

DEMUTH'S PATH TO RANDOMNESS

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Abstract. Oswald Demuth (1936–1988) studied constructive analysis from the viewpoint of the Russian school of constructive mathematics. In the course of his work he introduced various notions of effective null set which, when phrased in classical language, yield a number of major algorithmic randomness notions. In addition, he proved several results connecting constructive analysis and randomness that were rediscovered only much later.

In this paper, we trace the path that took Demuth from his constructivist roots to his deep and innovative work on the interactions between constructive analysis, algorithmic randomness, and computability theory. We will focus specifically on (i) Demuth's work on the differentiability of Markov computable functions and his study of constructive versions of the Denjoy alternative, (ii) Demuth's independent discovery of the main notions of algorithmic randomness, as well as the development of Demuth randomness, and (iii) the interactions of truth-table reducibility, algorithmic randomness, and semigenericity in Demuth's work.

§1. Introducing Demuth. The mathematician Oswald Demuth worked primarily on constructive analysis in the Russian style, which was initiated by Markov, Šanin, Ceitin, and others in the 1950s. Born in 1936 in Prague, Demuth graduated from the Faculty of Mathematics and Physics at Charles University in Prague in 1959 with the equivalent of a master's degree. Thereafter he studied constructive mathematics under the supervision of A.A. Markov Jr. in Moscow, where he successfully defended his doctoral thesis in 1964.

After completing his doctoral studies with Markov, he returned to Charles University, completing his Habilitation in 1968. He remained at Charles University until the end of his life in 1988. During this period of time, he faced intense persecution for his political views. After the Russian invasion of the Czech Republic in 1968, Demuth left the Communist Party in 1969, as he was opposed to the invasion. The consequences of this decision for Demuth's career were dire. From 1972–1978, he was forbidden to lecture at the university, although he continued his scientific work during this period. Moreover, he was not permitted to travel abroad until 1987. Lastly, he never achieved the rank of full professor, even though he clearly deserved this rank.

Despite these hardships, Demuth made a number of contributions to constructive analysis. His most significant work revealed deep and interesting connections between notions of typicality naturally occurring in constructive analysis and various notions of algorithmic randomness. These connections

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have only recently been rediscovered (see, for instance, [12] and [7]). He was also extremely productive, publishing nearly 60 research articles, including over 45 articles in the journal *Commentationes Mathematicae Universitatis Carolinae* during the period from 1968 to 1988. That journal imposed a page limit of 30 pages per year on him. Only a small number of Demuth's articles were written in collaboration with others.

Demuth's work, especially its connection to computability theory, has been largely underappreciated. One goal of this paper is to remedy this situation. We highlight the path that led Demuth from his initial work in constructive analysis to his later work that drew heavily upon the techniques of computability theory, work in which notions of algorithmic randomness feature prominently. As we will see, what is particularly noteworthy about this path is how Demuth's constructivism changed over time: initially, he worked primarily with constructive objects, but in later work, he considered larger classes of nonconstructive objects, such as Δ_2^0 reals (which he called *pseudonumbers*), then arithmetical real numbers, and eventually the collection of all real numbers. Even in this latter phase, however, Demuth did not abandon his constructivist roots, still framing his results in the language of constructive analysis (albeit extended to allow for reference to nonconstructive objects).

We also discuss a number of recent developments in algorithmic randomness that can be seen as extending Demuth's results. We will concentrate in particular on developments that link computable analysis, specifically differentiability and almost everywhere behavior, with notions of randomness. While some of the results we survey are not Demuth's contributions, they fit naturally into his program of studying constructive analysis through the lens of computability theory. We have not discussed similar results that link notions of algorithmic randomness and ergodic theory.

The outline of this paper is as follows. In Section 2, we will briefly discuss Demuth's constructivism as laid out in his survey "Remarks on constructive mathematical analysis" [38], co-authored with the first author of this paper and published in 1979. In Section 3, we will review the basics of Markov computability of real-valued functions. Section 4 concerns the notions of algorithmic randomness that appear in Demuth's work, especially in his study of the differentiability of Markov computable functions and the Denjoy alternative. In Section 5 we look closely at Demuth's own notions of randomness, nowadays known as *Demuth randomness* and *weak Demuth randomness*, outlining a number of facts that Demuth proved about these notions as well as some additional results that have been recently obtained. In Section 6, we consider Demuth's work on the interactions of truth-table reducibility, algorithmic randomness, and semigenericity, some of which was carried out jointly with the first author. In Section 7, we conclude with some remarks on Demuth's contributions.

Interpreting Demuth's results can be a difficult task that may involve some guesswork. This is evident from a quick look at one of his papers (see the electronic databases given in Section 7), or even at the sample of his writing given in Figure 2 below. The main problem is that the papers are written in

notationally heavy, formal constructive language. The constructive results have to be re-interpreted classically. So, when we attribute a result to Demuth that was later proved independently, it does not diminish the credit due for this rediscovery the way that it would if Demuth had written his work in the customary classical language of computable analysis.

This paper is a substantially extended version of the conference paper [54]. Here we cover Demuth's research more broadly and in more detail.

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§2. Demuth's constructivism. Before we survey Demuth's technical results, we will briefly review the basics of the Russian school of constructive mathematics and introduce Demuth's unique approach to constructive mathematics, which is laid out in the 1978 survey paper "Remarks on constructive mathematical analysis" [38], written by Demuth and Kučera.¹

Constructive mathematics in the Russian school (RUSS), like other versions of constructivism, namely Bishop's constructive mathematics (BISH) and Brouwer's intuitionism (INT), aims to put mathematics on a secure foundation. Like BISH and INT, in RUSS one rejects the general use of the law of excluded middle and thus double negation elimination. However, what distinguishes RUSS from BISH and INT is the central role that the notion of effectivity plays in the theory.

But why emphasize this notion of effectivity? According to Demuth and Kučera, there is a historical reason, for as they write [38, p. 81],

From the historical point of view, the development of mathematics was substantially influenced by applications of mathematics where solutions of problems consisted, de facto, in transformation of particular information coded by words.

They add, "The means necessary for algorithmic processing of words are indispensable for any sufficiently rich mathematical theory" [38, p. 81]. Demuth and Kučera further held that such means also prove to be *sufficient* for developing rich mathematical theories.

The distinctive features of RUSS as laid out by Demuth are the following.

1. The objects studied are constructive objects, coded as words in a finite alphabet.
2. The Church-Turing Thesis is accepted (see [20], [75]). That is, the algorithms by means of which the words coding constructive objects are

¹We should note that at the time of the publication of [38], its second author accepted the basic principles of constructivism. However, in the years that followed, he gradually turned to the use of classical nonconstructive methods.

transformed are precisely the Turing computable functions, or equivalently the Markov algorithms (which is a formalism often used in the Russian school; see, for instance, [56]).

3. The so-called “constructive interpretation of mathematical propositions” as developed by Šanin [70] is used. According to this interpretation, the existential quantifier and disjunction are interpreted constructively. That is, one is entitled to assert the existence of an object if there is an algorithmic procedure for constructing the object, and one is entitled to accept a disjunction of two formulas if there is an algorithmic procedure for determining which of the disjuncts is true.
4. The following principle, known as Markov's Principle, is allowed: if one has refuted the claim that some Markov algorithm A does not accept a given input x , then one can conclude that A accepts x . In modern notation,

$$\neg\neg A(x)\downarrow \Rightarrow A(x)\downarrow,$$

where $A(x)\downarrow$ means that the Markov algorithm A halts on input x . Thus, although double negation elimination is not permissible in general, it can be used in specific situations as laid out by Markov's Principle.

For more details on the basic features of RUSS, see [51, Chapter 1] or the survey [52].

Though Demuth is explicit about his commitment to the principles of RUSS, in surveying his work, one will find that Demuth routinely appeals to objects and techniques that are well beyond the scope of what is constructively admissible, at least according to principles accepted by most constructivists.

One finds that Demuth became more and more lenient about the objects to which he appealed in his theorems. That is, he gradually extended his domain of discourse to include more and more nonconstructive objects. Initially, he restricted his attention to the computable real numbers, but later he formulated certain of his results in terms of the more general collection of Δ_2^0 reals. Later yet he further extended his work to encompass the collection of arithmetical reals, and finally in his last papers, he even proved statements involving quantification over all real numbers. How did Demuth account for this use of nonconstructive objects?

Demuth's answer to this question is a subtle one. He formulated many of his later results in terms of the collection of all real numbers, proving, for example, that a number of computability-theoretic statements hold relative to any oracle. However, he was primarily concerned with these results insofar as they apply to arithmetical real numbers. As he and Kučera write,

It should be noted that we are interested, owing to the natural connection between concepts of constructive mathematical analysis and arithmetical predicates, only in the computability relative to jumps of the empty set. [38, p. 84]

Here Demuth appeals to Post's theorem, according to which a predicate P of the natural numbers is arithmetical if and only if there is some $n \in \mathbb{N}$ such that P is computable relative to $\emptyset^{(n)}$, the n -th Turing jump of the empty set. Of course, one who strictly adheres to the principles of RUSS would find such an appeal to $\emptyset^{(n)}$ completely unacceptable.

However, in Demuth's view, functions computable from $\emptyset^{(n)}$, where $n \in \mathbb{N}$, are still in some sense constructively grounded, as they "can be represented on the basis of recursive functions by means of noneffective limits" according to the strong form of Shoenfield's Limit Lemma (see, for instance [40, Corollary 2.6.3]). Demuth and Kučera further argue,

Without leaving [the] constructive program concerning effective processes we improve, by the use of relative computability, our ability to handle effective procedures. The advantage of the improvement consists in both substantial simplification and clearness of formulations [38, p. 84].

Thus, Demuth studied notions connected to relative computability from a constructive point of view. Although he himself was only concerned with arithmetical reals as potential inputs for algorithmic procedures, he left open the possibility of considering his results in terms of a broader class of inputs, even, potentially, the entire collection of real numbers.²

Whether or not one agrees that Demuth was still being faithful to the basic principles of RUSS, it is fair to characterize Demuth's approach as an *extended constructivism*. As we will see in the sections that follow, this extension was a gradual one, but it allowed him to bring techniques of constructive mathematics and classical computability theory together in interesting and often insightful ways.

§3. The basic definitions of computable analysis in the Russian school. As Demuth worked primarily in the field of computable analysis, we will review the basic definitions of this subject. In this paper, these definitions will be phrased in the language of modern computable analysis, as developed, for instance, in Brattka et al. [11], Pour-El and Richards [68] and Weihrauch [76]. See Aberth [1] for a more recent discussion of computable analysis in the Russian school.

One of the central notions of computable analysis in the Russian style is the notion of a *constructive* real number. Its modern analogue is the notion of a *computable* real number.

DEFINITION 3.1 (Turing [75]). A *computable real number* z is given by a computable Cauchy name, i.e., a computable sequence $(q_n)_{n \in \mathbb{N}}$ of rationals converging to z such that $|q_k - q_n| \leq 2^{-n}$ for each $k \geq n$.

A sequence $(x_n)_{n \in \mathbb{N}}$ of reals is *computable* if there is a computable double sequence $(q_{n,k})_{n,k \in \mathbb{N}}$ of rationals such that each x_n is a computable real as witnessed by its Cauchy name $(q_{n,k})_{k \in \mathbb{N}}$.

²It is known from private communication with the first author that Demuth would have also accepted hyperarithmetical reals, but he saw no need to work with them.

As is well-known, one can equivalently define a computable real number in terms of a computable sequence of rationals $(q_n)_{n \in \mathbb{N}}$ and a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every n and every $k \geq f(n)$, $|q_{f(n)} - q_k| \leq 2^{-n}$. In modern terminology, f is referred to as a *modulus function*, whereas Demuth referred to f as the *regulator of fundamentality* of the sequence $(q_n)_{n \in \mathbb{N}}$. We will write \mathbb{R}_c to denote the collection of computable real numbers.

