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Towards a descriptive theory of cb$_0$-spaces

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The paper tries to extend some results of the classical Descriptive Set Theory to as many countably based T_0-spaces (cb$_0$-spaces) as possible. Along with extending some central facts about Borel, Luzin and Hausdorff hierarchies of sets we also consider the more general case of k-partitions. In particular, we investigate the difference hierarchy of k-partitions and the fine hierarchy closely related to the Wadge hierarchy.

1. Introduction

Classical Descriptive Set Theory (Kechris 1995) is an important field of mathematics with numerous applications. It investigates descriptive complexity of sets, functions and equivalence relations in Polish (i.e., separable complete metrizable) spaces.

Although Polish spaces are sufficient for many fields of classical mathematics, they are certainly not sufficient for many fields of Theoretical Computer Science where non-Hausdorff spaces (in particular, ω-continuous domains) and non-countably-based spaces (in particular, Kleene–Kreisel continuous functionals) are of central importance. For this reason, the task of extending classical Descriptive Set Theory (DST) to as many non-Polish spaces as possible attracted attention of several researchers.

Some parts of DST for ω-continuous domains (which are typically non-Polish) were developed in Selivanov (2004, 2005b, 2006, 2008a). In de Brecht (2013), a good deal of DST was developed for the so called quasi-Polish spaces (see the next section for a definition of this class of cb$_0$-spaces) which include both the Polish spaces and the ω-continuous domains. For some attempts to develop DST for non-countably based spaces see e.g. Friedman et al. (2014), Jayne and Rogers (1982), Motto Ros and Semmes (2010), Pauly (2012) and Pauly and de Brecht (2013).

In this paper, we try to develop DST for some classes of cb$_0$-spaces beyond the class of quasi-Polish spaces. As is usual in classical DST, we put emphasis on the 'infinitary version' of hierarchy theory where people are concerned with transfinite (along with finite) levels of hierarchies. The 'finitary' version concentrating on the finite levels of hierarchies has a special flavor and is relevant to several fields of Logic and Computation Theory; it was systematized in Selivanov (2006, 2008b, 2012).

We extend some well-known facts about classical hierarchies in Polish spaces to natural classes of cb$_0$-spaces. Namely, we show that some levels of hierarchies of cb$_0$-spaces introduced in Schröder and Selivanov (2015, 2014) provide natural examples of classes of cb$_0$-spaces with reasonable DST (in particular, the classical Suslin, Hausdorff–Kuratowski and non-collapse theorems for the Borel, Luzin and Hausdorff hierarchies are true for such spaces). This portion of our results are technically easy and follow rather straightforwardly from the classical DST and some notions and results in Motto Ros et al. (2015) and Schröder and Selivanov (2015).

Along with the classical hierarchies of sets we are interested also in the difference and fine hierarchies of k-partitions (Selivanov 2006, 2007a,b, 2008a,b, 2011) which seem to be natural, non-trivial and useful generalizations of the corresponding hierarchies of sets. Also, along with the classical Wadge reducibility (Van Wesep 1976; Wadge 1972, 1984) we discuss its extension to k-partitions (Hertling 1993, 1996; Selivanov 2007b), and some of its weaker versions introduced and studied in Andretta (2006), Andretta and Martin (2003), Motto Ros (2009) and Motto Ros et al. (2015).

Already the extension of the Hausdorff difference hierarchy to k-partitions is a non-trivial task. The general 'right' finitary version of this hierarchy was found only recently in Selivanov (2012), although for some particular cases it was already in our previous publications. The general 'right' infinitary version of this hierarchy is new here, although the definition adequate for bases with the ω-reduction property was also found earlier (Selivanov 2007a,b, 2008a). That the definition in this paper is right follows from the nice properties of the difference hierarchy of k-partition (in particular, the natural version of the Hausdorff–Kuratowski theorem).

The situation with the fine hierarchy (which aims to extend the Wadge hierarchy to the case of sets and k-partitions in arbitrary spaces) is even more complicated. This task is not obvious even for the case of sets because the Wadge hierarchy is developed so far only for the Baire space (and some of its close relatives) in terms of m-reducibility by continuous functions and with a heavy use of Martin determinacy theorem (Van Wesep 1976; Wadge 1972, 1984). As a result, there is no clear explicit description of levels of the hierarchy in terms of set-theoretic operations which one could try to extend to other spaces (more precisely, some rather indirect descriptions presented in Wadge (1984) strongly depend on the ω-reduction property of the open sets which usually fails in non-zero-dimensional spaces). Probably, that was the reason why some authors tried to obtain alternative characterisations of levels of the Wadge hierarchy (Duparc 2001; Louveau 1983). In a series of our papers (see e.g. Selivanov (2006, 2008b)) a characterization of levels of an abstract version of Wadge hierarchy in the finitary case was achieved that was extended in Selivanov (2012) to the case of k-partitions. Here we develop an infinitary version of this approach and try to explain why the corresponding hierarchy is the 'right' extension of the Wadge hierarchy to arbitrary spaces and to the k-partitions. Since the notation and full proofs in this context are extremely involved, we concentrate here only on the partitions of finite Borel rank and avoid the complete proofs of some complicated results, giving only precise formulations and short proof hints with references to closely related earlier proofs in the finitary context. Thus, this part of the paper is a kind of 'extended announcement' of results on the infinitary version of the fine hierarchy of k-partitions; a detailed treatment shall appear in subsequent publication(s).

After recalling some notions and known facts in the next section, we discuss some basic properties of Borel and Luzin hierarchies in cb_0-spaces in Section 3. In Section 4 we establish some basic facts on the difference hierarchies of k-partitions in

; cb_0-spaces. The main result here is the Hausdorff–Kuratowski theorem for k-partitions in quasi-Polish spaces. In Section 5 we extend the difference hierarchies of k-partitions to the fine hierarchies of k-partitions. In particular, we extend the Hausdorff–Kuratowski theorem to the fine hierarchy. We conclude in Section 6 with sketching a possible further research on extending the classical Descriptive Set Theory.

2. Notation and preliminaries

In this section we recall some notation, notions and results used in the subsequent sections.

2.1. *cb_0-spaces and qcb_0-spaces*

Here we recall some topological notions and facts relevant to this paper.

We freely use the standard set-theoretic notation like $dom(f), rng(f)$ and $graph(f)$ for the domain, range and graph of a function f, respectively, $X \times Y$ for the Cartesian product, and $P(X)$ for the set of all subsets of X. For $A \subseteq X$, $\overline{A}$ denotes the complement $X \setminus A$ of A in X. We identify the set of natural numbers with the first infinite ordinal ω. The first uncountable ordinal is denoted by ω_1. The notation $f : X \to Y$ means that f is a total function from a set X to a set Y.

We assume the reader to be familiar with the basic notions of topology (Engelking 1989). The collection of all open subsets of a topological space X (i.e. the topology of X) is denoted by $\mathcal{O}(X)$; for the underlying set of X we will write X in abuse of notation. We will often abbreviate 'topological space' to 'space.' A space is *zero-dimensional* if it has a basis of clopen sets. Recall that a *basis* for the topology on X is a set $\mathcal{B}$ of open subsets of X such that for every $x \in X$ and open U containing x there is $B \in \mathcal{B}$ satisfying $x \in B \subseteq U$.

Let ω be the space of non-negative integers with the discrete topology. Of course, the spaces $\omega \times \omega = \omega^2$ (cartesian product), and $\omega \sqcup \omega$ (disjoint union) are homeomorphic to ω, the first homeomorphism is realized by the Cantor pairing function $\langle \cdot, \cdot \rangle$.

Let $\mathcal{N} = \omega^\omega$ be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi : \omega \to \omega$). Let ω^* be the set of finite sequences of elements of ω, including the empty sequence. For $\sigma \in \omega^*$ and $\xi \in \mathcal{N}$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ. By $\sigma\xi = \sigma \cdot \xi$ we denote the concatenation of σ and ξ, and by $\sigma \cdot \mathcal{N}$ the set of all extensions of σ in $\mathcal{N}$. For $x \in \mathcal{N}$, we can write $x = x(0)x(1)\dots$ where $x(i) \in \omega$ for each $i < \omega$. For $x \in \mathcal{N}$ and $n < \omega$, let $x^{<n} = x(0)\dots x(n-1)$ denote the initial segment of x of length n. Notations in the style of regular expressions like 0^ω, 0^*1 or 0^m1^n have the obvious standard meaning.

By endowing $\mathcal{N}$ with the product of the discrete topologies on ω, we obtain the so-called *Baire space*. The product topology coincides with the topology generated by the collection of sets of the form $\sigma \cdot \mathcal{N}$ for $\sigma \in \omega^*$. The Baire space is of primary importance for Descriptive Set Theory and Computable Analysis. The importance stems from the fact that many countable objects are coded straightforwardly by elements of $\mathcal{N}$, and it has very specific topological properties. In particular, it is a perfect zero-dimensional space and the spaces $\mathcal{N}^2$, $\mathcal{N}^\omega$, $\omega \times \mathcal{N} = \mathcal{N} \sqcup \mathcal{N} \sqcup \cdots$ (endowed with the product topology) are all homeomorphic to $\mathcal{N}$. Let $(x, y) \mapsto \langle x, y \rangle$ be a homeomorphism between $\mathcal{N}^2$ and

$\mathcal{N}$. The subspace $\mathcal{C} := 2^\omega$ of $\mathcal{N}$ formed by the infinite binary strings (endowed with the relative topology inherited from $\mathcal{N}$) is known as the *Cantor space*.

The *Sierpinski space* $\mathbb{S}$ is the two-point set $\{\bot, \top\}$ where the set $\{\top\}$ is open but not closed. The space $P\omega$ is formed by the set of subsets of ω equipped with the Scott topology. A countable base of the Scott topology is formed by the sets $\{A \subseteq \omega \mid F \subseteq A\}$, where F ranges over the finite subsets of ω. Note that $P\omega = \mathcal{O}(\omega)$. As is well known (Gierz et al. 2003), $P\omega$ is universal for cb$_0$-spaces:

Proposition 2.1 A topological space X embeds into $P\omega$ iff X is a cb_0-space.

Remember that a space X is *Polish* if it is countably based and metrizable with a metric d such that (X, d) is a complete metric space. Important examples of Polish spaces are ω, $\mathcal{N}$, $\mathcal{C}$, the space of reals $\mathbb{R}$ and its Cartesian powers $\mathbb{R}^n$ $(n < \omega)$, the closed unit interval $[0, 1]$, the Hilbert cube $[0, 1]^\omega$ and the Hilbert space $\mathbb{R}^\omega$. Simple examples of non-Polish spaces are $\mathbb{S}$, $P\omega$ and the space $\mathbb{Q}$ of rationals.

A space X is *quasi-Polish* (de Brecht 2013) if it is countably based and quasi-metrizable with a quasi-metric d such that (X, d) is a complete quasi-metric space. A *quasi-metric* on X is a function from $X \times X$ to the nonnegative reals such that $d(x, y) = d(y, x) = 0$ iff $x = y$, and $d(x, y) \leqslant d(x, z) + d(z, y)$. Since for the quasi-metric spaces different notions of completeness and of a Cauchy sequence are considered, the definition of quasi-Polish spaces should be made more precise (see de Brecht 2013 for additional details). We skip these details because we will in fact use some other characterizations of these spaces recalled below. Note that the spaces $\mathbb{S}$, $P\omega$ are quasi-Polish while the space $\mathbb{Q}$ is not.

A *representation* of a space X is a surjection of a subspace of the Baire space $\mathcal{N}$ onto X. A basic notion of Computable Analysis is the notion of admissible representation. A representation δ of X is *admissible*, if it is continuous and any continuous function $\nu : Z \to X$ from a subset $Z \subseteq \mathcal{N}$ to X is continuously reducible to δ, i.e. $\nu = \delta \circ g$ for some continuous function $g : Z \to \mathcal{N}$. A topological space is *admissibly representable*, if it has an admissible representation.

The notion of admissibility was introduced in Kreitz and Weihrauch (1985) for representations of cb$_0$-spaces (in a different but equivalent formulation) and was extensively studied by many authors. In Schröder (2002, 2003), the notion was extended to non-countably based spaces and a nice characterization of the admissibly represented spaces was achieved. Namely, the admissibly represented sequential topological spaces coincide with the qcb$_0$-spaces, i.e., T_0-spaces which are topological quotients of countably based spaces.

In de Brecht (2013), the following important characterization of quasi-Polish spaces was obtained: A cb$_0$-space X is quasi-Polish if there is a total admissible representation $\delta : \mathcal{N} \to X$ of X.

2.2. *Hierarchies of sets*

Here we briefly recall definitions and some properties of Borel and Luzin hierarchies in arbitrary topological spaces.

A *pointclass* in a space X is simply a collection $\mathbf{\Gamma}(X)$ of subsets of X. A *family of pointclasses* (Selivanov 2013) is a family $\mathbf{\Gamma} = \{\mathbf{\Gamma}(X)\}$ indexed by arbitrary topological spaces X such that each $\mathbf{\Gamma}(X)$ is a pointclass on X and $\mathbf{\Gamma}$ is closed under continuous preimages, i.e. $f^{-1}(A) \in \mathbf{\Gamma}(X)$ for every $A \in \mathbf{\Gamma}(Y)$ and every continuous function $f : X \to Y$. A basic example of a family of pointclasses is given by the family $\mathcal{O} = \{\mathcal{O}(X)\}$ of the topologies of all the spaces X.

We will use some operations on families of pointclasses. First, the usual set-theoretic operations will be applied to the families of pointclasses pointwise: for example, the union $\bigcup_i \mathbf{\Gamma}_i$ of the families of pointclasses $\mathbf{\Gamma}_0, \mathbf{\Gamma}_1, \dots$ is defined by $(\bigcup_i \mathbf{\Gamma}_i)(X) = \bigcup_i \mathbf{\Gamma}_i(X)$.

