

Diffeomorphism cocycles over partially hyperbolic systems

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Abstract. We consider Hölder continuous cocycles over an accessible partially hyperbolic system with values in the group of diffeomorphisms of a compact manifold \mathcal{M} . We obtain several results for this setting. If a cocycle is bounded in $C^{1+\gamma}$, we show that it has a continuous invariant family of γ -Hölder Riemannian metrics on \mathcal{M} . We establish continuity of a measurable conjugacy between two cocycles assuming bunching or existence of holonomies for both and pre-compactness in C^0 for one of them. We give conditions for existence of a continuous conjugacy between two cocycles in terms of their cycle weights. We also study the relation between the conjugacy and holonomies of the cocycles. Our results give arbitrarily small loss of regularity of the conjugacy along the fiber compared to that of the holonomies and of the cocycle.

Key words: Cocycle, diffeomorphism group, partially hyperbolic system, conjugacy, isometry

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1. *Introduction and statement of the results*

Cohomology of group-valued cocycles over hyperbolic and, later, partially hyperbolic systems has been extensively studied, starting with the work of Livšič [Liv71, Liv72], who obtained definitive results for commutative groups and some results for more general groups. The theory has many applications to rigidity of hyperbolic and partially hyperbolic systems and actions. The case of non-commutative groups such as $GL(n, \mathbb{R})$ is more complicated, and groups of diffeomorphisms present further difficulties. The study of diffeomorphism-valued cocycles over hyperbolic systems was started in [NT95, NT96], and continued in [NT98, KtN07, dILW10, dILW11, ASV13, BaK15, KP16, AKL18, S19, DX20].

In this paper we study cohomology of diffeomorphism-valued cocycles over accessible partially hyperbolic systems. The central question in this area is existence and regularity of a conjugacy, or transfer map, between two cocycles. One of our main theorems gives conditions for continuity of a measurable conjugacy between two cocycles. Such a result is new even for hyperbolic systems in the base. It yields, in particular, continuity of a measurable conjugacy to the identity for any bunched cocycle. Another theorem shows that bounded cocycles are isometric, which extends our recent results in [S19] to partially hyperbolic systems. We also give conditions for existence of a conjugacy between two arbitrary cocycles in terms of their cycle weights. Results of this type were established in [KtN07] for cohomology to a constant cocycle, and used to obtain certain cocycle rigidity for higher-rank hyperbolic abelian group actions in [KtN07, DX20].

In this paper we study cocycles depending Hölder continuously on the base point. When the dependence on the base point is smooth, one can apply the theory of smooth partially hyperbolic systems to the skew product, as in [NT98, KtN07, DX20]. Our approach is different, and an important role in the arguments is played by the *holonomies* of the cocycles and their relation with a conjugacy. We also use results from [ASV13] on continuity of invariant sections of fiber bundles over partially hyperbolic systems. In our setup, it is important to consider the regularity of the conjugacy along the fiber, which may be lower than that of the cocycles. For a conjugacy to the identity cocycle, the regularity can be bootstrapped to that of the cocycle [dILW11], but there are no such results for two general cocycles. We obtain a conjugacy almost as regular as the holonomies of the cocycle, which, in turn are almost as regular as the cocycle satisfying sufficient bunching.

We now formulate the main definitions and results.

1.1. Basic definitions. Let X be a compact connected manifold. A diffeomorphism f of X is *partially hyperbolic* if there exist a non-trivial Df -invariant splitting of the tangent bundle $TX = E^s \oplus E^c \oplus E^u$, a Riemannian metric on X , and positive continuous functions $\lambda < 1$, $\hat{\lambda} < 1$, ξ , $\hat{\xi}$ such that for any $x \in X$ and any unit vectors $\mathbf{v}^s \in E^s(x)$, $\mathbf{v}^c \in E^c(x)$, and $\mathbf{v}^u \in E^u(x)$,

$$\|Df_x(\mathbf{v}^s)\| < \lambda(x) < \xi(x) < \|Df_x(\mathbf{v}^c)\| < \hat{\xi}(x)^{-1} < \hat{\lambda}(x)^{-1} < \|Df_x(\mathbf{v}^u)\|. \quad (1.1)$$

The sub-bundles E^s , E^u , and E^c are called stable, unstable, and center. E^s and E^u are tangent to the stable and unstable foliations W^s and W^u , respectively. If the center bundle is trivial, f is called *Anosov*.

The diffeomorphism f is *center bunched* if the functions λ , $\hat{\lambda}$, ξ , $\hat{\xi}$ can be chosen so that $\lambda < \xi \hat{\xi}$ and $\hat{\lambda} < \xi \hat{\xi}$.

An *su-path* in X is a concatenation of finitely many subpaths which lie entirely in a single leaf of W^s or W^u . The diffeomorphism f is called *accessible* if any two points in X can be connected by an *su-path*.

We say that f is *volume-preserving* if it has an invariant probability measure μ in the measure class of a volume induced by a Riemannian metric. It was proved in [BW10]

that any essentially accessible center bunched C^2 partially hyperbolic diffeomorphism f is ergodic with respect to such μ .

Definition 1.1. Let f be a homeomorphism of a compact metric space X . Let \mathcal{M} be a compact manifold and let A be a function from X to $\text{Diff}^q(\mathcal{M})$. The $\text{Diff}^q(\mathcal{M})$ -valued cocycle over f generated by A is the map $\mathcal{A} : X \times \mathbb{Z} \rightarrow \text{Diff}^q(\mathcal{M})$ defined by $\mathcal{A}(x, 0) = \text{Id}$ and, for $n \in \mathbb{N}$,

$$\mathcal{A}(x, n) = \mathcal{A}_x^n = A(f^{n-1}x) \circ \dots \circ A(x) \quad \text{and} \quad \mathcal{A}(x, -n) = \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1}.$$

Clearly, \mathcal{A} satisfies the cocycle equation $\mathcal{A}_x^{n+k} = \mathcal{A}_{f^kx}^n \circ \mathcal{A}_x^k$.

In this paper we consider the group $\text{Diff}^0(\mathcal{M})$ of homeomorphisms of a compact manifold \mathcal{M} and the groups $\text{Diff}^q(\mathcal{M})$, $q \geq 1$, of diffeomorphisms of \mathcal{M} . We denote by $\|\cdot\|_{C^q}$ the usual C^q norm adapted to the manifold setting, set $|g|_{C^q} = \|g\|_{C^q} + \|g^{-1}\|_{C^q}$, and consider a distance d_{C^q} on $\text{Diff}^q(\mathcal{M})$; see §2.1.

We say that a $\text{Diff}^q(\mathcal{M})$ -valued cocycle \mathcal{A} is β -Hölder, $0 < \beta \leq 1$, if there exists a constant $c > 0$ such that

$$d_{C^q}(\mathcal{A}_x, \mathcal{A}_y) \leq c d_X(x, y)^\beta \quad \text{for all } x, y \in X. \tag{1.2}$$

1.2. *Hölder continuous cocycles with bounded set of values are isometric.* The following theorem gives a partially hyperbolic version of [S19, Theorem 1.3], where we considered $\text{Diff}^2(\mathcal{M})$ -valued cocycles with bounded periodic data over hyperbolic systems.

THEOREM 1.2. *Let $f : X \rightarrow X$ be an accessible center bunched C^2 partially hyperbolic diffeomorphism preserving a volume μ .*

Let $0 < \gamma \leq 1$, and let \mathcal{A} be a β -Hölder continuous $\text{Diff}^{1+\gamma}(\mathcal{M})$ -valued cocycle over (X, f) such that the set $\{|\mathcal{A}_x^n|_{C^{1+\gamma}} : x \in X, n \in \mathbb{Z}\}$ is bounded. Then there exists a family of Riemannian metrics $\{\tau_x : x \in X\}$ on \mathcal{M} such that:

- (a) $\mathcal{A}_x : (\mathcal{M}, \tau_x) \rightarrow (\mathcal{M}, \tau_{fx})$ is an isometry for each $x \in X$;
- (b) each τ_x is γ -Hölder continuous on \mathcal{M} , and
- (c) τ_x depends continuously on x in C^α distance for each $0 < \alpha < \gamma$;
- (d) for each $0 < \alpha < \gamma$, τ_x depends Hölder continuously on x along the leaves of W^s and W^u in C^α distance with exponent $\beta(\gamma - \alpha)$.

If (X, f) is a hyperbolic system and a cocycle \mathcal{A} is as above, then, additionally, for each $0 < \alpha < \gamma$ the metric τ_x depends Hölder continuously on $x \in X$ in C^α distance with exponent $\beta(\gamma - \alpha)$.

In the theorem above, by a hyperbolic system we mean either a transitive Anosov diffeomorphism, or a mixing diffeomorphism of a locally maximal hyperbolic set, or a mixing subshift of finite type; see [S19, §2.1] or [KtH] for definitions.

1.3. *Existence and regularity of holonomies.* An important role in the study of non-commutative cocycles, and in particular diffeomorphism-valued cocycles, is played by their holonomies. For example, in Theorem 1.2, essential invariance of τ under holonomies helps to obtain its regularity.

Definition 1.3. Let (X, f) be a hyperbolic or partially hyperbolic system, and let \mathcal{A} be a $\text{Diff}^q(\mathcal{M})$ -valued cocycle over (X, f) . We say that \mathcal{A} has *stable holonomies* $H_{x,y}^{\mathcal{A},s}$ in $\text{Diff}^r(\mathcal{M})$ if

$$H_{x,y}^{\mathcal{A},s} = \lim_{n \rightarrow +\infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \text{ exists in } \text{Diff}^r(\mathcal{M}) \text{ for every } x \in X \text{ and } y \in W^s(x),$$

and the map $(x, y) \mapsto H_{x,y}^{\mathcal{A},s}$ into $\text{Diff}^r(\mathcal{M})$ is continuous on the set of pairs (x, y) where $x \in X$ and $y \in W_{\text{loc}}^s(x)$. If $r = 0$, we also require that the homeomorphisms $H_{x,y}^{\mathcal{A},s}$ are Hölder continuous with uniform exponent and constant for all such pairs (x, y) .

The *unstable holonomies* $H_{x,y}^{\mathcal{A},u}$ are similarly defined as

$$H_{x,y}^{\mathcal{A},u} = \lim_{n \rightarrow -\infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \text{ where } y \in W^u(x).$$

We say that \mathcal{A} has holonomies if it has both stable and unstable ones.

Clearly, $H_{x,x}^{\mathcal{A},s/u} = \text{Id}$ for every $x \in X$. We say that the stable holonomies of \mathcal{A} are β' -Hölder along the stable leaves if, for some $c_1 > 0$,

$$d_{C^r}(H_{x,y}^{\mathcal{A},s}, \text{Id}) \leq c_1 d_X(x, y)^{\beta'} \text{ for all } x \in X \text{ and } y \in W_{\text{loc}}^s(x), \tag{1.3}$$

Hölder continuity of $H^{\mathcal{A},u}$ along the unstable leaves is defined similarly.

Existence of holonomies for cocycles has been extensively studied. We summarize the results for homeomorphism- and diffeomorphism-valued cocycles. They show that Hölder continuity and certain bunching, or domination, assumptions on the cocycle imply existence of its holonomies and their Hölder continuity along the stable and unstable leaves. We formulate results for stable holonomies. The statements for unstable holonomies are similar, with $\hat{\lambda}$ in place of λ .

Let (X, f) be a hyperbolic or partially hyperbolic system, and let

$$\lambda = \max \{ \lambda(x) : x \in X \} \text{ where } \lambda(x) \text{ is as in (1.1)}. \tag{1.4}$$

Let \mathcal{A} be a β -Hölder continuous $\text{Diff}^q(\mathcal{M})$ -valued cocycle, $q \geq 1$, and let

$$\sigma = \max_{x \in X} \max \{ \|D\mathcal{A}_x\|, \|D\mathcal{A}_x^{-1}\| \} \text{ where } \|D\mathcal{A}_x\| = \max_{t \in \mathcal{M}} \|D_t \mathcal{A}_x\|. \tag{1.5}$$

(E1) [KtN07, Proposition 3.3], [ASV13, Proposition 3.10], [Proposition 3.1]. If $q = 1$ and $\sigma \lambda^\beta < 1$, then \mathcal{A} has stable holonomies in $\text{Diff}^0(\mathcal{M})$, and they are β -Hölder along the leaves of W^s . Instead of σ as in (1.5), we can take σ such that, for some constant K ,

$$\|D\mathcal{A}_x^n\| \leq K \sigma^{|n|} \text{ for every } x \in X \text{ and } n \in \mathbb{Z}.$$

(E2) [BaK15, Proposition 3.1] If $2 \leq q \in \mathbb{N}$ such that $\sigma^{2q-1} \lambda^\beta < 1$, then \mathcal{A} has stable holonomies in $\text{Diff}^{q-1}(\mathcal{M})$, and they are β -Hölder along the leaves of W^s .