We should note one subtle difference between the constructive approach to computable real numbers and the modern approach. In the constructive approach, a computable real number is held to be a finite syntactic object, given by the pair consisting of the index of the sequence $(q_n)_{n \in \mathbb{N}}$ and the index of the modulus f . However, in the modern approach, one need not take a computable real number to be some finite object. Instead, a computable real number is simply a real number that has a computable name. This approach is compatible with a nonconstructive view of real numbers, according to which they are completed totalities.

According to the Russian school, the continuum should be understood constructively, in the sense that it consists entirely of computable real numbers. From this point of view, the continuum should not be seen as having gaps, since the constructive continuum is constructively complete, in the sense that every uniformly computable Cauchy sequence of real numbers with a computable modulus of Cauchy convergence converges to a computable real number (see [51]).

The notion of a Markov computable function was central to Russian-style constructivism. In the context of the constructive continuum, this is a natural notion of computability for a function.

For a computable Cauchy name $(q_n)_{n \in \mathbb{N}}$, if ϕ_i is a computable function such that $\phi_i(n) = q_n$ for every n , then we call i the *index* of $(q_n)_{n \in \mathbb{N}}$. The following definition is due to Markov [57]. In keeping with the constructive commitment to studying the transformation of finite words, a Markov computable function can be seen as uniformly transforming an algorithm for computing a given computable real into another algorithm for computing the output real.

DEFINITION 3.2. A function $g : \mathbb{R}_c \rightarrow \mathbb{R}_c$ defined on the computable reals is called *Markov computable* if from any index for a computable Cauchy name for x one can compute an index for a computable Cauchy name for $g(x)$.

Demuth referred to Markov computable functions as *constructive*. By a *c-function* he meant a constructive function that is constant on $(-\infty, 0]$ and on $[1, \infty)$. This in effect restricts the domain to the unit interval. Note that a constructivist cannot explicitly write this restriction to $[0, 1]$ into the definition since the relation $x \leq y$ is not decidable for computable reals and, thus, it is not decidable whether a given computable real is negative. Hereafter we will only make reference to Markov computable functions (we will assume when necessary that a given Markov computable function is constant outside of the unit interval).

By a result of Ceĭtin (see, for instance, [15], [16], and [17]) and also a similar result of Kreisel, Lacombe and Shoenfield [49], each Markov computable function is continuous on the computable reals (with respect to the subspace topology on \mathbb{R}_c). However, since such a function may only be defined on the computable reals, it is not necessarily uniformly continuous. This was first shown by Zaslavskii in [78].

Such an example of a Markov computable function that is not uniformly continuous can be produced by a typical construction in constructive analysis. In this construction, a Markov computable function is defined in terms of a Σ_1^0 class \mathcal{A} that contains all computable reals. In a natural way \mathcal{A} may be viewed as a c.e. set S of rational intervals. Now one may describe a Markov computable function on computable reals by defining it on all rational intervals from S . However, in general, for a computable real z we cannot find exactly one interval from S containing z . This is due to the fact that the relation $x \leq y$ is not decidable for computable reals and, thus, given an interval $[a, b]$ we cannot in general determine whether a given computable real belongs to $[a, b]$. At best, for a computable real z we can find rationals $a < b < c$ such that the intervals $[a, b]$ and $[b, c]$ belong to S and $a < z < c$. Thus, a Markov computable function f has to be defined consistently and continuously on computable reals from any open interval (a, c) such that $[a, b], [b, c]$ belong to S for some b .

In this way one may construct a Markov computable function f that is continuous on the computable reals but is not uniformly continuous: Let S be an infinite c.e. set of nonoverlapping rational intervals with the property that for every computable real x there is some $I \in S$ such that $x \in I$. Let $(I_n)_{n \in \mathbb{N}}$ be an effective enumeration of the intervals in S . We define f to be piecewise linear on each interval from S , so that for each n , f is equal to 0 at the endpoints of I_n and takes its local maximum with value n at the midpoint of I_n .

Now, for any real r not covered by any interval $I \in S$, f takes arbitrarily large values at computable reals sufficiently close to r . Hence f is not uniformly continuous — we cannot even continuously extend f to any real outside the union of the intervals in S .

The notion of a Markov computable function should be contrasted with the standard definition of a computable real-valued function from modern computable analysis, hereafter, a *standard* computable function, which is essentially due to Turing [75] (although Borel had formulated the basic ideas of computability of real-valued functions in [10]; see [2] for a helpful discussion of these developments). In this approach, $f : \mathbb{R} \rightarrow \mathbb{R}$ is computable if

- (i) for every computable sequence of real numbers $(x_k)_{k \in \mathbb{N}}$, the sequence $(f(x_k))_{k \in \mathbb{N}}$ is computable, and
- (ii) f is *effectively uniformly continuous*, i.e., there is a computable function $p : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $x, y \in \mathbb{R}$ and every $n \in \mathbb{N}$,

$$|x - y| \leq 2^{-p(n)} \Rightarrow |f(x) - f(y)| \leq 2^{-n}.$$

Every standard computable function is uniformly continuous, unlike the case with Markov computable functions, as mentioned above. However, a significant portion of Demuth's work was concerned with uniformly continuous Markov computable functions. Recall that a *modulus of uniform continuity* for a function f is a function θ on positive rationals such that $|x - y| \leq \theta(\varepsilon)$ implies $|f(x) - f(y)| \leq \varepsilon$ for each rational $\varepsilon > 0$.

From a constructive point of view, it is reasonable to study uniformly continuous Markov computable functions with a *computable* modulus of uniform continuity, which Demuth referred to as \emptyset -uniformly continuous functions. Note that the restriction of a standard computable real-valued function to the computable reals yields a \emptyset -uniformly continuous Markov computable function (see [11, 76]).

Even if we consider uniformly continuous Markov computable functions with a *noncomputable* modulus, such a modulus cannot have arbitrarily high complexity. Demuth proved that every uniformly continuous Markov computable function has a modulus that is computable in \emptyset' . Demuth thus referred to classically uniformly continuous Markov computable functions as \emptyset' -uniformly continuous. Demuth proved a more general result about uniformly continuous Markov computable functions. Before we state the result, we need one additional definition.

Let $f : \mathbb{R}_c \rightarrow \mathbb{R}_c$ be a Markov computable function. We define $R[f] : \mathbb{R} \rightarrow \mathbb{R}$ to be the classical function that is the maximal extension of f that is continuous on its domain. More precisely, for each noncomputable $r \in [0, 1]$, if $\ell = \lim_{x \rightarrow r} f(x)$ exists, then we set $R[f](r) = \ell$. Otherwise, $R[f](r)$ is undefined.

Recall that a real r is Δ_3^0 if and only if $r \leq_T \emptyset''$, i.e., there is a \emptyset'' -computable sequence $(q_n)_{n \in \mathbb{N}}$ of rationals converging to r such that $|q_k - q_n| \leq 2^{-n}$ for each $k \geq n$.

THEOREM 3.3 (Demuth, Kryl, Kučera [37], [33]). *Let f be a Markov computable function. Then the following are equivalent.*

1. f is uniformly continuous.
2. f is \emptyset' -uniformly continuous.
3. $R[f]$ is defined at all Δ_3^0 reals.
4. $R[f]$ is defined at all reals.

PROOF. (1 \Rightarrow 2): If f is uniformly continuous, then \emptyset' can compute a modulus of uniform continuity for f .

The implications (2 \Rightarrow 3), (2 \Rightarrow 4), and (4 \Rightarrow 3) are immediate. It remains to show (3 \Rightarrow 1).

We claim that if f is not uniformly continuous then there is a \emptyset'' -computable real x at which $R[f]$ is not defined, i.e., f cannot be extended continuously to x . For suppose that there is an n such that

$$(\diamond) \text{ for every } k \text{ there exist } x, y \text{ with } |x - y| < 2^{-k} \text{ and } |f(x) - f(y)| > 2^{-n+1}.$$

For each $\sigma \in \{0, 1\}^*$, the *interval represented by σ* , denoted $[\sigma]$, is defined to be the half-open interval $[0.\sigma, 0.\sigma + 2^{-|\sigma|})$. Now since f is defined and

continuous on all dyadic rationals the condition (\diamond) can be replaced with the following:

(\diamond') for every k there exist a string σ of length k and rationals x, y in the interval represented by σ (so that $|x - y| < 2^{-k}$) and $|f(x) - f(y)| > 2^{-n}$.

Indeed, for $x^*, y^* \in [0, 1]$ such that $|x^* - y^*| < 2^{-k}$ and $|f(x^*) - f(y^*)| > 2^{-n+1}$, if x^*, y^* do not belong to an interval $[\sigma]$ for some string σ of length k then for some j such that $0 < j < 2^k$, we have $x^* < j/2^{-k} \leq y^*$. But then either $|f(x^*) - f(j/2^{-k})| > 2^{-n}$ or $|f(y^*) - f(j/2^{-k})| > 2^{-n}$. Now using the continuity of f at $j/2^k$ we can easily find x, y for which the condition (\diamond') holds.

We can use condition (\diamond') to build a \emptyset' -computable tree such that $\sigma \in \{0, 1\}^*$ is on the tree if and only if there are x, y which belong to the interval represented by σ and $|f(x) - f(y)| > 2^{-n}$. By condition (\diamond') , this tree is infinite. Thus, using \emptyset'' as an oracle, we can compute an infinite path through this tree, which corresponds to a real x . At this real x , $R[f]$ is not defined. \dashv

Using Theorem 3.3, it is not difficult to verify that for every \emptyset -uniformly continuous Markov computable function f , the function $R[f]$ is a standard computable function. Indeed, since f is \emptyset -uniformly continuous, it is clearly \emptyset' -uniformly continuous, and so by Theorem 3.3, $R[f]$ is defined on all reals. Since $R[f](x) = f(x)$ for all computable reals, condition (i) in the definition of a standard computable function is satisfied. Furthermore, condition (ii) in the definition of a standard computable function is also satisfied, as $R[f]$ is \emptyset -uniformly continuous with the same modulus as f , since f is \emptyset -uniformly continuous on a dense subset of \mathbb{R} .

Thus, just as \emptyset -uniformly continuous Markov computable functions can be obtained by restricting standard computable functions to \mathbb{R}_c , standard computable functions can be obtained by extending \emptyset -uniformly continuous Markov computable functions from \mathbb{R}_c to \mathbb{R} via the operator R .

We should note that Theorem 3.3 was not originally formulated in terms of the operator R that extends a Markov computable function to a classical function but rather in terms of a more restricted operator. For instance, in [37] and [28], Demuth defined $Op[f]$ to be the maximal continuous extension of f that is defined on all *arithmetical* reals. However, in Demuth's later papers such as [35] and [33], we find the operator R that behaves like Op except that for a Markov computable function f the domain of the function $R[f]$ can potentially be defined on *all* real numbers. This is another example of Demuth's willingness to recast his results in terms of nonconstructive objects.

§4. Notions of randomness in Demuth's work. As discussed in the introduction, Demuth considered a number of different notions of effective null set. They are equivalent to several major randomness notions that have been introduced independently.

It is striking that Demuth never actually referred to random or nonrandom sequences. Instead, he characterized these classes in terms of nonapproximability in measure and approximability in measure, respectively. This reflects the fact that Demuth's motivation in introducing these classes differed significantly from the motivation of the recognized "fathers" of algorithmic randomness. Whereas the various randomness notions were introduced and developed by Martin-Löf, Kolmogorov, Levin, Schnorr, Chaitin, and others in the context of classical probability, statistics, and information theory, Demuth developed these notions in the context of and for application in constructive analysis, where the notion of approximability plays a central role.

