambda) = \mathbf{0}$. We generalize this example.

PROPOSITION 3.12. *If* $\mathbf{a}, \mathbf{b}$ *are ω_1-c.e. degrees and* $\mathbf{a} \leq \mathbf{b}$, *then there is an ω_1-computable weakly separable order-type λ with uncountably many adjacencies such that* $\mathbf{min}(\lambda) = \mathbf{a}$ *and* $\mathbf{max}(\lambda) = \mathbf{b}$.

PROOF. Let $A \in \mathbf{a}$ and $B \in \mathbf{b}$ be c.e., disjoint subsets of the collection of cuts of the rationals $\mathbb{Q}$. Define a computable linear order $\mathcal{L}$ by starting with $\mathbb{Q}$, and defining $(Q_1, Q_2)_{\mathcal{L}}$ for every cut (Q_1, Q_2) of $\mathbb{Q}$:

$$(Q_1, Q_2)_{\mathcal{L}} \cong \begin{cases} \mathbb{Q}, & \text{if } (Q_1, Q_2) \in A; \\ 3, & \text{if } (Q_1, Q_2) \in B; \textit{ and} \\ 2, & \text{if } (Q_1, Q_2) \notin A \cup B. \end{cases}$$

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

Then $I^{\mathbb{Q}}_{>1}(\mathcal{L})$ is computable, $I^{\mathbb{Q}}_{\infty}(\mathcal{L}) = A$, and

$$\bigoplus_{n\geq 2} I^{\mathbb{Q}}_n(\mathcal{L}) \equiv_{\mathrm{T}} B \oplus (\omega_1 \setminus (A \cup B)) \equiv_{\mathrm{T}} B. \qquad \dashv$$

Corollary 3.13. *For every ω_1-c.e. degree* $\mathbf{d}$ *there is an ω_1-computable order-type such that* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) = \{\mathbf{d}\}$.

We note that Corollary 3.13 fails for ω-computability: By the Downey-Lempp-Wu theorem, if λ is an ω-computable order-type and $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ is a singleton, then it must be $\{\mathbf{0}'\}$. Downey and Moses [6] constructed an ω-computable order-type such that $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) = \{\mathbf{0}'\}$ (a computable linear ordering with an *intrinsically complete* successor relation). Their construction is much more difficult than ours.

We turn to investigate how many of the intermediate degrees in the interval $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$ must be contained in $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$.

Theorem 3.14. *There is an ω_1-computable, weakly separable order-type λ with uncountably many adjacencies such that* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) \neq [\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$. *Indeed, there is an ω_1-c.e. set M with* $\mathbf{min}(\lambda) \leq_{\mathrm{T}} M \leq_{\mathrm{wtt}} \mathrm{max}_{\mathrm{wtt}}(\lambda)$ *but* $\deg_{\mathrm{T}}(M) \notin \mathrm{DegSpec}_{\mathrm{Succ}}(\mathcal{L})$.

Proof. We build an ω_1-computable linear ordering $\mathcal{L}$ by starting with $\mathbb{Q}$ and inserting either two or three points into every cut of $\mathbb{Q}$. This means that every cut of $\mathbb{Q}$ is in $I^{\mathbb{Q}}_{>1}(\mathcal{L})$, so $I^{\mathbb{Q}}_{>1}(\mathcal{L})$ is computable. Also $I^{\mathbb{Q}}_{\infty}(\mathcal{L})$ is empty. So $\mathbf{min}(\lambda) = \mathbf{0}$, and $\mathbf{max}_{\mathrm{wtt}}(\lambda) = \deg_{\mathrm{wtt}}(I^{\mathbb{Q}}_3(\mathcal{L}))$.

Hence, it is sufficient to build $\mathcal{L}$ and a c.e. set M such that $M \leq_{\mathrm{wtt}} I^{\mathbb{Q}}_3(\mathcal{L})$, but $\deg_{\mathrm{T}}(M) \notin \mathrm{DegSpec}_{\mathrm{Succ}}(\mathcal{L})$. We build $\mathcal{L}$ by enumerating $I^{\mathbb{Q}}_3(\mathcal{L})$. That is, we enumerate a c.e. set P of cuts of $\mathbb{Q}$ with $P = I^{\mathbb{Q}}_3(\mathcal{L})$.