(E3) [Propositions 3.3] If $k \leq r < q < k + 1$, where $k \in \mathbb{N}$, and there exist η and K such that $\eta^{2(r+1)/(q-r)} \cdot \lambda^\beta < 1$ and $|\mathcal{A}_x^n|_{C^q} \leq K \eta^n$ for all $x \in X$ and $n \in \mathbb{N}$, then \mathcal{A} has stable holonomies in $\text{Diff}^r(\mathcal{M})$, and they are $\beta(q - r)$ -Hölder along the leaves of W^s .

(E3') [Remark 3.4] If $k \leq r < q < k + 1$, where $k \in \mathbb{N}$, \mathcal{A} is a $\text{Diff}^{k+1}(\mathcal{M})$ -valued cocycle with bounded $|\mathcal{A}_x|_{C^{k+1}}$, and $\sigma^{2(r+1)(k+1)/(q-r)} \cdot \lambda^\beta < 1$, then \mathcal{A} has stable

holonomies in $\text{Diff}^r(\mathcal{M})$, and they are $\beta(q - r)$ -Hölder continuous along the leaves of W^s .

1.4. *Continuity of a measurable conjugacy between two cocycles.* We consider a conjugacy, or transfer map, between two cocycles. If it exists, the cocycles are called *cohomologous*.

Definition 1.4. Let \mathcal{A} and \mathcal{B} be $\text{Diff}^q(\mathcal{M})$ -valued cocycles over (X, f) . A conjugacy between \mathcal{A} and \mathcal{B} is a function $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ such that

$$\mathcal{A}_x^n = \Phi_{f^n x} \circ \mathcal{B}_x^n \circ \Phi_x^{-1} \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in X, \tag{1.6}$$

or equivalently, $\mathcal{A}_x = \Phi_{f x} \circ \mathcal{B}_x \circ \Phi_x^{-1}$ for all $x \in X$.

A conjugacy can be considered in various regularities, for example continuous, Hölder continuous, or measurable. In the latter case we understand that Φ is defined and satisfies (1.6) on a set of full measure.

A key step in proving regularity of a measurable conjugacy is showing that it intertwines the holonomies of the cocycles.

Definition 1.5. Let \mathcal{A} and \mathcal{B} be cocycles with holonomies, and let Φ be a conjugacy between them. We say that Φ *intertwines the holonomies* of \mathcal{A} and \mathcal{B} on a set $Y \subseteq X$ if

$$H_{x,y}^{\mathcal{A},s/u} = \Phi_y \circ H_{x,y}^{\mathcal{B},s/u} \circ \Phi_x^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W^{s/u}(x). \tag{1.7}$$

In the following theorem we establish continuity of a measurable conjugacy between cocycles over partially hyperbolic diffeomorphisms. If the holonomies of the cocycles are Hölder continuous along the leaves of W^s and W^u , as we have in (E1)–(E3'), then the conjugacy is also Hölder continuous along the leaves. We note that without suitable assumptions a measurable conjugacy may not be continuous even if f is an Anosov diffeomorphism and the cocycles are linear, close to identity, and one of them is constant [PWa01, §9].

THEOREM 1.6. *Let $f : X \rightarrow X$ be an accessible center bunched C^2 partially hyperbolic diffeomorphism preserving a volume μ . Let \mathcal{A} and \mathcal{B} be $\text{Diff}^q(\mathcal{M})$ -valued cocycles over (X, f) . Suppose that the set $\{\mathcal{B}_x^n : x \in X, n \in \mathbb{Z}\}$ has compact closure in $\text{Diff}^0(\mathcal{M})$ and that \mathcal{A} and \mathcal{B} have holonomies in $\text{Diff}^r(\mathcal{M})$, where either $r = 0$ or $1 \leq r \leq q$.*

- (a) *Let $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ be a μ -measurable conjugacy between \mathcal{A} and \mathcal{B} . Then Φ coincides on a set of full measure with a bounded conjugacy $\tilde{\Phi} : X \rightarrow \text{Diff}^r(\mathcal{M})$ which intertwines the holonomies of \mathcal{A} and \mathcal{B} . The function $\tilde{\Phi} : X \rightarrow \text{Diff}^p(\mathcal{M})$ is continuous for $p = r$ if r is an integer, and any $p < r$ otherwise.*
- (b) *Suppose that $r > 1$ and the stable and unstable holonomies of \mathcal{A} and \mathcal{B} are β' -Hölder along the stable and unstable leaves respectively in the sense of (1.3). Then the conjugacy $\tilde{\Phi} : X \rightarrow \text{Diff}^{p'}(\mathcal{M})$ is Hölder continuous along the stable and unstable leaves with exponent $\beta'(r - p')$ for any p' such that $r - 1 \leq p' < r$ if r is an integer, and $\lfloor r \rfloor \leq p' < r$ otherwise.*

By the results on existence of holonomies, instead of assuming existence and Hölder regularity of the holonomies we can assume that \mathcal{A} and \mathcal{B} are Hölder continuous cocycles with suitable bunching as in (E1)–(E3’).

Remark 1.7. In the case of $\mathcal{B} \equiv \text{Id}$, all assumptions on \mathcal{B} are satisfied, and we obtain continuity of a measurable conjugacy to the identity cocycle. Results of this type are often referred to as *measurable Livšic theorems*.

To the best of our knowledge, Theorem 1.6 is the first result of this type for diffeomorphism-valued cocycles even over a hyperbolic system and with $\mathcal{B} \equiv \text{Id}$.

The theorem applies to a volume-preserving Anosov diffeomorphism since it is accessible by the local product structure of the stable and unstable manifolds, and trivially center bunched. In this case we also obtain Hölder continuity of Φ on X .

COROLLARY 1.8. *If f in Theorem 1.6 is an Anosov diffeomorphism, $r > 1$, and the stable and unstable holonomies of \mathcal{A} and \mathcal{B} are β' -Hölder along the stable and unstable leaves respectively, then $\tilde{\Phi} : X \rightarrow \text{Diff}^{p'}(\mathcal{M})$ is Hölder continuous on X with exponent $\beta'(r - p')$ for any p' as in the theorem.*

1.5. *Existence and properties of a conjugacy intertwining holonomies.* We begin with the result that a Hölder continuous conjugacy between sufficiently bunched cocycles intertwines their holonomies.

PROPOSITION 1.9. *Let (X, f) be a hyperbolic or partially hyperbolic system, and let λ be as in (1.4). Let \mathcal{A} and \mathcal{B} be $\text{Diff}^1(\mathcal{M})$ -valued cocycles over (X, f) so that $\mathcal{A}_x, \mathcal{B}_x : X \rightarrow \text{Diff}^0(\mathcal{M})$ are β -Hölder continuous. If there exist constants K and σ such that*

$$\sigma\lambda^\beta < 1 \quad \text{and} \quad \|D\mathcal{A}_x^n\| \leq K\sigma^{|n|}, \quad \|D\mathcal{B}_x^n\| \leq K\sigma^{|n|} \quad \text{for every } x \in X \text{ and } n \in \mathbb{Z},$$

then \mathcal{A} and \mathcal{B} have stable holonomies in $\text{Diff}^0(\mathcal{M})$, and any β -Hölder continuous conjugacy $\Phi : X \rightarrow \text{Diff}^0(\mathcal{M})$ between \mathcal{A} and \mathcal{B} intertwines the stable holonomies.

We note that Hölder continuity of the conjugacy with exponent less than β does not guarantee the intertwining, even for linear cocycles over hyperbolic systems; see [KaS16, Proposition 4.4] based on examples in [dIL92, NT98].

Since intertwining of the holonomies is a ‘pointwise’ property, it suffices to obtain it in the lowest regularity. Once the intertwining is established, further properties of the conjugacy can be obtained, as stated in Proposition 1.10 below. Also, if f is an Anosov diffeomorphism and the cocycles are smooth along the base X , the main result of [NT98] can be used to obtain smoothness of an intertwining conjugacy.

Let $f : X \rightarrow X$ be an accessible partially hyperbolic diffeomorphism, and let \mathcal{A} be a $\text{Diff}^q(\mathcal{M})$ -valued cocycle over (X, f) with the stable and unstable holonomies $H^{\mathcal{A},s}$ and $H^{\mathcal{A},u}$. An *su*-cycle in X is a closed *su*-path, which we view as a sequence of points

$$P = P_{x_0} = \{x_0, x_1, \dots, x_{k-1}, x_k = x_0\} \quad \text{where } x_{i+1} \in W^{s/u}(x_i), \quad i = 0, \dots, k - 1.$$

We define the *cycle weight* of P as

$$\mathcal{H}_{x_0}^{A,P} = H_{x_{k-1},x_k} \circ \dots \circ H_{x_1,x_2} \circ H_{x_0,x_1}, \tag{1.8}$$

where $H_{x_i,x_{i+1}} = H_{x_i,x_{i+1}}^{A,s/u}$ if $x_{i+1} \in W^{s/u}(x_i)$. One can similarly consider the weight $\mathcal{H}_{x_0,x_k}^{A,P}$ for an su -path $P = P_{x_0,x_k}$ from x_0 to x_k .

PROPOSITION 1.10. *Let $f : X \rightarrow X$ be an accessible C^1 partially hyperbolic diffeomorphism. Let \mathcal{A} and \mathcal{B} be $\text{Diff}^0(\mathcal{M})$ -valued cocycles over (X, f) with the stable and unstable holonomies $H^{A,s/u}$ and $H^{B,s/u}$ in $\text{Diff}^0(\mathcal{M})$. Let $\Phi : X \rightarrow \text{Diff}^0(\mathcal{M})$ be any conjugacy between \mathcal{A} and \mathcal{B} which intertwines their holonomies. Then the following statements hold.*

- (a) Φ conjugates cycle weights, that is, $\mathcal{H}_x^{A,P_x} = \Phi_x \circ \mathcal{H}_x^{B,P_x} \circ \Phi_x^{-1}$ for any su -cycle P_x .
- (b) More generally, $\mathcal{H}_{x,y}^{A,P} = \Phi_y \circ \mathcal{H}_{x,y}^{B,P} \circ \Phi_x^{-1}$ for any su -path $P = P_{x,y}$.
- (c) Φ is uniquely determined by its value at one point.
- (d) $\Phi : X \rightarrow \text{Diff}^0(\mathcal{M})$ is continuous.
- (e) If for some $r \geq 1$, $H^{A,s/u}$ and $H^{B,s/u}$ are in $\text{Diff}^r(\mathcal{M})$ and $\Phi_{x_0} \in \text{Diff}^r(\mathcal{M})$ for some $x_0 \in X$, then Φ is a bounded function from X to $\text{Diff}^r(\mathcal{M})$, and $\Phi : X \rightarrow \text{Diff}^p(\mathcal{M})$ is continuous for $p = r$ if r is an integer, and for any $p < r$ otherwise.

The next theorem gives a sufficient condition for existence of a continuous conjugacy intertwining holonomies. By the previous proposition, condition (b) is also necessary.

THEOREM 1.11. *Let $f : X \rightarrow X$ be an accessible C^1 partially hyperbolic diffeomorphism. Let \mathcal{A} and \mathcal{B} be $\text{Diff}^q(\mathcal{M})$ -valued cocycles over (X, f) with holonomies in $\text{Diff}^r(\mathcal{M})$, where $r = 0$ or $1 \leq r \leq q$.*

- (a) Suppose that there exist a fixed point $x_0 \in X$ and $\Phi_{x_0} \in \text{Diff}^r(\mathcal{M})$ such that
 - (a1) $\mathcal{H}_{x_0}^{A,P} = \Phi_{x_0} \circ \mathcal{H}_{x_0}^{B,P} \circ \Phi_{x_0}^{-1}$ for every su -cycle P_{x_0} , and
 - (a2) $\mathcal{A}_{x_0} = \Phi_{x_0} \circ \mathcal{B}_{x_0} \circ \Phi_{x_0}^{-1}$.
- (b) More generally, suppose that there exist $x_0 \in X$ and $\Phi_{x_0} \in \text{Diff}^r(\mathcal{M})$ satisfying
 - (b1) = (a1) and
 - (b2) $\mathcal{A}_{x_0} = \Phi_{f x_0} \circ \mathcal{B}_{x_0} \circ \Phi_{x_0}^{-1}$, where $\Phi_{f x_0} = \mathcal{H}_{x_0, f x_0}^{A, \tilde{P}} \circ \Phi_{x_0} \circ (\mathcal{H}_{x_0, f x_0}^{B, \tilde{P}})^{-1}$ for some su -path $\tilde{P} = \tilde{P}_{x_0, f x_0}$ from x_0 to $f x_0$.