For the sake of readability, we will review the main definitions of algorithmic randomness that Demuth introduced. We will refer to them in the text that follows. See [40] or [64] for details. In the following, λ denotes the Lebesgue measure.

- **Martin-Löf randomness** (Martin-Löf [58]): A Martin-Löf test is a computable sequence of effectively open sets $(G_m)_{m \in \mathbb{N}}$ such that $\lambda(G_m) \leq 2^{-m}$ for every m . A real $x \in [0, 1]$ is *Martin-Löf random* if $x \notin \bigcap_{m \in \mathbb{N}} G_m$ for every Martin-Löf test $(G_m)_{m \in \mathbb{N}}$. A Solovay test [73] is a computable sequence of effectively open sets $(G_m)_{m \in \mathbb{N}}$ such that $\sum \lambda(G_m) < \infty$. A real x passes the test if $x \in G_m$ for at most finitely many m . Solovay proved that a real passes all Solovay tests if and only if it is Martin-Löf random (see, e.g., [40, Theorem 6.2.8] or [64, Proposition 3.2.19]).
- **Schnorr randomness** (Schnorr [71]): A Schnorr test is a computable sequence of effectively open sets $(G_m)_{m \in \mathbb{N}}$ such that (i) $\lambda(G_m) \leq 2^{-m}$ for every m and (ii) $\lambda(G_m)$ is a computable real uniformly in m . Furthermore, a real x is *Schnorr random* if and only if $x \notin \bigcap_m G_m$ for every Schnorr test $(G_m)_{m \in \mathbb{N}}$. Note that every Schnorr test is a Martin-Löf test. This implies that every Martin-Löf random real is Schnorr random. However, not every Martin-Löf test is a Schnorr test, as we do not require that $\lambda(G_m)$ be computable in the definition of a Martin-Löf test. Moreover, there are Schnorr random reals that are not Martin-Löf random.
- **Computable randomness** (Schnorr [71]): A computable martingale is a computable rational-valued function $M : \{0, 1\}^* \rightarrow \mathbb{Q}$ such that $2M(\sigma) = M(\sigma 0) + M(\sigma 1)$ for every $\sigma \in \{0, 1\}^*$. A computable martingale M succeeds on $X \in 2^{\mathbb{N}}$ if $\sup_n M(X \upharpoonright n) = \infty$. We say that $X \in 2^{\mathbb{N}}$ is *computably random* if no computable martingale succeeds on X . A real $x \in [0, 1]$ is computably random if the sequence X such that $0.X = x$ is computably random (we assume here x is not a dyadic rational, so X is unique). Every Martin-Löf random real is computably random. Every computably random real is Schnorr random. Neither of the implications can be reversed.

Demuth considered notions of randomness other than the four listed above. These notions include what are now known as Demuth randomness and weak Demuth randomness. They will be introduced in Section 4.4 and further discussed in Section 5.

4.1. Measurability and randomness. We first consider the earliest appearance of a randomness notion in Demuth’s work, which was in the context of constructive measurability.

In the papers [22] and [23] published in 1969, Demuth defines what it means for a property to hold for “almost every” computable real number. Demuth’s definition is given in terms of what he calls S_σ sets. A computable sequence $\{H_n\}_{n \in \mathbb{N}}$ of nonoverlapping intervals with rational endpoints is an S_σ set if $\sum_{n \in \mathbb{N}} |H_n|$ is a computable real number.

A property \mathcal{P} of computable reals holds for “almost every computable real” if there exists a computable sequence of S_σ sets $(S_n)_{n \in \mathbb{N}}$ such that for every n , $\lambda(S_n) \leq 2^{-n}$ and for any computable real x , if $x \notin S_n$ for some n , then x satisfies the property \mathcal{P} . It is immediate that such a collection $(S_n)_{n \in \mathbb{N}}$ is a Schnorr test.

In formulating his definition of a property holding for almost every computable real, Demuth drew on earlier work of Ceĭtin and Zaslavskii [19] from 1962. In this context, it is interesting to note that the absence of a universal Schnorr test follows from a result of Ceĭtin and Zaslavskii from that paper, where it is proved (in different terminology) that a Π_1^0 class of computable measure has a computable path.

Demuth later defined what it means for a property \mathcal{P} to hold for “almost every” pseudonumber (i.e., Δ_2^0 real) in [25, page 584]. In [38], it is stated that such a definition can be obtained by directly relativizing to \emptyset' the definition of a property holding for almost every computable real or “without using relativized concepts.” It is not clear whether he took these two approaches to be equivalent. The reference Demuth gives for the unrelativized definition contains the definition given in Figure 1, but he does not state that this definition is equivalent to the relativized definition.

We rephrase this definition in modern language. Demuth introduces a notion of tests; let us call them *interval sequence tests*. In the following let m, r, k range over the set of positive integers. An interval sequence test uniformly in a number $m \in \mathbb{N}$ provides a computable sequence of rational

Определение. Пусть \mathcal{S} свойство псевдоцисел, а $x \Delta y$ сегмент. Мы скажем, что $\mathcal{S}(\xi)$ выполнено для почти всех псевдоцисел ξ (соотв. ξ из $x \Delta y$), если для всякого $n \in \mathbb{N}$ существуют последовательность последовательностей рациональных сегментов $\{I_{k, \ell}^m\}_{k, \ell \in \mathbb{N}}$ и последовательность неинфинитных рекурсивно перечислимых (р.п.) множеств $\{D_m\}_{m \in \mathbb{N}}$ такие, что $\forall m, \ell (1 \leq \ell \leq \sum_{k: \neg(k \in D_m)} |I_{k, \ell}^m| < \frac{1}{2^{m+\ell}})$ и для всякого ПЧ ξ (соотв. ПЧ ξ из $x \Delta y$) верно $(\neg \exists m, \ell (\neg (\ell \in D_m) \& \xi \in I_{k, \ell}^m)) \supset \mathcal{S}(\xi)$ (см. лемму 7 из [2]).

FIGURE 1. [25, page 584]: Definition of interval sequence tests.

intervals $(Q_r^m(k))_{r,k \in \mathbb{N}}$, and a uniformly c.e. sequence of finite sets $(E_r^m)_{r \in \mathbb{N}}$, such that

$$\lambda\left(\bigcup\{Q_r^m(k) : k \notin E_r^m\}\right) \leq 2^{-(m+r)}. \tag{1}$$

The idea here is that each finite set E_r^m consists of indices $\{k_1, \dots, k_n\}$ for rational intervals $Q_r^m(k_1), \dots, Q_r^m(k_n)$ such that those $z \in Q_r^m(k)$ for some $k \notin E_r^m$ are contained in a fairly small set, i.e., one with measure less than $2^{-(m+r)}$.

A real z fails the interval sequence test if for each m there is r such that for some $k \notin E_r^m$ we have $z \in Q_r^m(k)$. In other words, for each m ,

$$z \in \bigcup_r \bigcup_{k \notin E_r^m} Q_r^m(k). \tag{2}$$

Note that the class in (2) has measure at most 2^{-m} , hence the reals z failing the test form a null set. If z does not fail the test we say that z passes the test. Demuth says that a property \mathcal{P} holds for almost all reals z if there is an interval sequence test (depending on \mathcal{P}) such that \mathcal{P} holds for all z passing the test.

We now show that this unrelativized definition of a property holding for almost every pseudonumber is equivalent to the relativization of Demuth's definition of a property holding for almost all computable reals.

PROPOSITION 4.1 (with Hirschfeldt). *Interval sequence tests are uniformly equivalent to Schnorr tests relative to \emptyset' . That is, given a test of one kind, we can effectively determine a test of the other kind so that a real fails the first test if and only if it fails the second test.*

PROOF. Firstly, suppose we are given an interval sequence test

$$(Q_r^m(k))_{r,k \in \mathbb{N}}, (E_r^m)_{r \in \mathbb{N}} \ (m \in \mathbb{N}).$$

Let G_m be the class in (2). Then G_m is $\Sigma_1^0(\emptyset')$ uniformly in m , and $\lambda(G_m)$ is computable relative to \emptyset' by (1).

Secondly, suppose we are given a Schnorr test $(G_m)_{m \in \mathbb{N}}$ relative to \emptyset' . Uniformly in m , using \emptyset' , we can compute $\lambda(G_m)$ for each $m \in \mathbb{N}$. Hence we can for each $r, m \in \mathbb{N}$ determine $u_r \in \mathbb{N}$ and, by possibly splitting into pieces some intervals from G_m , a finite sequence of rational intervals $P_r^m(i), u_r < i \leq u_{r+1}$, such that $\lambda(\bigcup_{u_r < i \leq u_{r+1}} P_r^m(i)) \leq 2^{-(m+r)}$ and $G_m = \bigcup_r \bigcup_{u_r < i \leq u_{r+1}} P_r^m(i)$. By the Limit Lemma we have a computable sequence of intervals $P_r^m(i, t)$ and a computable sequence $u_r(t), t \in \mathbb{N}$, such that for large enough $t, u_r(t) = u_r$ and $P_r^m(i, t) = P_r^m(i)$ for $i \leq u_r$. From this we can build an interval sequence test as required: the uniformly c.e. finite sets E_r^m correspond to the intervals we want to remove because of the mind changes of the approximations $u_r(t)$ and $P_r^m(i, t)$ for $i \leq u_r(t)$. \dashv

In later work [35], Demuth defined a fully relativized version of measure zero sets. For a set of natural numbers B , a set $\mathcal{S} \subseteq [0, 1]$ has B -measure zero if there is a B -Schnorr test $(G_m^B)_{m \in \mathbb{N}}$ such that $\mathcal{S} \subseteq \bigcap_m G_m^B$. In keeping with his extended constructivism, Demuth only applied the notion of a B -measure zero set for sets B such that $B \leq_T \emptyset^{(n)}$ for some n . Nevertheless, the definition is stated in full generality. In fact, Demuth defined the

more general notion of *B-measurability* for a given $B \subseteq \mathbb{N}$, of which the notion of a *B-measure zero set* is a special case.

4.2. Demuth’s version of Martin-Löf-randomness. Several other randomness notions arose in Demuth’s study of the differentiability of Markov computable functions. It was natural for Demuth to consider a broader class of reals than just the computable reals, as computable reals do not suffice to study the points of differentiability of these functions. For instance, Demuth proved that the derivative of a Markov computable function at a computable real need not be computable. He also proved the existence of an absolutely continuous Markov computable function that is not pseudodifferentiable at any computable real (where pseudodifferentiability is defined below). Demuth further showed that this function is only pseudodifferentiable at Martin-Löf random reals, and, as we will discuss in the next subsection, at all of them.

In a 1975 paper [24], Demuth introduced a randomness notion equivalent to Martin-Löf randomness. At the time of the publication of [24], Demuth was not aware of Martin-Löf’s earlier definition in [58] dating from 1966. Demuth originally considered only Martin-Löf random pseudonumbers, which he called Π_2 -numbers. As a constructivist, Demuth found it more natural to define the non-Martin-Löf random pseudonumbers first. He called them Π_1 -numbers.

DEFINITION 4.2. A Δ_2^0 real x is a Π_1 -number if there is a computable sequence of rationals $(q_n)_{n \in \mathbb{N}}$ with $x = \lim_{n \rightarrow \infty} q_n$ and a computable sequence of finite computable sets $(C_m)_{m \in \mathbb{N}}$ such that $\lambda(\bigcup_{n \notin C_m} [q_n, q_{n+1}]) < 2^{-m}$.