We can effectively list all "partial" computable orderings, that is, computable linear orders of c.e. domains. We use this to get a list $\langle \mathcal{A}_i, \Phi_i, \Psi_i, \pi_i \rangle$ of all quadruples consisting of a partial computable linear order, two Turing functionals, and an injective countable function π_i whose domain is $\mathbb{Q}$. The intended oracle of Ψ_i is $\mathrm{Succ}(\mathcal{A}_i)$; we require that any query Ψ_i makes to the oracle does not mention pairs involving elements in the range of π_i.

For all $i < \omega_1$, the requirement R_i states that one of three outcomes must happen:

(a) There is no isomorphism from $\mathcal{L}$ to $\mathcal{A}_i$ extending π_i.
(b) $\Phi_i(M) \neq \mathrm{Succ}(\mathcal{A}_i)$.
(c) $M \neq \Psi_i(\mathrm{Succ}(\mathcal{A}_i))$.

If every requirement R_i is met, then $\deg_{\mathrm{T}}(M) \notin \mathrm{DegSpec}_{\mathrm{Succ}}(\mathcal{L})$. For suppose that $\mathcal{A}$ is a computable copy of $\mathcal{L}$, and that $\mathrm{Succ}(\mathcal{A}) \equiv_{\mathrm{T}} M$. Let $F\colon \mathcal{L} \to \mathcal{A}$ be an isomorphism. The point is that there is a reduction of M to $\mathrm{Succ}(\mathcal{A})$ which does not query any pairs containing elements of $F \restriction \mathbb{Q}$, as there are only countably many such pairs. This shows that there is some i for which R_i fails.

The construction is a priority argument. A requirement R_i may be assigned a witness—a cut $(Q_1(i), Q_2(i))$ of $\mathbb{Q}$—to work with. If we act for requirement R_i at stage s, then the witnesses $(Q_1(j), Q_2(j))$ for $j > i$ are all canceled, and will need to be later redefined (with large value). In this way, the requirement R_i imposes restraint on weaker requirements R_j. If not reset by stronger requirements, the

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

witness persists to the next stage and across limit stages. A requirement R_i may also appoint a *follower* $m(i)$, targeted for M; the same rules apply.

We say that $\mathcal{A}_i$ *appears correct* at stage s if range $\pi_i \subseteq \mathcal{A}_{i,s}$, π_i is an embedding of $\mathbb{Q}$ into $\mathcal{A}_i$, and for all cuts $(Q_1, Q_2) <_{\omega_1} s$ of $\mathbb{Q}$, $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,s}}$ contains two points if $(Q_1, Q_2) \notin P_s$, and three points if $(Q_1, Q_2) \in P_s$. The point, of course, is that if F is an isomorphism from $\mathcal{L}$ to $\mathcal{A}_i$ which extends π_i, then for all cuts (Q_1, Q_2) of $\mathbb{Q}$, $F(Q_1, Q_2)_{\mathcal{L}} = (\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$, and so the latter contains two points if $(Q_1, Q_2) \notin P$, and three otherwise.

If the witness $(Q_1(i), Q_2(i))$ is defined at stage $s >_{\omega_1} (Q_1(i), Q_2(i))$, and $\mathcal{A}_i$ appears correct at stage s, then the interval $(\pi_i[Q_1(i)], \pi_i[Q_2(i)])_{\mathcal{A}_i}$ contains at least two points; we let $a(i)$ and $b(i)$ be the two points which are first enumerated in this interval.

A requirement R_i *requires attention* at stage s if $\mathcal{A}_i$ appears correct at stage s, and one of the following hold:

(1) A witness $(Q_1(i), Q_2(i))$ is not defined at stage s.
(2) A witness $(Q_1(i), Q_2(i)) <_{\omega_1} s$ is defined, $\Phi_i(M, (a(i), b(i)))\downarrow = 1\,[s]$, and a follower $m(i)$ is not defined at stage s.
(3) A follower $m(i)$ is defined, $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i))\downarrow = 0\,[s]$, and $m(i) \notin M_s$.