Then there exists a unique conjugacy Φ between \mathcal{A} and \mathcal{B} with value Φ_{x_0} at x_0 that intertwines H^A and H^B . The function $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ is bounded and $\Phi : X \rightarrow \text{Diff}^p(\mathcal{M})$ is continuous for $p = r$ if r is an integer, and for any $p < r$ otherwise.

Case (a) can be viewed as a sufficient condition for extending a conjugacy from a given value at a fixed point. The value $\Phi_{f x_0}$ in (b2) does not depend on the choice of a path $P_{x_0, f x_0}$ due to the first assumption. If x_0 is a fixed point for f , then for the trivial path from x_0 to $f x_0 = x_0$ the condition in (b2) becomes $\mathcal{A}_{x_0} = \Phi_{x_0} \circ \mathcal{B}_{x_0} \circ \Phi_{x_0}^{-1}$, so (b) indeed generalizes (a).

As a corollary of Theorem 1.11(b) we obtain the following result on conjugacy to a constant cocycle. A similar result was established in [KtN07, Proposition 5.6] for cocycles

which depend smoothly on the base point over partially hyperbolic systems satisfying a stronger accessibility assumption.

COROLLARY 1.12. *Let $f : X \rightarrow X$ be an accessible C^1 partially hyperbolic diffeomorphism. Let \mathcal{A} be $\text{Diff}^q(\mathcal{M})$ -valued cocycle over (X, f) with holonomies in $\text{Diff}^r(\mathcal{M})$, where $r = 0$ or $1 \leq r \leq q$. Suppose that*

$$\mathcal{H}_{x_0}^{\mathcal{A}, P_{x_0}} = \text{Id} \quad \text{for every } su\text{-cycle } P_{x_0} \text{ based at some point } x_0 \in X. \tag{1.9}$$

Then there exists a bounded conjugacy $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ between \mathcal{A} and a constant cocycle such that $\Phi : X \rightarrow \text{Diff}^p(\mathcal{M})$ is continuous for $p = r$ if r is an integer, and for any $p < r$ otherwise, and Φ satisfies

$$\Phi_y \circ \Phi_x^{-1} = H_{x,y}^{\mathcal{A}, s/u} \quad \text{for all } x, y \in X \text{ such that } y \in W^{s/u}(x). \tag{1.10}$$

In particular, if $\mathcal{A}_{x_0} = \text{Id}$ at a fixed point x_0 and (1.9) holds, then \mathcal{A} is conjugate to the identity cocycle via Φ as above with $\Phi(x_0) = \text{Id}$.

If condition (1.9) holds for some $x_0 \in X$, then it holds for every $x \in X$, since by accessibility for any su -cycle based at x one can consider a corresponding su -cycle based at x_0 .

We note that a constant cocycle conjugate to \mathcal{A} is not unique in general. Also, if \mathcal{A} is conjugate to a constant cocycle via a conjugacy intertwining their holonomies, then (1.10) holds and (1.9) follows.

This paper is organized as follows. In §2 we define distances on the spaces $\text{Diff}^r(\mathcal{M})$ and give estimates for norms and distances between compositions of diffeomorphisms. In §3 we formulate and prove results on existence and properties of holonomies of $\text{Diff}^q(\mathcal{M})$ -valued cocycles. In §4 we prove Theorem 1.2; in §5 we prove Proposition 1.10, Theorem 1.6 and Corollary 1.8; and in §6 we prove Proposition 1.9, Theorem 1.11, and Corollary 1.12.

2. Distances on $\text{Diff}^r(\mathcal{M})$ and estimates

2.1. *Distances on the space of diffeomorphisms $\text{Diff}^r(\mathcal{M})$.* This subsection draws on [dlw10, §5] and [S19, §2.2].

We fix a smooth background Riemannian metric and the corresponding distance $d_{\mathcal{M}}$ on \mathcal{M} .

We denote the space of homeomorphisms of \mathcal{M} by $\text{Diff}^0(\mathcal{M})$, and for $g, h \in \text{Diff}^0(\mathcal{M})$ we set

$$d_0(g, h) = \max_{t \in \mathcal{M}} d_{\mathcal{M}}(g(t), h(t)) \quad \text{and} \quad d_{C^0}(g, h) = d_0(g, h) + d_0(g^{-1}, h^{-1}). \tag{2.1}$$

We consider $r \geq 1$. The C^r topology on the group of diffeomorphisms $\text{Diff}^r(\mathcal{M})$ can be defined using coordinate patches and the C^r norm in the Euclidean space. For any $g \in \text{Diff}^r(\mathcal{M})$, $r \in \mathbb{N} = \{1, 2, \dots\}$, we define its C^r size as

$$\|g\|_{C^r} = \|g\|_{C^r} + \|g^{-1}\|_{C^r} \quad \text{where} \quad \|g\|_{C^r} = \max_{t \in \mathcal{M}} d_{\mathcal{M}}(g(t), t) + \max_{1 \leq i \leq r} \max_{t \in \mathcal{M}} \|D_t^i g\|,$$

where $D_t^i g$ is the derivative of g of order i at t , and its norm is defined as the norm of the corresponding multilinear form from $T_t \mathcal{M}$ to $T_{g(t)} \mathcal{M}$ with respect the Riemannian metric. For $r = k + \alpha$ with $k \in \mathbb{N}$ and $0 < \alpha \leq 1^-$, where $\alpha = 1^-$ means Lipschitz, the definition is similar, with

$$\|g\|_{C^r} = \|g\|_{C^k} + \sup\{\|D_t^k g - D_{t'}^k g\| \cdot d_{\mathcal{M}}(t, t')^{-\alpha} : t, t' \in \mathcal{M}, 0 < d_{\mathcal{M}}(t, t') < \epsilon_0\}.$$

Here ϵ_0 is chosen small compared to the injectivity radius of \mathcal{M} so that the tangent bundle is locally trivialized via parallel transport and the difference makes sense.

A natural distance $d_{C^r}(g, h)$ on $\text{Diff}^r(\mathcal{M})$, $r \in \mathbb{N}$, was defined in [dILW10] as the infimum of the lengths of piecewise C^1 paths in $\text{Diff}^r(\mathcal{M})$ connecting h with g and h^{-1} with g^{-1} , where the length of a path p_s is

$$\ell_{C^r}(p_s) = \max_{0 \leq i \leq r} \max_{t \in \mathcal{M}} \int \left\| \frac{d}{ds} (D_t^i p_s) \right\| ds.$$

For $r = k + \alpha$, one also adds the corresponding Hölder term:

$$\ell_{C^r}(p_s) = \ell_{C^k}(p_s) + \max_{t \in \mathcal{M}} \int \left| \frac{d}{ds} \|(D^k p_s)\|_{\alpha,t} \right| ds \quad \text{where}$$

$$\|D^k g\|_{\alpha,t} = \sup\{\|D_{t'}^k g - D_t^k g\| \cdot d_{\mathcal{M}}(t', t)^{-\alpha} : t' \in \mathcal{M}, 0 < d_{\mathcal{M}}(t, t') < \epsilon_0\}.$$

For sufficiently C^r -close diffeomorphisms, the distance $d_{C^r}(g, h)$ is Lipschitz equivalent to $\|g - h\|_{C^r} + \|g^{-1} - h^{-1}\|_{C^r}$, where the difference is understood using local trivialization. Specifically, there exist constants κ and $\delta_0 > 0$ depending only on r and the Riemannian metric so that

$$\begin{aligned} \kappa^{-1} d_{C^r}(g, h) &\leq \|g - h\|_{C^r} + \|g^{-1} - h^{-1}\|_{C^r} \leq \kappa d_{C^r}(g, h) \quad \text{provided that} \\ \text{either } d_{C^r}(g, h) &< \delta_0 \|g\|_{C^r}^{-1} \quad \text{or } \|g - h\|_{C^r} + \|g^{-1} - h^{-1}\|_{C^r} < \delta_0 \|g\|_{C^r}^{-1}. \end{aligned} \tag{2.2}$$

2.2. *Estimates of norms and distances.* Lemma 2.1 follows directly from [dILO98, Proposition 5.5], and Lemma 2.2 relies on further results in that paper.

LEMMA 2.1. [dILO98] *For any $r \geq 1$ there exists a constant M_r such that, for any $h, g \in C^r(\mathcal{M})$,*

$$\|h \circ g\|_{C^r} \leq M_r \|h\|_{C^r} (1 + \|g\|_{C^r})^r.$$

LEMMA 2.2. [S19, Lemma 3.6] *Let $q = k + \gamma$, $r = k + \alpha$, and $\rho = q - r$, where $k \in \mathbb{N}$ and $0 \leq \alpha < \gamma \leq 1^-$. There exists a constant $M = M(r, q, \mathcal{M}, K)$ such that, for any $g, \tilde{g} \in \text{Diff}^q(\mathcal{M})$ and $h_1, h_2 \in \text{Diff}^r(\mathcal{M})$ with $\|h_1\|_{C^r}, \|h_2\|_{C^r} \leq K$,*

$$\begin{aligned} d_{C^r}(g \circ h_1 \circ \tilde{g}, g \circ h_2 \circ \tilde{g}) \\ \leq M (\|g\|_{C^q} (1 + \|\tilde{g}\|_{C^r})^r + \|\tilde{g}^{-1}\|_{C^q} (1 + \|g^{-1}\|_{C^r})^r) \cdot d_{C^r}(h_1, h_2)^\rho \end{aligned} \tag{2.3}$$

provided that $d_{C^r}(h_1, h_2) \leq \delta_0 \|h_1\|_{C^r}^{-1}$ and the right-hand side of (2.3) is less than

$$\delta_0 (M_r^2 (1 + \|h_1\|_{C^r})^r (\|g\|_{C^r} (1 + \|\tilde{g}\|_{C^r})^r + \|\tilde{g}^{-1}\|_{C^r} (1 + \|g^{-1}\|_{C^r})^r))^{-1}, \tag{2.4}$$

where δ_0 is as in (2.2).

3. Existence and properties of the holonomies

In this section we state and prove some results on existence and properties of the holonomies, first in $\text{Diff}^0(\mathcal{M})$ and then in $\text{Diff}^r(\mathcal{M})$, $r \geq 1$. We formulate the results for stable holonomies. We note that we only consider pairs of points lying on the same stable leaf, and so only the contracting property of the stable leaves plays a role. The statements and proofs for the unstable holonomies are similar.

The local stable manifold of x , $W_{\text{loc}}^s(x)$, is a ball in $W^s(x)$ centered at x of a small radius ρ . We choose ρ sufficiently small so that, in particular, for λ as in (1.4),

$$d_X(f^n x, f^n y) \leq \lambda^n d_X(x, y) \quad \text{for all } x \in X, y \in W_{\text{loc}}^s(x), \text{ and } n \in \mathbb{N}. \tag{3.1}$$

Results similar to the next proposition can be found in [KtN07, Proposition 3.3] and [ASV13, Proposition 3.10]. We state and prove the result, including Hölder continuity along the leaves, joint continuity, and uniqueness, under a weaker assumption of Hölder continuity of \mathcal{A}_x as a function into $\text{Diff}^0(\mathcal{M})$ rather than $\text{Diff}^1(\mathcal{M})$.

PROPOSITION 3.1. *Let (X, f) be a hyperbolic or partially hyperbolic system, and let \mathcal{A} be a $\text{Diff}^1(\mathcal{M})$ -valued cocycle over (X, f) so that $\mathcal{A}_x : X \rightarrow \text{Diff}^0(\mathcal{M})$ is β -Hölder continuous. Suppose that there exist constants K and σ such that*

$$\sigma \lambda^\beta < 1 \quad \text{and} \quad \|D\mathcal{A}_x^n\| \leq K \sigma^{|n|} \quad \text{for every } x \in X \text{ and } n \in \mathbb{Z}. \tag{3.2}$$

Then for any $x \in X$ and $y \in W^s(x)$, the limit $H_{x,y}^{\mathcal{A},s} = \lim_{n \rightarrow +\infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$ exists in $\text{Diff}^0(\mathcal{M})$ and satisfies the following statements.

- (H1) $H_{x,x}^{\mathcal{A},s} = \text{Id}$ and $H_{y,z}^{\mathcal{A},s} \circ H_{x,y}^{\mathcal{A},s} = H_{x,z}^{\mathcal{A},s}$, which imply $(H_{x,y}^{\mathcal{A},s})^{-1} = H_{y,x}^{\mathcal{A},s}$.
- (H2) $H_{x,y}^{\mathcal{A},s} = (\mathcal{A}_y^n)^{-1} \circ H_{f^n x, f^n y}^{\mathcal{A},s} \circ \mathcal{A}_x^n$ for all $n \in \mathbb{N}$.
- (H3⁰) There exists a constant c such that

$$d_{C^0}(H_{x,y}^s, \text{Id}) \leq c d_X(x, y)^\beta \quad \text{for all } x \in X \text{ and } y \in W_{\text{loc}}^s(x).$$

- (H4⁰) The map $(x, y) \mapsto H_{x,y}^{\mathcal{A},s}$ into $\text{Diff}^0(\mathcal{M})$ is continuous on the set of pairs (x, y) , where $x \in X$ and $y \in W_{\text{loc}}^s(x)$.
- (H5) The homeomorphisms $H_{x,y}^{\mathcal{A},s}$ are Hölder continuous with uniform exponent and constant for all pairs (x, y) as above.

