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TRANSFINITE RECURSION IN HIGHER REVERSE MATHEMATICS

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Abstract. In this paper we investigate the reverse mathematics of higher-order analogues of the theory ATR_0 within the framework of higher order reverse mathematics developed by Kohlenbach [11]. We define a theory RCA_0^3, a close higher-type analogue of the classical base theory RCA_0 which is essentially a conservative subtheory of Kohlenbach's base theory RCA_0^ω. Working over RCA_0^3, we study higher-type analogues of statements classically equivalent to ATR_0, including open and clopen determinacy, and examine the extent to which ATR_0 remains robust at higher types. Our main result is the separation of open and clopen determinacy for reals, using a variant of Steel's tagged tree forcing; in the presentation of this result, we develop a new, more flexible framework for Steel-type forcing.

§1. Introduction.

The question

"What role do incomputable sets play in mathematics?"

has been a central theme in modern logic for almost as long as modern logic has existed. Six years before Alan Turing formalized the notion of computability, van der Waerden [26] showed that the splitting set of a field is not uniformly computable from the field; put another way, van der Waerden demonstrated the *necessity* of certain incomputable sets for Galois theory. Other results, especially Turing's solution to the Entscheidungsproblem and the solution by Davis, Matiyasevitch, Putnam, and Robinson of Hilbert's Tenth Problem, established the incomputability of particular sets of natural numbers of interest. In 1975, Friedman [2] initiated the axiomatic study of this question, dubbed "Reverse Mathematics."

Reverse mathematics requires the choice of both a common language in which to express all analyzed theorems, and a base theory in that language over which all equivalences and nonimplications are to be proved. The natural choice of language is that of second-order arithmetic, since it is in this language that computability-theoretic principles are most naturally expressed. The base theory is taken to be RCA_0, a precise definition of which is contained in [22]; as a base theory, RCA_0 is justified by the fact that it captures exactly "computable" mathematics, in the sense that the ω-models of RCA_0 are precisely the Turing ideals. One notable feature of reverse mathematics is the existence of the "Big Five," five subtheories of second-order arithmetic — RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0 — each of which is "robust," in the sense that the same theory results when small changes are made to its

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exact statement (or to the precise coding mechanisms used), and which correspond to the exact strength, over RCA_0, of the vast majority of theorems studied by reverse mathematics.

However, there is a significant amount of classical mathematics, including parts of measure theory and most of general topology, which resists any natural coding into the language of second-order arithmetic. This was already recognized by Friedman in [1]. Somewhat later, Victor Harnik [7] developed a higher-order version of RCA_0 in order to study the axiomatic strength of various results from stability theory. At the time, however, the higher-order program failed to draw mathematical attention comparable to that of second-order reverse mathematics.

Recently, however, there has been a return to this subject. The framework of finite types — in which objects of arbitrary finite order, such as sets of reals, are treated directly — has begun to emerge as a natural setting for a higher reverse mathematics, following Ulrich Kohlenbach's paper on the subject [11].[1] Kohlenbach expands the language of second-order arithmetic to all finite types, and extends the system RCA_0 to include a version of primitive recursion for arbitrary finite-type functionals. The resulting system, RCA_0^ω, is a proof-theoretically natural conservative extension of RCA_0. (From the point of view of computability theory, however, the choice of base theory may not be so clear; see the discussion at the end of this paper.)

Work on reverse mathematics in finite types has so far proceeded along one or the other of two general avenues: the analysis of classical theorems about objects not naturally codeable within second-order arithmetic, such as ultrafilters or general topological spaces ([8], [12], [25]), or the analysis of higher-type "uniformizations" of classical theorems of second-order arithmetic ([11], [24]). The present paper instead looks at the higher-type analogues of theorems studied by classical reverse mathematics, focusing in particular on what old patterns hold or fail and what new patterns emerge.[2]

One natural question along these lines is the following: to what extent do the robust subsystems of second-order arithmetic have robust analogues at higher types? It is this question which the present paper addresses, focusing on the system ATR_0. In the classical case, much of the robustness of ATR_0 comes from the fact that being a well-ordering is Π^1_1-complete. For instance, this is what drives the method of "pseudohierarchies" by which ill-founded linear orders which appear well-founded, such as those constructed in [6], are used to prove a large number of equivalences at the level of ATR_0; see [22]. Moving up a type, however, changes the situation completely: since we can code an infinite sequence of reals by a single real, the class of well-orderings of subsets of $\mathbb{R}$ is again Π^1_1, instead of being Π^2_1 complete. This causes the entire method of pseudohierarchies to break down, and raises doubt that the higher-type analogues of various theorems classically equivalent to ATR_0 are still equivalent.

We begin by presenting in Section 2 a base theory, RCA_0^3, which is essentially equivalent to, yet simpler to use than, RCA_0^ω. We then study the complexity over

[1]Although it is by no means the only one — see [21] for an approach via α-recursion theory instead, and also [4] for a closely-related α-recursive structure theory. Shore also suggests other approaches which could be interesting, such as via E-recursion or the computation theory of Blum-Shub-Smale.

[2]This is also the approach taken in [21], there with respect to α-recursion rather than finite types.

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RCA_0^3 of several higher-type analogues of several principles classically equivalent to ATR_0: comparability of well-orderings, clopen determinacy, open determinacy, Σ^1_1 separation, and definition by recursion along a well-founded tree. In Section 2, we prove some basic implications and nonimplications. At the bottom of this hierarchy lies the principle asserting the comparability of well-orderings of sets of reals, which we show is remarkably weak at higher types relative to the other principles; above clopen determinacy, a higher-type version of the separation principle Σ^1_1-Sep. We also examine the role of the axiom of choice in higher determinacy principles.