At stage s we act on behalf of the strongest requirement which requires attention. Say we act for R_i at stage s. In case (1), we define a new witness $(Q_1(i), Q_2(i))$ with large value. In case (2), we appoint a new follower $m(i)$ with large value. In case (3), we enumerate $m(i)$ into M_{s+1}, and enumerate $(Q_1(i), Q_2(i))$ into P_{s+1}. This construction defines M and P, and so defines λ.

We first show that $M \leq_{\mathrm{wtt}} I_3^{\mathbb{Q}}(\mathcal{L})$. Observe that $x \in M$ only if x is chosen as a follower for some requirement by stage x. If x is a follower for R_i at stage x, then $x \in M$ if and only if the interval $(Q_1(i), Q_2(i))_{\mathcal{L}}$ contains three points. The cut $(Q_1(i), Q_2(i))$ is obtained effectively from x, and so the use of this reduction is computably bounded. We note that this reduction is the only driver for making intervals of size 3; the requirements R_i would be easily met if every $\mathbb{Q}$-interval has two elements, making $\mathrm{Succ}(\mathcal{L})$ intrinsically computable and making M noncomputable.

Finally, we see that every requirement is met. An inductive "countable injury" argument shows that for every $i < \omega_1$, R_i is only reset countably many times. For if R_i is never injured after stage s, then we act for R_i at most three times after stage s, possibly once at step (1), maybe later at step (2), and then maybe later at step (3).

Fix $i < \omega_1$; we show that the requirement R_i is met. Let r^* be a stage after which R_i is never reset. Suppose that there is an isomorphism F from $\mathcal{L}$ to $\mathcal{A}_i$ extending π_i, so R_i is not satisfied by clause (a) above. The regularity of ω_1 shows that the set of stages at which $\mathcal{A}_i$ looks correct is closed and unbounded in ω_1.

This means that there is some stage $s > r^*$ at which we act for R_i by step (1), appointing a witness cut $(Q_1(i), Q_2(i))$. This witness is never canceled. Since F extends π_i, the interval $(\pi_i[Q_1(i)], \pi_i[Q_2(i)])_{\mathcal{A}_i}$ has the same number of points as the interval $(Q_1, Q_2)_{\mathcal{L}}$, namely three if R_i ever reaches step (3) after stage r^*, and two otherwise. Let $a(i)$ and $b(i)$ be the two points which are enumerated earliest into $(\pi_i[Q_1(i)], \pi_i[Q_2(i)])_{\mathcal{A}_i}$.

If R_i never reaches step (2) after stage r^*, then $(a(i), b(i)) \in \mathrm{Succ}(\mathcal{A}_i)$, but it is not the case that $\Phi_i(M, (a(i), b(i))) = 1$. In this case, R_i is satisfied by clause (b)

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

above. Suppose that R_i reaches step (2) at some stage $s' > r^*$. The resetting of weaker requirements at stage s', the fact that $s' > r^*$, and the fact that at step (2), $m(i)$ is chosen to be large, show that $\Phi_i(M, (a(i), b(i))) = 1$.

At step (2), R_i appoints a follower $m(i)$ which is never canceled. If R_i never reaches step (3) after that, then $m(i) \notin M$ but $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i)) \neq 0$, so R_i is satisfied by clause (c) above. Suppose that R_i reaches step (3) at some stage $t > r^*$. Then $m(i) \in M$; we argue that $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i)) = 0$, which would mean that R_i is satisfied by clause (c).

At stage t, $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i)) = 0$. We show that $\mathrm{Succ}(\mathcal{A}_i)$ and $\mathrm{Succ}(\mathcal{A}_{i,t})$ agree on the use of the computation at stage t. Since Ψ_i does not query pairs involving elements of $\pi_i[\mathbb{Q}]$, and since pairs of elements from distinct $\pi_i[\mathbb{Q}]$-intervals of $\mathcal{A}_i$ are not successor pairs in either $\mathcal{A}_i$ or $\mathcal{A}_{i,t}$, it suffices to show that for all $\mathbb{Q}$-cuts $(Q_1, Q_2) <_{\omega_1} t$, for all $a <_{\mathcal{A}_i} b$ in $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,t}}$, $(a, b) \in \mathrm{Succ}(\mathcal{A}_{i,t})$ if and only if $(a, b) \in \mathrm{Succ}(\mathcal{A}_i)$.