We provide a sketch of the proof that a Δ_2^0 real $x \in [0, 1]$ is a Π_1 -number if and only if x is not Martin-Löf random. For one implication, from a computable sequence of rationals $(q_n)_{n \in \mathbb{N}}$ with $x = \lim_{n \rightarrow \infty} q_n$ and a computable sequence of finite computable sets $(C_m)_{m \in \mathbb{N}}$ such that $\lambda(\bigcup_{n \notin C_m} [q_n, q_{n+1}]) < 2^{-m}$, we can construct a Martin-Löf-test $(B_m)_{m \in \mathbb{N}}$ by setting

$$B_m = \{y : \exists n, k[|y - q_n| < 2^{-m-1-k} \wedge \#\{j : j \leq n, j \in C_{m+1}\} = k]\}.$$

It is not hard to verify that $\lambda(B_m) \leq 2^{-m}$ for every m and that $x \in \bigcap_{m \in \mathbb{N}} B_m$.

For the reverse implication, let $(U_m)_{m \in \mathbb{N}}$ be a universal Martin-Löf-test, i.e. a Martin-Löf test $(U_m)_{m \in \mathbb{N}}$ such that for any $z \in [0, 1]$, z is Martin-Löf random if and only if $z \notin \bigcap_{m \in \mathbb{N}} U_m$.

Recall that $A \subseteq \{0, 1\}^*$ is prefix-free if for every $\sigma, \tau \in \{0, 1\}^*$, if $\sigma \in A$ and τ properly extends σ , then $\tau \notin A$. Now let $(V_m)_{m \in \mathbb{N}}$ be a prefix-free subset of $\{0, 1\}^*$ for which $U_m = \bigcup_{\sigma \in V_m} [\sigma]$ for any $m \in \mathbb{N}$, where $[\sigma]$ is the interval $[0.\sigma, 0.\sigma + 2^{-|\sigma|})$ as in the proof of Theorem 3.3.

Suppose $x \in [0, 1]$ is a Δ_2^0 non-Martin-Löf random real and $x = \lim_{n \rightarrow \infty} q_n$ for a computable sequence of rationals $(q_n)_{n \in \mathbb{N}}$ from $[0, 1]$. We let

$$C_m = \{n : \text{Hop}_m(q_n, q_{n+1})\},$$

where the condition $\text{Hop}_m(q_n, q_{n+1})$ means that q_n and q_{n+1} belong to intervals represented by two strings from V_m that are not contiguous. That is, $q_n \in [\sigma]$ and $q_{n+1} \in [\tau]$ for some $\sigma, \tau \in V_m$ such that $0.\sigma + 2^{-|\sigma|} \neq 0.\tau$ and $0.\tau + 2^{-|\tau|} \neq 0.\sigma$. C_m is clearly a computable set uniformly in m . Since there is a $\sigma \in V_m$ such that $x \in [\sigma]$, it is easy to verify that each C_m is finite. This concludes our sketch of the equivalence.

In addition to defining Π_1 -numbers and their complement, the Π_2 -numbers, Demuth constructed a universal Martin-Löf-test [24, Theorems 2 and 6], albeit in different terminology: he built a computable sequence of rational intervals $(\mathcal{K}_s^t)_{t,s \in \mathbb{N}}$ for which $\lambda(\bigcup_s \mathcal{K}_s^t) < 2^{-t}$ for all t and such that any Δ_2^0 real x is a Π_1 -number if and only if $x \in \bigcup_s \mathcal{K}_s^t$ for all t . Furthermore, he showed that the property of a Δ_2^0 real x to be a Π_1 -number does not depend on the choice of a computable sequence $(q_n)_{n \in \mathbb{N}}$ with $x = \lim_{n \rightarrow \infty} q_n$ (see [24, Corollary 1 of Theorem 5]).

Demuth also studied an analogue of Solovay tests in [24]. As stated at the beginning of this section, a real is Solovay random if and only if it is Martin-Löf random. Significantly, a restricted version of this result was also established by Demuth, who proved that a Δ_2^0 real x is a Π_2 -number if and only if it is Solovay random [24, Corollary 2 of Theorem 5]. We should note that Demuth's proof is easily extendible to hold for all reals, not just the Δ_2^0 reals.

In [13], it is shown that the Martin-Löf random Δ_2^0 -reals are precisely the finitely bounded random reals, which are defined in terms of Martin-Löf tests $(G_m)_{m \in \mathbb{N}}$ where each G_m is a finite union of intervals. In [24], Demuth anticipated this result by proving that his definition of Π_2 -number is equivalent to one given in terms of finitely bounded tests.

Another topic that Demuth investigated was the extent to which Π_1 -numbers are preserved under basic arithmetical operations. His main result on the subject, given in [24] is that for every Δ_2^0 real α , there exist Π_1 -numbers β_1 and β_2 such that $\alpha = \beta_1 + \beta_2$. Thus, since such an α can be a Π_2 -number, the sum of two Π_1 -numbers need not be a Π_1 -number.

Recall that a real α is left-c.e. if α is the limit of a computable, nondecreasing sequence of rational numbers. Demuth further showed that the situation differs significantly if we consider pseudonumbers that are left-c.e.: if β_1 and β_2 are left-c.e. Π_1 -numbers, then $\beta_1 + \beta_2$ is also a left-c.e. Π_1 -number (see [24]). In other words if α is left-c.e. and Martin-Löf random, and $\alpha = \beta_1 + \beta_2$ for left-c.e. reals β_1 and β_2 , then at least one of β_1, β_2 is Martin-Löf random. This is one of the earliest results in the theory of left-c.e. reals, a subject developed by Solovay in [73] that has been of much interest in recent years (see, for instance, [41] where Demuth's result is rediscovered, as well as Chapters 5 and 9 of [40]).

Beginning in 1978, Demuth was willing to countenance arithmetical reals. For instance, in [37], Demuth, Kryl, and Kučera prove that pseudonumbers relative to $\emptyset^{(n)}$ correspond to computable reals relative to $\emptyset^{(n+1)}$. In [30] he refers to the arithmetical *non*-Martin-Löf-random reals as \mathcal{A}_1 numbers and the arithmetical Martin-Löf random reals \mathcal{A}_2 numbers. For instance, the definition of \mathcal{A}_1 can be found in [30, page 457]. By then, Demuth knew

of Martin-Löf’s work: he defined \mathcal{A}_1 to be $\bigcap_k [W_{g(k)}]^\prec$, where g is a computable function determining a universal Martin-Löf test, and $[X]^\prec$ is the set of arithmetical reals extending a string in X . In the English language papers such as [35], the non-Martin-Löf random reals were called AP (for approximable in measure), and the Martin-Löf random reals were called NAP (for nonapproximable in measure).

4.3. Differentiability and randomness. Before discussing the application of Demuth’s version of Martin-Löf-randomness to differentiability of Markov computable functions, we will review some definitions and provide some terminology. For a function f , the *slope* at a pair a, b of distinct reals in its domain is

$$S_f(a, b) = \frac{f(a) - f(b)}{a - b}.$$

Recall that if z is in the domain of f then

$$\begin{aligned} \overline{D}f(z) &= \limsup_{h \rightarrow 0} S_f(z, z + h), \\ \underline{D}f(z) &= \liminf_{h \rightarrow 0} S_f(z, z + h). \end{aligned}$$

Note that we allow the values $\pm\infty$. By the definition, a function f is differentiable at z if $\underline{D}f(z) = \overline{D}f(z)$ and this value is finite. We will denote the derivative of f at z by $f'(z)$.

If one wants to study the differentiability of Markov computable functions, one immediately runs into the problem that these functions are only defined on the computable reals. So one has to introduce upper and lower “pseudoderivatives” at a real z , taking the limit of slopes close to z where the function is defined. This is precisely what Demuth did. Consider a function g defined on $I_{\mathbb{Q}}$, the rationals in $[0, 1]$. For $z \in [0, 1]$ let

$$\begin{aligned} \widetilde{D}g(z) &= \limsup_{h \rightarrow 0^+} \{S_g(a, b) : a, b \in I_{\mathbb{Q}} \wedge a \leq z \leq b \wedge 0 < b - a \leq h\}, \\ \underline{D}g(z) &= \liminf_{h \rightarrow 0^+} \{S_g(a, b) : a, b \in I_{\mathbb{Q}} \wedge a \leq z \leq b \wedge 0 < b - a \leq h\}. \end{aligned}$$

DEFINITION 4.3. We say that a function f with domain containing $I_{\mathbb{Q}}$ is *pseudodifferentiable* at x if $-\infty < \underline{D}f(x) = \widetilde{D}f(x) < \infty$, in which case the value $\underline{D}f(x) = \widetilde{D}f(x)$ will be denoted $f'(x)$.

Since Markov computable functions are continuous on the computable reals, it does not matter which dense set of computable reals one takes in the definition of these upper and lower pseudoderivatives. For instance, one could take all computable reals, or only the dyadic rationals. For a total continuous function g , we have $\underline{D}g(z) = \underline{D}g(z)$ and $\widetilde{D}g(z) = \overline{D}g(z)$. The last section of the extended arXiv version of [12] contains more detail on pseudoderivatives.

Initially, Demuth studied the pseudodifferentiability of Markov computable functions at computable reals in the 1969 paper [21]. For reasons mentioned above, in this limited setting, the resulting theory of pseudodifferentiability was not adequate. However, in a second 1975 paper [25], Demuth considered pseudodifferentiability at pseudonumbers, which enabled him

to prove a number of significant results on the pseudodifferentiability of Markov computable functions of bounded variation. The abstract of the paper, translated literally, is as follows:

It is shown that every constructive function f which cannot fail to be a function of weakly bounded variation is finitely pseudodifferentiable on each Π_2 -number.

For almost every pseudonumber ξ there is a pseudonumber which is a value of pseudoderivative of the function f on ξ , where the differentiation is almost uniform.

We rephrase Demuth's result in modern terminology.

THEOREM 4.4 ([25]). *Let f be a Markov computable function of bounded variation.*

- (i) *f is pseudodifferentiable at any Δ_2^0 Martin-Löf random real.*
- (ii) *Furthermore, there is a Schnorr test relative to \emptyset' such that for any Δ_2^0 real ξ passing the test, ξ is Martin-Löf random, $f'(\xi)$ exists, and $f'(\xi)$ is a Δ_2^0 real which can be computed uniformly in \emptyset' and the representation of ξ as a Δ_2^0 real.*

To prove that a classical function f of bounded variation is almost everywhere differentiable one usually expresses f as a difference of two nondecreasing functions f_1, f_2 . In the constructive setting, this approach no longer works, since a Markov computable function of bounded variation need not be expressible as a difference of two nondecreasing Markov computable functions, as proved by Ceitin and Zaslavskii in [19].

A function f is called interval-c.e. [44] if $f(0) = 0$ and $f(y) - f(x)$ is a left-c.e. real, uniformly in rationals $x < y$. If we wanted to allow a little more leeway, any Markov computable function of bounded variation is expressible as $f_1 - f_2$, where f_1, f_2 are nondecreasing *interval-c.e. functions*. Unfortunately, functions of this more general type need not be differentiable at each Martin-Löf random real as shown recently by Nies in [66, Theorem 7], so a different approach is needed.

For a c.e. set C of closed rational intervals, we will write $\mathcal{H}(C)$ to denote that the intervals in C are nonoverlapping, i.e., they have at most endpoints in common, and that for any k one can compute a stage s in the enumeration of C such that any interval enumerated into C after stage s has size less than 2^{-k} .

The approach Demuth took to prove part (i) of Theorem 4.4 is roughly as follows. First, for a given Markov computable function f and a c.e. set C of rational intervals such that $\mathcal{H}(C)$ holds, Demuth defines the function $[f, C]$ to be the Markov computable function such that for each interval I in C , $[f, C]$ is equal to f on the endpoints of I , is linear on the interior of I , and is equal to f otherwise.