Moreover, a family of maps $\{H_{x,y}^{\mathcal{A},s} : x \in X, y \in W^s(x)\}$ in $\text{Diff}^0(\mathcal{M})$ satisfying (H2) and (H3⁰) is unique.

With $\lambda = \max_{x \in X} \hat{\lambda}(x)$, a similar result holds for the unstable holonomies that satisfy the following in place of (H2):

$$(H2') \quad H_{x,y}^{\mathcal{A},u} = (\mathcal{A}_y^{-n})^{-1} \circ H_{f^{-n}x, f^{-n}y}^{\mathcal{A},u} \circ \mathcal{A}_x^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Remark 3.2. Holonomies of a cocycle are sometimes defined as any family of homeomorphisms $\tilde{H}_{x,y}^{\mathcal{A},s}$ satisfying properties (H1), (H2), (H4⁰), and (H5) for nonlinear cocycles (see, for example, [ASV13]), and the holonomies $H^{\mathcal{A},s}$ as in Definition 1.3 are referred to as *standard holonomies* to distinguish them [KaS16]. Without condition (H3⁰), uniqueness of holonomies may fail even for linear cocycles, as discussed in [KaS16] after Corollary 4.9.

Proof. The maps $H_{x,y}^s$ are constructed for $x \in X$ and $y \in W_{\text{loc}}^s(x)$, and then extended to the whole leaf $W^s(x)$ by the invariance property (H2). We fix x and $y \in W_{\text{loc}}^s(x)$ and consider the sequence of diffeomorphisms $((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n)_{n \geq 0}$.

For $g_1, g_2 \in \text{Diff}^0(\mathcal{M})$, we consider $d_0(g_1, g_2)$ and $d_{C^0}(g_1, g_2)$ as in (2.1). We write x_n for $f^n x$ and y_n for $f^n y$. Since \mathcal{A} is β -Hölder continuous and (3.1) holds, we obtain

$$\begin{aligned} d_0(\mathcal{A}_{y_n}^{-1} \circ \mathcal{A}_{x_n}, \text{Id}) &= d_0(\mathcal{A}_{y_n}^{-1} \circ \mathcal{A}_{x_n}, \mathcal{A}_{x_n}^{-1} \circ \mathcal{A}_{x_n}) \\ &= d_0(\mathcal{A}_{y_n}^{-1}, \mathcal{A}_{x_n}^{-1}) \leq d_{C^0}(\mathcal{A}_{y_n}, \mathcal{A}_{x_n}) \\ &\leq K_1 d_X(x_n, y_n)^\beta \leq K_1 d_X(x, y)^\beta \cdot \lambda^{n\beta}. \end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned} &d_0((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n, (\mathcal{A}_y^{n+1})^{-1} \circ \mathcal{A}_x^{n+1}) \\ &= d_0((\mathcal{A}_y^n)^{-1} \circ \text{Id} \circ \mathcal{A}_x^n, (\mathcal{A}_y^n)^{-1} \circ (\mathcal{A}_{y_n})^{-1} \circ \mathcal{A}_{x_n} \circ \mathcal{A}_x^n) \\ &= d_0((\mathcal{A}_y^n)^{-1} \circ \text{Id}, (\mathcal{A}_y^n)^{-1} \circ ((\mathcal{A}_{y_n})^{-1} \circ \mathcal{A}_{x_n})) \leq K \sigma^n \cdot d_0(\mathcal{A}_{y_n}^{-1} \circ \mathcal{A}_{x_n}, \text{Id}) \\ &\leq K \sigma^n \cdot K_2 d_X(x, y)^\beta \cdot \lambda^{n\beta} = K_3 d_X(x, y)^\beta \cdot \theta^n \quad \text{where } \theta = \sigma \lambda^\beta < 1. \end{aligned}$$

The same estimate holds for d_0 between the inverses, and so there exists a constant K_4 such that, for all $x \in X$ and $y \in W_{\text{loc}}^s(x)$,

$$d_{C^0}((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n, (\mathcal{A}_y^{n+1})^{-1} \circ \mathcal{A}_x^{n+1}) \leq K_3 d_X(x, y)^\beta \cdot \theta^n.$$

Thus $((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n)$ is a Cauchy sequence in $\text{Diff}^0(\mathcal{M})$, and so it has a limit $H_{x,y}^{\mathcal{A},s}$ there. The convergence is uniform in all (x, y) with $y \in W_{\text{loc}}^s(x)$. For each $n \in \mathbb{N}$, $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$ depends continuously on (x, y) , hence so does the limit $H_{x,y}^{\mathcal{A},s}$ and we obtain (H4⁰).

Properties (H1) and (H2) are easy to verify. For (H3⁰), we note that $(\mathcal{A}_y^0)^{-1} \circ \mathcal{A}_x^0 = \text{Id}$, so for every $n \in \mathbb{N}$ we have

$$\begin{aligned} d_{C^0}((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n, \text{Id}) &\leq \sum_{i=0}^{n-1} d_{C^0}((\mathcal{A}_y^i)^{-1} \circ \mathcal{A}_x^i, (\mathcal{A}_y^{i+1})^{-1} \circ \mathcal{A}_x^{i+1}) \\ &\leq K_3 d_X(x, y)^\beta \cdot \sum_{i=0}^{\infty} \theta^i \leq c d_X(x, y)^\beta, \quad \text{and hence } d_{C^0}(H_{x,y}^s, \text{Id}) \leq c d_X(x, y)^\beta. \end{aligned}$$

A proof of (H5) is given in [ASV13, Proposition 3.10], and it uses only the assumptions of this proposition.

Finally, we prove uniqueness. Let $H = \{H_{x,y}\}$ and $\tilde{H} = \{\tilde{H}_{x,y}\}$ be two such families. By property (H2) it suffices to verify that $H_{x,y} = \tilde{H}_{x,y}$ for any $x \in X$ and $y \in W_{\text{loc}}^s(x)$. We fix such x and y . Then, using property (H2), assumption (3.2), and property (H3⁰), we obtain

$$\begin{aligned} d_0(H_{x,y}, \tilde{H}_{x,y}) &= d_0((\mathcal{A}_y^n)^{-1} \circ H_{x_n, y_n} \circ \mathcal{A}_x^n, (\mathcal{A}_y^n)^{-1} \circ \tilde{H}_{x_n, y_n} \circ \mathcal{A}_x^n) \\ &= d_0((\mathcal{A}_y^n)^{-1} \circ H_{x_n, y_n}, (\mathcal{A}_y^n)^{-1} \circ \tilde{H}_{x_n, y_n}) \leq K \sigma^n \cdot d_0(H_{x_n, y_n}, \tilde{H}_{x_n, y_n}) \end{aligned}$$

$$\begin{aligned} &\leq K\sigma^n \cdot 2c(d_X(x_n, y_n))^\beta \leq K\sigma^n \cdot 2c(d_X(x, y)\lambda^n)^\beta \\ &= K'(\sigma\lambda^\beta)^n \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

since $\sigma\lambda^\beta < 1$. Thus $H_{x,y} = \tilde{H}_{x,y}$. □

The next proposition establishes existence and regularity of holonomies in $\text{Diff}^r(\mathcal{M})$, where $r \geq 1$.

PROPOSITION 3.3. *Let \mathcal{A} be a β -Hölder $\text{Diff}^q(\mathcal{M})$ -valued cocycle over a hyperbolic or partially hyperbolic system, where $q = k + \gamma$, with $k \in \mathbb{N}$ and $0 < \gamma \leq 1^-$. Let $k \leq r < q$ and let $\rho = q - r$. Suppose that there exist constants η and K such that*

$$\eta^{2(r+1)}\lambda^{\beta\rho} < 1 \quad \text{and} \quad |\mathcal{A}_x^n|_{C^q} \leq K\eta^n \quad \text{for all } x \in X \text{ and } n \in \mathbb{N}. \tag{3.4}$$

Then, for any $x \in X$ and $y \in W^s(x)$, the limit $H_{x,y}^{A,s} = \lim_{n \rightarrow +\infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$ exists in $\text{Diff}^r(\mathcal{M})$ and satisfies (H1) and (H2) as in Proposition 3.1, and the following statements also hold.

(H3^r) *There exists a constant c_1 such that*

$$d_{C^r}(H_{x,y}^{A,s}, \text{Id}) \leq c_1 d_X(x, y)^{\beta\rho} \quad \text{for all } x \in X \text{ and } y \in W_{\text{loc}}^s(x).$$

(H4^r) *The map $(x, y) \mapsto H_{x,y}^{A,s}$ into $\text{Diff}^r(\mathcal{M})$ is continuous on the set of pairs (x, y) , where $x \in X$ and $y \in W_{\text{loc}}^s(x)$.*

Under the same assumptions, a similar result holds for the unstable holonomies.

Parts (H1), (H2), and (H3^r) were established for cocycles over hyperbolic systems in [S19, Proposition 3.3 and Remark 3.4]. The same argument applies in the partially hyperbolic case since it involves only points on a stable manifold. The key estimate is that, for all sufficiently close x, y with $y \in W_{\text{loc}}^s(x)$ and all $n \in \mathbb{N}$,

$$d_{C^r}((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n, (\mathcal{A}_y^{n+1})^{-1} \circ \mathcal{A}_x^{n+1}) \leq K' d_X(x, y)^{\beta\rho} \cdot \tilde{\theta}^n \quad \text{where } \tilde{\theta} < 1.$$

This yields that $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$ converges to $H_{x,y}^{A,s}$ in $\text{Diff}^r(\mathcal{M})$ uniformly in such (x, y) . Hence $H_{x,y}^{A,s}$ depends continuously on (x, y) and we obtain (H4^r).

Remark 3.4. Suppose that $k \in \mathbb{N}$ and \mathcal{A} is a $\text{Diff}^{k+1}(\mathcal{M})$ -valued cocycle with bounded $|\mathcal{A}_x|_{C^{k+1}}$. Then condition (3.4) can be deduced from the bunching assumption that

$$\sigma = \max_{x \in X} \max\{\|D\mathcal{A}_x\|, \|D(\mathcal{A}_x)^{-1}\|\} \text{ satisfies } \sigma^{2(r+1)(k+1)/\rho} \cdot \lambda^\beta < 1.$$

Indeed, By Lemma 5.5 in [dLLW10], there exists a constant c such that

$$\|D^m \mathcal{A}_x^n\| \leq c \sigma^{|m|n|} \quad \text{for every } x \in X \text{ and } 1 \leq m \leq k + 1.$$

Since $|\mathcal{A}_x|_{C^0}$ is bounded, it follows that there exists a constant c' such that

$$|\mathcal{A}_x^n|_{C^q} \leq c' |\mathcal{A}_x^n|_{C^{k+1}} \leq c'' \sigma^{(k+1)n} \quad \text{for every } x \in X \text{ and } n \in \mathbb{N}.$$

If $\sigma^{2(r+1)(k+1)/\rho} \cdot \lambda^\beta < 1$, then $\sigma^{2(r+1)(k+1)} \cdot \lambda^{\beta\rho} < 1$, and so condition (3.4) is satisfied for $\eta = \sigma^{k+1}$.

4. Proof of Theorem 1.2

The first part of the proof is essentially the same for the hyperbolic and partially hyperbolic cases, and it follows the arguments in [S19], so we just outline it. We will give a detailed last part of the proof in the partially hyperbolic case.