Let $(Q_1, Q_2) <_{\omega_1} t$ be a cut of $\mathbb{Q}$, and let $a <_{\mathcal{A}_i} b$ be elements of the interval $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,t}}$. There are two cases. If $(Q_1, Q_2) \neq (Q_1(i), Q_2(i))$, then the fact that $t > r^*$, and the fact that R_i resets weaker requirements at stage s (and later these requirements choose large witnesses) means that $(Q_1, Q_2) \in P$ if and only if $(Q_1, Q_2) \in P_t$. At stage t, $\mathcal{A}_i$ appears correct, so $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,t}}$ contains three points if and only if $(Q_1, Q_2) \in P_t$; and since F extends π_i, the interval $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$ contains three points if and only if $(Q_1, Q_2) \in P$. It follows that $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i,t} = (\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$, so $\mathrm{Succ}(\mathcal{A}_i)$ cannot change on (a, b) after stage t.

If $(Q_1, Q_2) = (Q_1(i), Q_2(i))$, then as $(Q_1, Q_2) \notin P_t$, we must have $a = a(i)$ and $b = b(i)$. We have $(a(i), b(i)) \in \mathrm{Succ}(\mathcal{A}_{i,t})$, and by assumption, $(a(i), b(i)) \in \mathrm{Succ}(\mathcal{A}_i)$. In other words, the third point enumerated into $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$ after stage t does not break the adjacency $(a(i), b(i))$, or otherwise R_i is already satisfied by clause (b) as explained above. ⊣

At the opposite extreme, there is an ω_1-computable linear order $\mathcal{L}$ such that the degree spectrum of its successor relation contains every ω_1-c.e. degree. This follows from Theorem 3.4, applying it to any ω_1-computable linear ordering $\mathcal{L}$ which is not weakly separable but such that $\mathrm{Succ}(\mathcal{L})$ is ω_1-computable; an example for such an ordering is $(\omega_1, <)$. Here, we show that the example can be weakly separable.

Theorem 3.15. *There is an ω_1-computable, weakly separable order-type λ such that the degree spectrum* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ *contains every c.e. degree. Further, every ω_1-c.e. set is weak truth-table equivalent to* $\mathrm{Succ}(\mathcal{L})$ *for some ω_1-computable presentation of $\mathcal{L}$.*

Proof. The idea is to effectively encode the set W_α into $\mathcal{L}$ by replacing the $(\alpha, \beta)^{\text{th}}$ irrational with the order-type 2 or 3 depending on whether $\beta \in W_\alpha$. Fix an effective list $\langle r_{\alpha,\beta} \rangle_{\alpha,\beta<\omega_1}$ of all the irrational numbers.

The order $\mathcal{L}$ is obtained from $\mathbb{R}$ by replacing $r_{\alpha,\beta}$ by two points if $\beta \notin W_\alpha$ and by three points if $\beta \in W_\alpha$. Then $\mathcal{L}$ is computable, and $\mathbb{Q} \subseteq \mathcal{L}$ witnesses that $\mathcal{L}$ is weakly separable.

For any $\gamma < \omega_1$, we construct a computable $\mathcal{A} \cong \mathcal{L}$ such that $\mathrm{Succ}(\mathcal{A}) \equiv_{\mathrm{wtt}} W_\gamma$. We start with $\mathbb{Q}$; for any irrational number r, let C_r be the $\mathbb{Q}$-interval of $\mathcal{A}$ replacing r.

We start by enumerating two points into each C_r. If β enters W_α, and $\alpha \neq \gamma$, we enumerate a third point into $C_{r_{\alpha,\beta}}$ to the right of the existing two points. If β enters W_γ, then we enumerate a third point into $C_{r_{\gamma,\beta}}$ between the existing two points.