Second, a large class of such operations is induced by the set-theoretic operations of Kantorovich and Livenson (see e.g. Selivanov 2013 for the general definition). Among them are the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\sigma$, where $\mathbf{\Gamma}(X)_\sigma$ is the set of all countable unions of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\delta$, where $\mathbf{\Gamma}(X)_\delta$ is the set of all countable intersections of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_c = \check{\mathbf{\Gamma}}$, where $\mathbf{\Gamma}(X)_c$ is the set of all complements of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_d$, where $\mathbf{\Gamma}(X)_d$ is the set of all differences of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\exists$ defined by $\mathbf{\Gamma}_\exists(X) := \{\exists^{\mathcal{N}}(A) \mid A \in \mathbf{\Gamma}(\mathcal{N} \times X)\}$, where $\exists^{\mathcal{N}}(A) := \{x \in X \mid \exists p \in \mathcal{N}.(p,x) \in A\}$ is the projection of $A \subseteq \mathcal{N} \times X$ along the axis $\mathcal{N}$, and finally the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\forall$ defined by $\mathbf{\Gamma}_\forall(X) := \{\forall^{\mathcal{N}}(A) \mid A \in \mathbf{\Gamma}(\mathcal{N} \times X)\}$, where $\forall^{\mathcal{N}}(A) := \{x \in X \mid \forall p \in \mathcal{N}.(p,x) \in A\}$.

The operations on families of pointclasses enable to provide short uniform descriptions of the classical hierarchies in arbitrary spaces. For example, the Borel hierarchy is the family of pointclasses $\{\mathbf{\Sigma}^0_\alpha\}_{\alpha<\omega_1}$ defined by induction on α as follows: (de Brecht 2013; Selivanov 2006):

$$\mathbf{\Sigma}^0_0 := \{\varnothing\},\ \mathbf{\Sigma}^0_1 := \mathcal{O},\ \mathbf{\Sigma}^0_2 := (\mathbf{\Sigma}^0_1)_{d\sigma},\ \text{and}\ \mathbf{\Sigma}^0_\alpha := (\bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta)_{c\sigma}$$

for $\alpha > 2$. The sequence $\{\mathbf{\Sigma}^0_\alpha(X)\}_{\alpha<\omega_1}$ is called *the Borel hierarchy* in X. We also let $\mathbf{\Pi}^0_\beta(X) := (\mathbf{\Sigma}^0_\beta(X))_c$ and $\mathbf{\Delta}^0_\alpha(X) := \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$. The classes $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ are called the *levels* of the Borel hierarchy in X.

Borel hierarchy in $P\omega$ enables the following alternative characterization of quasi-Polish spaces from de Brecht (2013):

Proposition 2.2 A space is quasi-Polish iff it is homeomorphic to a $\mathbf{\Pi}^0_2$-subset of $P\omega$ with the induced topology.

We recall from Kechris (1995) and Selivanov (2013) an important structural property of $\mathbf{\Sigma}$-levels of the Borel hierarchy. Let Γ be a family of pointclasses. A pointclass $\Gamma(X)$ has the *ω-reduction property* if for each countable sequence $A_0, A_1, \dots$ in $\Gamma(X)$ there is a countable sequence $D_0, D_1, \dots$ in $\Gamma(X)$ such that $D_i \subseteq A_i$, $D_i \cap D_j = \varnothing$ for all $i \neq j$ and $\bigcup_{i<\omega} D_i = \bigcup_{i<\omega} A_i$.

Proposition 2.3 For any space X and any $2 \leqslant \alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X)$ has the ω-reduction properties. If X is zero-dimensional, the same holds for the class $\mathbf{\Sigma}^0_1(X)$ of open sets.

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The *hyperprojective hierarchy* is the family of pointclasses $\{\mathbf{\Sigma}^1_\alpha\}_{\alpha<\omega_1}$ defined by induction on α as follows: $\mathbf{\Sigma}^1_0 = \mathbf{\Sigma}^0_2$, $\mathbf{\Sigma}^1_{\alpha+1} = (\mathbf{\Sigma}^1_\alpha)_{c\exists}$, $\mathbf{\Sigma}^1_\lambda = (\mathbf{\Sigma}^1_{<\lambda})_{\delta\exists}$, where $\alpha, \lambda < \omega_1$, λ is a limit ordinal, and $\mathbf{\Sigma}^1_{<\lambda}(X) := \bigcup_{\alpha<\lambda} \mathbf{\Sigma}^1_\alpha(X)$.

In this way, we obtain for any topological space X the sequence $\{\mathbf{\Sigma}^1_\alpha(X)\}_{\alpha<\omega_1}$, which we call here *the hyperprojective hierarchy in* X. The pointclasses $\mathbf{\Sigma}^1_\alpha(X)$, $\mathbf{\Pi}^1_\alpha(X) := (\mathbf{\Sigma}^1_\alpha(X))_c$ and $\mathbf{\Delta}^1_\alpha(X) := \mathbf{\Sigma}^1_\alpha(X) \cap \mathbf{\Pi}^1_\alpha(X)$ are called *levels of the hyperprojective hierarchy in* X. The finite non-zero levels of the hyperprojective hierarchy coincide with the corresponding levels of the Luzin's projective hierarchy (de Brecht 2013; Schröder and Selivanov 2015). The class of *hyperprojective sets* in X is defined as the union of all levels of the hyperprojective hierarchy in X. For more information on the hyperprojective hierarchy, see Kechris (1983, 1995) and Schröder and Selivanov (2014). Below we will also consider some other hierarchies, in particular the Hausdorff difference hierarchy.

2.3. *k-Partitions and hierarchies over well posets*

Here we discuss a more general notion of a hierarchy (compared with the notion of hierarchy of sets (Selivanov 2008b, 2012)) which applies, in particular, to the hierarchies of k-partitions.

Let $2 \leqslant k < \omega$. By a *k-partition of a space X* we mean a function $A : X \to k = \{0,\dots,k-1\}$ often identified with the sequence $(A_0,\dots,A_{k-1})$ where $A_i = A^{-1}(i)$. Obviously, 2-partitions of X are identified with the subsets of X using the characteristic functions. The set of all k-partitions of X is denoted k^X. For $\mathbf{\Gamma} \subseteq P(X)$, let $(\mathbf{\Gamma})_k$ denote the set of k-partitions $A \in k^X$ such that $A_0,\dots,A_{k-1} \in \mathbf{\Gamma}(X)$. In particular, $(\mathbf{\Sigma}_{<\omega}(X))_k$ is the set of k-partitions of finite Borel rank which will be considered in Section 5.2.

The Wadge reducibility on subsets of X is naturally extended to k-partitions: for $A, B \in k^X$, $A \leqslant_W B$ means that $A = B \circ f$ for some continuous function f on X. In this way, we obtain the preorder $(k^X; \leqslant_W)$ which for $k \geqslant 3$ turns out much more complicated than the structure of Wadge degrees, even for the simple case $X = \mathcal{N}$ (Hertling 1993, 1996; Selivanov 2006, 2011).

To find the 'right' extensions of the classical difference and Wadge hierarchies from the case of sets to the case of k-partitions is a quite non-trivial task. A reason is that levels of hierarchies of sets are always semi-well-ordered by inclusion (in particular, there are no three levels which are pairwise incomparable by inclusion) while the structure of hierarchies of k-partitions for $k \geqslant 3$ is usually more complicated than the structure of the hierarchies of sets (in particular, for $k \geqslant 3$ the poset of levels of difference hierarchies of k-partitions under inclusion usually has antichains with any finite number of elements).

Here we recall from Selivanov (2012) a very general notion of a hierarchy that covers all hierarchies we discuss in this paper.

Definition 2.4

1. For any poset P and any set A, by a P-hierarchy in A we mean a family $\{H_p\}_{p\in P}$ of subsets of A such that $p \leqslant q$ implies $H_p \subseteq H_q$.

2. Levels (resp. constituents) of a P-hierarchy $\{H_p\}$ are the sets $H_{p_0} \cap \cdots \cap H_{p_n}$ (resp. the sets $C_{p_0,\ldots,p_n} = (H_{p_0} \cap \cdots \cap H_{p_n}) \setminus \bigcup\{H_q \mid q \in P \setminus \uparrow \{p_0,\ldots,p_n\}\})$ where $n \geqslant 0$ and $\{p_0,\ldots,p_n\}$ is an antichain in P.
3. A P-hierarchy $\{H_p\}$ is precise if $p \leqslant q$ is equivalent to $H_p \subseteq H_q$.

Note that the classical hierarchies of sets are obtained from the above definition if $A = P(X)$ and $P = \bar{2} \cdot \eta$ is the poset obtained by replacing any element of the ordinal η by an antichain with two elements, and that the notion of preciseness extends the non-collapse property of hierarchies. Note that levels of the classical hierarchies coincide with levels in the sense of the definition above. The constituents of say, Borel hierarchy, are $\mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$, $\mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$, $\mathbf{\Delta}^0_{\alpha+1} \setminus (\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha)$, and $\mathbf{\Delta}^0_\lambda \setminus \bigcup_{\alpha<\lambda} \mathbf{\Sigma}^0_\alpha$, where λ is a limit countable ordinal.

As it was already mentioned, for hierarchies of k-partitions (obtained when $A = k^X$) we cannot hope to deal only with semi-well-ordered posets $P = \bar{2} \cdot \eta$ in the definition above. Fortunately, a slight weakening of this property is sufficient for our purposes: we can confine ourselves with the so called well posets (wpo) or, more generally well preorders (wqo). Recall that a wqo is a preorder P that has neither infinite descending chains nor infinite antichains. The theory of wqo (widely known as the wqo-theory) is a well-developed field with several deep results and applications, see e.g. Kruskal (1972). It is also of great interest to hierarchy theory. An important role in wqo-theory belongs to the rather technical notion of a better preorder (bqo). Bqo's form a subclass of wqo's with good closure properties.

Note that if P is a wpo then the structure $(\{H_p \mid p \in P\};\subseteq)$ of levels of a P-hierarchy under inclusion is also a wpo, hence some important features of the hierarchies of sets hold also for the hierarchies of partitions. Moreover, for such hierarchies we have some important properties of constituents, in particular the constituents form a partition of the set $\bigcup\{H_p \mid p \in P\}$ (see also Section 7 of Selivanov 2012 for additional details).

Although well posets are very simple compared with arbitrary posets, they are still much more complicated than semi-well-orders which essentially reduce to the ordinals. Obviously, there are a lot of isomorphism types of well posets of a fixed rank. Below we consider some examples of well posets especially designed for naming levels of the difference and fine hierarchies of k-partitions.

2.4. *Hierarchies of cb_0-spaces and qcb_0-spaces*

Here we recall some classifications of qcb_0-spaces induced by the classical hierarchies of sets.

For any representation δ of a space X, let $EQ(\delta) := \{\langle p,q\rangle \in \mathcal{N} \mid p,q \in dom(\delta) \wedge \delta(p) = \delta(q)\}$. Let $\mathbf{\Gamma}$ be a family of pointclasses. A qcb_0-space X is called $\mathbf{\Gamma}$-*representable*, if X has an admissible representation δ with $EQ(\delta) \in \mathbf{\Gamma}(\mathcal{N})$. The class of all $\mathbf{\Gamma}$-representable spaces is denoted $\mathsf{QCB}_0(\mathbf{\Gamma})$. A cb_0-space X is called a $\mathbf{\Gamma}$-*space*, if X is homeomorphic to a $\mathbf{\Gamma}$-subspace of $P\omega$. The class of all $\mathbf{\Gamma}$-spaces is denoted $\mathsf{CB}_0(\mathbf{\Gamma})$.

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These notions from Schröder and Selivanov (2015) enable to transfer hierarchies of sets to the corresponding hierarchies of cb_0- and qcb_0-spaces. In particular, we arrive at the following definition.

Definition 2.5 The sequence $\{\mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha)\}_{\alpha<\omega_1}$ (resp. the sequence $\{\mathsf{QCB}_0(\mathbf{\Sigma}^0_\alpha)\}_{\alpha<\omega_1}$) is called the *Borel hierarchy* of cb_0-spaces (resp. of qcb_0-spaces). By *levels* of this hierarchy we mean the classes $\mathsf{CB}_0(\mathbf{\Sigma}^0_\alpha)$ as well as the classes $\mathsf{CB}_0(\mathbf{\Pi}^0_\alpha)$ and $\mathsf{CB}_0(\mathbf{\Delta}^0_\alpha)$. In a similar way one can define the hyperprojective hierarchies of cb_0- and of qcb_0-spaces.

The following fact from Schröder and Selivanov (2015, 2014) shows that the introduced hierarchies agree on cb_0-spaces:

Proposition 2.6 For any $\mathbf{\Gamma} \in \{\mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_\beta, \mathbf{\Pi}^0_\beta, \mathbf{\Sigma}^1_\alpha, \mathbf{\Pi}^1_\alpha \mid 1 \leqslant \alpha < \omega_1, 3 \leqslant \beta < \omega_1\}$, we have $\mathsf{QCB}_0(\mathbf{\Gamma}) \cap \mathsf{CB}_0 = \mathsf{CB}_0(\mathbf{\Gamma})$, where CB_0 is the class of all cb_0-spaces.

Note that, by Proposition 2.2, $\mathsf{CB}_0(\mathbf{\Pi}^0_2)$ coincides with the class of quasi-Polish spaces.

3. Borel and Luzin hierarchies

In this section we extend some classical facts on the Borel and Luzin hierarchies in Polish spaces on larger classes of cb_0-spaces.

3.1. *Some reducibilities and isomorphisms*

Here we provide some information on versions of the Wadge reducibility and of the notion of homeomorphism relevant to this paper.

Let $\mathbf{\Gamma}$ be a family of pointclasses and X, Y be topological spaces. By $\mathbf{\Gamma}(X, Y)$ we denote the class of functions $f : X \to Y$ such that $f^{-1}(A) \in \mathbf{\Gamma}(X)$ whenever $A \in \mathbf{\Gamma}(Y)$. A set $A \subseteq X$ is $\mathbf{\Gamma}$-*reducible* to a set $B \subseteq X$ (in symbols, $A \leqslant_{\mathbf{\Gamma}} B$) if $A = f^{-1}(B)$ for some $f \in \mathbf{\Gamma}(X, X)$.

Note that the $\mathbf{\Sigma}^0_1$-functions coincide with the continuous functions and the $\mathbf{\Sigma}^0_1$-reducibility coincides with the classical Wadge reducibility. $\mathbf{\Sigma}^0_\alpha$-Functions and $\mathbf{\Sigma}^0_\alpha$-reducibilities were investigated in Andretta (2006); Andretta and Martin (2003); Motto Ros (2009).

We say that topological spaces X, Y are $\mathbf{\Gamma}$-*isomorphic* if there is a bijection f between X and Y such that $f \in \mathbf{\Gamma}(X, Y)$ and $f^{-1} \in \mathbf{\Gamma}(Y, X)$. It is a classical fact of Descriptive Set Theory that every two uncountable Polish spaces X, Y are $\mathbf{\Delta}^1_1$-isomorphic (see e.g. Kechris 1995, Theorem 15.6). The next result from Motto Ros et al. (2015) extends this fact to the context of uncountable quasi-Polish spaces and computes an upper bound for the complexity of the Borel isomorphism.