Let (X, f) be a hyperbolic or partially hyperbolic system. We consider the vector bundle \mathcal{V} over $X \times \mathcal{M}$ with fiber $\mathcal{V}_{(x,t)} = T_t \mathcal{M}$ and the linear cocycle

$$\mathcal{D}_{(x,t)} = D_t \mathcal{A}_x \text{ on } \mathcal{V} \text{ over the skew product } F(x, t) = (f(x), \mathcal{A}_x(t)).$$

The iterates of \mathcal{D} are given by $\mathcal{D}_{(x,t)}^n : T_t \mathcal{M} \rightarrow T_{\mathcal{A}_x^n(t)} \mathcal{M}$, where $\mathcal{D}_{(x,t)}^n = D_t \mathcal{A}_x^n$. Since the value set of the cocycle \mathcal{A} is bounded in $\text{Diff}^{1+\gamma}(\mathcal{M})$, $\|\mathcal{D}_{(x,t)}^n\|$ is uniformly bounded in $(x, t) \in X \times \mathcal{M}$ and $n \in \mathbb{Z}$, and there exists a constant c_2 such that

$$\|\mathcal{D}_{(x,t)}^n - \mathcal{D}_{(x,t')}^n\| \leq c_2 d_{\mathcal{M}}(t, t')^\gamma \text{ for all nearby } t, t' \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

The space \mathcal{T}^m of inner products on \mathbb{R}^m identifies with the space of real symmetric positive definite $m \times m$ matrices, which is isomorphic to $\text{GL}(m, \mathbb{R})/\text{SO}(m, \mathbb{R})$. The group $\text{GL}(m, \mathbb{R})$ acts transitively on \mathcal{T}^m via $A[E] = A^T E A$. The space \mathcal{T}^m with a certain $\text{GL}(m, \mathbb{R})$ -invariant metric is a Riemannian symmetric space of non-positive curvature [La, Ch. XII, Theorem 1.2]. Using a background Riemannian metric on \mathcal{V} , we identify an inner product with a symmetric linear operator. For each $(x, t) \in X \times \mathcal{M}$, we denote the space of inner products on $\mathcal{V}_{(x,t)}$ by $\mathcal{T}_{(x,t)}$, and so we obtain a bundle \mathcal{T} over $X \times \mathcal{M}$ with fiber $\mathcal{T}_{(x,t)}$. We equip the fibers of \mathcal{T} with the Riemannian metric $d_{\mathcal{T}}$ as above. A measurable (continuous) Riemannian metric on \mathcal{V} is a measurable (continuous) section of \mathcal{T} . A metric τ is called *bounded* if the distance between $\tau_{(x,t)}$ and $\tilde{\tau}_{(x,t)}$ is uniformly bounded on $X \times \mathcal{M}$ for some continuous metric $\tilde{\tau}$ on \mathcal{V} . The pushforward of an inner product $\tau_{(x,t)}$ on $\mathcal{V}_{(x,t)}$ to $\mathcal{V}_{F(x,t)}$ by the linear cocycle \mathcal{D} is given by

$$((\mathcal{D}_{(x,t)})_*(\tau_{(x,t)}))(v_1, v_2) = \tau_{(x,t)}(\mathcal{D}_{(x,t)}^{-1}(v_1), \mathcal{D}_{(x,t)}^{-1}(v_2)) \text{ for } v_1, v_2 \in \mathcal{V}_{F(x,t)}.$$

We say that a metric τ is *\mathcal{D} -invariant* if $\mathcal{D}_*(\tau) = \tau$.

First, [S19, §4.1] gives an everywhere defined bounded measurable section τ of \mathcal{V} invariant under the cocycle \mathcal{D} . Then [S19, Proposition 4.2] yields that for each $x \in X$ the metric τ_x is γ -Hölder continuous on \mathcal{M} ; more precisely, there exists a constant c_3 such that

$$d_{\mathcal{T}}(\tau_{(x,t)}, \tau_{(x,t')}) \leq c_3 d_{\mathcal{M}}(t, t')^\gamma \text{ for all } x \in X \text{ and } t, t' \in \mathcal{M}. \tag{4.1}$$

We take any $0 < \alpha < \gamma$. By Proposition 3.3, boundedness of the set of values of \mathcal{A} in $\text{Diff}^{1+\gamma}(\mathcal{M})$ gives existence and regularity of the stable and unstable holonomies $H^s = H^{\mathcal{A},s}$ and $H^u = H^{\mathcal{A},u}$ in $\text{Diff}^{1+\alpha}(\mathcal{M})$. By property (H3^r) of holonomies, for all $x \in X$ and $y \in W_{\text{loc}}^{s/u}(x)$,

$$d_{C^r}(H_{x,y}^{s/u}, \text{Id}) \leq c_1 d_X(x, y)^{\beta\rho} \text{ where } \rho = \gamma - \alpha. \tag{4.2}$$

We define the stable sets \tilde{W}^s for the map $F(x, t)$ of $X \times \mathcal{M}$ using the stable holonomies. For any $(x, t) \in X \times \mathcal{M}$,

$$\tilde{W}^s(x, t) = \{(y, t') \in X \times \mathcal{M} : y \in W^s(x), t' = H_{x,y}^s(t)\}.$$

These sets satisfy the contraction property $d_{X \times \mathcal{M}}(F^n(x, t), F^n(y, t')) \rightarrow 0$ as $n \rightarrow \infty$ for any $(x, t) \in X \times \mathcal{M}$ and $(y, t') \in \tilde{W}^s(x, t)$. The unstable sets \tilde{W}^u are defined similarly using the unstable holonomies.

The next proposition establishes *essential invariance* of τ under the derivatives of H^s along the stable sets in $X \times \mathcal{M}$. Similar invariance holds for the unstable holonomies.

PROPOSITION 4.1. [S19, Proposition 4.4] *Let ν be an ergodic F -invariant measure on $X \times \mathcal{M}$. If τ is a ν -measurable \mathcal{D} -invariant metric on \mathcal{V} , then there exists an F -invariant set $E \subset X \times \mathcal{M}$ with $\nu(E) = 1$ such that*

$$\tau(y, t') = (D_t H_{x,y}^s)_*(\tau(x, t)) \quad \text{for all } (x, t), (y, t') \in E \text{ with } (y, t') \in \tilde{W}_{\text{loc}}^s(x, t).$$

In the *hyperbolic case*, we denote the *measure of maximal entropy* on X by μ , and in the *partially hyperbolic case* we denote the *invariant volume* on X by μ . Let m_x be the normalized volume induced by the metric τ along the fiber \mathcal{M}_x . We define a measure $\hat{\mu}$ on $X \times \mathcal{M}$ by $\hat{\mu} = \int m_x d\mu(x)$. This measure is F -invariant, but not necessarily ergodic. Applying Proposition 4.1 to its ergodic components, we obtain the following corollary.

COROLLARY 4.2. [S19, Corollary 4.5] *There exists a set $\hat{G} \subset X \times \mathcal{M}$ with $\hat{\mu}(\hat{G}) = 1$ such that τ on \hat{G} is invariant under the holonomies, that is,*

$$\tau(y, t') = (D_t H_{x,y}^s)_*(\tau(y, t)) \quad \text{for all } (x, t) \in \hat{G} \text{ and all } (y, t') \in \hat{G} \cap \tilde{W}_{\text{loc}}^s(x, t).$$

The next proposition gives μ -essential invariance of τ_x , as a Riemannian metric on the whole fiber $\mathcal{M}_x = \mathcal{M}$, under the stable and unstable holonomies of \mathcal{A} .

PROPOSITION 4.3. [S19, Proposition 4.6] *There exists a set $G \subset X$ with $\mu(G) = 1$ such that, for any $x, y, y' \in G$ with $y \in W_{\text{loc}}^s(x)$ and $y' \in W_{\text{loc}}^u(x)$, the diffeomorphisms*

$$H_{x,y}^s : (\mathcal{M}, \tau_x) \rightarrow (\mathcal{M}, \tau_y) \quad \text{and} \quad H_{x,y'}^u : (\mathcal{M}, \tau_x) \rightarrow (\mathcal{M}, \tau_{y'}) \quad \text{are isometries.}$$

We denote by $\mathcal{T}^\alpha(\mathcal{M})$ the space of α -Hölder continuous Riemannian metrics on \mathcal{M} equipped with C^α distance $d_{\mathcal{T}^\alpha}$. The μ -essential invariance of τ under the holonomies together with (4.2) yields μ -essential Hölder continuity of τ as a function from X to $\mathcal{T}^\alpha(\mathcal{M})$ along the stable and unstable leaves in X .

COROLLARY 4.4. [S19, Corollary 4.7] *The function $x \mapsto \tau_x$ is $\beta\rho$ -Hölder continuous on G along the stable and unstable leaves in X as a function from X to $\mathcal{T}^\alpha(\mathcal{M})$, that is,*

$$d_{\mathcal{T}^\alpha}(\tau_x, \tau_y) \leq c_4 d_X(x, y)^{\beta\rho} \quad \text{for all } x, y \in G \text{ with } y \in W_{\text{loc}}^{s/u}(x). \tag{4.3}$$

Then in the *hyperbolic case* Hölder continuous dependence of τ_x on $x \in X$ follows using local product structure of the measure of maximal entropy μ and of the stable and unstable manifolds, as done in [S19, §4.5].

In the *partially hyperbolic case* we use the following results from [ASV13]. We formulate them using our notation.

Definition 4.5. [ASV13, Definition 2.9] Let (X, f) be a partially hyperbolic system, and let \mathcal{N} be a continuous fiber bundle over X . A *stable holonomy* on \mathcal{N} is a family

$h_{x,y}^s : \mathcal{N}_x \rightarrow \mathcal{N}_y$ of γ -Hölder homeomorphisms, with uniform $\gamma > 0$, defined for all x, y in the same stable leaf of f and satisfying:

- (a) $h_{y,z}^s \circ h_{x,y}^s = h_{x,z}^s$ and $h_{x,x}^s = \text{Id}$;
- (b) the map $(x, y, \eta) \mapsto h_{x,y}^s(\eta)$ is continuous when (x, y) varies in the set of pairs of points in the same local stable leaf.

Unstable holonomy is defined analogously, for pairs of points in the same unstable leaf.

We consider the fiber bundle \mathcal{N} over X with fiber $\mathcal{N}_x = \mathcal{T}^0(\mathcal{M})$ of continuous Riemannian metrics on \mathcal{M} and the maps $h_{x,y}^s$ induced by $H_{x,y}^s$ on these metrics. Property (a) in the definition above holds by (H1) in Proposition 3.3. By (H4'), the diffeomorphisms $H_{x,y}^s$ depend continuously in $\text{Diff}^{1+\alpha}(\mathcal{M})$ on (x, y) , where $x \in X$ and $y \in W_{\text{loc}}^s(x)$, which yields property (b). By the continuity, $H_{x,y}^s$ are uniformly bounded in $\text{Diff}^{1+\alpha}(\mathcal{M})$ over (x, y) as above. It follows that the maps $h_{x,y}^s$ are Hölder homeomorphisms of \mathcal{N} with uniform Hölder exponent and constant.

Definition 4.6. [ASV13, Definition 2.10] A measurable section $\Psi : X \rightarrow \mathcal{N}$ of the fiber bundle \mathcal{N} is called *s-invariant* if $h_{x,y}^s(\Psi(x)) = \Psi(y)$ for every x, y in the same stable leaf and *essentially s-invariant* if this relation holds restricted to some full measure subset. The definition of *u-invariance* is analogous. Finally, Ψ is *bi-invariant* if it is both *s-invariant* and *u-invariant*, and it is *bi-essentially invariant* if it is both *essentially s-invariant* and *essentially u-invariant*.

A set in X is called *bi-saturated* if it consists of full stable and unstable leaves.

THEOREM 4.7. [ASV13, Theorem D] *Let $f : X \rightarrow X$ be a C^2 partially hyperbolic center bunched diffeomorphism preserving a volume μ , and let \mathcal{N} be a continuous fiber bundle with stable and unstable holonomies and with refinable fiber. Then:*

- (a) *for every bi-essentially invariant section $\Psi : X \rightarrow \mathcal{N}$, there exist a bi-saturated set X_Ψ with full measure, and a bi-invariant section $\tilde{\Psi} : X_\Psi \rightarrow \mathcal{N}$ that coincides with Ψ at μ almost every point;*
- (b) *if f is accessible then $X_\Psi = X$ and $\tilde{\Psi}$ is continuous.*

By the remark after [ASV13, Definition 2.10], every Hausdorff topological space with a countable basis of topology is refinable. Since the space $\mathcal{T}^0(\mathcal{M})$ is separable, the fiber $\mathcal{N}_x = \mathcal{T}^0(\mathcal{M})$ of \mathcal{N} is refinable. The section $\Psi(x) = \tau_x$ of \mathcal{N} is bi-essentially invariant by Proposition 4.3. Applying Theorem 4.7, we and conclude that, up to modification on a set of measure zero, τ is continuous as a function from X to $\mathcal{T}^0(\mathcal{M})$ and bi-invariant on X . The latter implies that Corollary 4.4 hold on $G = X$, and part (d) follows.