Demuth next proves the following lemma (see [25, Lemma 4]).

LEMMA 4.5. *Let f be a Markov computable function and w, z computable reals such that $w < z$, both w, z are not equal to $(f(b) - f(a))/(b - a)$ for*

any rationals $a, b \in [0, 1]$, and $f(1) - f(0) < z$. Then there is a c.e. set C of rational intervals with the following properties:

- (i) the condition $\mathcal{H}(C)$ holds;
- (ii) $w(b - a) < f(b) - f(a)$ for any interval $[a, b]$ from C ; and
- (iii) $[f, C](y) - [f, C](x) < z(y - x)$ for any computable reals x, y from $[0, 1]$.

An analogous statement holds if replace each occurrence of “ $<$ ” with “ $>$ ” (including “ $w > z$ ” instead of “ $w < z$ ”).

It follows that the function $g(x) = [f, C](x) - z \cdot x$ is strictly monotone on $[0, 1]$ and $[f, C]$ is uniformly continuous of bounded variation.

In particular, if f is a Markov computable function of bounded variation, then for any k , we can compute sufficiently large w and z with $w < z$ such that the measure of the intervals from the set C guaranteed to exist by Lemma 4.5 is less than 2^{-k} (and similarly for $z < w$).

Moreover, if we apply Lemma 4.5 twice, first to some appropriately chosen $w < z$ and then to some appropriately chosen $w' > z'$, we can truncate a Markov computable function of bounded variation f into a Markov computable Lipschitz function $[f, C]$ for some c.e. set C of rational intervals with $\mathcal{H}(C)$. Combining this with the statement in the previous paragraph, we can find a c.e. set C_k of rational intervals effectively in k such that $\mathcal{H}(C_k)$ holds and the measure of the intervals in C_k is less than 2^{-k} .

Next, by using a series of complicated approximations of Markov computable Lipschitz functions by Markov computable polygonal functions, Demuth proves that every Markov computable Lipschitz function is differentiable at every Martin-Löf random real [25, Theorems 1 and 2].

Demuth then proves that if f is a Markov computable function of bounded variation and C is a c.e. set of rational intervals such that $\mathcal{H}(C)$ holds and such that the function $[f, C]$ is Lipschitz, the function $f - [f, C]$ is differentiable at any Martin-Löf random real outside of any interval in C . But since $f = [f, C]$ outside of any interval in C , it follows that f is differentiable at any Martin-Löf random real outside of any interval in C .

Lastly, using the fact that we can control the measure of the intervals in the above set C , we produce a uniformly c.e. collection $(C_k)_{k \in \mathbb{N}}$ of rational intervals such that for every k , (a) $\mathcal{H}(C_k)$ holds, (b) the function $[f, C_k]$ is Lipschitz, and (c) the measure of the intervals in C_k less than 2^{-k} , where condition (c) implies that $(C_k)_{k \in \mathbb{N}}$ defines a Martin-Löf test. Thus for every Martin-Löf random real x , there is some k such that x not in any interval in C_k , and since $[f, C_k]$ is differentiable at x , it follows that f is differentiable at x as well. This concludes the proof of (i).

A more general version of Theorem 4.4 (i), which holds for *all* Martin-Löf random reals, has been recently reproved in [12, Thm. 6.7] in an indirect way. It relies on a similar result for computable randomness: each Markov computable nondecreasing function is differentiable at each computably random real. The latter result is in the same paper [12, Theorem 4.1], taking into account the extension of the theorem in the last section of the arXiv version.

According to part (ii) of Theorem 4.4, there is a single Schnorr test relative to \emptyset' such that for any Δ_2^0 real ξ passing the test, ξ is Martin-Löf random, $f'(\xi)$ exists and is Δ_2^0 , and $f'(\xi)$ can be uniformly computed from \emptyset' and the representation of ξ as a Δ_2^0 real.

To prove this, Demuth carried out detailed calculations to produce the desired Schnorr test relative to \emptyset' . We can reprove Theorem 4.4 (ii) as follows. First, since f is Markov computable, it is easy to verify that

$$f'(z) \leq_T z',$$

namely, the value of the pseudoderivative of f at z is computable in the Turing jump of z whenever this pseudoderivative exists. Thus $f'(z)$ is Δ_2^0 whenever z is low. Moreover, by [35, Remark 10, part 3b], or [64, Theorem 3.6.26], there is a single Schnorr test relative to \emptyset' (in fact, a Demuth test as defined in Definition 4.12 below) such that each real z passing it is generalized low, i.e., $z' \leq_T z \oplus \emptyset'$. Thus, the only reals z for which $f'(z)$ is not Δ_2^0 are captured by this Schnorr test relative to \emptyset' .

Moreover, there is a fixed effective procedure for computing z' from $z \oplus \emptyset'$ for any z passing this Schnorr test (see the proof of Theorem 3.6.26 in [64]). This yields the desired uniform computability of $f'(z)$ from \emptyset' and the representation of z as a Δ_2^0 real for any Δ_2^0 real z passing the test.

4.4. The Denjoy alternative. Demuth also closely studied the Denjoy alternative for Markov computable functions. One simple version of the Denjoy alternative for a function f defined on the unit interval says that

$$\text{either } f'(z) \text{ exists, or } \overline{D}f(z) = \infty \text{ and } \underline{D}f(z) = -\infty. \tag{3}$$

The full result is given in terms of left and right upper and lower Dini derivatives, but we consider only the more compact version here.

It is a consequence of the classical Denjoy (1907), Young (1912), and Saks (1937) Theorem that for *any* function defined on the unit interval, the Denjoy alternative holds at almost every z . Denjoy himself obtained the Denjoy alternative for continuous functions, Young for measurable functions, and Saks for all functions. For a proof see for instance Bogachev [9, p. 371].

Here we formulate the Denjoy alternative in terms of pseudoderivatives.

DEFINITION 4.6. Suppose the domain of a partial function f contains $I_{\mathbb{Q}}$. We say that the *Denjoy alternative* holds for f at z if

$$\text{either } \tilde{D}f(z) = \underline{D}f(z) < \infty, \text{ or } \tilde{D}f(z) = \infty \text{ and } \underline{D}f(z) = -\infty. \tag{4}$$

$$\underline{D}f(z). \tag{5}$$

This is equivalent to (3) if the function is total and continuous.

For any function $g: [0, 1] \rightarrow \mathbb{R}$, the reals z such that $\underline{D}g(z) = \infty$ form a null set. This well-known fact from classical analysis is usually proved via covering theorems, such as Vitali's or Sierpinski's. (Cater [14] has given an alternative proof of a stronger fact: the reals z where the right lower derivative $\underline{D}_+g(z)$ is infinite form a null set.)

Demuth was interested in determining which type of null class is needed to make an analog of this classic fact hold for Markov computable functions (see Definition 3.2). The following notion can be found in [28] (although a variant was given in the earlier [27]). As usual, for functions not defined everywhere we have to work with pseudoderivatives as defined in Subsection 4.3.

DEFINITION 4.7. A real $z \in [0, 1]$ is called *Denjoy random* (or a *Denjoy set*) if for no Markov computable function g do we have $\underline{D}g(z) = \infty$.

We should emphasize here that Demuth only used the term “Denjoy set” in the preprint and final version of his paper “Remarks on Denjoy sets” [34]. The preprint was based on a talk Demuth gave at the Logic Colloquium 1988 in Padova, Italy (close to the end of communist era in 1989, as it became easier to travel to the “West”). He later turned the preprint survey into the paper [36] with the same title, but it contains only part of the preprint survey.

As reported in the preprint survey [34, p. 6], in [27] it is shown that if $z \in [0, 1]$ is Denjoy random, then for every \emptyset -uniformly continuous Markov computable $f : [0, 1] \rightarrow \mathbb{R}$ the Denjoy alternative (3) holds at z . Combining this with the results in [12] we can now determine precisely what Denjoy randomness is, and also obtain a pleasing new characterization of computable randomness of reals through differentiability of standard computable functions.

THEOREM 4.8 ([7]). *The following are equivalent for a real $z \in [0, 1]$.*

- (i) z is Denjoy random,
- (ii) z is computably random,
- (iii) for every standard computable $f : [0, 1] \rightarrow \mathbb{R}$ the Denjoy alternative (3) holds at z .

PROOF. (i) \rightarrow (iii) is Demuth’s result (see [28, Theorem 1] and [27, Theorem 3]). For (iii) \rightarrow (ii), let f be a nondecreasing standard computable function. Then f satisfies the Denjoy alternative at z . Since $\underline{D}f(z) \geq 0$, this means that $f'(z)$ exists. This implies that z is computably random by [12, Theorem, 4.1].

The implication (ii) \rightarrow (i) is proved by contraposition: if g is Markov computable and $\underline{D}g(z) = \infty$ then one builds a computable martingale that witnesses that z is not computably random. See [6, Theorem, 15] or [7] for the details of the proof. \dashv

REMARK 4.9. For the contraposition of the implication (ii) \rightarrow (i), it suffices to use the weaker hypothesis on g that $g(q)$ is a computable real uniformly in a rational $q \in I_{\mathbb{Q}}$.

We do not fully understand how Demuth obtained (i) \rightarrow (iii) of the theorem; a proof of this using classical language would be useful. We can, however, obtain a direct proof of the contraposition of (i) \rightarrow (ii) that uses techniques from modern algorithmic randomness (which can be found in [12, Thm. 3.6]): if z is not computably random then a martingale M with the so-called “savings property” succeeds on (the binary expansion of) a real z .

Recall that M has the savings property if $M(\tau) \geq M(\sigma) - 2$ for every pair of strings $\tau \succeq \sigma$. The authors now build a standard computable function g such that $\underline{D}g(z) = \underline{D}g(z) = \infty$.

Together with Remark 4.9 we obtain:

COROLLARY 4.10. *The following are equivalent for a real z :*

- (i) *For no function g such that $g(q)$ is uniformly computable for $q \in I_{\mathbb{Q}}$ do we have $\underline{D}g(z) = \infty$.*
- (ii) *z is Denjoy random, i.e., for no Markov computable function g do we have $\underline{D}g(z) = \infty$.*
- (iii) *For no standard computable function g do we have $\underline{D}g(z) = \infty$.*

This implies that the particular choice of Markov computable functions in Definition 4.7 is irrelevant. Similar equivalences stating that the exact level of effectivity of functions does not matter have been obtained in the article [12]. For instance, the version of Theorem 4.4 (i) from [12] holds for any functions of bounded variation with any of the three particular effectiveness properties above: standard computable, Markov computable, and uniformly computable on the rationals. For nondecreasing *continuous* functions, the three effectiveness properties coincide as observed in [12, Prop. 2.2].

Because of Theorem 4.8 one could assert that Demuth studied computable randomness indirectly via his Denjoy sets. Presumably he didn't know the notion of computable randomness, which was independently introduced by Schnorr in [71] (see also [64, Ch. 7] or [40, Section 7.1]). Demuth also proved in [35, Thm. 2] that every Denjoy set that is AP (i.e., non-Martin-Löf random) must be high. The analogous result for computable randomness was later obtained in [67]. There, the authors also show a kind of converse: each high degree contains a computably random set that is not Martin-Löf random. This fact was apparently not known to Demuth (although he did prove a closely related result, as we will see in Section 6.3 in our discussion of semigenericity).

As mentioned above, Demuth knew that Denjoy randomness of a real z implies the Denjoy alternative at z for all standard computable functions. It was thus natural for Demuth to ask the following question:

How much randomness for a real z is needed to ensure the Denjoy alternative at z for all Markov computable functions?