Proposition 3.1 Let X, Y be two uncountable quasi-Polish spaces. Then X and Y are $\mathbf{\Delta}^0_{<\omega}$-isomorphic. If the inductive dimensions $dim(X)$, $dim(Y)$ of X, Y are distinct from ∞ then X and Y are $\mathbf{\Delta}^0_3$-isomorphic.

Let again $\mathbf{\Gamma}$ be a family of pointclasses. By a $\mathbf{\Gamma}$-*family of pointclasses* we mean a family $\{E(X)\}_X$ indexed by arbitrary spaces such that $E(X)$ is a pointset in X, and $f^{-1}(A) \in E(X)$ for all $A \in E(Y)$ and $f \in \mathbf{\Gamma}(X, Y)$. Obviously, the $\mathbf{\Sigma}^0_1$-families of pointclasses are precisely the 'usual' families of pointclasses.

Lemma 3.2 Let $\mathbf{\Gamma}$ be a family of pointclasses. Then $\mathbf{\Gamma}$ is a $\mathbf{\Gamma}$-family of pointclasses, any continuous function $f : X \to Y$ is in $\mathbf{\Gamma}(X, Y)$, and any $\mathbf{\Gamma}$-family of pointclasses is a family of pointclasses.

Proof. The first assertion is obvious. For the second assertion, let $f : X \to Y$ be continuous and $A \in \mathbf{\Gamma}(Y)$. Since $\mathbf{\Gamma}$ is a family of pointclasses, $f^{-1}(A) \in \mathbf{\Gamma}(X)$. Since A was arbitrary, $f \in \mathbf{\Gamma}(X, Y)$. The third assertion follows from the second one. □

Lemma 3.3 Let $\alpha < \beta < \omega_1$. Then any $\mathbf{\Sigma}^0_\alpha$-function is a $\mathbf{\Sigma}^0_\beta$-function, and any $\mathbf{\Sigma}^1_\alpha$-function is a $\mathbf{\Sigma}^1_\beta$-function.

Proof hint. Proof is straightforward by induction on β, so we consider only the first assertion for the case $\beta = \alpha + 1$, as an example. Let $A \in \mathbf{\Sigma}^0_\beta(Y)$ and $f \in \mathbf{\Sigma}^0_\alpha(X, Y)$. Then $A = \bigcup_n (Y \setminus A_n)$ for some $A_0, A_1, \ldots \in \mathbf{\Sigma}^0_\alpha(Y)$. Then $f^{-1}(A) = \bigcup_n (X \setminus f^{-1}(A_n)) \in \mathbf{\Sigma}^0_\beta(X)$. Since A was arbitrary, f is a $\mathbf{\Sigma}^0_\beta$-function. □

Proposition 3.4

1. Let $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_\beta, \mathbf{\Pi}^1_\beta \mid \omega \leqslant \alpha < \omega_1, 1 \leqslant \beta < \omega_1\}$ and $X \in \mathsf{CB}_0(\mathbf{\Gamma})$. Then X is $\mathbf{\Gamma}$-isomorphic to a subspace S of $\mathcal{N}$ such that $S \in \mathbf{\Gamma}(\mathcal{N})$.
2. Let $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Sigma}^1_\beta, \mathbf{\Pi}^1_\beta \mid 3 \leqslant \alpha < \omega_1, 1 \leqslant \beta < \omega_1\}$ and $X \in \mathsf{CB}_0(\mathbf{\Gamma})$, $dim(X) \neq \infty$. Then X is $\mathbf{\Gamma}$-isomorphic to a subspace S of $\mathcal{N}$ such that $S \in \mathbf{\Gamma}(\mathcal{N})$.

Proof. Both items are checked in the same way, so consider only the first one. Assume without loss of generality that $X \in \mathbf{\Gamma}(P\omega)$. By Proposition 3.1, the spaces $P\omega$ and $\mathcal{N}$ are $\mathbf{\Delta}^0_{<\omega}$-isomorphic, hence also $\mathbf{\Sigma}^0_\omega$-isomorphic. By Lemma 3.3 $P\omega$ and $\mathcal{N}$ are $\mathbf{\Gamma}$-isomorphic, let $f : P\omega \to \mathcal{N}$ be a $\mathbf{\Gamma}$-isomorphism. Then $f|_X$ is a $\mathbf{\Gamma}$-isomorphism between X and $S = f(X) \in \mathbf{\Gamma}(\mathcal{N})$. □

3.2. *Borel and Luzin hierarchies in cb_0-spaces*

Here we extend some well-known facts on the Borel and Luzin hierarchies in Polish and quasi-Polish spaces.

As is well-known, any uncountable Polish (or quasi-Polish) space is of continuum cardinality. The next fact extends this to many cb_0-spaces:

Proposition 3.5 Any uncountable space X in $\mathsf{CB}_0(\mathbf{\Sigma}^1_1)$ is of continuum cardinality.

Proof. By Proposition 3.4, X is $\mathbf{\Sigma}^1_1$-isomorphic to a subspace S of $\mathcal{N}$ such that $S \in \mathbf{\Sigma}^1_1(\mathcal{N})$, so it suffices to show that S is of continuum cardinality. But this follows from a well-known fact of classical DST (Theorem 29.1 in Kechris (1995)). □

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Remark 3.6 The last result cannot be improved within ZFC because, as is well known, it is consistent with ZFC that there is a non-countable set $S \in \mathbf{\Pi}^1_1(\mathcal{N})$ of cardinality less than continuum. It would be nice to have similar precise estimates of consistency with ZFC for other results of this paper (we do not deal with this question in the sequel).

Next we establish an extension of the Suslin theorem which equates the Borel sets to the $\mathbf{\Delta}^1_1$-sets. This is a classical result of DST for the case of Polish spaces, and it was extended to quasi-Polish spaces in de Brecht (2013).

For this we need the following version of a well-known easy fact:

Lemma 3.7 Let $X \subseteq Y$ be topological spaces and $1 \leqslant \alpha < \omega_1$. Then $\mathbf{\Sigma}^1_\alpha(X) = \{X \cap A \mid A \in \mathbf{\Sigma}^1_\alpha(Y)\}$, $\mathbf{\Pi}^1_\alpha(X) = \{X \cap A \mid A \in \mathbf{\Pi}^1_\alpha(Y)\}$, and similarly for the Borel hierarchy.

Proposition 3.8 The Suslin theorem holds for any space X in $\mathsf{CB}_0(\mathbf{\Delta}^1_1)$, i.e. $\mathbf{\Delta}^1_1(X) = \bigcup\{\mathbf{\Sigma}^0_\alpha(X) \mid \alpha < \omega_1\}$.

Proof. It suffices to show the inclusion $\mathbf{\Delta}^1_1(X) \subseteq \bigcup\{\mathbf{\Sigma}^0_\alpha(X) \mid \alpha < \omega_1\}$. Assume without loss of generality that $X \in \mathbf{\Delta}^1_1(P\omega)$. Let $A \in \mathbf{\Delta}^1_1(X)$, then A is in both $\mathbf{\Sigma}^1_1(X)$ and $\mathbf{\Pi}^1_1(X)$. By Lemma 3.7, $A = X \cap B = X \cap C$ for some $B \in \mathbf{\Sigma}^1_1(P\omega)$ and $C \in \mathbf{\Pi}^1_1(P\omega)$. Then $A \in \mathbf{\Delta}^1_1(P\omega)$. By Suslin theorem for $P\omega$, $A \in \mathbf{\Sigma}^0_\alpha(P\omega)$ for some $\alpha < \omega_1$. By Lemma 3.7, $A \in \mathbf{\Sigma}^0_\alpha(X)$. □

As is well known (Kechris 1995), the Borel and Luzin hierarchies do not collapse in any Polish uncountable space X (for the Borel hierarchy, for instance, this means that $\mathbf{\Sigma}^0_\alpha(X) \neq \mathbf{\Pi}^0_\alpha(X)$ for any $\alpha < \omega_1$). In de Brecht (2013), this was extended to the quasi-Polish spaces (which coincide with the spaces in $\mathsf{CB}_0(\mathbf{\Pi}^0_2)$). We conclude this section with a further extension of the non-collapse property. For this we need the following lemma:

Lemma 3.9 Let $1 \leqslant \alpha < \omega_1$, X, Y be $\mathbf{\Sigma}^0_\alpha$-isomorphic topological spaces, and the Borel hierarchy (resp. the hyperprojective hierarchy) in X does not collapse. Then the Borel hierarchy (resp. the hyperprojective hierarchy) in Y does not collapse.

Proof. Both hierarchies are treated similarly, so consider only the Borel hierarchy. Suppose for a contradiction that $\mathbf{\Sigma}^0_\beta(Y) = \mathbf{\Pi}^0_\beta(Y)$ for some $\beta < \omega_1$. By the definition of the Borel hierarchy, $\mathbf{\Sigma}^0_\gamma(Y) = \mathbf{\Pi}^0_\gamma(Y)$ for all countable ordinals $\gamma \geqslant \beta$, in particular for $\gamma = sup\{\alpha, \beta\}$. By Lemma 3.3, $\mathbf{\Sigma}^0_\gamma(X) = \mathbf{\Pi}^0_\gamma(X)$. A contradiction. □

Proposition 3.10 The Borel and hyperprojective hierarchies do not collapse for any uncountable space X in $\mathsf{CB}_0(\mathbf{\Delta}^1_1)$.

Proof. By Proposition 3.4 and Lemma 3.9, we can without loss of generality assume that X is a subspace of $\mathcal{N}$ such that $X \in \mathbf{\Delta}^1_1(\mathcal{N})$. By Theorem 29.1 in Kechris (1995), there is a subspace $C \subseteq X$ homeomorphic to the Cantor space. Since the Borel and hyperprojective hierarchies in C do not collapse, by Lemma 3.7 they also do not collapse in X. □

4. Difference hierarchies

In this section we extend some classical facts on the Hausdorff difference hierarchy (DH) of sets, like the Hausdorff–Kuratowski theorem and the non-collapse property in Polish spaces, to larger classes of cb$_0$-spaces and to the case of k-partitions.

4.1. *Difference hierarchies of sets*

Here we recall definition and basic properties of the Hausdorff difference hierarchy of sets, and extend some facts on the DH in Polish spaces to larger classes of cb$_0$-spaces.

An ordinal α is *even* (resp. *odd*) if $\alpha = \lambda + n$ where λ is either zero or a limit ordinal and $n < \omega$, and the number n is even (resp., odd). For an ordinal α, let $r(\alpha) = 0$ if α is even and $r(\alpha) = 1$, otherwise. For any ordinal α, define the operation D_α sending sequences of sets $\{A_\beta\}_{\beta<\alpha}$ to sets by

$$D_\alpha(\{A_\beta\}_{\beta<\alpha}) = \bigcup\{A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma \mid \beta < \alpha,\ r(\beta) \neq r(\alpha)\}.$$

For any ordinal $\alpha < \omega_1$ and any pointclass $\mathcal{L}$ in X, let $D_\alpha(\mathcal{L})$ be the class of all sets $D_\alpha(\{A_\beta\}_{\beta<\alpha})$, where $A_\beta \in \mathcal{L}$ for all $\beta < \alpha$. By the *difference hierarchy over* $\mathcal{L}$ we mean the sequence $\{D_\alpha(\mathcal{L})\}_{\alpha<\omega_1}$. Usually we assume that $\mathcal{L}$ is a *base* which by definition means that $\mathcal{L}$ is closed under finite intersection and countable union (note that in finitary versions of the DH we used the term 'σ-base' to denote such pointclasses but, since we are interested here only in such pointclasses, we simplify the terminology). As usual, classes $D_\alpha(\mathcal{L})$, $\check{D}_\alpha(\mathcal{L})$ are called non-self-dual levels while $D_\alpha(\mathcal{L}) \cap \check{D}_\alpha(\mathcal{L})$ are called self-dual levels of the DH.

Over bases, the difference hierarchy really looks as a hierarchy, i.e., any level and its dual are contained in all higher levels. The most interesting cases for Descriptive Set Theory are difference hierarchies over non-zero levels of the Borel hierarchy, whose $\mathbf{\Sigma}$-levels are $\mathbf{\Sigma}^{-1,\theta}_\alpha(X) = D_\alpha(\mathbf{\Sigma}^0_\theta(X))$, for any space X and for all $\alpha, \theta < \omega$, $\theta > 0$. For $\theta = 1$, we simplify $\mathbf{\Sigma}^{-1,\theta}_\alpha$ to $\mathbf{\Sigma}^{-1}_\alpha$.

A classical result of DST is the following Hausdorff–Kuratowski theorem:

Theorem 4.1 Let X be a Polish space. For any non-zero ordinal $\theta < \omega_1$, $\bigcup\{\mathbf{\Sigma}^{-1,\theta}_\alpha(X) \mid \alpha < \omega_1\} = \mathbf{\Delta}^0_{\theta+1}(X)$.

In de Brecht (2013), this result was extended to the quasi-Polish spaces. This extension is an easy corollary of the following nice result (Theorem 68 in de Brecht (2013) based on Lemma 17 in Saint Raymond (2007)):

Theorem 4.2 Let X be a cb$_0$-space, $\delta : D \to X$ an admissible representation of X ($D \subseteq \mathcal{N}$), $A \subseteq X$ X, $\alpha, \theta < \omega_1$ and $\theta \geqslant 1$. Then $A \in D_\alpha(\mathbf{\Sigma}^0_\theta(X))$ iff $\delta^{-1}(A) \in D_\alpha(\mathbf{\Sigma}^0_\theta(D))$.

One of the aims of this section is to extend these results from sets to k-partitions. This needs some information on k-forests and h-preorders which are recalled in the next subsection.

In this subsection we establish a partial extension of the Hausdorff–Kuratowski theorem for quasi-Polish spaces (which coincide with the spaces in $\mathsf{CB}_0(\mathbf{\Pi}^0_2)$) to a larger class of cb_0-spaces:

Proposition 4.3 For any space X in $\mathsf{CB}_0(\mathbf{\Delta}^1_1)$ there is a non-zero ordinal $\beta < \omega_1$ such that the Hausdorff–Kuratowski theorem holds in X for each countable ordinal $\theta \geqslant \beta$, i.e. $\bigcup\{\mathbf{\Sigma}^{-1,\theta}_\alpha(X) \mid \alpha < \omega_1\} = \mathbf{\Delta}^0_{\theta+1}(X)$.