It remains to show that τ is continuous as a function from X to $\mathcal{T}^\alpha(\mathcal{M})$. Since τ is a continuous function from a compact set X to $\mathcal{T}^0(\mathcal{M})$, it is bounded in $\mathcal{T}^0(\mathcal{M})$. The estimate (4.1) implies that

$$\sup\{d_{\mathcal{T}}(\tau_{(x,t)}, \tau_{(x,t')}) \cdot (d_{\mathcal{M}}(t, t'))^{-\gamma} : x \in X \text{ and } t \neq t' \in \mathcal{M}\} \leq c_3.$$

Hence the set $\{\tau_x : x \in X\}$ is bounded in $\mathcal{T}^\gamma(\mathcal{M})$. Since $\alpha < \gamma$, the embedding of $\mathcal{T}^\gamma(\mathcal{M})$ into $\mathcal{T}^\alpha(\mathcal{M})$ is compact (see, for example, [dILO98, Proposition 3.3]), and

hence the set $\{\tau_x\}$ has compact closure in $\mathcal{T}^\alpha(\mathcal{M})$. It follows that $\tau : X \rightarrow \mathcal{T}^\alpha(\mathcal{M})$ is continuous. Indeed, suppose that $x_n \rightarrow x$ in X , but $\tau_{x_n} \not\rightarrow \tau_x$ in $\mathcal{T}^\alpha(\mathcal{M})$. Then (τ_{x_n}) has a subsequence converging to some $\hat{\tau} \neq \tau_x$ in $\mathcal{T}^\alpha(\mathcal{M})$ and hence in $\mathcal{T}^0(\mathcal{M})$, which contradicts the convergence of (τ_{x_n}) to τ_x in $\mathcal{T}^0(\mathcal{M})$. Thus $\tau_{x_n} \rightarrow \tau_x$ in $\mathcal{T}^\alpha(\mathcal{M})$.

This completes the proof for the partially hyperbolic case.

5. Proofs of Proposition 1.10, Theorem 1.6 and Corollary 1.8

5.1. Proof of Proposition 1.10. The last part of the proposition will be used in the proof of Theorem 1.6. Parts (a) and (b) follow immediately from the Definitions 1.5 and (1.8).

To prove (c), we fix $x \in X$. Since f is accessible, for every $y \in X$ there is an su -path $P = P_{x,y}$ from x to y . Then by (b) we have

$$\Phi_y = \mathcal{H}_{x,y}^{A,P} \circ \Phi_x \circ (\mathcal{H}_{x,y}^{B,P})^{-1}. \tag{5.1}$$

Turning to (d) and (e), suppose that $H^{A,s/u}$ and $H^{B,s/u}$ are in $\text{Diff}^r(\mathcal{M})$, where $r = 0$ or $r \geq 1$. Then so are $\mathcal{H}^{A,P}$ and $\mathcal{H}^{B,P}$ for any su -path P , and accessibility together with (5.1) implies that if $\Phi_{x_0} \in \text{Diff}^r(\mathcal{M})$, then $\Phi(y) \in \text{Diff}^r(\mathcal{M})$ for all $y \in X$.

Now for $r \geq 1$ we show that the function $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ is bounded. The holonomies H^A and H^B are uniformly bounded in $\text{Diff}^r(\mathcal{M})$ over all $x \in X$ and $y \in W_{\text{loc}}^s(x)$ by the continuity property (H4), and we set

$$K_H = \sup\{\|H_{x,y}^{\mathcal{D},s}\|_{C^r} : x \in X, y \in W_{\text{loc}}^s(x), \mathcal{D} = A, B\}.$$

For any $x \in X$ and $y \in W_{\text{loc}}^s(x)$ we have $\Phi_y = H_{x,y}^{A,s/u} \circ \Phi_x \circ H_{y,x}^{B,s/u}$. For $r \geq 1$, we use Lemma 2.1 twice to obtain the estimate

$$\begin{aligned} \|\Phi_y\|_{C^r} &= \|H_{x,y}^A \circ \Phi_x \circ H_{y,x}^B\|_{C^r} \leq M_r^2 \|H_{x,y}^A\|_{C^r} (1 + \|\Phi_x\|_{C^r})^r (1 + \|H_{y,x}^B\|_{C^r})^r \\ &\leq M_r^2 K_H \cdot (1 + \|\Phi_x\|_{C^r})^r (1 + K_H)^r = K' \cdot (1 + \|\Phi_x\|_{C^r})^r. \end{aligned}$$

Since f is accessible, there exist constants L and K such that, for any $x, y \in X$, there exists an su -path from x to y with at most L subpaths of length at most K , each lying in a single leaf of W^s or W^u [W13, Lemma 4.5]. For any $y \in X$, we consider such a path from x_0 to y and, starting with Φ_{x_0} , apply the above estimate a bounded number of times. Thus we obtain

$$\|\Phi_y\|_{C^r} \leq K'' = K''(K_H, \|\Phi_{x_0}\|_{C^r}, K, L, r) \quad \text{for all } y \in X.$$

Now we establish continuity of Φ . For $r = 0$ or $r \geq 1$, we let $\ell = [r]$ be the integer part of r , and consider the fiber bundle \mathcal{N} over X with fiber $\text{Diff}^\ell(\mathcal{M})$. For any $x \in X$, $y \in W^s(x)$, and $y' \in W^u(x)$ we define maps $h_{x,y}^s : \mathcal{N}_x \rightarrow \mathcal{N}_y$ and $h_{x,y'}^u : \mathcal{N}_x \rightarrow \mathcal{N}_{y'}$ by

$$h_{x,y}^s(g) = H_{x,y}^{A,s} \circ g \circ H_{y,x}^{B,s} \quad \text{and} \quad h_{x,y'}^u(g) = H_{x,y'}^{A,u} \circ g \circ H_{y',x}^{B,u}.$$

Since Φ intertwines H^A and H^B we have $\Phi(y) = h_{x,y}^s(\Phi_x)$ and $\Phi(y') = h_{x,y'}^u(\Phi_x)$, that is, Φ is invariant under h^s and h^u . It is well known that, for any $\ell \in \mathbb{N} \cup \{0\}$, $\text{Diff}^\ell(\mathcal{M})$ is a topological group and so the map $(g, h) \mapsto g \circ h$ from $\text{Diff}^\ell(\mathcal{M}) \times \text{Diff}^\ell(\mathcal{M})$ to

$\text{Diff}^\ell(\mathcal{M})$ is continuous. Also, by property (H4) of the holonomies, the maps $(x, y) \mapsto H_{x,y}^{\mathcal{A}/\mathcal{B},s/u}$ into $\text{Diff}^\ell(\mathcal{M})$ are continuous on the set of pairs (x, y) where $y \in W_{\text{loc}}^s(x)$, and it follows that the maps $(x, y, g) \mapsto h_{x,y}^{s/u}(g)$ into $\text{Diff}^\ell(\mathcal{M})$ are continuous. Therefore, by its invariance, Φ a bi-continuous section of \mathcal{N} in the sense of the definition below.

Definition 5.1. [ASV13, Definition 2.12] A measurable section $\Psi : X \rightarrow \mathcal{N}$ of a continuous fiber bundle \mathcal{N} is s -continuous if the map $(x, y, \Psi(x)) \mapsto \Psi(y)$ is continuous on the set of pairs of points (x, y) in the same local stable leaf. The u -continuity is defined similarly using unstable leaves. Finally, Ψ is bi-continuous if it is both s -continuous and u -continuous.

We apply the theorem below to conclude that $\Phi : X \rightarrow \text{Diff}^\ell(\mathcal{M})$ is continuous, completing the proof for the case of an integer $r = \ell$.

THEOREM 5.2. [ASV13, Theorem E] *Let $X \rightarrow X$ be a C^1 partially hyperbolic accessible diffeomorphism and \mathcal{N} be a continuous fiber bundle. Then every bi-continuous section $\Psi : X \rightarrow \mathcal{N}$ is continuous on X .*

The argument above does not apply with a non-integer r in place of its integer part ℓ since the composition of C^r maps does not depend continuously on the terms in C^r distance in general; see [dILO98, Example 6.4].

Finally, suppose that r is not an integer. As we showed above, $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ is bounded and $\Phi : X \rightarrow \text{Diff}^\ell(\mathcal{M})$ is continuous. We take p such that $\ell < p < r$. Since the embedding of $\text{Diff}^r(\mathcal{M})$ into $\text{Diff}^p(\mathcal{M})$ is compact, it follows as at the end of the proof of Theorem 1.2 that $\Phi : X \rightarrow \text{Diff}^p(\mathcal{M})$ is continuous.

5.2. Proof of Theorem 1.6. Part (a) of Theorem 1.6 follows from Propositions 5.3 and 5.4 below, and Proposition 1.10(e). In Proposition 5.3 we prove that a measurable conjugacy intertwines the holonomies of \mathcal{A} and \mathcal{B} on a set of full measure. Then in Proposition 5.4 we show that it coincides on a set of full measure with a continuous conjugacy which intertwines the holonomies on X . Finally, we apply Proposition 1.10(e) to obtain the regularity of the conjugacy.

PROPOSITION 5.3. *Let (X, f) be either a partially hyperbolic diffeomorphism or a hyperbolic system, and let μ be an ergodic f -invariant measure. Let \mathcal{A} and \mathcal{B} be $\text{Diff}^0(\mathcal{M})$ -valued cocycles over (X, f) . Suppose that the set $\{\mathcal{B}_x^n : x \in X, n \in \mathbb{Z}\}$ has compact closure in $\text{Diff}^0(\mathcal{M})$ and that, for any $x \in X$ and $y \in W^s(x)$,*

$$H_{x,y}^{\mathcal{A},s} = \lim_{n \rightarrow +\infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \text{ and } H_{x,y}^{\mathcal{B},s} = \lim_{n \rightarrow +\infty} (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n \text{ exist in } \text{Diff}^0(\mathcal{M}).$$

Let $\Phi : X \rightarrow \text{Diff}^0(\mathcal{M})$ be a μ -measurable conjugacy between \mathcal{A} and \mathcal{B} . Then Φ intertwines the stable holonomies $H^{\mathcal{A},s}$ and $H^{\mathcal{B},s}$ of \mathcal{A} and \mathcal{B} on a set of full measure.

A similar statement holds for the unstable holonomies.

We note that continuity of the map $(x, y) \mapsto H_{x,y}^{\mathcal{A},s}$ is not assumed in this proposition.

Proof. We will give a proof for the stable holonomies. We will show that, for all x and $y \in W^s(x)$ in a set of full measure,

$$\Phi_y^{-1} \circ H_{x,y}^{\mathcal{A},s} \circ \Phi_x = H_{x,y}^{\mathcal{B},s}. \tag{5.2}$$

Since the map Φ is μ -measurable and the space $\text{Diff}^0(\mathcal{M})$ is separable, by Lusin’s theorem there exists a compact set $S \subset X$ with $\mu(S) > 1/2$ such that $\Phi : X \rightarrow \text{Diff}^0(\mathcal{M})$ is uniformly continuous on S .

Let Y be the set of points in X for which the frequency of visiting the set S equals $\mu(S) > 1/2$. By Birkhoff’s ergodic theorem, $\mu(Y) = 1$. Let x and $y \in W^s(x)$ be in Y . Then there exists a sequence $\{n_i\}$ such that $f^{n_i}x$ and $f^{n_i}y$ are in S for all i . We denote $x_n = f^{n_i}x$ and $y_n = f^{n_i}y$. Since $d_X(x_{n_i}, y_{n_i}) \rightarrow 0$ as $i \rightarrow \infty$ and Φ is uniformly continuous on S ,

$$d_{C^0}(\Phi_{x_{n_i}}, \Phi_{y_{n_i}}) \rightarrow 0 \text{ as } i \rightarrow \infty.$$

For d_0 as in (2.1) it follows that

$$d_0(\Phi_{y_{n_i}}^{-1} \circ \Phi_{x_{n_i}}, \text{Id}) \rightarrow 0 \text{ as } i \rightarrow \infty. \tag{5.3}$$

Now we establish (5.2). Since $\mathcal{A}_x^{n_i} = \Phi_{x_{n_i}} \circ \mathcal{B}_x^{n_i} \circ \Phi_x^{-1}$, we have

$$\Phi_y^{-1} \circ (\mathcal{A}_y^{n_i})^{-1} \circ \mathcal{A}_x^{n_i} \circ \Phi_x = (\mathcal{B}_y^{n_i})^{-1} \circ \Phi_{y_{n_i}}^{-1} \circ \Phi_{x_{n_i}} \circ \mathcal{B}_x^{n_i}. \tag{5.4}$$

We show that the left-hand side converges to $\Phi_y^{-1} \circ H_{x,y}^{\mathcal{A},s} \circ \Phi_x$ and the right-hand side converges to $H_{x,y}^{\mathcal{B},s}$ in d_0 .