Demuth showed the following (see the preprint survey, [34, p. 7, Theorem 5, item 4], which refers to [26]).

THEOREM 4.11. *There is a Markov computable function f such that the Denjoy alternative fails at some Martin-Löf random real z . Moreover, f is extendable to a continuous function on $[0, 1]$.*

This theorem has been reproved by Bienvenu, Hölzl, Miller and Nies [6, 7]. In their proof, z can be taken to be the least element of an arbitrary effectively closed set of reals containing only Martin-Löf random reals. In particular, one can make z left-c.e.

It was now clear to Demuth that a randomness notion stronger than Martin-Löf’s was needed. Such a notion was introduced in the paper “Some classes of arithmetical reals” [30, p. 458]. The definition is reproduced in the preprint survey [34, p. 4]. In modern language the definitions are as follows.

DEFINITION 4.12. A *Demuth test* is a sequence of c.e. open sets $(S_m)_{m \in \mathbb{N}}$ such that $\forall m \lambda(S_m) \leq 2^{-m}$, and there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ with $f \leq_{\text{wtt}} \emptyset'$ such that $S_m = [W_{f(m)}]^\frown$.

A set Z passes the test if $Z \notin S_m$ for almost every m . We say that Z is *Demuth random* if Z passes each Demuth test.

Recall that $f \leq_{\text{wtt}} \emptyset'$ if and only if f is ω -c.e., namely, $f(x) = \lim_t g(x, t)$ for some computable function g such that the number of stages t with $g(x, t) \neq g(x, t - 1)$ is computably bounded in x . Hence the idea is that we can change the m -th component S_m a computably bounded number of times.

Fig. 2 shows the definition of Demuth randomness as it appears in the 1982 paper [30, p. 458]. For a given index q of a binary computable function $\phi_q(k, x)$, Demuth defines the set

$$\mathcal{U}_q = \{Z : (\forall m)(\exists k \geq m) Z \in [W_{\lim(s_1^1(q,k))}]^\frown\},$$

provided that $\lim(s_1^1(q, k))$ (which simply means $\lim_x \phi_q(k, x)$, the final version r of the test) exists. A further condition $\mathcal{K}(p, q)$, involving an index p for a computable unary function, yields the bound $\phi_p(k)$ on the number of changes. The bound 2^{-k} on measures of the k -th component can be found in part a) of Fig. 2. The notation $Mis(s_1^1(q, k))$ in Fig. 2 refers to the number

$$\begin{aligned} \text{a) } S_0(q) &\equiv \forall k \ell (!\langle q \rangle(k, \ell) \& ! \lim_{t \rightarrow \infty} \langle q \rangle(k, t)), \\ \hat{S}_0(q) &\equiv (S_0(q) \& \forall k (\hat{\mu}_1(\lim(s_1^1(q, k))) \leq 2^{-k})), \\ \bar{S}_0(q) &\equiv (S_0(q) \& \forall k (\mu_0(\lim(s_1^1(q, k+1))) \leq 2^{-k-1})), \\ \mathcal{K}(p, q) &\equiv (\mathcal{K}_0(q) \& \forall k (!\langle p \rangle(k) \& Mis(s_1^1(q, k)) \leq \langle p \rangle(k))), \\ \text{где } \mathcal{K} &\text{ одно из выражений } S, \hat{S} \text{ и } \bar{S}, \\ \text{б) если верно } S_0(q), &\text{ то} \\ \mathcal{U}_q^* &\equiv \bigcap_m \hat{\bigcup}_{m \geq m} [W_{\lim(s_1^1(q, m))}] , \\ \mathcal{U}_q^{**} &\equiv \hat{\bigcup}_m \bigcap_{m \geq m} [W_{\lim(s_1^1(q, m))}] , \\ \mathcal{U}_q^c &\equiv \hat{\bigcup}_k [\hat{\bigcap}_{\ell} [W_{\lim(s_1^1(q, \ell))}]^c] , \\ \text{в) } \mathcal{A}_\alpha &\equiv \wedge X (\neg \exists m (\hat{S}(p, m) \& X \in \mathcal{U}_m^*)); \\ \text{2) } \mathcal{A}_\alpha &\equiv \wedge X (\neg \exists p q (\hat{S}(p, q) \& X \in \mathcal{U}_q^*)), \\ \mathcal{A}_\alpha^* &\equiv \wedge X (\neg \exists p q (\hat{S}(p, q) \& X \in \mathcal{U}_q^{**})), \\ \mathcal{A}_\beta &\equiv \mathcal{A} \setminus \mathcal{A}_\alpha . \end{aligned}$$

FIGURE 2. [30, p. 458]: \mathcal{A}_β is the definition of Demuth randomness.

of “mistakes”, i.e. changes, and Demuth requires it be bounded by $\langle p \rangle(k)$, meaning $\phi_p(k)$.

If we apply the usual passing condition for tests, we obtain the following notion which only occurs in [30, p. 458].

DEFINITION 4.13. We say that a set $Z \subseteq \mathbb{N}$ is *weakly Demuth random* if for each Demuth test $(S_m)_{m \in \mathbb{N}}$ there is an m such that $Z \notin S_m$.

In [30] weak Demuth randomness is defined in terms of a set Υ_q^* , where the quantifiers are switched compared to the definition of Υ_q :

$$\Upsilon_q^* = \{Z : (\exists m)(\forall k \geq m) Z \in [W_{\lim(s_1^1(q,k))}]^{\prec}\},$$

again provided that $\lim(s_1^1(q,k))$ exists.

Note, however, that there is a slight difference between the definition of weak Demuth randomness as given in [30] and that given by Definition 4.13 above. If we set $S_k = [W_{\lim(s_1^1(q,k))}]^{\prec}$, then $Z \notin \Upsilon_q^*$ means $Z \notin S_m$ for infinitely many m . However, the two definitions are equivalent, since for a given Demuth test $(S_m)_{m \in \mathbb{N}}$, for each $m \in \mathbb{N}$, $(S_k)_{k \geq m}$ also yields a Demuth test.

The class of arithmetical non-Demuth randoms is denoted \mathcal{A}_α , and the class of arithmetical nonweakly Demuth randoms is denoted \mathcal{A}_α^* . The complement of \mathcal{A}_α within the arithmetical reals is denoted \mathcal{A}_β and, similarly, the complement of \mathcal{A}_α^* within the arithmetical reals is called \mathcal{A}_β^* . Later on, in the preprint survey, Demuth used the terms WAP sets (weakly approximable in measure) for the non-Demuth randoms, and NWAP for the Demuth randoms and the terms WAP* sets and NWAP* sets for the nonweakly Demuth randoms and the weakly Demuth randoms, respectively.

In the preprint survey [34, p. 7, Theorem 5, item 5)], Demuth states that Demuth randomness is sufficient to guarantee that the Denjoy alternative for Markov computable functions holds (referring to [31, Theorem 2]).

THEOREM 4.14. *Let z be a Demuth random real. Then the Denjoy alternative holds at z for every Markov computable function.*

To derive this result, Demuth constructs a single Demuth test $(\mathcal{S}_n)_{n \in \mathbb{N}}$ containing all non-Martin-Löf random reals such that for any Markov computable function and any real x one of the following holds:

- (1) $\tilde{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$;
- (2) either $\underline{D}f(x) > -\infty$ or $\tilde{D}f(x) < +\infty$, and one of the following holds:
 - (i) $\tilde{D}f(x) = \underline{D}f(x)$, $\lim_{r \rightarrow x} f(r) = y$ exists, $y \notin \mathcal{S}_n$ for almost every n , and $\tilde{D}f(x) \neq 0$;
 - (ii) $\lim_{r \rightarrow x} f(r) = y$ exists but $y \in \mathcal{S}_n$ for infinitely many n , i.e., y does not pass the test $(\mathcal{S}_n)_{n \in \mathbb{N}}$;
 - (iii) $\lim_{r \rightarrow x} f(r)$ does not exist.

If x is Demuth random and f is a Markov computable function it is possible to show that

- condition (2)(iii) cannot hold for x . More precisely, as claimed in [31, Remark 7], condition (2)(iii) implies that x is either a left-c.e. or a right-c.e. real (where a real is right-c.e. if it is the limit of a computable nonincreasing sequence of rationals), which cannot be Demuth random;
- condition (2)(ii) reduces to the situation where f is differentiable at x with the value $f'(x)$ equal to 0;
- condition (2)(i) reduces to the situation where the value $\tilde{D}f(x) = Df(x)$ is finite and $f'(x) \neq 0$.

Thus, the Denjoy alternative for f holds at any Demuth random real x .

REMARK 4.15. Franklin and Ng [43] introduced difference randomness, a concept much weaker than even weak Demuth randomness, but still stronger than Martin-Löf randomness. Bienvenu, Hölzl, Miller and Nies [6, Theorem. 1] have shown that difference randomness is sufficient as a hypothesis on the real z in Theorem 4.14. No converse holds. They also show that the “randomness notion” to make the Denjoy alternative hold for each Markov computable function is incomparable with Martin-Löf randomness!

§5. Further results on Demuth randomness. The notions of Demuth and weak Demuth randomness have proven to be very fruitful, being studied in a number of recent papers. However, due to the relative inaccessibility of Demuth’s work, many researchers in the field have been unaware of just how much Demuth proved about these notions. In this section, we review some of Demuth’s results on his notions of randomness.

5.1. Computability-theoretic properties of Demuth randomness. In the mid-1970s, the mathematics department at Charles University held a seminar on computability theory based on Rogers’ book [69], which had been translated into Russian in 1972. As a result of this seminar, Demuth became more interested in computability theory and the computational complexity of random reals.

In particular, Demuth thoroughly studied the relationship between Demuth randomness and the Turing degrees. For instance, in [35] he proved the following, which was already implicit in [29, Theorem 6].

PROPOSITION 5.1.

- Every Demuth random real is generalized low, i.e., $z' \leq_T z \oplus \emptyset'$.
- There is a single Demuth test $(S_m)_{m \in \mathbb{N}}$ such that for every z for which $z \in S_m$ for at most finitely many m , z is generalized low.

Demuth actually proved a stronger result. Recall that a truth-table reduction (tt-reduction for short) is a Turing reduction given in terms of a computable sequence of truth-tables that determine the outputs of the reduction. Equivalently, a tt-reduction is a Turing reduction Φ such that Φ^X is total for all oracles X (Nerode [62]). A \emptyset' -tt-reduction is thus a reduction given in terms of a \emptyset' -computable sequence of truth-tables. Demuth proved that for any Demuth random z , z' is \emptyset' -tt-reducible to z . Note that Demuth’s

result does not imply that $z' \equiv_{tt} z \oplus \emptyset'$, since this latter statement implies that the use of \emptyset' in the reduction is bounded by a computable function, which need not be the case for a \emptyset' -tt-reduction.

Demuth also proved results about the growth rate of functions computable from Demuth random reals. First, he showed that every Demuth random real has hyperimmune degree (i.e. that every Demuth random computes a function not dominated by any computable function). In contrast, he also proved the following.

THEOREM 5.2 (Demuth [33]). *There is a \emptyset' -computable function g such that for every Demuth random z and every z -partial computable function f , $f(n) \leq g(n)$ for almost every n .*

In modern terminology, this result implies that \emptyset' is *uniformly almost everywhere dominating*, a result established earlier by Kurtz in [50]. What Kurtz showed is that there is a measure one set of reals \mathcal{S} such that every total function computable from a member of \mathcal{S} is dominated by a fixed \emptyset' -computable function. Demuth was unaware of this result, but improved it in two ways, (1) by showing that \mathcal{S} includes every Demuth random real, and (2) by showing the function g dominates every *partial* function computable from every Demuth random.