Proof. Assume without loss of generality that $X \in \mathbf{\Delta}^1_1(P\omega)$. By Suslin theorem for $P\omega$, $X \in \mathbf{\Sigma}^0_\beta(P\omega)$ for some $\beta < \omega_1$. It remains to show that $\mathbf{\Delta}^0_{\theta+1}(X) \subseteq \bigcup\{\mathbf{\Sigma}^{-1,\theta}_\alpha(X) \mid \alpha < \omega_1\}$ for all $\theta \geqslant \beta$. Let $A \in \mathbf{\Delta}^0_{\theta+1}(X)$, then A is in both $\mathbf{\Sigma}^0_{\theta+1}(X)$ and $\mathbf{\Pi}^0_{\theta+1}(X)$. By Lemma 3.7, $A = X \cap B = X \cap C$ for some $B \in \mathbf{\Sigma}^0_{\theta+1}(P\omega)$ and $C \in \mathbf{\Pi}^0_{\theta+1}(P\omega)$. Then $A \in \mathbf{\Delta}^0_{\theta+1}(P\omega)$. By the Hausdorff–Kuratowski theorem for $P\omega$, $A \in \mathbf{\Sigma}^{-1,\theta}_\alpha(P\omega)$ for some $\alpha < \omega_1$. Since $\theta \geqslant \beta$ and $X \in \mathbf{\Sigma}^0_\beta(P\omega)$, $A \in \mathbf{\Sigma}^{-1,\theta}_\alpha(X)$. □

The problem of non-collapse of the DHs is more subtle (compared with the problem of non-collapse of the Borel and Luzin hierarchies) but it is again possible to prove at least a partial result about this property. First we formulate an analogue of Lemma 3.9 which is proved essentially by the same argument:

Lemma 4.4 Let $1 \leqslant \alpha < \omega_1$, X, Y be $\mathbf{\Sigma}^0_\alpha$-isomorphic topological spaces, $\alpha \leqslant \theta < \omega_1$, and the DH $\{\mathbf{\Sigma}^{-1,\theta}_\beta(X)\}_{\beta<\omega_1}$ does not collapse. Then the DH $\{\mathbf{\Sigma}^{-1,\theta}_\beta(Y)\}_{\beta<\omega_1}$ does not collapse.

Once we have this lemma and note that Lemma 3.7 holds also for the DHs, we easily deduce the following:

Proposition 4.5 For any uncountable space X in $\mathsf{CB}_0(\mathbf{\Delta}^1_1)$ there is a non-zero ordinal $\alpha < \omega_1$ such that the DH $\{\mathbf{\Sigma}^{-1,\theta}_\beta(X)\}_{\beta<\omega_1}$ does not collapse for each countable ordinal $\theta \geqslant \alpha$.

4.2. *h-Preorder*

Here we discuss some posets which serve as notation systems for levels of the DHs of k-partitions. All notions and facts of this subsection are contained (at least, implicitly) in Selivanov (2007a,b).

Posets considered here are assumed to be (at most) countable and without infinite chains. The absence of infinite chains in a poset $(P;\leqslant)$ is of course equivalent to well-foundedness of both $(P;\leqslant)$ and $(P;\geqslant)$. By a *forest* we mean a poset without infinite chains in which every upper cone $\{y \mid x \leqslant y\}$ is a chain. A *tree* is a forest having a biggest element (called *the root* of the tree).

Let $(T;\leqslant)$ be a tree without infinite chains; in particular, it is well founded. As for each well-founded partial order, there is a canonical rank function rk_T from T to ordinals. The rank $rk(T)$ of $(T;\leqslant)$ is by definition the ordinal $rk_T(r)$, where r is the root of $(T;\leqslant)$. It is well known that the rank of any countable tree without infinite chains is a countable ordinal, and any countable ordinal is the rank of such a tree.

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A *k-poset* is a triple $(P;\leqslant,c)$ consisting of a poset $(P;\leqslant)$ and a labelling $c: P \to k$. Rank of a k-poset $(T;\leqslant,c)$ is by definition the rank of $(T;\leqslant)$. A *morphism* $f:(P;\leqslant,c) \to (P';\leqslant',c')$ between k-posets is a monotone function $f:(P;\leqslant) \to (P';\leqslant')$ respecting the labellings, i.e. satisfying $c = c' \circ f$. Let $\widetilde{\mathcal{F}}_k$ and $\widetilde{\mathcal{T}}_k$ denote the classes of all countable k-forests and countable k-trees without infinite chains, respectively. Note that we use tilde in our notation in order to distinguish the introduced objects from their finitary versions extensively studied in Selivanov (2006, 2008b, 2012).

The h-preorder $\leqslant_h$ on $\widetilde{\mathcal{P}}_k$ is defined as follows: $(P,\leqslant,c) \leqslant_h (P',\leqslant',c')$, if there is a morphism from $(P,\leqslant,c)$ to $(P',\leqslant',c')$. Let $\widetilde{\mathbb{F}}_k, \widetilde{\mathbb{T}}_k$ be the quotient posets of $\widetilde{\mathcal{F}}_k$ and $\widetilde{\mathcal{T}}_k$ under h-equivalence, respectively. Let $\widetilde{\mathbb{F}}'_k$ be obtained from $\widetilde{\mathbb{F}}_k$ by adjoining a new smallest element $\bot$ (corresponding to the empty forest).

Let $P \sqcup Q$ be the disjoint union of k-posets P, Q and $\bigsqcup_i P_i = P_0 \sqcup P_1 \sqcup \cdots$ the disjoint union of an infinite sequence $P_0, P_1, \ldots$ of k-posets. For a k-forest F and $i < k$, let $p_i(F)$ be the k-tree obtained from F by adjoining a new biggest element and assigning the label i to this element. It is clear that the introduced operations respect h-equivalence and that any countable k-forest is h-equivalent to a countable term of signature $\{\sqcup, p_0, \ldots, p_{k-1}, 0, \ldots, k-1\}$ without free variables (the constant symbol i in the signature is interpreted as the singleton tree carrying the label i).

Proposition 4.6

1. For any $k \geqslant 2$, the structures $(\widetilde{\mathcal{F}}_k;\leqslant)$ and $(\widetilde{\mathcal{T}}_k;\leqslant)$ are well preorders of rank ω_1.
2. The posets $\widetilde{\mathbb{F}}_2$ and $\widetilde{\mathbb{T}}_2$ have width 2 (i.e., they have no antichains with more than 2 elements).
3. The poset $\widetilde{\mathbb{F}}'_k$ is a distributive lattice where any countable set of elements have a supremum.
4. The set $\sigma ji(\widetilde{\mathbb{F}}'_k)$ of σ-join-irreducible elements of the lattice $\widetilde{\mathbb{F}}'_k$ (i.e., the elements x such that $x \leqslant \bigsqcup\{y_n \mid n < \omega\}$ implies that $y \leqslant y_n$ for some n) coincides with $\widetilde{\mathbb{T}}_k$.
5. The set $ji(\widetilde{\mathbb{F}}'_k)$ of join-irreducible elements of the lattice $\widetilde{\mathbb{F}}'_k$ coincides with $\widetilde{\mathbb{T}}_k \cup S$ where S is the set of supremums of infinite increasing sequences of elements in $\widetilde{\mathbb{T}}_k$.
6. Any element of $\widetilde{\mathbb{F}}'_k$ is the infimum of finitely many elements of $\widetilde{\mathbb{T}}_k$.

For a result in the next subsection we need the following canonical representatives for the structures $\widetilde{\mathcal{F}}_2$ and $\widetilde{\mathcal{T}}_2$. Define by induction the sequence $\{T_\alpha\}_{\alpha<\omega_1}$ as follows: $T_0 = 0$, $T_{\alpha+1} = p_0(\overline{T}_\alpha)$ where $\overline{T}_\alpha$ is obtained from T_α by changing any label $l < 2$ by the label $1-l$, and $T_\lambda = p_0(\overline{T}_{\alpha_0} \sqcup \overline{T}_{\alpha_1} \sqcup \cdots)$ for a limit ordinal λ where $\alpha_0 < \alpha_1 < \cdots$ is a sequence of odd ordinals satisfying $sup\{\alpha_n \mid n < \omega\} = \lambda$. The next assertion follows from the proof of the corresponding result in Selivanov (2007a,b).

Proposition 4.7

1. For all $\alpha < \beta < \omega_1$, $T_\alpha \sqcup \overline{T}_\alpha <_h T_\beta$ and $T_\alpha, \overline{T}_\alpha$ are $\leqslant_h$-incomparable.
2. Any element of $\widetilde{\mathcal{T}}_2$ (resp. of $\widetilde{\mathcal{F}}_2$) is h-equivalent to precisely one of $T_\alpha, \overline{T}_\alpha$, (resp. to precisely one of $T_\alpha, \overline{T}_\alpha, T_\alpha \sqcup \overline{T}_\alpha, \bigsqcup_{\alpha<\lambda} T_\alpha$ where λ is a limit countable ordinal).
3. $T_\alpha \sqcup \overline{T}_\alpha \equiv_h T_{\alpha+1} \sqcap \overline{T}_{\alpha+1}$ and $\bigsqcup_{\alpha<\lambda} T_\alpha \equiv_h T_\lambda \sqcap \overline{T}_\lambda$ for each limit countable ordinal λ, where $\sqcap$ is the binary operation of k-forests inducing the infimum operation in $\widetilde{\mathbb{F}}'_k$.

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4.3. *Difference hierarchies of k-partitions*

Here we extend the difference hierarchy of sets to that of k-partitions. Note that similar hierarchies were considered e.g. in Selivanov (2007a,b) but in fact the definition here is slightly different which results in equivalence of the corresponding DHs only over the bases with ω-reduction properties. In general, the definition here is better than the previous one because it yields, for example, Theorem 4.14 for $\theta = 1$ which is not always the case for the previous definition.

Let $\mathcal{L}$ be a base in X. Recall that a k-partition of X is a function $A : X \to k$ often written as a tuple $(A_0, \ldots, A_{k-1})$ where $A_i = \{x \in X \mid A(x) = i\}$. By a *partial k-partition of X* we mean a function $A : Y \to k$ for some $Y \in \mathcal{L}$. Let $P \in \widetilde{\mathcal{F}}_k$. We say that a partial k-partition A is *defined by a P-family $\{B_p\}_{p\in P}$ of $\mathcal{L}$-sets* if $A_i = \bigcup_{p\in P_i} \tilde{B}_p$ for each $i < k$ where $\tilde{B}_p = B_p \setminus \bigcup_{q<p} B_q$ and $P_i = c^{-1}(i)$; note that in this case $A \in k^Y$ where $Y = \bigcup_{p\in P} B_p \in \mathcal{L}$.

We denote by $\mathcal{L}^Y(P)$ the set of partitions $A : Y \to k$ defined by some P-family $\{B_p\}_{p\in P}$ of $\mathcal{L}$-sets. In case $Y = X$ we omit the superscript X and call the classes $\mathcal{L}(P)$, for $P \in \widetilde{\mathcal{F}}_k$, *levels of the DH of k-partitions over $\mathcal{L}$*. We formulate some basic properties of the levels. We omit the proofs because they are quite similar to the corresponding proofs for the finitary version of the DH in Selivanov (2012).

Proposition 4.8

1. If $A \in \mathcal{L}^Y(P)$ then $A|_Z \in \mathcal{L}^Z(P)$ for each $Z \subseteq Y$, $Z \in \mathcal{L}$.
2. Any $A \in \mathcal{L}^Y(P)$ is defined by a monotone P-family $\{C_p\}$ (monotonicity means that $C_q \subseteq C_p$ for $q \leqslant p$).
3. Let f be a function on X such that $f^{-1}(A) \in \mathcal{L}$ for each $A \in \mathcal{L}$. Then $A \in \mathcal{L}^Y(P)$ implies $f^{-1}(A) = (f^{-1}(A_0), \ldots, f^{-1}(A_{k-1})) \in \mathcal{L}^{f^{-1}(Y)}(P)$.
4. If $P \leqslant_h Q$ then $\mathcal{L}^Y(P) \subseteq \mathcal{L}^Y(Q)$.
5. For all $F, G \in \widetilde{\mathcal{F}}_k$, $\mathcal{L}(F) \cap \mathcal{L}(G) = \mathcal{L}(F \sqcap G)$.

Proposition 4.8(5) and Proposition 4.6(6) show that any level $\mathcal{L}(F)$ is a finite intersection of levels $\mathcal{L}(T)$, $T \in \widetilde{\mathcal{T}}_k$. This remark, together with the results below, suggest that the levels $\mathcal{L}(T)$, $T \in \widetilde{\mathcal{T}}_k$, are analogs for the DH of k-partitions of the non-self-dual levels $D_\alpha(\mathcal{L}), \check{D}_\alpha(\mathcal{L})$ of the DH of sets. Therefore, the precise analog of the DH of sets is the family $\{\mathcal{L}(T)\}_{T\in\widetilde{\mathcal{T}}_k}$ rather than the family $\{\mathcal{L}(F)\}_{F\in\widetilde{\mathcal{F}}_k}$.

The meaning of the last paragraph might be not clear because it is not even obvious that the DH of k-partitions really extends the DH of sets; we have at least to show that the DH of 2-partitions essentially coincides with the DH of sets. We do this in the next proposition where we employ the 2-trees T_α from Section 4.2.

Proposition 4.9 Let $\mathcal{L}$ be a base in X. Then $\mathcal{L}(T_\alpha) = D_\alpha(\mathcal{L})$ for each $\alpha < \omega_1$.

Proof. Let $A \in \mathcal{L}(T_\alpha)$ be defined by a family $\{B_p\}_{p\in T_\alpha}$ where $B_p \in \mathcal{L}$. Define the sequence $\{A_\beta\}_{\beta<\alpha}$ as follows: if $r(\beta) = r(\alpha)$ then $A_\beta = \bigcup\{B_p \mid rk(p) \leqslant \beta \wedge c(p) = 0\}$, otherwise $A_\beta = \bigcup\{B_p \mid rk(p) \leqslant \beta \wedge c(p) = 1\}$. Then $A_\beta \in \mathcal{L}$ and $A = \bigcup\{\tilde{A}_\beta \mid r(\beta) \neq r(\alpha)\}$ where $\tilde{A}_\beta = A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma$, hence $A \in D_\alpha(\mathcal{L})$.