For a homeomorphism g of \mathcal{M} and $\delta > 0$ we define

$$\omega_g(\delta) = \sup\{d_{\mathcal{M}}(g^*(t), g^*(t')) : * = 1, -1, t, t' \in \mathcal{M} \text{ and } d(t, t') \leq \delta\}.$$

Since g is uniformly continuous on \mathcal{M} , $\omega_g(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Since $\{\mathcal{B}_x^n : x \in X, n \in \mathbb{Z}\}$ has compact closure in $\text{Diff}^0(\mathcal{M})$, the family $\{\mathcal{B}_x^n\}$ is uniformly equicontinuous. It follows that

$$\omega_{\mathcal{B}}(\delta) = \sup\{\omega_{\mathcal{B}_x^n}(\delta) : x \in X, n \in \mathbb{Z}\} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

We observe that if $g, h_n, k \in \text{Diff}^0(\mathcal{M})$ and $h_n \rightarrow h$ in $\text{Diff}^0(\mathcal{M})$, then

$$d_0(g \circ h_n \circ k, g \circ h \circ k) = d_0(g \circ h_n, g \circ h) \leq \omega_g(d_0(h_n, h)) \rightarrow 0.$$

Since $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \rightarrow H_{x,y}^{\mathcal{A},s}$ in $\text{Diff}^0(\mathcal{M})$, it follows that

$$d_0(\Phi_y^{-1} \circ (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \circ \Phi_x, \Phi_y^{-1} \circ H_{x,y}^{\mathcal{A},s} \circ \Phi_x) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Denoting $g_{n_i} = (\mathcal{B}_y^{n_i})^{-1}$, $h_{n_i} = \Phi(y_{n_i})^{-1} \circ \Phi(x_{n_i})$, and $k_{n_i} = \mathcal{B}_x^{n_i}$, we estimate

$$d_0(g_{n_i} \circ h_{n_i} \circ k_{n_i}, g_{n_i} \circ \text{Id} \circ k_{n_i}) = d_0(g_{n_i} \circ h_{n_i}, g_{n_i} \circ \text{Id}) \leq \omega_{\mathcal{B}}(d_0(h_{n_i}, \text{Id})).$$

Since $d_{C^0}(h_{n_i}, \text{Id}) \rightarrow 0$ by (5.3), we obtain

$$d_0((\mathcal{B}_y^{n_i})^{-1} \circ \Phi_{y_{n_i}}^{-1} \circ \Phi_{x_{n_i}} \circ \mathcal{B}_x^{n_i}, (\mathcal{B}_y^{n_i})^{-1} \circ \mathcal{B}_x^{n_i}) \rightarrow 0.$$

Finally, as $d_0((\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n, H_{x,y}^{\mathcal{B},s}) \rightarrow 0$,

$$d_0((\mathcal{B}_y^{n_i})^{-1} \circ \Phi_{y^{n_i}}^{-1} \circ \Phi_{x^{n_i}} \circ \mathcal{B}_x^{n_i}, H_{x,y}^{\mathcal{B},s}) \rightarrow 0.$$

Therefore, (5.4) together with the above estimates implies that

$$\Phi_y^{-1} \circ H_{x,y}^{\mathcal{A},s} \circ \Phi_x = H_{x,y}^{\mathcal{B},s} \quad \text{or equivalently, } H_{x,y}^{\mathcal{A},s} = \Phi_y \circ H_{x,y}^{\mathcal{B},s} \circ \Phi_x^{-1},$$

and we conclude that Φ intertwines the stable holonomies of \mathcal{A} and \mathcal{B} on the set Y . □

PROPOSITION 5.4. *Let $f : X \rightarrow X$ be an accessible center bunched C^2 partially hyperbolic diffeomorphism preserving a volume μ . Let \mathcal{A} and \mathcal{B} be $\text{Diff}^q(\mathcal{M})$ -valued cocycles over (X, f) with holonomies in $\text{Diff}^0(\mathcal{M})$.*

Let $\Phi : X \rightarrow \text{Diff}^0(\mathcal{M})$ be a μ -measurable conjugacy between \mathcal{A} and \mathcal{B} which intertwines their holonomies on a set $Y \subseteq X$ of full measure. Then Φ coincides on a set of full measure with a continuous conjugacy $\tilde{\Phi} : X \rightarrow \text{Diff}^0(\mathcal{M})$ which intertwines the holonomies of \mathcal{A} and \mathcal{B} on X .

Proof. For every $x \in Y, y \in W^s(x) \cap Y$ and $y' \in W^u(x) \cap Y$, we have

$$\Phi_y = H_{x,y}^{\mathcal{A},s} \circ \Phi_x \circ (H_{x,y}^{\mathcal{B},s})^{-1} = H_{x,y}^{\mathcal{A},s} \circ \Phi_x \circ H_{y,x}^{\mathcal{B},s} \quad \text{and} \quad \Phi_{y'} = H_{x,y'}^{\mathcal{A},u} \circ \Phi_x \circ H_{y',x}^{\mathcal{B},u}. \tag{5.5}$$

Now we apply Theorem 4.7. We consider the fiber bundle \mathcal{N} over X with fiber $\text{Diff}^0(\mathcal{M})$. Since the space $\text{Diff}^0(\mathcal{M})$ is separable, the fiber bundle is refinable. As in the proof of Proposition 1.10, for any $x \in X, y \in W^s(x)$, and $y' \in W^u(x)$ we consider the maps $h_{x,y}^s : \mathcal{N}_x \rightarrow \mathcal{N}_y$ and $h_{x,y'}^u : \mathcal{N}_x \rightarrow \mathcal{N}_{y'}$ given by

$$h_{x,y}^s(g) = H_{x,y}^{\mathcal{A},s} \circ g \circ H_{y,x}^{\mathcal{B},s} \quad \text{and} \quad h_{x,y'}^u(g) = H_{x,y'}^{\mathcal{A},u} \circ g \circ H_{y',x}^{\mathcal{B},u}.$$

The family $\{h_{x,y}^s\}$ is a stable holonomy in the sense of Definition 4.5. Indeed, by property (H1) of $H^{\mathcal{A},s}$ and $H^{\mathcal{B},s}$ we have $h_{x,x}^s = \text{Id}$, and for any $y, z \in W^s(x)$,

$$(h_{y,z}^s \circ h_{x,y}^s)(g) = H_{y,z}^{\mathcal{A},s} \circ H_{x,y}^{\mathcal{A},s} \circ g \circ H_{y,x}^{\mathcal{B},s} \circ H_{z,y}^{\mathcal{B},s} = H_{x,z}^{\mathcal{A},s} \circ g \circ H_{z,x}^{\mathcal{B},s} = h_{x,z}^s(g).$$

Since the map $(g, h) \mapsto g \circ h$ from $\text{Diff}^0(\mathcal{M}) \times \text{Diff}^0(\mathcal{M})$ to $\text{Diff}^0(\mathcal{M})$ is continuous, and $H_{x,y}^{\mathcal{A},s}$ and $H_{x,y}^{\mathcal{B},s}$ depend continuously on (x, y) with $y \in W_{\text{loc}}^s(x)$, it follows that the map $(x, y, g) \mapsto h_{x,y}^s(g)$ is continuous. Also, by property (H5) the maps $H_{x,y}^{\mathcal{A},s}$ and $H_{x,y}^{\mathcal{B},s}$ are Hölder with a uniform constant K and exponent γ . Then, for any $g, g' \in \text{Diff}^0(\mathcal{M})$, we have

$$d_0(H_{x,y}^{\mathcal{A},s} \circ g \circ H_{y,x}^{\mathcal{B},s}, H_{x,y}^{\mathcal{A},s} \circ g' \circ H_{y,x}^{\mathcal{B},s}) = d_0(H_{x,y}^{\mathcal{A},s} \circ g, H_{x,y}^{\mathcal{A},s} \circ g') \leq K d_0(g, g')^\gamma.$$

A similar estimate holds for the inverses, and we obtain

$$d_{C^0}(h_{x,y}^s(g), h_{x,y}^s(g')) \leq K d_{C^0}(g, g')^\gamma.$$

Thus the homeomorphisms $h_{x,y}^s$ are also Hölder with uniform constant and exponent. Similarly, we see that $\{h_{x,y'}^u\}$ is an unstable holonomy.

Since (5.5) can be restated as $\Phi_y = h_{x,y}^s(\Phi_x)$ and $\Phi_{y'} = h_{x,y'}^u(\Phi_x)$, the conjugacy Φ is a bi-essentially invariant section of \mathcal{N} as in Definition 4.6. Then Theorem 4.7 yields that Φ coincides on a set of full measure with a continuous conjugacy $\tilde{\Phi} : X \rightarrow \text{Diff}^0(\mathcal{M})$ that is invariant under h^s and h^u . The latter means that is $\tilde{\Phi}$ intertwines the holonomies of \mathcal{A} and \mathcal{B} on X . □

For $r = 0$, the two propositions above yield part (a) of the theorem. To complete the proof for $r \geq 1$, we apply the last part of Proposition 1.10 to the continuous conjugacy $\tilde{\Phi} : X \rightarrow \text{Diff}^0(\mathcal{M})$ which intertwines the holonomies. This completes the proof of (a).

We now turn to (b). We assume that $r > 1$ and the holonomies of \mathcal{A} and \mathcal{B} satisfy the Hölder condition (1.3). We write Φ for $\tilde{\Phi}$ to simplify the notation. We give a proof of Hölder continuity of Φ along W^s ; the argument for W^u is similar, using the unstable holonomies. The stable holonomies of \mathcal{A} and \mathcal{B} are uniformly bounded in $\text{Diff}^r(\mathcal{M})$ over all $x \in X$ and $y \in W_{\text{loc}}^s(x)$ by (H4'), and we showed already that $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ is bounded. So we set

$$K_H = \sup\{|H_{x,y}^{\mathcal{D},s}|_{C^r} : x \in X, y \in W_{\text{loc}}^s(x), \mathcal{D} = \mathcal{A}, \mathcal{B}\},$$

$$K_\Phi = \sup\{|\Phi_x|_{C^r} : x \in X\}.$$

For any $x \in X$ and $y \in W_{\text{loc}}^s(x)$, we use Lemma 2.1 to estimate

$$\|H_{x,y}^{\mathcal{A},s} \circ \Phi_x\|_{C^r} \leq M_r \|H_{x,y}^{\mathcal{A},s}\|_{C^r} (1 + \|\Phi_x\|_{C^r})^r \leq M_r K_H (1 + K_\Phi)^r =: K_1,$$

$$\|\Phi_x^{-1} \circ (H_{x,y}^{\mathcal{A},s})^{-1}\|_{C^r} \leq M_r \|\Phi_x^{-1}\|_{C^r} (1 + \|(H_{x,y}^{\mathcal{A},s})^{-1}\|_{C^r})^r \leq M_r K_\Phi (1 + K_H)^r =: K_2.$$

If r is an integer, we take $r - 1 \leq p' < r$, if r is not an integer, we take $\lfloor r \rfloor \leq p' < r$, and we set $\rho = r - p'$. In the estimates below, we use Lemma 2.2 with $q = r$, $r = p'$, and either $g = \text{Id}$ or $\tilde{g} = \text{Id}$:

$$d_{C^{p'}}(h_1 \circ \tilde{g}, h_2 \circ \tilde{g}) \leq M(2^{p'} \|\tilde{g}^{-1}\|_{C^r} + (1 + \|\tilde{g}\|_{C^{p'}})^{p'}) \cdot d_{C^{p'}}(h_1, h_2)^\rho,$$

$$d_{C^{p'}}(g \circ h_1, g \circ h_2) \leq M(2^{p'} \|g\|_{C^r} + (1 + \|g^{-1}\|_{C^{p'}})^{p'}) \cdot d_{C^{p'}}(h_1, h_2)^\rho.$$

Since Φ intertwines the holonomies, $\Phi_y = H_{x,y}^{\mathcal{A},s} \circ \Phi_x \circ H_{y,x}^{\mathcal{B},s}$, and we estimate

$$\begin{aligned} d_{C^{p'}}(\Phi_x, \Phi_y) &= d_{C^{p'}}(\Phi_x, H_{x,y}^{\mathcal{A},s} \circ \Phi_x \circ H_{y,x}^{\mathcal{B},s}) \\ &\leq d_{C^{p'}}(\text{Id} \circ \Phi_x, H_{x,y}^{\mathcal{A},s} \circ \Phi_x) + d_{C^{p'}}((H_{x,y}^{\mathcal{A},s} \circ \Phi_x) \circ \text{Id}, (H_{x,y}^{\mathcal{A},s} \circ \Phi_x) \circ H_{y,x}^{\mathcal{B},s}) \\ &\leq M \cdot (2^{p'} \|\Phi_x^{-1}\|_{C^r} + (1 + \|\Phi_x\|_{C^{p'}})^{p'}) \cdot (d_{C^{p'}}(\text{Id}, H_{x,y}^{\mathcal{A},s}))^\rho \\ &\quad + M \cdot (2^{p'} \|H_{x,y}^{\mathcal{A},s} \circ \Phi_x\|_{C^r} + (1 + \|(H_{x,y}^{\mathcal{A},s} \circ \Phi_x)^{-1}\|_{C^{p'}})^{p'}) \cdot (d_{C^{p'}}(\text{Id}, H_{y,x}^{\mathcal{B},s}))^\rho \\ &\leq K_3 \cdot (d_{C^r}(\text{Id}, H_{y,x}^{\mathcal{A},s}))^\rho + K_4 \cdot (d_{C^r}(\text{Id}, H_{y,x}^{\mathcal{B},s}))^\rho \leq K_5 \cdot d_X(x, y)^{\beta \cdot \rho}. \end{aligned}$$

In each of the two applications of Lemma 2.2 above, the assumptions of the lemma are satisfied. Indeed, since $h_1 = \text{Id}$ and $h_2 = H_{x,y}^{\mathcal{A}/\mathcal{B},s}$ we have $|h_1|_{C^r}, |h_2|_{C^r} \leq K_H$, and by property (1.3) we have $d_{C^{p'}}(h_1, h_2) \leq \delta_0 |h_1|_{C^{p'}}^{-1}$ for all sufficiently close x and $y \in W_{\text{loc}}^s(x)$. Also, the expression in (2.4) is uniformly bounded below by some $c'' > 0$, and so the second assumption also holds provided that $d_X(x, y)$ is small enough.