Demuth proved a further result of which a variant of which was only recently rediscovered.

THEOREM 5.3 (Demuth [35]). *Let y be Demuth random and x Martin-Löf random. If $x \leq_T y$ then x is Demuth random.*

Miller and Yu [61] proved that for every 2-random y (i.e. y is Martin-Löf random relative to \emptyset') and Martin-Löf random x , $x \leq_T y$ implies that x is 2-random (see also [40, Theorem 8.5.3] or [64, Corollary 3.6.20]). This follows from their more general result that for any z , every Martin-Löf random Turing below a z -Martin-Löf random is also z -Martin-Löf random.

Demuth's proof is very similar to the proof of the result of Miller and Yu given in [61]. For a Turing functional Φ and $n > 0$, consider the open set

$$S_{\Phi,n}^A = [\{\sigma \in \{0,1\}^* : A \upharpoonright_n \preceq \Phi^\sigma\}]^c.$$

Miller and Yu proved that if A is Martin-Löf random then there is a constant c such that $\forall n \lambda(S_{\Phi,n}^A) \leq 2^{-n+c}$ (see [40, Lemma 10.3.7] or [64, Theorem 5.1.14]). This method works for most test notions of randomness stronger than Martin-Löf randomness. An equivalent result (given in slightly different terminology) was obtained by Demuth and Kučera [39, Theorem 18], which Demuth used in his proof of Theorem 5.3.

Demuth also proved a version of the jump inversion theorem for Demuth random reals.

THEOREM 5.4 (Demuth Jump Inversion, [35]). *For every $z \geq_T \emptyset'$, there is a Demuth random real x such that $x' \equiv_T z$.*

An immediate corollary of Theorem 5.4 is that there exists a Δ_2^0 Demuth random real [35, Theorem 12]. For a direct proof of this corollary, see [40, Theorem 7.6.3] or [64, Theorem 3.6.25].

To prove Theorem 5.4, Demuth appealed to the following result.

THEOREM 5.5 (Demuth, [35]). *For $y, z \in [0, 1]$ and any $\mathcal{E} \subseteq [0, 1]$ of y -measure zero, there is $x \notin \mathcal{E}$ such that $x \leq_T y \oplus z$ and $z \leq_T x \oplus y$.*

The Demuth Jump Inversion theorem can be derived from Theorem 5.5 as follows. Let $z \geq_T \emptyset'$ be given, and let $y = \emptyset'$. Demuth proved that there is a single Schnorr test $(G_m^{\emptyset'})_{m \in \mathbb{N}}$ relative to \emptyset' that contains every non-Demuth random. We set $\mathcal{E} = \bigcap_{m \in \mathbb{N}} G_m^{\emptyset'}$, so that \mathcal{E} has \emptyset' -measure zero. By Theorem 5.5, there is some Demuth random x such that $x \leq_T z \oplus \emptyset' \leq_T z$ and $z \leq_T x \oplus \emptyset'$. It follows that $z \equiv_T x \oplus \emptyset'$. Then, since every Demuth random is generalized low, we have $z \equiv_T x'$.

5.2. Weak Demuth randomness and density. Another surprising result that Demuth proved involves the relationship between weak Demuth randomness and density in the sense of Lebesgue. Only recently have researchers in the field recognized the significance of the relationship between randomness and Lebesgue density. For instance, density considerations were used to solve a long-standing open problem known as the covering problem, originally due to F. Stephan, and posed in print e.g. in [60]. This problem asks whether every K -trivial set is Turing below an incomplete ML-random set. A survey of the affirmative solution is given in [3]. Anticipating this connection between randomness and density, already in 1982, Demuth [29] proved a remarkable result. Recall that the lower density of a measurable set \mathcal{P} at a real z is

$$\rho(\mathcal{P} \mid z) = \liminf_{h \rightarrow 0} \{ \lambda(\mathcal{P} \cap I) / \lambda(I) : I \text{ is an open interval, } z \in I \text{ \& } |I| < h \}.$$

DEFINITION 5.6. A real z is a *density-one point* if for every effectively closed class \mathcal{P} containing z , $\rho(\mathcal{P} \mid z) = 1$.

THEOREM 5.7 (Demuth, [29]). *Every weakly Demuth random is a density-one point.*

Demuth actually proves a stronger result: there is a single Demuth test $(S_m)_{m \in \mathbb{N}}$ such that every real for which $z \notin S_m$ for infinitely many m is a density-one point. A further strengthening was obtained by a group of researchers working at Oberwolfach at the beginning of 2012, who introduced a new notion of randomness that they called *Oberwolfach randomness* (see [5]). We give a definition equivalent to the original one in terms of left-c.e. bounded tests.

DEFINITION 5.8.

- (i) A *left-c.e. bounded test* is an effective descending sequence $(U_m)_{m \in \mathbb{N}}$ of open sets in $[0, 1]$ together with computable increasing sequence of rationals $(\beta_m)_{m \in \mathbb{N}}$ with limit β such that $\lambda(U_m) \leq \beta - \beta_m$ for every m .
- (ii) A real z is *Oberwolfach random* if and only if it passes every left-c.e. bounded test.

By definition, β is a left-c.e. real. As the rate at which $(\beta_m)_{m \in \mathbb{N}}$ converges to β may not be bounded by a computable function, not every left-c.e. bounded test is a Martin-Löf test. However, since every Martin-Löf test is a left-c.e. bounded test, it follows that every Oberwolfach random real is

Martin-Löf random. Moreover, one can show that every weakly Demuth random real is Oberwolfach random. The implication is strict.

The Oberwolfach group proved the following, unaware of the fact that they were strengthening a result of Demuth.

THEOREM 5.9 (Bienvenu, Greenberg, Kučera, Nies, Turetsky [5]). *Every Oberwolfach random is a density-one point.*

Determining the precise relationship between the following three classes is still open:

- (i) the Martin-Löf random reals that are not LR-hard, where a real z is *LR-hard* if every z -Martin-Löf random real is \emptyset' -Martin-Löf random,
- (ii) the Oberwolfach random reals,
- (iii) the collection of Martin-Löf random density-one points.

The known implications for Martin-Löf random z are as follows:

z is not LR-hard $\rightarrow z$ is Oberwolfach random $\rightarrow z$ is a density-one point.

5.3. Demuth randomness and lowness notions. As discussed at the end of Section 4.4, the Demuth randomness of a real is much too strong for its original purpose, namely, ensuring that the Denjoy alternative holds at this real for all Markov computable functions. However, since Demuth randomness is stronger than Martin-Löf randomness but still compatible with being Δ_2^0 , it interacts well with certain computability-theoretic notions. In particular, Demuth randomness has recently turned out to be very useful for the study of lowness notions.

A lowness notion is given by a collection of sequences that are in some sense computationally weak. Many lowness notions take the following form: For a relativizable collection $\mathcal{S} \subseteq 2^{\mathbb{N}}$, we say that A is *low for \mathcal{S}* if $\mathcal{S} \subseteq \mathcal{S}^A$. For instance, a sequence A such that every Demuth random sequence is Demuth random relative to A is *low for Demuth randomness*.

Another lowness notion is that of being a *base for randomness*. For a randomness notion \mathcal{R} , A is a *base for \mathcal{R} -randomness* if $A \leq_T Z$ for some Z that is \mathcal{R} -random relative to A . If we let \mathcal{R} be Demuth randomness, this yields the definition of being a base for Demuth randomness.

One additional lowness notion that has received much attention recently is known as *strong jump traceability*. Recall that a computable order h is a nondecreasing, unbounded computable function such that $h(0) > 0$. If we let $J^A(n)$ denote $\Phi_n^A(n)$, then $A \in 2^{\mathbb{N}}$ is *h -jump traceable* for a computable order h if there is a uniformly c.e. collection of sets $(T_e)_{e \in \mathbb{N}}$ such that $|T_n| \leq h(n)$ and $J^A(n) \downarrow$ implies that $J^A(n) \in T_n$ for all n (Nies, [63]). Moreover, A is *strongly jump traceable* if it is h -jump traceable for all computable orders h (Figueira, Nies, Stephan, [42]). The notion of tracing is due to Zambella [77] and Terwijn [74].

Some of the main results on Demuth randomness and lowness notions are as follows:

- (i) Kučera and Nies [53] proved that every c.e. set Turing below a Demuth random is strongly jump traceable. Greenberg and Turetsky [45] have recently provided a converse of this result: every c.e. strongly jump traceable set has a Demuth random set Turing above.
- (ii) Nies [65] showed that each base for Demuth randomness is strongly jump traceable. Greenberg and Turetsky [45] proved that this inclusion is proper.
- (iii) Lowness for Demuth randomness and weak Demuth randomness have been characterized by Bienvenu et al. [4]. The former is given by a notion called BLR-traceability (first defined by Cole and Simpson in [72]), in conjunction with being computably dominated. The latter is the same as being computable.

§6. Randomness, semigenericity, and tt-reducibility. In this last section, we discuss Demuth's work published near the end of his life, namely his work on tt-reducibility in [35] and [33] and his work on semigenericity in [32] and [39], the latter paper written jointly with Kučera.

6.1. Reducibilities from constructive analysis. In [33], Demuth proved a number of results connecting truth-table reducibility and various reducibilities from constructive analysis. These results can be seen as providing bridge principles between certain concepts from computability theory and concepts from constructive analysis.

Recall from Section 3 that the operator R maps a Markov computable function g to the maximal continuous extension $R[g]$ of g . Using this operator, Demuth defines the following reduction for pairs of reals.

DEFINITION 6.1. Given $\alpha, \beta \in [0, 1]$, α is f -reducible to β , denoted $\alpha \leq_f \beta$, if there is a Markov computable function g such that

$$R[g](\beta) = \alpha.$$

In this case, we say that α is f -reducible to β via g .

The relation \leq_f is transitive. This follows from the fact that for any Markov computable functions g_1, g_2 , $R[g_1 \circ g_2] = R[g_1] \circ R[g_2]$, which can be routinely verified.

Even though α and β may be highly nonconstructive reals, the reduction from α to β is in a sense constructively grounded, being witnessed by the extension of a Markov computable function.

Demuth then introduces three variants of f -reducibility:

DEFINITION 6.2.

1. α is \emptyset -ucf-reducible to β , denoted $\alpha \leq_{\emptyset\text{-ucf}} \beta$, if α is f -reducible to β via a Markov computable function g that is \emptyset -uniformly continuous.
2. α is \emptyset' -ucf-reducible to β , denoted $\alpha \leq_{\emptyset'\text{-ucf}} \beta$, if α is f -reducible to β via a Markov computable function g that is \emptyset' -uniformly continuous.
3. α is mf-reducible to β , denoted $\alpha \leq_{\text{mf}} \beta$, if α is f -reducible to β via a Markov computable function g that is monotonically increasing.

In order to compare these reducibilities to those from classical computability theory, Demuth identifies infinite sequences in $2^{\mathbb{N}}$ with reals in $[0, 1]$. Moreover, Demuth excludes a set $\mathcal{C} \subseteq 2^{\mathbb{N}}$ of \emptyset -measure zero that contains all finite and cofinite sequences.

Demuth then proves the following:

THEOREM 6.3 (Demuth [33]).