Conversely, let $A \in D_\alpha(\mathcal{L})$, then $A = \bigcup\{\tilde{A}_\beta \mid r(\beta) \neq r(\alpha)\}$ for some sequence $\{A_\beta\}_{\beta<\alpha}$ of $\mathcal{L}$-sets. Define the family $\{B_p\}_{p\in T_\alpha}$ of $\mathcal{L}$-sets as follows: if p is the root of T_α then $B_p = X$, otherwise $B_p = A_{rk(p)}$ where $rk : T_\alpha \to \alpha+1$ is the rank function. Since rk is a surjection for each α, $\tilde{B}_p = \tilde{A}_{rk(p)}$.

Note that $\tilde{B}_p, \tilde{B}_q$ are disjoint whenever $c(p) \neq c(q)$ because $\tilde{A}_\beta, \tilde{A}_\gamma$ are disjoint for distinct $\beta, \gamma < \alpha$. If p is the root of T_α then $\tilde{B}_p = X \setminus \bigcup_{\beta<\alpha} \tilde{A}_\beta \subseteq \overline{A}$. If p is not the root then, by the definition of T_α, $r(rk(p)) = r(\alpha)$ iff $c(p) = 0$. Then for each $x \in \bigcup_{\beta<\alpha} \tilde{A}_\beta$ we have: $x \in A$ iff $r(\beta) \neq r(\alpha)$ (where β is the unique ordinal with $x \in \tilde{A}_\beta$) iff $c(p) = 1$. Thus, $A \in \mathcal{L}(T_\alpha)$ is defined by $\{B_p\}_{p\in T_\alpha}$ and therefore $A \in \mathcal{L}(T_\alpha)$. □

For $F \in \widetilde{\mathcal{F}}_k$, by a *reduced F-family of $\mathcal{L}$-sets* we mean a monotone F-family $\{B_p\}_{p\in F}$ of $\mathcal{L}$-sets such that $B_p \cap B_q = \emptyset$ for all incomparable $p, q \in F$. Let $\mathcal{L}_r^Y(F)$ be the set of partial k-partitions defined by reduced F-families $\{B_p\}_{p\in F}$ of $\mathcal{L}$-sets such that $\bigcup_p B_p = Y$. The next result is an infinitary version of Proposition 7.15 (Selivanov 2012) and is proved by essentially the same argument.

Proposition 4.10 Let $\mathcal{L}$ have the ω-reduction property, $Y \in \mathcal{L}$ and $F \in \mathcal{F}_k$. Then $\mathcal{L}^Y(F) = \mathcal{L}_r^Y(F)$, in particular $\mathcal{L}(F) = \mathcal{L}_r(F)$.

In Selivanov (2013), principal total representations (TR) of pointclasses were introduced and studied. The results in Selivanov (2013) naturally extend to k-partitions. By a *family of k-partition classes* we mean a family $\mathbf{\Gamma} = \{\mathbf{\Gamma}(X)\}$ indexed by arbitrary topological spaces X such that $\mathbf{\Gamma}(X) \subseteq k^X$ and $A \circ f \in \mathbf{\Gamma}(X)$ for any continuous function $f : X \to Y$ and any k-partition $A : Y \to k$ from $\mathbf{\Gamma}(Y)$.

We note that, by Proposition 4.8(3), the levels of the DHs over any $\mathbf{\Sigma}$-level of the Borel or Luzin hierarchy are families of k-partition classes. Let $\mathbf{\Sigma}^0_\theta(X, F)$ be the F-level ($F \in \widetilde{\mathcal{F}}_k$) of the DH of k-partitions over $\mathbf{\Sigma}^0_\theta(X)$. The next fact is obvious:

Proposition 4.11 Let θ be a non-zero countable ordinal and $F \in \widetilde{\mathcal{F}}_k$. Then $\mathbf{\Sigma}^0_\theta(F) = \{\mathbf{\Sigma}^0_\theta(X, F)\}_X$ is a family of k-partition classes.

Let $\{\mathbf{\Gamma}(X)\}$ be a family of k-partition classes. A function $\nu : \mathcal{N} \to \mathbf{\Gamma}(X)$ is a *$\mathbf{\Gamma}$-TR* if the k-partition $\lambda a, x.\nu(a)(x)$ is in $\mathbf{\Gamma}(\mathcal{N} \times X)$. Such ν is a *principal $\mathbf{\Gamma}$-TR* if any $\mathbf{\Gamma}$-TR $\mu : \mathcal{N} \to \mathbf{\Gamma}(X)$ is continuously reducible to ν.

According to Theorem 5.2 in Selivanov (2013), the non-self-dual levels $\mathbf{\Gamma}$-TR of the classical hierarchies in cb$_0$-spaces have $\mathbf{\Gamma}(X)$-TRs. This result extends to the 'non-self-dual' levels of the DHs of k-partitions but only under the additional assumption that the corresponding bases have the ω-reduction property:

Proposition 4.12 Let X be a cb$_0$-space, $\theta \geqslant 2$ a countable ordinal and $T \in \widetilde{\mathcal{T}}_k$. Then $\mathbf{\Sigma}^0_\theta(X, T)$ has a principal $\mathbf{\Sigma}^0_\theta(T)$-TR. If X is in addition zero-dimensional then $\mathbf{\Sigma}^0_1(X, T)$ has a principal $\mathbf{\Sigma}^0_1(T)$-TR.

Proof hint. Modulo Theorem 5.2 in Selivanov (2013), the proof is straightforward, so we give only informal proof hints. Note that, by Proposition 2.3, the assumptions on θ guarantee that the class $\mathbf{\Sigma}^0_\theta(X)$ has the ω-reduction property, so by Proposition 4.10 the elements of $\mathbf{\Sigma}^0_\theta(X,T)$ are precisely those defined by the monotone reduced T-families $\{B_p\}$ of $\mathbf{\Sigma}^0_\theta(X)$-sets with $B_p = X$ where p is the root of T. The principal $\mathbf{\Sigma}^0_\theta$-TR of $\mathbf{\Sigma}^0_\theta(X)$ induces a natural representation of all T-families $\{C_p\}$ of $\mathbf{\Sigma}^0_\theta(X)$-sets with $C_p = X$. The problem is that such a family typically does not define any k-partition, hence we do not in general have an induced TR of $\mathbf{\Sigma}^0_\theta(X,T)$. But the ω-reduction property gives a uniform procedure of transforming $\{C_p\}$ to a monotone reduced T-family $\{B_p\}$ of $\mathbf{\Sigma}^0_\theta(X)$-sets with $B_p = X$, such that $\{B_p\} = \{C_p\}$ whenever $\{C_p\}$ already has this property. Since any such $\{B_p\}$ defines an element of $\mathbf{\Sigma}^0_\theta(X,T)$, this induces a TR of $\mathbf{\Sigma}^0_\theta(X,T)$. It is straightforward to check that this TR has the desired properties. □

We conclude this section with extending the Hausdorff–Kuratowski theorem to k-partitions. First we extend Theorem 4.2 to k-partitions:

Theorem 4.13 Let X be a cb$_0$-space, $\delta : D \to X$ an admissible representation of X ($D \subseteq \mathcal{N}$), $A : X \to k$ a k-partition of X, $\theta \geqslant 1$ a countable ordinal and $F \in \widetilde{\mathcal{F}}_k$. Then $A \in \mathbf{\Sigma}^0_\theta(X,F)$ iff $A \circ \delta \in \mathbf{\Sigma}^0_\theta(D,F)$.

Proof. is similar to the proof of Theorem 68 in de Brecht (2013). Let first $A \in \mathbf{\Sigma}^0_\theta(X,T)$, then A is defined by an F-family $\{B_p\}$ of $\mathbf{\Sigma}^0_\theta(X)$-sets. Then $A \circ \delta$ is defined by the F-family $\{\delta^{-1}(B_p)\}$ of $\mathbf{\Sigma}^0_\theta(D)$-sets, hence $A \circ \delta \in \mathbf{\Sigma}^0_\theta(D,T)$.

Conversely, let $A \circ \delta \in \mathbf{\Sigma}^0_\theta(D,T)$, then $A \circ \delta$ is defined by an F-family $\{C_p\}$ of $\mathbf{\Sigma}^0_\theta(D)$-sets, so $\delta^{-1}(A_i) = \bigcup\{\tilde{C}_p \mid c(p) = i\}$ for each $i < k$. By the proof of Theorem 68 in de Brecht (2013), we can without loss of generality assume that δ has Polish fibers, i.e., $\delta^{-1}(x)$ is Polish for each $x \in X$. For any $p \in F$, let B_p consist of the elements $x \in X$ such that the set $C_p \cap \delta^{-1}(x)$ is non-meager in $\delta^{-1}(x)$. By the proof of Theorem 68 in de Brecht (2013), $B_p \in \mathbf{\Sigma}^0_\theta(X)$, hence it suffices to show that A is defined by the F-family $\{B_p\}$.

First we check that $\tilde{B}_p \subseteq \delta(\tilde{C}_p)$. Let $x \in \tilde{B}_p$, so $C_p \cap \delta^{-1}(x)$ is non-meager in $\delta^{-1}(x)$ and $C_q \cap \delta^{-1}(x)$ is meager in $\delta^{-1}(x)$ for each $q < p$, hence also $(\bigcup_{q<p} C_q) \cap \delta^{-1}(x)$ is meager in $\delta^{-1}(x)$. Since

$$C_p \cap \delta^{-1}(x) = (\tilde{C}_p \cap \delta^{-1}(x)) \cup (\bigcup_{q<p} C_q) \cap \delta^{-1}(x),$$

$\tilde{C}_p \cap \delta^{-1}(x)$ is non-meager in $\delta^{-1}(x)$, in particular $\tilde{C}_p \cap \delta^{-1}(x)$ is non-empty. Let $a \in \tilde{C}_p \cap \delta^{-1}(x)$, then $x = \delta(a) \in \delta(\tilde{C}_p)$, as desired.

We have to show that $A_i = \bigcup\{\tilde{B}_p \mid c(p) = i\}$ for each $i < k$. Let first $x \in \tilde{B}_p$, $c(p) = i$. Then $x = \delta(a)$ for some $a \in \tilde{C}_p \subseteq \delta^{-1}(A_i)$. Thus, $x \in A_i$.

Conversely, let $x \in A_i$. Choose $a \in D$ with $x = \delta(a)$. Then $a \in \delta^{-1}(A_i)$, so $a \in \tilde{C}_p$ for some $p \in F$, $c(p) = i$. Note that $x \in B_q$ for some $q \in F$ (otherwise, $C_q \cap \delta^{-1}(x)$ is meager in $\delta^{-1}(x)$ for each $q \in F$, hence also $\delta^{-1}(x)$ is meager, a contradiction). Then $x \in \tilde{B}_q$ for some $q \in F$, hence $x = \delta(b)$ for some $b \in \tilde{C}_q$. Let $j = c(q)$, then $a \in \delta^{-1}(A_i)$ and $b \in \delta^{-1}(A_j)$, then $x \in A_i \cap A_j$, so $i = j$ and $c(q) = i$. □

As an immediate corollary, we obtain the Hausdorff–Kuratowski theorem for k-partitions in quasi-Polish spaces.

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Theorem 4.14 Let X be a quasi-Polish space and $\theta \geqslant 1$ a countable ordinal. Then $\bigcup\{\mathbf{\Sigma}^0_\theta(X,F) \mid F \in \widetilde{\mathcal{F}}_k\} = (\mathbf{\Delta}^0_{\theta+1}(X))_k$.

Proof. For $\theta \geqslant 2$, the assertion follows from Theorem 5.1 in Selivanov (2008a) and Theorem 4.2 but for $\theta = 1$ the result is new. Let $A \in (\mathbf{\Delta}^0_2(X))_k$, we have to show that $A \in \bigcup\{\mathbf{\Sigma}^0_1(X,T) \mid F \in \widetilde{\mathcal{F}}_k\}$. Let δ be an admissible total representation of X. Then $A \circ \delta \in (\mathbf{\Delta}^0_2(\mathcal{N}))_k$. Since $\mathbf{\Sigma}^0_1(\mathcal{N})$ has the ω-reduction property, $A \circ \delta \in \bigcup\{\mathbf{\Sigma}^0_\alpha(\mathcal{N},F) \mid F \in \widetilde{\mathcal{F}}_k\}$ by Theorem 5.1 in Selivanov (2008a). By Theorem 4.13, $A \in \bigcup\{\mathbf{\Sigma}^0_1(X,T) \mid F \in \widetilde{\mathcal{F}}_k\}$. □

Remarks 4.15

1. Using the argument of Proposition 4.3, it is straightforward to extend the last result to spaces X in $\mathsf{CB}_0(\mathbf{\Delta}^1_1)$.
2. The collection $\{\mathbf{\Sigma}^0_\theta(X,T) \mid 1 \leqslant \theta < \omega_1, T \in \widetilde{\mathcal{T}}_k\}$ consists of all 'non-self-dual' levels of the DH of k-partitions over any non-zero level of the Borel hierarchy, so it must contain also analogs of non-self-dual levels of the Borel hierarchy of k-partitions. As noticed in Proposition 8.21 of Selivanov (2012) (for the finitary case), these are precisely the levels $\mathbf{\Sigma}^0_\theta(X,T_0),\dots,\mathbf{\Sigma}^0_\theta(X,T_{k-1})$ where T_i is defined by $T_i := i * (0 \sqcup \cdots \sqcup (k-1))$ for any $i < k$. Note that for any distinct $i,j < k$ we have $\mathbf{\Sigma}^0_\theta(X,T_i) \cap \mathbf{\Sigma}^0_\theta(X,T_j) = (\mathbf{\Delta}^0_{\theta+1}(X))_k$.
3. That the DH of k-partitions over $\mathbf{\Sigma}^0_1$ in the Baire space is the right extension of the DH of sets over $\mathbf{\Sigma}^0_1$ is confirmed by the result in Selivanov (2007b) that the constituents of this hierarchy of k-partitions coincide with the Wadge degrees of $\mathbf{\Delta}^0_2$-measurable k-partitions. Using extensions of the jump operations in Motto Ros (2009) it can be shown that this result extends to any non-zero countable ordinal θ in the following sense: the constituents of the DH of k-partitions over $\mathbf{\Sigma}^0_\theta$ in the Baire space coincide with the $\mathbf{\Delta}^0_\theta$-degrees of $\mathbf{\Delta}^0_{\theta+1}$-measurable k-partitions (cf. Selivanov (2011) and remarks in Section 5.3 below).