We conclude that, for any sufficiently close $x \in X$ and $y \in W_{\text{loc}}^s(x)$,

$$d_{C^{p'}}(\Phi_x, \Phi_y) \leq K_5 \cdot d_X(x, y)^{\beta' \rho}. \tag{5.6}$$

The same estimate holds for any sufficiently close $x \in X$ and $y \in W_{\text{loc}}^u(x)$. This concludes the proof of Theorem 1.6.

Proof of Corollary 1.8. We have already established Hölder continuity of Φ along W^s and W^u . Let $x \in X$, and let z be sufficiently close to x so that the intersection $W_{\text{loc}}^s(x) \cap W_{\text{loc}}^u(z)$ consists of a single point, which we denote by y . Then, by (5.6),

$$d_{C^{p'}}(\Phi_x, \Phi_y) \leq K_5 \cdot d_X(x, y)^{\beta' \rho} \quad \text{and} \quad d_{C^{p'}}(\Phi_y, \Phi_z) \leq K_5 \cdot d_X(y, z)^{\beta' \rho},$$

and it follows that $d_{C^{p'}}(\Phi_x, \Phi_z) \leq K_6 \cdot d_X(x, z)^{\beta' \rho}$.

6. Proofs of Proposition 1.9, Theorem 1.11 and Corollary 1.12

6.1. *Proof of Proposition 1.9.* Both cocycles have stable holonomies in $\text{Diff}^0(\mathcal{M})$ by Proposition 3.1.

In the proof we use only Hölder continuity of Φ along W^s ; specifically, that there exists a constant K_1 such that

$$d_{C^0}(\Phi_x, \Phi_y) \leq K_1 d_X(x, y)^\beta \quad \text{for all } x \in X \text{ and } y \in W_{\text{loc}}^s(x).$$

By the invariance property (H2) of holonomies, it suffices to prove the intertwining for $y \in W_{\text{loc}}^s(x)$. We fix $x \in X$ and $y \in W_{\text{loc}}^s(x)$. As in (3.3), we obtain that, for all $n \in \mathbb{N}$,

$$d_0(\Phi_{y_n}^{-1} \circ \Phi_{x_n}, \text{Id}) \leq K_1 d_X(x, y)^\beta \cdot \lambda^{n\beta}.$$

Since $\mathcal{A}_x^n = \Phi_{f_{x_n}} \circ \mathcal{B}_x^n \circ \Phi_x^{-1}$, we have

$$\Phi_y^{-1} \circ (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \circ \Phi_x = (\mathcal{B}_y^n)^{-1} \circ \Phi_{y_n}^{-1} \circ \Phi_{x_n} \circ \mathcal{B}_x^n. \tag{6.1}$$

Since Φ_y is a homeomorphism of a compact manifold, Φ_y^{-1} is uniformly continuous on \mathcal{M} . Since $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \rightarrow H_{x,y}^{\mathcal{A},s}$ uniformly on \mathcal{M} , it follows that

$$\begin{aligned} & d_0(\Phi_y^{-1} \circ (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n \circ \Phi_x, \Phi_y^{-1} \circ H_{x,y}^{\mathcal{A},s} \circ \Phi_x) \\ &= d_0(\Phi_y^{-1} \circ ((\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n), \Phi_y^{-1} \circ H_{x,y}^{\mathcal{A},s}) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Also,

$$\begin{aligned} & d_0((\mathcal{B}_y^n)^{-1} \circ \Phi_{y_n}^{-1} \circ \Phi_{x_n} \circ \mathcal{B}_x^n, (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n) \\ &= d_0((\mathcal{B}_y^n)^{-1} \circ (\Phi_{y_n}^{-1} \circ \Phi_{x_n}), (\mathcal{B}_y^n)^{-1} \circ \text{Id}) \leq K \sigma^n \cdot d_0((\Phi_{y_n}^{-1} \circ \Phi_{x_n}), \text{Id}) \\ &\leq K \sigma^n \cdot K_1 d_X(x, y)^\beta \cdot \lambda^{n\beta} = K K_1 d_X(x, y)^\beta \cdot (\sigma \lambda^\beta)^n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since $d_0((\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n, H_{x,y}^{\mathcal{B},s}) \rightarrow 0$, it follows that

$$d_0((\mathcal{B}_y^n)^{-1} \circ \Phi_{y_n}^{-1} \circ \Phi_{x_n} \circ \mathcal{B}_x^n, H_{x,y}^{\mathcal{B},s}) \rightarrow 0.$$

Thus the left-hand side of (6.1) converges to $\Phi_y^{-1} \circ H_{x,y}^{A,s} \circ \Phi_x$ and the right-hand side converges to $H_{x,y}^{B,s}$ in $C^0(\mathcal{M})$, and we conclude that Φ intertwines the stable holonomies of \mathcal{A} and \mathcal{B} .

6.2. *Proof of Theorem 1.11.* We will prove (b), and then (a) follows.

We write x in place of x_0 . For every $y \in X$, we define

$$\Phi_y = \mathcal{H}_{x,y}^{A,P} \circ \Phi_x \circ (\mathcal{H}_{x,y}^{B,P})^{-1} \quad \text{where } P = P_{x,y} \text{ is an } su\text{-path from } x \text{ to } y. \quad (6.2)$$

The value Φ_y does not depend on the choice of a path from x to y , and hence is well defined. Indeed, let $(P_{x,y})^{-1} = \{y = x_k, x_{k-1}, \dots, x_1, x_0 = x\}$, let $\tilde{P}_{x,y}$ be another su -path $P_{x,y}$ from x to y , and let $\tilde{\Phi}_y$ be the corresponding value. Then $(P_{x,y})^{-1}\tilde{P}_{x,y}$ is an su -cycle, and, using (b1), we obtain

$$\begin{aligned} \Phi_y^{-1} \circ \tilde{\Phi}_y &= \mathcal{H}_{x,y}^{B,P} \circ \Phi_x^{-1} \circ (\mathcal{H}_{x,y}^{A,P})^{-1} \circ \mathcal{H}_{x,y}^{A,\tilde{P}} \circ \Phi_x \circ (\mathcal{H}_{x,y}^{B,\tilde{P}})^{-1} \\ &= \mathcal{H}_{x,y}^{B,P} \circ \Phi_x^{-1} \circ \mathcal{H}_x^{A,P^{-1}\tilde{P}} \circ \Phi_x \circ (\mathcal{H}_{x,y}^{B,\tilde{P}})^{-1} \\ &= \mathcal{H}_{x,y}^{B,P} \circ \mathcal{H}_x^{B,P^{-1}\tilde{P}} \circ (\mathcal{H}_{x,y}^{B,\tilde{P}})^{-1} = \text{Id}. \end{aligned}$$

Let $z \in W^{s/u}(y)$. Then

$$\Phi_z = H_{y,z}^{A,s/u} \circ \mathcal{H}_{x,y}^{A,P} \circ \Phi_x \circ (\mathcal{H}_{x,y}^{B,P})^{-1} \circ (H_{y,z}^{B,s/u})^{-1} = H_{y,z}^{A,s/u} \circ \Phi_y \circ (H_{y,z}^{B,s/u})^{-1},$$

and so Φ intertwines the holonomies. Then it follows by Proposition 1.10(e) that $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ is bounded and $\Phi : X \rightarrow \text{Diff}^p(\mathcal{M})$ is continuous for $p = r$ if r is an integer, and for any $p < r$ otherwise.

It remains to show that Φ is a conjugacy, that is, it satisfies

$$\mathcal{A}_y = \Phi_{fy} \circ \mathcal{B}_y \circ \Phi_y^{-1} \quad \text{for all } y \in X.$$

We consider a point $y \in \mathcal{M}$ and an su -path $P = P_{x,y}$ from x to y . Then $f(P)$ is an su -path from fx to fy . It follows from the definition of Φ that, for any $z, w \in \mathcal{M}$ and any su -path $P_{z,w}$ from z to w , we have $\Phi_w = \mathcal{H}_{z,w}^{A,P} \circ \Phi_z \circ (\mathcal{H}_{z,w}^{B,P})^{-1}$, and in particular for $z = fx$ and $w = fy$,

$$\Phi_{fy} = \mathcal{H}_{fx,fy}^{A,f(P)} \circ \Phi_{fx} \circ (\mathcal{H}_{fx,fy}^{B,f(P)})^{-1}. \quad (6.3)$$

By properties (H2, H2') of the holonomies, for any $z \in X, w \in W^{s/u}(z)$,

$$H_{fz,fw}^{A,s/u} = \mathcal{A}_w \circ H_{z,w}^{A,s/u} \circ (\mathcal{A}_z)^{-1},$$

and it follows that

$$\mathcal{H}_{fx,fy}^{A,f(P)} = \mathcal{A}_y \circ \mathcal{H}_{x,y}^{A,P} \circ (\mathcal{A}_x)^{-1} \quad \text{and, similarly,} \quad \mathcal{H}_{fx,fy}^{B,f(P)} = \mathcal{B}_y \circ \mathcal{H}_{x,y}^{B,P} \circ (\mathcal{B}_x)^{-1}. \quad (6.4)$$

Since the definition of Φ_{fx} in (b2) is consistent with (6.2), by (b2) we have

$$(\mathcal{A}_x)^{-1} \circ \Phi_{fx} \circ \mathcal{B}_x = \Phi_x. \quad (6.5)$$

Combining (6.3), (6.4), and (6.5), we obtain

$$\begin{aligned} \Phi_{fy} &= \mathcal{A}_y \circ \mathcal{H}_{x,y}^{A,P} \circ \mathcal{A}_x^{-1} \circ \Phi_{fx} \circ \mathcal{B}_x \circ (\mathcal{H}_{x,y}^{B,P})^{-1} \circ \mathcal{B}_y^{-1} \\ &= \mathcal{A}_y \circ \mathcal{H}_{x,y}^{A,P} \circ \Phi_x \circ (\mathcal{H}_{x,y}^{B,P})^{-1} \circ \mathcal{B}_y^{-1} = \mathcal{A}_y \circ \Phi_y \circ \mathcal{B}_y^{-1}. \end{aligned}$$

6.3. *Proof of Corollary 1.12.* Let \mathcal{B} be a constant cocycle. Then its stable and unstable holonomies are trivial, that is, $H_{x,y}^{B,s/u} = \text{Id}$, and hence $\mathcal{H}^{\mathcal{B},P} = \text{Id}$ for every su -cycle P . So in this case condition (b1) in Theorem 1.11 is (1.9), and condition (b2) can be rewritten as

$$\mathcal{B}_{x_0} = \Phi_{x_0}^{-1} \circ (\mathcal{H}_{x_0,fx_0}^{A,\tilde{P}})^{-1} \circ \mathcal{A}_{x_0} \circ \Phi_{x_0} \quad \text{for some path } \tilde{P} = \tilde{P}_{x_0,fx_0}.$$

We choose any $\Phi_{x_0} \in \text{Diff}^r(\mathcal{M})$, for example $\Phi_{x_0} = \text{Id}$, and define a constant cocycle $\mathcal{B} \equiv \mathcal{B}_{x_0}$. Then it follows by Theorem 1.11(b) that \mathcal{A} is conjugate to \mathcal{B} via a bounded function $\Phi : X \rightarrow \text{Diff}^r(\mathcal{M})$ such that $\Phi : X \rightarrow \text{Diff}^p(\mathcal{M})$ is continuous. Also, Φ intertwines the holonomies of \mathcal{A} and \mathcal{B} , which in the case of constant \mathcal{B} means (1.10).

We note that in this construction a constant cocycle \mathcal{B} is determined by the choice of Φ_{x_0} and does not depend on \tilde{P} by the assumption (1.9).

In the case when x_0 is fixed and $\mathcal{A}_{x_0} = \text{Id}$, we obtain $\mathcal{B} \equiv \mathcal{B}_{x_0} = \text{Id}$.

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