To compute $\mathrm{Succ}(\mathcal{A})$ from W_γ, we take $a <_{\mathcal{A}} b$; again, we may assume that $a, b \notin \mathbb{Q}$, and that they lie in the same $\mathbb{Q}$-interval $C_{r_{\alpha,\beta}}$; α and β are effectively obtained from a and b. If $\alpha \neq \gamma$, then $(a,b) \in \mathrm{Succ}(\mathcal{A})$ if and only if $(a,b) \in \mathrm{Succ}(\mathcal{A}_s)$ for any stage s at which $a, b \in \mathcal{A}_s$. If $\alpha = \gamma$, then W_γ tells us the size of $C_{r_{\gamma,\beta}}$, and so a stage s at which $C_{r_{\gamma,\beta},s} = C_{r_{\gamma,\beta}}$; then, of course, $(a,b) \in \mathrm{Succ}(\mathcal{A})$ if and only if $(a,b) \in \mathrm{Succ}(\mathcal{A}_s)$.

To compute W_γ from $\mathrm{Succ}(\mathcal{A})$, for $\beta < \omega_1$, we let $a <_{\mathcal{A}} b$ be the first two points enumerated into $C_{r_{\gamma,\beta}}$; these are obtained effectively from β. Then $\beta \in W_\gamma$ if and only if $(a,b) \in \mathrm{Succ}(\mathcal{A})$. ⊣

3.1. Open Questions on Spectra of Relations. We close this section with some open questions concerning the spectra of relations on ω_1-computable linear orders.

QUESTION 3.16. *Is there an ω_1-computable weakly separable linear order such that* $\mathbf{min}(\lambda) < \mathbf{max}(\lambda)$ *but* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) = \{\mathbf{min}(\lambda), \mathbf{max}(\lambda)\}$? *In general, what are the possible relations between* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ *and the interval* $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$?

QUESTION 3.17. *What can be said about the degree spectra of the block relation "$a <_{\mathcal{L}} b$ and $(a,b)_{\mathcal{L}}$ is finite" or the countable-distance relation "$a <_{\mathcal{L}} b$ and $(a,b)_{\mathcal{L}}$ is countable" in ω_1-computable linear orderings? These are Π^0_1 and Σ^0_2, respectively; when are the degree spectra of these relations upwards closed in the appropriate degrees?*

§4. Acknowledgments. Greenberg's research was supported by a grant from the Marsden Fund of New Zealand and by a Rutherford Discovery Fellowship and a Turing Research Fellowship (mind, mechanism and mathematics). Kach's research was partially supported by a grant from the Marsden Fund of New Zealand via a Post-Doctoral Fellowship. Lempp's research was partially supported by NSF grant DMS-0555381, Grant # 13407 by the John Templeton Foundation entitled "Exploring the Infinite by Finitary Means" as well as AMS Simons Foundation Collaboration Grant 209087. Turetsky's research was supported by the Graduate School of the University of Wisconsin-Madison through a research assistantship. Lempp and Turetsky would like to thank Victoria University of Wellington for its hospitality during the time this work was carried out.

REFERENCES

[1] CHRISTOPHER J. ASH, CARL G. JOCKUSCH, JR., and JULIA F. KNIGHT, *Jumps of orderings*. ***Transactions of the American Mathematical Society***, vol. 319 (1990), no. 2, pp. 573–599.

[2] CHRISTOPHER J. ASH and JULIA F. KNIGHT, *Pairs of recursive structures*. ***Annals of Pure and Applied Logic***, vol. 46 (1990), no. 3, pp. 211–234.

[3] CHI TAT CHONG and WEI WANG, *Hyperimmune-free degrees beyond ω*, in preparation.

[4] RODNEY G. DOWNEY and JULIA F. KNIGHT, *Orderings with αth jump degree $\mathbf{0}^{(\alpha)}$*. ***Proceedings of the American Mathematical Society***, vol. 114 (1992), no. 2, pp. 545–552.

[5] RODNEY G. DOWNEY, STEFFEN LEMPP, and GUOHUA WU, *On the complexity of the successivity relation in computable linear orderings*. ***Journal of Mathematical Logic***, vol. 10 (2010), vol. 1–2, pp. 83–99.