5. Fine hierarchies

In this section we extend the DH of k-partitions to the FH of k-partitions. Many results and proofs here extend the ones for the DH from the previous section or the corresponding results on the finitary version of the FH from Selivanov (2012), so we concentrate only on the new moments and try to be concise whenever the material is a straightforward extension of the previous one. We will see that the FH of k-partitions is in a sense an 'iterated version' of the DH of k-partitions which is far from obvious for the particular case of the Wadge hierarchy, under the classical definition.

As we explain below, the FH of sets in the Baire space does coincide with the Wadge hierarchy. To our knowledge, the extension of this hierarchy to non-zero-dimensional spaces is new here (so far such an extension was known only for the finitatry version of the FH (Selivanov 2008b, 2012)). Interestingly, in our approach here the definition of the FH of sets is in fact not simpler than for the k-partitions for arbitrary $k \geqslant 2$.

Since even the definition of the FH is technically very involved, we concentrate here on a slightly easier case of sets and k-partitions of finite Borel rank and provide only proof

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hints for some long proofs, appealing to the analogy with the finitary version in Selivanov (2012).

5.1. *More on the h-preorders*

Here we extend some notions and facts from Subsection 4.2 about the h-preorders, in order to describe notation systems for levels of the FHs of k-partitions. We omit the proofs which are easy variations of their finitary versions in Selivanov (2012).

Let $(Q;\leqslant)$ be a poset. A *Q-poset* is a triple $(P,\leqslant,c)$ consisting of a countable nonempty poset $(P;\leqslant)$, $P\subseteq\omega$ without infinite chains, and a labeling $c:P\to Q$. By default, we denote the labeling in a Q-poset by c. A *morphism* $f:(P,\leqslant,c)\to(P',\leqslant',c')$ of Q-posets is a monotone function $f:(P;\leqslant)\to(P';\leqslant')$ satisfying $\forall x\in P(c(x)\leqslant c'(f(x)))$. Let $\widetilde{\mathcal{P}}_Q$, $\widetilde{\mathcal{F}}_Q$ and $\widetilde{\mathcal{T}}_Q$ denote the sets of all countable Q-posets, Q-forests and Q-trees, respectively.

The *h-preorder* $\leqslant_h$ on $\widetilde{\mathcal{P}}_Q$ is defined as follows: $P\leqslant_h P'$, if there is a morphism from P to P'. The quotient-posets of $\widetilde{\mathcal{P}}_Q$, $\widetilde{\mathcal{F}}_Q$, $\widetilde{\mathcal{T}}_Q$ are denoted $\widetilde{\mathbb{P}}_Q$, $\widetilde{\mathbb{F}}_Q$, $\widetilde{\mathbb{T}}_Q$, respectively. Note that for the particular case $Q=\bar{k}$ of the antichain with k elements we obtain the preorders $\widetilde{\mathcal{P}}_k$, $\widetilde{\mathcal{F}}_k$, $\widetilde{\mathcal{T}}_k$ from the previous section.

Next we formulate some lattice-theoretic properties of the h-preorders. By a *partial lower semilattice* we mean a poset in which any two elements that have a lower bound have a (unique) greatest lower bound. For any poset Q, let Q' be the poset obtained from Q by adjoining the new element $\perp$ which is smaller than all elements in Q. Define the function $s:Q\to\widetilde{\mathcal{P}}_Q$ as follows: $s(q)$ is the singleton tree labeled by $q\in Q$. If Q is an upper σ-semilattice (i.e., any countable subset of Q has a supremum), define the function $l:\widetilde{\mathcal{P}}_Q\to Q$ by $l(P)=\bigcup\{c(p)\mid p\in P\}$ where $\cup$ is the supremum operation in Q. For Q-posets P and R (resp. $P_0,P_1,\ldots$), let $P\sqcup R$ (resp. $\bigsqcup_i P_i$) denote their disjoint union (resp. countable disjoint union), so that $P\sqcup R,\bigsqcup_i P_i\in\widetilde{\mathcal{P}}_Q$.

For posets P and Q we write $P\subseteq Q$ (resp. $P\sqsubseteq Q$) if P is a substructure of Q (resp. P is an initial segment of Q). The next assertion is essentially contained in Selivanov (2012).

Proposition 5.1

1. If Q is a partial lower semilattice then Q' is a lower semilattice.
2. $(\widetilde{\mathcal{P}}_Q;\leqslant_h,\sqcup)$ is a distributive upper σ-semilattice that contains $(\widetilde{\mathcal{F}}_Q;\leqslant_h,\sqcup)$ as a distributive upper σ-subsemilattice.
3. If Q is a partial lower semilattice then $(\widetilde{\mathcal{P}}'_Q;\leqslant_h)$ and $(\widetilde{\mathcal{F}}'_Q;\leqslant_h)$ are distributive lattices.
4. If Q is σ-directed (i.e. any countable set of elements has an upper bound) then $s(Q)$ is a cofinal subset of $\widetilde{\mathcal{P}}_Q$ (i.e. any $x\in\widetilde{\mathcal{P}}_Q$ is below $s(q)$ for some $q\in Q$).
5. The mapping s is an isomorphic embedding of Q into $\widetilde{\mathcal{P}}_Q$.
6. If Q is a partial lower semilattice then s preserves the greatest lower bound operations in Q and $\widetilde{\mathcal{P}}_Q$ (and similarly for $\widetilde{\mathcal{F}}_Q$).
7. If Q is an upper σ-semilattice then $l:\widetilde{\mathcal{P}}_Q\to Q$ is a homomorphism of upper σ-semilattices and $q=l(s(q))$ for each $q\in Q$.
8. If Q is a bqo then $(\widetilde{\mathcal{F}}_Q;\leqslant_h)$, $(\widetilde{\mathcal{T}}_Q;\leqslant_h)$ are bqo's.
9. For arbitrary posets P and Q, $P\subseteq Q$ implies $\widetilde{\mathbb{F}}_P\subseteq\widetilde{\mathbb{F}}_Q$, and $P\sqsubseteq Q$ implies $\widetilde{\mathbb{F}}_P\sqsubseteq\widetilde{\mathbb{F}}_Q$.

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Now we can iterate the construction $Q \mapsto \widetilde{\mathbb{F}}_Q$ starting with the antichain $\overline{k}$ of k elements $\{0, \ldots, k-1\}$. Define the sequence $\{\widetilde{\mathcal{F}}_k(n)\}_{n<\omega}$ of preorders by induction on n as follows: $\widetilde{\mathcal{F}}_k(0) = \overline{k}$ and $\widetilde{\mathcal{F}}_k(n+1) = \widetilde{\mathcal{F}}_{\widetilde{\mathcal{F}}_k(n)}$. Identifying the elements $i < k$ of $\overline{k}$ with the corresponding minimal elements $s(i)$ of $\widetilde{\mathcal{F}}_k(1)$, we may think that $\widetilde{\mathcal{F}}_k(0) \sqsubseteq \widetilde{\mathcal{F}}_k(1)$. By items (8,9) of Proposition 5.1, $\widetilde{\mathcal{F}}_k(n) \sqsubseteq \widetilde{\mathcal{F}}_k(n+1)$ for each $n < \omega$, and $\widetilde{\mathcal{F}}_k(\omega) = \bigcup_{n<\omega} \widetilde{\mathcal{F}}_k(n)$ is a wqo. For any $n \leqslant \omega$, let $\widetilde{\mathbb{F}}_k(n)$ be the quotient-poset of $\widetilde{\mathcal{F}}_k(n)$.

Of course, similar constructions can be done with $\widetilde{\mathcal{T}}$ in place of $\widetilde{\mathcal{F}}$. The preorders $\widetilde{\mathcal{F}}_k(\omega)$, $\widetilde{\mathcal{T}}_k(\omega)$ and the set $\widetilde{\mathcal{T}}_k^{\sqcup}(\omega)$ of countable joins of elements in $\widetilde{\mathcal{T}}_k(\omega)$, play an important role in the study of the FH of k-partitions because they provide convenient naming systems for the levels of this hierarchy (similar to the previous section where $\widetilde{\mathcal{F}}_k$ and $\widetilde{\mathcal{T}}_k$ were used to name the levels of the DH of k-partitions). Note that $\widetilde{\mathcal{F}}_k(1) = \widetilde{\mathcal{F}}_k$ and $\widetilde{\mathcal{T}}_k(1) = \widetilde{\mathcal{T}}_k$.

By Proposition 5.1, for any $n < \omega$ there is an embedding $s = s_n$ of $\widetilde{\mathbb{F}}_k(n)$ into $\widetilde{\mathbb{F}}_k(n+1)$, and s_{n+1} coincides with s_n on $\widetilde{\mathbb{F}}_k(n)$. This induces the embedding $s = \bigcup_{n<\omega} s_n$ of $\mathbb{F}_k(\omega)$ into itself such that s coincides with s_n on $\widetilde{\mathbb{F}}_k(n)$ for each $n < \omega$. Similarly, for any $n < \omega$ we have the function l from $\widetilde{\mathbb{F}}_k(n+2)$ onto $\widetilde{\mathbb{F}}_k(n+1)$ which induces the function (denoted also by l) from $\widetilde{\mathbb{F}}_k(\omega)$ onto $\widetilde{\mathbb{F}}_k(\omega)$.

Define the binary operation $*$ on $\widetilde{\mathcal{P}}_k(\omega)$ as follows: $F * G$ is the labeled poset obtained from G by adjoining a new largest (root) element and assigning the label F to that element. It is easy to see that the operation $*$ respects the h-equivalence relation and hence induces the binary operation on $\widetilde{\mathbb{P}}_k(\omega)$ also denoted by $*$. Note that for $F = s(i) = i < k$ we have $F * G = p_i(G)$ and that $\widetilde{\mathcal{F}}_k(\omega)$ is closed under $*$.

We formulate some properties of the introduced objects illustrating a rich algebraic structure of $\widetilde{\mathcal{F}}_k(\omega)$. They are again almost the same as their corresponding finitary versions in Selivanov (2012).

Proposition 5.2

1. For any n with $1 \leqslant n \leqslant \omega$, $\widetilde{\mathbb{F}}'_k(n)$ is a well distributive lattice which is an upper σ-semilattice for $n < \omega$.
2. Any element of $\widetilde{\mathbb{F}}'_k(\omega)$ is the value of a variable-free term (countable joins are allowed) of signature $\{\sqcup, *, \bot, 0, \ldots, k-1\}$.
3. For any $0 < n \leqslant \omega$, $\widetilde{\mathbb{T}}_k(n)$ generates $\widetilde{\mathbb{F}}'_k(n)$ under $\sqcap$.
4. The set of σ-join-irreducible elements of $\widetilde{\mathbb{T}}_k^{\sqcup}(\omega)$ coincides with $\widetilde{\mathbb{T}}_k(\omega)$.
5. The set of join-irreducible elements $\mathbb{T}_k^{\sqsubseteq}(\omega)$ by of $\tilde{\mathbb{T}}_k^{\sqsubseteq}(\omega)$. coincides with $\widetilde{\mathbb{T}}_k(\omega) \cup S$, where S is the set of supremums of infinite increasing sequences in $\widetilde{\mathbb{T}}_k(\omega)$.

Next we provide a characterization of $\widetilde{\mathbb{P}}_k(\omega)$ (and the related substructures) which is sometimes more convenient when dealing with the FH in the next subsection. For $P, Q \in \widetilde{\mathcal{P}}_k(n)$, an *explicit morphism* $\varphi : P \to Q$ is a sequence $(\varphi_0, \ldots, \varphi_{n-1})$ where φ_0 is a morphism from P to Q, $\varphi_1 = \{\varphi_{1,p_0}\}_{p_0 \in P}$ is a family of morphisms from $c(p_0)$ to $c(\varphi_0(p_0))$, $\varphi_2 = \{\varphi_{2,p_0,p_1}\}_{p_0 \in P, p_1 \in c(p_0)}$ is a family of morphisms from $c(p_1)$ to $c(\varphi_{1,p_0}(p_1))$ and so on (this notion makes use of the convention that $i = s(i)$ for each $i < k$). Note that $P \leqslant_h Q$ iff there is an explicit morphism from P to Q, and that for $n = 1$ the explicit morphisms essentially coincide with the morphisms. In the next assertion we treat $\widetilde{\mathbb{P}}_k(\omega)$ as the category whose morphisms are the explicit morphisms.

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For each positive $n < \omega$, a *k-labelled n-preorder* (Selivanov 2008b) is a countable structure $(S; d, \leqslant_0, \ldots, \leqslant_{n-1})$ where $d : S \to k$ is a k-partition of S and $\leqslant_0, \ldots, \leqslant_{n-1}$ are preorders on S such that $\leqslant_{n-1}$ is a partial order, $x \leqslant_{i+1} y$ implies $x \equiv_i y$, the quotient-poset of $(S; \leqslant_0)$ has no infinite chains, for each $x \in S$ the quotient-poset of $([x]_0; \leqslant_1)$ has no infinite chains for each $y \in [x]_0$ the quotient-poset of $([y]_1; \leqslant_2)$ has no infinite chains, and so on. Let $\mathcal{S}_n(k)$ be the category of k-labelled n-posets as objects where the morphisms are functions that preserve the labellings and are monotone w.r.t. all the preorders.

Proposition 5.3 For any positive $n < \omega$, the categories $\widetilde{\mathcal{P}}_k(n)$ and $\mathcal{S}_n(k)$ are equivalent.

Proof hint. The equivalence is witnessed (cf. Proposition 8.8 of Selivanov (2012)) by the functors defined as follows. Relate to any object $(P; \leqslant, c)$ of $\widetilde{\mathcal{P}}_k(n)$ the object $P^\circ = (X; \leqslant_0, \ldots, \leqslant_{n-1}, d)$ of $\mathcal{S}_n(k)$ where X is formed by the elements $p = (p_0, \ldots, p_{n-1})$ such that $p_0 \in P, p_1 \in c(p_0), \ldots$, the preorders between such p and $r = (r_0, \ldots, r_{n-1})$ are defined by $p \leqslant_0 r$ iff $p_0 \leqslant r_0$, $p \leqslant_1 r$ iff $p_0 = r_0$ and $p_1 \leqslant r_1$ and so on, and the labeling $d : X \to k$ is defined by $d(p) = c(p_{n-1})$. Relate to any explicit morphism $\varphi = (\varphi_0, \ldots, \varphi_{n-1}) : P \to Q$ of $\widetilde{\mathcal{P}}_k(n)$ the morphism $\varphi^\circ : P^\circ \to Q^\circ$ by $\varphi^\circ(p_0, p_1, \ldots) = (\varphi_0(p_0), \varphi_{1,p_0}(p_1), \ldots)$.