1. For any \emptyset -uniformly continuous Markov computable function f , one can uniformly obtain an index of a tt-functional Φ such that for every $A, B \in 2^{\mathbb{N}}$ such that $B \notin \mathcal{C}$,

$$A \leq_{\emptyset\text{-ucf}} B \text{ via } f \text{ if and only if } A \leq_{\text{tt}} B \text{ via } \Phi.$$

2. For any tt-functional Φ , one can uniformly obtain the index of a \emptyset -uniformly continuous Markov computable function $f : [0, 1] \rightarrow [0, 1]$ such that for any $A, B \in 2^{\mathbb{N}}$ such that $A, B \notin \mathcal{C}$

$$A \leq_{\emptyset\text{-ucf}} B \text{ via } f \text{ if and only if } A \leq_{\text{tt}} B \text{ via } \Phi.$$

Demuth also proved that Theorem 6.3 can be relativized to \emptyset' by using the notions of \emptyset' -ucf-reducibility and \emptyset' -tt-reducibility, excluding sequences from a set $\widehat{\mathcal{C}} \subseteq 2^{\mathbb{N}}$ of \emptyset' -measure zero. In addition, Demuth proved similar results for tt-reducibility and mf-reducibility (see Theorem 13 and 14 of [33]).

These theorems relating tt-reducibility and the reducibilities from constructive analysis were essentially used in Demuth's proof of a theorem on the behavior of Martin-Löf random reals under tt-reducibility, to which we now turn.

6.2. Truth-table reductions to random sequences. The following is one of the most well-studied of Demuth's results (for instance, in [47]), which is referred to as "Demuth's Theorem" in [40]. We formulate the result here in terms of members of $2^{\mathbb{N}}$.

THEOREM 6.4 (Demuth [35]). *If B is noncomputable and tt-reducible to a Martin-Löf random A , then there is a Martin-Löf random C such that*

$$B \leq_{\text{tt}} C \leq_T B.$$

Following Kautz's reconstruction in [47], in which he only proves that $B \equiv_T C$, recent proofs of this result are given in terms of computable measures (see [40, Section 8.6]). A measure μ on $2^{\mathbb{N}}$ is computable if $\mu([\sigma]^\prec)$ is a computable real uniformly in $\sigma \in 2^{<\mathbb{N}}$.

While the standard definition of Martin-Löf randomness is formulated in terms of the Lebesgue measure, for any computable measure μ one can also define Martin-Löf randomness with respect to μ simply by replacing the condition $\lambda(U_i) \leq 2^{-i}$ with $\mu(U_i) \leq 2^{-i}$ for each Martin-Löf test $(U_n)_{n \in \mathbb{N}}$.

Kautz recognized that Demuth's result follows from several facts about randomness and measures. First, for any tt-functional Φ , the measure λ_Φ defined by $\lambda_\Phi([\sigma]^\prec) = \lambda(\Phi^{-1}([\sigma]^\prec))$ is a computable measure. One can show this using Nerode's characterization of tt-functionals as total Turing functionals (see [62]). Second, for any tt-functional Φ and any Martin-Löf

random A , $\Phi(A)$ is Martin-Löf random with respect to λ_Φ (a result due to Levin [55]). Third, as shown by Kautz (and independently and earlier by Levin; see [55]), for any computable measure μ , if A is Martin-Löf random with respect to μ and is not computable, then there is a Martin-Löf random B (with respect to λ) such that $A \equiv_T B$.

On the surface, Demuth's proof of his result takes a very different approach. A rough sketch of his proof is as follows. First, Demuth applies part 2 of Theorem 6.3 from the previous section to replace the initial tt-reduction Φ with an \emptyset -ucf reduction from B to A given by some Markov computable function f . From this function f , Demuth then defines a monotone Markov computable function g , which allows him to construct (effectively in B) the set C and an mf-reduction from B to C . Lastly, by part 1 of Theorem 6.3, this mf-reduction yields the desired tt-reduction from B to C .

Close examination of Demuth's proof shows that the function g witnessing the mf-reduction in his proof is the distribution function of the computable measure induced by the initial tt-functional Φ . The use of distribution functions is at the heart of Kautz's proof, which shows that Demuth's proof is not too dissimilar from Kautz's reconstruction.

Demuth's result is, in a sense, the best possible. One might hope to improve the theorem by showing the existence of a Martin-Löf random C such that $B \leq_{\text{tt}} C \leq_{\text{wt}} B$, or even $B \equiv_{\text{tt}} C$. But this cannot be achieved, as shown by the following theorem.

THEOREM 6.5 (Bienvenu, Porter [8]). *There is a Martin-Löf random A and a tt-functional Φ such that $\Phi(A)$ is noncomputable and cannot wtt-compute any Martin-Löf random.*

In this same paper [8], using the technique discussed above, the following result was shown without the authors being aware that Demuth had already proved it.

THEOREM 6.6 (Demuth [32]). *There is a tt-degree containing both a c.e. set and a Martin-Löf random set. Thus there is some c.e. set $S \in 2^{\mathbb{N}}$ that is Martin-Löf random with respect to some computable measure.*

6.3. Semigenericity. Researchers in algorithmic randomness are interested in the relationship between notions of effective randomness and effective genericity. Demuth too was interested in this relationship, studying a notion he referred to as semigenericity.

DEFINITION 6.7 ([32]). A noncomputable set Z is called *semigeneric* if every Π_1^0 class containing Z has a computable member.

Intuitively, to be semigeneric means to be close to computable in the sense that the set cannot be separated from the computable sets by a Π_1^0 class.

The notion of semigenericity was studied independently though much later in Joseph Miller's thesis [59], who referred to the notion as *unavoidability*, although Miller also counted the computable points as unavoidable. As noted in [59], noncomputable unavoidable points were also studied by Kalantari and Welch in [46], who referred to these points as *shadow points*.

It is particularly natural to study semigenericity from the point of view of constructive mathematical analysis. As discussed in Section 3, one can define a Markov computable function in terms of a Σ_1^0 class \mathcal{A} that contains every computable set. Since the complement of \mathcal{A} is a Π_1^0 class with no computable members, it follows that \mathcal{A} contains every semigeneric real. From this fact, one can show that for every Markov computable function g , the classical extension $R[g]$ of g is continuous at every semigeneric real.

Demuth and Kučera [39] studied semigenericity and its relationship with other types of genericity. We review some of their results. First, Demuth and Kučera showed that semigenericity is closely related to a notion studied by Čeitin.

DEFINITION 6.8 ([18]). A set Z is called *strongly undecidable* if there is a partial computable function p such that for any computable set M and any index v of its characteristic function, $p(v)$ is defined and $Z \upharpoonright_{p(v)} \neq M \upharpoonright_{p(v)}$.

In [39, Cor. 2], Demuth and Kučera proved that a noncomputable set Z is semigeneric if and only if Z is not strongly undecidable. This same result was also obtained by Miller (see [59, Proposition 4.2.4]). Miller also studied a variant of strong undecidability in which one requires that the function p be total; he referred to this stronger notion as *hyperavoidability*. Interestingly, the hyperavoidable reals were recently shown by Kjos-Hanssen, Merkle, and Stephan to be equivalent to an important class in algorithmic randomness known as the *complex reals*. A nondyadic rational $x \in [0, 1]$ is complex if for the sequence $X \in 2^{\mathbb{N}}$ such that $x = 0.X$, there is some computable order such that $C(X \upharpoonright n) \geq f(n)$ for all n . Here $C(\sigma)$ is the plain Kolmogorov complexity of $\sigma \in \{0, 1\}^*$. See [48, Section 3] for more details.

Demuth and Kučera also proved that strong undecidability can be characterized by some kind of “uniform nonhyperimmunity”: by [39, Thm. 5], a set Z is strongly undecidable if and only if there is a computable function f such that for each computable set M and any index v of its characteristic function, the symmetric difference $M \Delta Z$ is infinite and its listing in order of magnitude is dominated by the computable function with index $f(v)$.

In [39, Thm. 14], Demuth and Kučera also characterized the sets Z such that the Turing degree of Z contains a strongly undecidable set: this happens precisely when there is a Π_2^0 class containing Z but no computable sets. Thus we have a weaker form of separation from the computable sets than for noncomputable sets that are not semigeneric, where the separating class is Π_1^0 by definition.

This result was actually proved in terms of so-called V -coverings (where V stands for Vitali). A set Z is *V -covered* by a c.e. set of strings \mathcal{A} if for all k there is a string $\sigma \in \mathcal{A}$ such that $|\sigma| \geq k$ and $\sigma \prec Z$. It is easy to see that a class of sets \mathcal{A} is a Π_2^0 class if and only if there is a c.e. set of strings \mathcal{B} such that \mathcal{A} is equal to the class of sets V -covered by \mathcal{B} (see [64, Proposition 1.8.60]).

Demuth also studied the relationship between semigenericity and weak 1-genericity, which was introduced by Kurtz in [50]. Recall that a set Z is

weakly 1-generic if Z is in each dense Σ_1^0 class. Clearly any weakly 1-generic set is semigeneric. However, the converse fails; for instance, Demuth proved in [32, Theorem 9.2] that if Z is weakly 1-generic, then $Z \oplus Z$ is semigeneric but not weakly 1-generic.

We conclude this section with a discussion of a number of results that Demuth obtained on the relationship between randomness and semigenericity. As shown by Demuth, one immediate consequence of the definition of semigenericity is that no Martin-Löf random is semigeneric, since every Martin-Löf random is contained in a Π_1^0 class with no computable members, given by the complement of some finite level of the universal Martin-Löf test. We should note, however, that there is a computable measure μ such that some Martin-Löf random sequence with respect to μ is semigeneric. For instance, the real $\Phi(A)$ in the statement of Theorem 6.5 is Martin-Löf random with respect to the induced measure λ_Φ and is also semigeneric.

One interesting similarity between Martin-Löf randomness and semigenericity is the following. Demuth's Theorem 6.4 implies that the class of noncomputable reals that are Martin-Löf random with respect to some computable measure is closed downwards under tt-reducibility. Similarly, semigenericity is closed downwards under tt-reducibility: Demuth proved in [32, Thm. 9] that if a set Z is semigeneric then any set B such that $\emptyset <_{\text{tt}} B \leq_{\text{tt}} Z$ is also semigeneric. In particular, its tt-degree only contains semigeneric sets.

Demuth also proved that no semigeneric real can tt-compute a Martin-Löf random real. The example from Theorem 6.5 shows that the converse does not hold: $\Phi(A)$ is semigeneric and tt-reducible to the Martin-Löf random real A . On a similar note, Demuth and Kučera also proved that no 1-generic (a notion slightly stronger than weak 1-genericity) can compute a Martin-Löf random (see [39, Corollary 9 of Theorem 8]).

A number of connections between semigenericity and Denjoy randomness were also established by Demuth in [36]. As discussed in Section 4.4, Demuth proved that every non-Martin-Löf Denjoy random real is high but not the converse result that every high degree contains such a real (as proved in [67]). However, he was close, as he showed that every real of high degree can compute a semigeneric Denjoy random real. In this same work, Demuth proved that there is a minimal Turing degree containing a semigeneric Denjoy random real, and that every semigeneric Denjoy random real is tt-reducible to a Denjoy random that is neither semigeneric nor Martin-Löf random.

§7. Concluding remarks. As we have seen, Demuth's contribution to the study of constructive mathematics, and in particular, his work on the various definitions of randomness in the context of constructive analysis, is remarkable in the depth and breadth of ideas that it contains. Despite working largely in isolation, Demuth produced an enormous number of results, some of which have been subsequently rediscovered, and some of which have yet to be fully understood. As our discussion has shown, many recent developments in algorithmic randomness can be seen as extending Demuth's



FIGURE 3. Osvald Demuth.

larger project of bringing the tools of computability theory to bear on the study of constructive analysis.

The searchable database at

<http://cca-net.de/publications/unclassified.html>

provides references for most of Demuth's papers, and the searchable database at

<http://www.dml.cz>

contains papers of Demuth published in *Commentationes Mathematicae Universitatis Carolinae* (CMUC) or *Acta Universitatis Carolinae* (AUC).

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