[6] RODNEY G. DOWNEY and MICHAEL F. MOSES, *Recursive linear orders with incomplete successivities*. ***Transactions of the American Mathematical Society***, vol. 326 (1991), no. 2, pp. 653–668.

https://doi.org/10.1017/jsl.2014.69 Published online by Cambridge University Press

[7] Andrey N. Frolov, *Presentations of the adjacency relation of a computable linear order*. ***Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika***, vol. 7 (2010), pp. 64–74.

[8] Andrey N. Frolov, Iskander Sh. Kalimullin, Valentina S. Harizanov, Oleg V. Kudinov, and Russell G. Miller, *Spectra of high$_n$ and non-low$_n$ degrees*. ***Journal of Logic and Computation***, vol. 22 (2012), no. 4, pp. 755–777.

[9] Sergey S. Goncharov, Valentina S. Harizanov, Julia F. Knight, Charles F. D. McCoy, Russell G. Miller, and D. Reed Solomon, *Enumerations in computable structure theory*. ***Annals of Pure and Applied logic***, vol. 136 (2005), no. 3, pp. 219–246.

[10] Noam Greenberg, Asher M. Kach, Steffen Lempp, and Daniel D. Turetsky, *Computability and uncountable linear orders, part I: computable categoricity*, this Journal, vol. 80 (2015), no. 1, pp. 116–144.

[11] Noam Greenberg and Julia F. Knight, *Computable structure theory using admissible recursion theory on ω_1*, ***Effective mathematics of the uncountable*** (Greenberg, Hirschfeldt, Miller, Hamkins, editors), Lecture Notes in Logic, Association for Symbolic Logic and Cambridge University Press, pp. 50–80, 2013, to appear.

[12] Carl G. Jockusch, Jr. and Robert I. Soare, *Degrees of orderings not isomorphic to recursive linear orderings*. ***Annals of Pure and Applied Logic***, vol. 52 (1991), no. 1-2, 39–64. International Symposium on Mathematical Logic and its Applications (Nagoya, 1988).

[13] Julia F. Knight, *Degrees coded in jumps of orderings*, this Journal, vol. 51 (1986), no. 4, pp. 1034–1042.

[14] Russell G. Miller, *The Δ^0_2-spectrum of a linear order*, this Journal, vol. 66 (2001), no. 2, pp. 470–486.

[15] Michael F. Moses, *Recursive linear orders with recursive successivities*. ***Annals of Pure and Applied Logic***, vol. 27 (1984), no. 3, pp. 253–264.

[16] H. Gordon Rice, *Recursive and recursively enumerable orders*. ***Transactions of the American Mathematical Society***, vol. 83 (1956), pp. 277–300.

[17] Linda Jean Richter, *Degrees of structures*, this Journal, vol. 46 (1981), no. 4, pp. 723–731.

[18] Gerald E. Sacks, ***Higher recursion theory***, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990.

[19] Gaisi Takeuti, *On the recursive functions of ordinal numbers*. ***Journal of the Mathematical Society of Japan***, vol. 12 (1960), pp. 119–128.

[20] Vladimir A. Uspenskii, *Some notes on recursively enumerable sets*. ***Zeitschrift fur mathematische Logik und Grundlagen der Mathematik***, vol. 3 (1957), pp. 157–170.

[21] Stephan Wehner, *Enumerations, countable structures and Turing degrees*. ***Proceedings of the American Mathematical Society***, vol. 126 (1998), no. 7, pp. 2131–2139.

DEPARTMENT OF MATHEMATICS,
VICTORIA UNIVERSITY OF WELLINGTON
WELLINGTON, NEW ZEALAND
E-mail: noam.greenberg@msor.vuw.ac.nz
URL: http://homepages.mcs.vuw.ac.nz/~greenberg/

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, IL 60637, USA
E-mail: asher.kach@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN MADISON,
WI 53706-1388, USA
E-mail: lempp@math.wisc.edu
URL: http://www.math.wisc.edu/~lempp/

KURT GÖDEL RESEARCH CENTER
UNIVERSITY OF VIENNA
1090 VIENNA, AUSTRIA
E-mail: turetsd4@univie.ac.at
URL: http://tinyurl.com/dturetsky