Conversely, relate to any object X of $\mathcal{S}_n(k)$ as above the object $X^+ = (P; \leqslant, c)$ of $\widetilde{\mathcal{P}}_k(n)$ where $(P; \leqslant)$ is the quotient-poset of $(X; \leqslant_0)$ and $c([x]_0) = ([x]_0; \leqslant_1, \ldots, \leqslant_{n-1}, d|_{[x_0]})^+$; we can suppose by induction that $c([x]_0)$ is an object of $\widetilde{\mathcal{P}}_k(n-1)$ if $n > 1$. Relate to any morphism $\psi : X \to Y$ of $\mathcal{S}_n(k)$ the explicit morphism $\psi^+ = (\varphi_0, \ldots, \varphi_{n-1}) : X^+ \to Y^+$ where $\varphi_0([x]_0) = [\psi(x)]_0$, $\varphi_{1,[x]_0}([z]_1) = [\psi(z)]_1$ for each $z \in [x]_0$ and so on. □

Note that the notion of an explicit morphism does not in fact depend on the number n because the explicit morphism $\varphi : P \to Q$ of $P, Q \in \widetilde{\mathcal{P}}_k(n)$ is uniquely extended to an explicit morphism of P, Q considered as objects of $\widetilde{\mathcal{P}}_k(n+1)$. Thus, we can consider the category $\widetilde{\mathcal{P}}_k(\omega)$ with the explicit morphism. Similarly, we can consider the category $\mathcal{S}_\omega(k) = \bigcup_n \mathcal{S}_n(k)$ because $\mathcal{S}_n(k)$ may be considered as a subcategory of $\mathcal{S}_{n+1}(k)$ (just add the equality relation as $\leqslant_n$). In this way, we obtain the equivalence of categories $\widetilde{\mathcal{P}}_k(\omega)$ and $\mathcal{S}_\omega(k)$.

The full subcategories $\widetilde{\mathcal{F}}_k(\omega)$ and $\widetilde{\mathcal{T}}_k(\omega)$ of $\widetilde{\mathcal{P}}_k(\omega)$ are then equivalent to suitable full subcategories $\mathcal{U}_\omega(k)$ and $\mathcal{V}_\omega(k)$ of $\mathcal{S}_\omega(k)$ (e.g., the objects of $\mathcal{U}_\omega(k)$ are $(X; \leqslant_0, \leqslant_1, \ldots, d)$ where $(X; \leqslant_0)$ is a forest, $([x]_0; \leqslant_1)$ is a forest for each $x \in X$, and so on).

We conclude this subsection with extending Proposition 4.7 to the structure $\widetilde{\mathcal{T}}_2^{\sqcup}(\omega)$. For this we need the ordinal $\xi = sup\{\omega_1, \omega_1^{\omega_1}, \omega_1^{(\omega_1^{\omega_1})}, \ldots\}$. According to the Cantor normal form, any non-zero ordinal $\alpha < \xi$ is uniquely representable in the form $\alpha = \omega_1^{\gamma_0} \cdot \alpha_0 + \cdots + \omega_1^{\gamma_l} \cdot \alpha_l$ for some $l < \omega$, $\alpha > \gamma_0 > \cdots > \gamma_l$ and non-zero ordinals $\alpha_0, \ldots, \alpha_l < \omega_1$). For $F \in \widetilde{\mathcal{F}}_2(\omega)$, let $\bar{F} \in \widetilde{\mathcal{F}}_2(\omega)$ be obtained from F by interchanging $\{0, 1\}$ in all the labels.

Definition 5.4 We define the sequence $\{T_\alpha\}_{\alpha<\xi}$ of trees in $\widetilde{\mathcal{T}}_2(\omega)$ by induction on α as follows:

1. For $\alpha < \omega_1$, use the definition from the end of Subsection 4.2.
2. For any non-zero ordinal $\gamma < \xi$, $T_{\omega_1^\gamma} = s(T_\gamma)$.

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3. For any limit uncountable ordinal $\lambda < \xi$ of countable cofinality, $T_\lambda = 0 * (\bigsqcup_n T_{\alpha_n})$ where $\alpha_0 < \alpha_1 < \cdots$ and $\lambda = sup\{\alpha_0, \alpha_1, \ldots\}$.
4. For any ordinal $\beta \geqslant \omega_1$, $T_{\beta+1} = 0 * (T_\beta \sqcup \bar{T}_\beta)$.
5. For all δ, γ such that $1 \leqslant \delta < \omega_1$ and $1 \leqslant \gamma < \xi$, $T_{\omega_1^\gamma(\delta+1)} = T_\gamma * \bar{T}_{\omega_1^\gamma \delta}$.
6. For all β, γ such that $1 \leqslant \gamma < \xi$ and $\beta = \omega_1^{\gamma_1} \cdot \beta_1$ for some $\beta_1 > 0$ and $\gamma_1 > \gamma$, $T_{\beta+\omega_1^\gamma} = T_\gamma * (T_\beta \sqcup \bar{T}_\beta)$.
7. For all δ, β, γ such that $1 \leqslant \delta < \omega_1$, $1 \leqslant \gamma < \xi$ and $\beta = \omega^{\gamma_1} \cdot \beta_1$ for some $\beta_1 > 0$ and $\gamma_1 > \gamma$, $T_{\beta+\omega_1^\gamma(\delta+1)} = T_\gamma * \bar{T}_{\beta+\omega_1^\gamma \delta}$.

The next assertion is the extension of Proposition 4.7. We omit the proof which is very similar to that of Proposition 8.29 in Selivanov (2012) being a finitary version of this assertion.

Proposition 5.5

1. For all $\alpha < \beta < \xi$, T_α is correctly defined up to $\equiv_h$ (in particular, it does not depend on the choice of the ordinals α_n in Definition 5.4(3)), T_α and $\bar{T}_\alpha$ are $\leqslant_h$-incomparable elements of $\widetilde{\mathcal{T}}_2(\omega)$ that satisfy $T_\alpha \sqcup \bar{T}_\alpha <_h T_\beta$.
2. Any $T \in \widetilde{\mathcal{T}}_2(\omega)$ is h-equivalent to precisely one of $T_\alpha, \bar{T}_\alpha$ for some $\alpha < \xi$.
3. Any $T \in \widetilde{\mathcal{T}}_2^{\sqcup}(\omega)$ is h-equivalent to precisely one of $T_\alpha, \overline{T}_\alpha, T_\alpha \sqcup \overline{T}_\alpha, \bigsqcup_n T_{\alpha_n}$ where $\{\alpha_n\}$ is an increasing sequence of ordinals converging to a limit ordinal (of countable cofinality).
4. For any $\alpha < \omega_1$, $T_\alpha \sqcup \overline{T}_\alpha \equiv_h T_{\alpha+1} \sqcap \overline{T}_{\alpha+1}$, where $\sqcap$ is a binary operation of k-forests inducing the infimum operation in $\widetilde{\mathbb{F}}_k'(\omega)$.
5. For any limit countable ordinal λ of countable cofinality and any increasing sequence of ordinals converging to λ, $\bigsqcup_n T_{\alpha_n} \equiv_h T_\lambda \sqcap \overline{T}_\lambda$.
6. The ranks of $\widetilde{\mathbb{T}}_2(\omega)$ and of $\widetilde{\mathbb{T}}_2^{\sqcup}(\omega)$ coincide with ξ.

5.2. *Fine hierarchies of k-partitions*

Here we define the FH of k-partitions and formulate its basic properties. The proofs are almost the same as those for the finitary case in Selivanov (2012).

Let X be a space. By an *ω-base* (cf. Selivanov (2008b, 2012)) in X we mean a sequence $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ of bases such that $\mathcal{L}_n \cup \check{\mathcal{L}}_n \subseteq \mathcal{L}_{n+1}$ for each $n < \omega$. The main example of an ω-base is of course $\{\mathbf{\Sigma}^0_{n+1}\}_{n<\omega}$ but also other examples are interesting, in particular the m-shifts $\{\mathbf{\Sigma}^0_{m+n}\}_{n<\omega}$ for any fixed $1 \leqslant m < \omega$.

Let $P \in \mathcal{F}_k(n)$ for some positive $n < \omega$. By a *P-family over $\mathcal{L}$* we mean a family $\{B_{p_0}, B_{p_0,p_1}, \ldots, B_{p_0,\ldots,p_{n-1}}\}$ where $p = (p_0, \ldots, p_{n-1}) \in P^\circ$ (see the end of Section 5.1), $B_{p_0} \in \mathcal{L}_0, B_{p_0,p_1} \in \mathcal{L}_1, \ldots, B_{p_0 \ldots, p_{n-1}} \in \mathcal{L}_{n-1}$, and the sets

$$\tilde{B}_{p_0} = B_{p_0} \setminus \bigcup\{B_r \mid r \leqslant_0 p\},\ \tilde{B}_{p_0,p_1} = B_{p_0,p_1} \setminus \bigcup\{B_r \mid r \leqslant_1 p\}, \ldots$$

satisfy

$$\tilde{B}_{p_0} = \bigcup\{B_{p_0,p_1} \mid p_1 \in c(p_0)\},\ \tilde{B}_{p_0,p_1} = \bigcup\{B_{p_0,p_1,p_2} \mid p_1 \in c(p_0), p_2 \in c(p_1)\}, \ldots.$$

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To simplify notation, we often denote families just by $\{B_p\}$. Note that $d(p) = c(p_{n-1})$ is always in $\widetilde{\mathcal{F}}_k(0) = \{0, \ldots, k-1\}$,

$$\tilde{B}_{p_0} = \bigcup\{\tilde{B}_{p_0,p_1} \mid p_1 \in c(p_0)\},\ \tilde{B}_{p_0,p_1} = \bigcup\{\tilde{B}_{p_0,p_1,p_2} \mid p_1 \in c(p_0), p_2 \in c(p_1)\}, \ldots$$

and that for $n = 1$ the P-families over $\mathcal{L}$ essentially coincide with the P-families of $\mathcal{L}_0$-sets in Section 4.3. Obviously, $\bigcup_{p_0} B_{p_0} = \bigcup_{p \in P^\circ} \tilde{B}_p$. We call a P-family $\{B_p\}$ over $\mathcal{L}$ *consistent* if $d(p) = d(q)$ whenever the components $\tilde{B}_p$ and $\tilde{B}_q$ have a nonempty intersection. Any such consistent P-family determines the k-partition $A : \bigcup_{p_0} B_{p_0} \to k$ where $A(x) = d(p)$ for some (equivalently, for any) $p \in P^\circ$ with $x \in \tilde{B}_p$; we also say in this case that A is defined by $\{B_p\}$. Note that this k-partition is determined by the defining P-family and it does not depend on the number n with $P \in \widetilde{\mathcal{F}}_k(n)$.

Let $\mathcal{L}^Y(P)$ be the set of k-partitions $A : Y \to k$ defined by some P-family over $\mathcal{L}$. In case $Y = X$ we omit the superscript X and call (temporarily) the family $\{\mathcal{L}(P)\}_{P \in \widetilde{\mathcal{F}}_k(\omega)}$ the FH of k-partitions over $\mathcal{L}$.

For $F \in \widetilde{\mathcal{F}}_k(\omega)$, by a *reduced F-family over $\mathcal{L}$* we mean a monotone F-family $\{B_p\}$ over $\mathcal{L}$ such that $B_{p_0} \cap B_{q_0} = \varnothing$ for all incomparable $p_0, q_0 \in F$, $B_{p_0,p_1} \cap B_{p_0,q_1} = \varnothing$ for all incomparable $p_1, q_1 \in c(p_0)$ and so on. Let $\mathcal{L}_r^Y(F)$ be the set of partial k-partitions defined by the reduced F-families $\{B_p\}$ over $\mathcal{L}$ such that $\bigcup_{p_0} B_{p_0} = Y$.

The next assertion is a a straightforward extension of Proposition 4.8 proved similarly to its finitary version in Selivanov (2012).

Proposition 5.6

1. If $A \in \mathcal{L}^Y(P)$ then $A|_Z \in \mathcal{L}^Z(P)$ for each $Z \subseteq Y$, $Z \in \mathcal{L}_0$.
2. Any $A \in \mathcal{L}^Y(P)$ is defined by a monotone P-family $\{C_p\}$ (monotonicity means that $C_{q_0} \subseteq C_{p_0}$ for $q_0 \leqslant p_0$, $C_{p_0,q_1} \subseteq C_{p_0,p_1}$ for $q_1 \leqslant p_1$ and so on).
3. Let $f : X_1 \to X$ be a morphism of ω-spaces and $A \in \mathcal{L}^Y(P)$ in X. Then $f^{-1}(A) \in \mathcal{L}^{f^{-1}(Y)}(P)$.
4. If $P \leqslant_h Q$ then $\mathcal{L}^Y(P) \subseteq \mathcal{L}^Y(Q)$.
5. The collection $\{\mathcal{L}(P) \mid P \in \widetilde{\mathcal{F}}_k(\omega)\}$ is well partially ordered by inclusion.
6. For all $F, G \in \widetilde{\mathcal{F}}'_k(\omega)$, $\mathcal{L}(F) \cap \mathcal{L}(G) = \mathcal{L}(F \sqcap G)$.
7. Any level $\mathcal{L}(F)$, $F \in \widetilde{\mathcal{F}}_k(\omega)$, of the FH is the intersection of finitely many 'non-self-dual' levels $\mathcal{L}(T)$, $T \in \widetilde{\mathcal{T}}_k(\omega)$.
8. Let $\mathcal{L}_n$ have the ω-reduction property for each $n < \omega$, $Y \in \mathcal{L}_0$ and $F \in \widetilde{\mathcal{F}}_k(\omega)$. Then $\mathcal{L}^Y(F) = \mathcal{L}_r^Y(F)$, in particular, $\mathcal{L}(F) = \mathcal{L}_r(F)$.
9. Any constituent of the hierarchy $\{\mathcal{L}(x)\}_{x \in \widetilde{\mathbb{F}}_k(\omega)}$ is a constituent of the hierarchy $\{\mathcal{L}(x)\}_{x \in \widetilde{\mathbb{T}}_k(\omega)}$.

Item (9) shows that we can simplify the hierarchy $\{\mathcal{L}(F)\}_{F \in \widetilde{\mathcal{F}}_k(\omega)}$ to the hierarchy $\{\mathcal{L}(T)\}_{T \in \widetilde{\mathcal{T}}_k(\omega)}$. Similar to the DH of k-partitions, levels $\mathcal{L}(T)$, $T \in \widetilde{\mathcal{T}}_k(\omega)$, generalize the non-self-dual levels $\mathbf{\Sigma}, \mathbf{\Pi}$ of the hierarchies of sets while finite intersections of these levels correspond to the self-dual levels of hierarchies of sets. This is illustrated by the following extension of Propositions 4.11, 4.12 and Theorem 4.13.

Proposition 5.7

1. Let $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+1}\}_{n<\omega}$ and $F \in \widetilde{\mathcal{F}}_k(\omega)$. Then $\{\mathcal{L}(X, F)\}_X$ is a family of k-partition classes.
2. Let X be a cb$_0$-space, $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+2}(X)\}_{n<\omega}$, and $T \in \widetilde{\mathcal{T}}_k(\omega)$. Then $\mathcal{L}(X, T)$ has a principal $\mathcal{L}(T)$-TR. If X is in addition zero-dimensional and $\mathcal{M} = \{\mathbf{\Sigma}^0_{n+1}(X)\}_{n<\omega}$ then $\mathcal{M}(X, T)$ has a principal $\mathcal{M}(T)$-TR.

The next result is an infinitary version of Proposition 8.19 in Selivanov (2012). It extends the Hausdorff–Kuratowski theorem to all levels of the FH. We call the ω-base $\mathcal{L}$ is *interpolable* if, for each $n \geqslant 1$, $\mathcal{L}_n$ has the ω-reduction property and the Hausdorff–Kuratowski theorem holds for any non-zero level of $\mathcal{L}$. In particular, the base $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+1}(X)\}_{n<\omega}$ is interpolable in any quasi-Polish space X.

Theorem 5.8 Let $\mathcal{L}$ be an interpolable base in X, $n < \omega$, and $T_0, \ldots, T_n \in \widetilde{\mathcal{T}}_k(\omega)$. Then

$$\mathcal{L}(T_0) \cap \cdots \cap \mathcal{L}(T_n) = \bigcup \{\mathcal{L}(S) \mid S \in \mathcal{T}_k^{\sqcup}(\omega), S \leqslant T_0 \sqcap \cdots \sqcap T_n\}.$$

The last theorem informally means that the FH of k-partitions is the finest possible in the sense that it cannot be further refined in ‘non-self-dual’ levels because any such level is exhausted by the lower levels. The Hausdorff–Kuratowski theorem for k-partitions (Theorem 4.14 for finite θ) is a very particular case of the last theorem because for any positive integer n we have (using Remark 4.15) $(\mathbf{\Delta}^0_{n+1}(X))_k = \mathcal{L}(s^n(0 \sqcup \cdots \sqcup (k-1)))$, $\mathbf{\Sigma}^0_n(T) = \mathcal{L}(s^{n-1}(T))$, and $\{\mathcal{L}(s^{n-1}(T)) \mid T \in \widetilde{\mathcal{T}}_k\}$ is cofinal in $\{\mathcal{L}(S) \mid S \in \mathcal{T}_k^{\sqcup}(\omega), S \leqslant s^n(0 \sqcup \cdots \sqcup (k-1))\}$.

The next result extends (with essentially the same proof) Theorem 4.13 to the FH of k-partitions.

Theorem 5.9 Let X be a cb$_0$-space, $\delta : D \to X$ an admissible representation of X ($D \subseteq \mathcal{N}$), $A : X \to k$ a k-partition of X, $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+1}\}_{n<\omega}$, and $T \in \widetilde{\mathcal{T}}_k(\omega)$. Then $A \in \mathcal{L}(X, T)$ iff $A \circ \delta \in \mathcal{L}(D, T)$.

Corollary 5.10 Let X be a quasi-Polish space, $\delta : \mathcal{N} \to X$ an admissible TR of X, $A : X \to k$ a k-partition of X, $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+1}\}_{n<\omega}$, and $T \in \widetilde{\mathcal{T}}_k(\omega)$. Then $A \in \mathcal{L}(X, T)$ iff $A \circ \delta \in \mathcal{L}(\mathcal{N}, T)$.

For the DH of k-partitions, it was easy to demonstrate that it really extends the DH of sets (which coincides with the DH of 2-partitions by Proposition 4.9). For the FH of 2-partitions the same task is more complicated. The reason is that this hierarchy should generalize the Wadge hierarchy which was so far defined and relatively well understood only for the Baire space (and some other closely related spaces). Moreover, the most popular definition of this hierarchy is in terms of the Wadge reducibility rather than in set-theoretic terms (in fact, there are also set-theoretic definitions (Louveau 1983; Wadge 1984) but they are very indirect and hard to deal with). Nevertheless, we claim that the FH of 2-partitions in the Baire space coincides with the Wadge hierarchy. We discuss this (rather informally) in the next subsection.

5.3. *FH of k-partitions and Wadge-like reducibilities*

Here we discuss the relation of the FH of k-partitions to Wadge-like reducibilities. Let us first briefly recall some relevant facts about the Wadge reducibility in the Baire space. In Wadge (1972, 1984) Wadge (with a heavy use of the Martin determinacy theorem) proved that the structure $(\mathbf{\Delta}^1_1(\mathcal{N});\leqslant_W)$ is semi-well-ordered (i.e., it is well-founded and for all $A,B \in \mathbf{\Delta}^1_1(\mathcal{N})$ we have $A \leqslant_W B$ or $\overline{B} \leqslant_W A$. He has also computed the rank ν of this structure which is a rather large ordinal.

In Steel (1980); Van Wesep (1976), the following deep relation of the Wadge reducibility to the separation property was established: For any Borel set A which is non-self-dual (i.e., $A \not\leqslant_W \overline{A}$) exactly one of the principal ideals $\{X \mid X \leqslant_W A\}$, $\{X \mid X \leqslant_W \overline{A}\}$ has the separation property.

The mentioned results give rise to the *Wadge hierarchy in the Baire space* which is, by definition, the sequence $\{\mathbf{\Sigma}_\alpha\}_{\alpha<\nu}$ of all non-self-dual principal ideals of $(\mathbf{\Delta}^1_1(\mathcal{N});\leqslant_W)$ that do not have the separation property and satisfy for all $\alpha < \beta < \nu$ the strict inclusion $\mathbf{\Sigma}_\alpha \subset \mathbf{\Delta}_\beta$. As usual, we set $\mathbf{\Pi}_\alpha = \{\overline{A} \mid A \in \mathbf{\Sigma}_\alpha\}$ and $\mathbf{\Delta}_\alpha = \mathbf{\Sigma}_\alpha \cap \mathbf{\Pi}_\alpha$. Note that the constituents of the Wadge hierarchy are precisely the equivalence classes induced by $\leqslant_W$ on Borel subsets of the Baire space (i.e., the Wadge degrees).

As shown in Wadge (1984), $\mathbf{\Sigma}_\alpha = \mathbf{\Sigma}^{-1}_\alpha(\mathcal{N})$ for each $\alpha < \omega_1$, i.e. the DH over open sets is an initial segment of the Wadge hierarchy. In order to see how much finer is the Wadge hierarchy compared with the Borel hierarchy, we mention the equalities from Wadge (1984) relating both hierarchies: $\mathbf{\Sigma}_1 = \mathbf{\Sigma}^0_1(\mathcal{N})$, $\mathbf{\Sigma}_{\omega_1} = \mathbf{\Sigma}^0_2(\mathcal{N})$, $\mathbf{\Sigma}_{\omega_1^{\omega_1}} = \mathbf{\Sigma}^0_3$ and so on. Thus, the sets of finite Borel rank coincide with the sets of Wadge rank less than $\xi = sup\{\omega_1, \omega_1^{\omega_1}, \omega_1^{(\omega_1^{\omega_1})}, \ldots\}$. Note that ξ is the smallest solution of the ordinal equation $\omega_1^\varkappa = \varkappa$. Hence, we warn the reader not to mistake $\mathbf{\Sigma}_\alpha$ with $\mathbf{\Sigma}^0_\alpha$. To give an impression about the Wadge ordinal we note that the rank of the preorder $(\mathbf{\Delta}^0_\omega;\leqslant_W)$ is the ω_1-st solution of the ordinal equation $\omega_1^\varkappa = \varkappa$ Wadge (1984).

The structure of Wadge degrees is known to have the following properties: at the bottom (i.e., zero) level and at the limit levels of uncountable cofinality we have non-self-dual pairs of degrees; at the limit levels of countable cofinality we have self-dual degrees; at successor levels the self-dual degrees and non-self-dual pairs alternate. Remembering Proposition 5.5 we immediately see that the structure of Wadge degrees of finite Borel rank is isomorphic to the structure $\widetilde{\mathbb{T}}_2^\sqcup(\omega)$! This observation makes more plausible our claim that the FH of k-partitions extends the Wadge hierarchy.

To explain this more precisely, we note that it is possible to relate to any $F \in \widetilde{\mathcal{T}}_k^\sqcup(\omega)$ a k-partition A_F of the Baire space in such a way that A_F is Wadge complete in $\mathcal{L}(F)$ where $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+1}(\mathcal{N})\}_{n<\omega}$. Since the proof of this result is too technical for this paper, we postpone it to a subsequent publication and only note that very relevant particular cases are considered in Selivanov (2007a,b, 2011) (in fact, from these papers only the proof for the initial segment $\widetilde{\mathcal{T}}_k^\sqcup(2)$ is easily extracted, while for the general result one has to employ (suitable extensions of) some jump operators from Andretta (2006); Motto Ros (2009) in order to relate A_F to arbitrary $F \in \widetilde{\mathcal{T}}_k^\sqcup(\omega)$. In this way one obtains the following result showing, in particular, that the FH of sets really extends the Wadge hierarchy of sets of finite Borel rank:

Theorem 5.11 For any $F \in \widetilde{\mathcal{T}}_k^{\sqcup}(\omega)$, A_F is Wadge complete in $\mathcal{L}(\mathcal{N}, F)$ where $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+1}(\mathcal{N})\}_{n<\omega}$ and, moreover, $F \leqslant_h G$ iff $\mathcal{L}(\mathcal{N}, F) \subseteq \mathcal{L}(\mathcal{N}, G)$. In fact, $F \mapsto A_F$ induces an isomorphism between $\widetilde{\mathbb{T}}_k^{\sqcup}(\omega)$ and the structure of Wadge degrees of k-partitions of the Baire space of finite Borel rank.

Our definition of the FH of k-partitions is thus a natural extension of the Wadge hierarchy to arbitrary spaces. Interestingly, for the important particular case of quasi-Polish spaces this hierarchy is naturally induced by the Wadge hierarchy of k-partitions of the Baire space via admissible representations. This follows immediately from Corollary 5.10 and maybe considered as an alternative definition of the FH of k-partitions: Let X be a quasi-Polish space, $\mathcal{L} = \{\mathbf{\Sigma}^0_{n+1}(\mathcal{N})\}_{n<\omega}$, $\mathcal{M} = \{\mathbf{\Sigma}^0_{n+1}(X)\}_{n<\omega}$, and $T \in \widetilde{\mathcal{T}}_k(\omega)$. Then $A \in \mathcal{M}(X, T)$ iff $A \circ \delta \in \mathcal{L}(\mathcal{N}, T)$ where δ is some (equivalently, any) admissible TR of X.

As is well known, the structure of Wadge degrees in many natural non-zero-dimensional cb$_0$-spaces is very complicated (see e.g. Motto Ros et al. 2015 and references therein) so it seems hopeless to understand these structures completely. Nevertheless, if we slightly weaken the notion of Wadge reducibility by extending the class of reducing functions (this is similar to the relativization process in Computability Theory) we obtain natural versions of Wadge reducibility which behave similar to the classical one in many natural spaces. This also applies to reducibilities of k-partition. We illustrate this with the following assertion:

Proposition 5.12 Let X be a quasi-Polish space such that $dim(X) \neq \infty$. Then the structure of $\mathbf{\Sigma}^0_3$-degrees of k-partitions of X of finite Borel rank is isomorphic to $\widetilde{\mathcal{T}}_k^{\sqcup}(\omega)$.

Proof hint. By Proposition 3.1, there is a $\mathbf{\Sigma}^0_3$-isomorphism f between X and $\mathcal{N}$. Clearly, f induces an isomorphism of the quotient-structures of $(k^X; \leqslant_{\mathbf{\Sigma}^0_3})$ and $(k^{\mathcal{N}}; \leqslant_{\mathbf{\Sigma}^0_3})$ which preserves the initial segments of k-partitions of finite Borel rank. Therefore, it suffices to prove the assertion for $X = \mathcal{N}$. But this is just the $\mathbf{\Sigma}^0_3$-relativization of Theorem 5.11. Note that similar relativizations are employed in Motto Ros (2009) and, in the context of Computability Theory, in Selivanov (1983). □

6. Conclusion

The results of this paper suggest that DST in cb$_0$-spaces (or at least in some rich classes of cb$_0$-spaces like $\mathsf{CB}_0(\mathbf{\Delta}^1_1)$) resembles in many respects the classical DST in Polish spaces. Also, the methods of classical DST seem also to work well in this context, although a more systematic treatment of DST in cb$_0$-spaces is desirable. In particular, the classical theory of equivalence relations and descriptive theory of functions on cb$_0$-spaces (more complicated than the k-partitions considered here) seem to be interesting.

Of course, many details about the FH of k-partitions in cb$_0$-spaces should be elaborated much more carefully than in this paper (here we have only given precise definitions and formulations and very short proof hints). In fact, even the structure of Wadge degrees of Borel k-partitions of the Baire space should be described much more carefully; we plan to do this in subsequent publications (of course one cannot expect to fulfil this task in a short single paper because even the much easier case of sets is technically very involved

(Van Wesep 1976; Wadge 1984), and the game-theoretic technique does not apply to non-zero-dimensional spaces and to the case of k-partitions in the Baire space (at least, in a straightforward way)). The results of this paper suggest that the FH of k-partitions in arbitrary quasi-Polish spaces, and even in more general cb_0-spaces, is nevertheless tractable.

A special challenge is the systematic development of DST in non-countably based spaces, in particular, in reasonable rich enough classes of qcb_0-spaces. This task could require new methods compared with the case of cb_0-spaces.

Another interesting direction is the development of effective DST in effective (in some reasonable sense) spaces. As is well known from Computability Theory, this task is highly non-trivial even for 'simple' spaces like the Baire space. For topologically more complicated spaces this direction is still widely open, although it seems of principal importance, in particular for Computable Analysis.

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