The main result of this paper, to which Section 3 is devoted, concerns the two determinacy principles. In classical reverse mathematics, clopen determinacy fails in HYP, the model consisting of the hyperarithmetic sets, despite hyperarithmetic clopen games having hyperarithmetic winning strategies, since the method of pseudohierarchies allows us to construct games which are "hyperarithmetically clopen" but are undetermined in HYP. This method, as noted above, is no longer valid at higher types, while the complexities of winning strategies for clopen games on reals can still be bounded by a transfinite iteration of an appropriate jump-like operator. This suggests that at higher types, open determinacy becomes strictly stronger than clopen determinacy; using an uncountable version of Steel's tagged tree forcing, we show that this is indeed the case.

1.1. Background and Conventions. We refer the reader to [13] for the relevant background in set theory; for descriptive set theory, [18] and [9] are the standard sources. For background on reverse mathematics, see [22]. Finally, for background in finite types, as well as the various computability-theoretic concerns which arise in higher-type settings, see [14].

There are several notational conventions we adopt for simplicity. Throughout, we use $\mathbb{R}$ to refer to the Baire space, the set of functions from ω to ω; this is because, during the main result, ordinals will be used as tags, and for this reason a symbol other than "ω^ω" is preferable. If σ is a nonempty finite string, we write σ^- for the immediate $\preccurlyeq$-predecessor of σ, and if f is an infinite string we write f^- for the string $n \mapsto f(n+1)$.

When writing formulas in many-sorted logic, we use the convention that the first time a variable occurs it is decorated with the appropriate sort symbol; for example,

$$\exists x^1 \forall y^0 (xy = 2)$$

is the statement "There is a function from naturals to naturals which is identically 2." (See Section 2.1 for a discussion of types.) If φ is a sentence, then $[\![\varphi]\!]$ is the truth value of φ: 1 if φ holds, and 0 if φ does not. We will denote the constant function $n \mapsto i$ by $\underline{i}$.

If $\Sigma, \Pi\colon A^{<\omega} \to A$, we write $\Sigma \otimes \Pi$ for the element of A^ω built by alternately applying Σ and Π:

$$\Sigma \otimes \Pi = \langle \Sigma(\langle\rangle), \Pi(\langle \Sigma(\langle\rangle)\rangle), \Sigma(\langle \Sigma(\langle\rangle), \Pi(\langle\Sigma(\langle\rangle)\rangle)\rangle), \dots \rangle.$$

We write $(\Sigma\otimes\Pi)_k$ for the length-k initial segment of $\Sigma\otimes\Pi$. A game is said to be a *win for player X* if that player has a winning strategy. A *quasistrategy* for a game played

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on a set A (so, viewed as a subtree of $A^{<\omega}$) is a multi-valued map from $A^{<\omega}$ to A; a quasistrategy is said to be *winning* if each element of A^ω which is compatible with the quasistrategy is a win for the corresponding player.

Finally, our main theorem 3.6 rely heavily on the method of set-theoretic forcing. For completeness, we present here a brief summary of this method; for details and proofs, see chapter VII of [13].

Given a model V of ZFC and a poset $\mathbb{P} \in V$, a *filter* is a subset F of $\mathbb{P}$ which is closed upwards, and such that any two elements of F have a common lower bound in F; a set $D \subseteq \mathbb{P}$ is *dense* if every element of $\mathbb{P}$ has a lower bound in D. The $\mathbb{P}$-*names* are defined inductively to be the sets $\{(p_i, \gamma_i) : i \in I\}$ of pairs with first coordinate an element of the partial order $\mathbb{P}$, and second coordinate a $\mathbb{P}$-name. If G is a filter meeting every dense subset of $\mathbb{P}$ which is in V — that is, G is $\mathbb{P}$-*generic over* V — and γ is a $\mathbb{P}$-name, we let $\gamma[G] = \{\theta[G] : \exists p \in G((p, \theta) \in \gamma)\}$ (this is of course a recursive definition). Crucially, the definition of $\gamma[G]$ is made inside V, although G will itself will never be in V.

We then define the *generic extension of* V *by* G to be

$$V[G] = \{\gamma[G] : \gamma \text{ is a } \mathbb{P}\text{-name in } V\}.$$

If $V[G] \models \varphi$ whenever $p \in G$, we write $p \Vdash \varphi$; the relation $\Vdash$ is the *forcing relation* given by $\mathbb{P}$. The essential properties of set-theoretic forcing are that the generic extension $V[G]$ is a model of ZFC; that the forcing relation is definable in the ground model; and that any statement true in the generic extension is forced by some condition in the generic filter. These are Theorems VII.4.2, VII.3.6(1), and VII.3.6(2) of [13], respectively.

Additionally, the forcing used in the proof of 3.6 will be *countably closed*:

Definition 1.1. $\mathbb{P}$ is *countably closed* if any chain of countably many conditions $\cdots \leq p_2 \leq p_1 \leq p_0$ has a common strengthening $p \leq p_0, p_1, p_2, \ldots$.

Countable closure yields a strong restriction on how a forcing notion can alter the set-theoretic universe, which will be crucial in 3.6:

Fact 1.2. *If $\mathbb{P}$ is countably closed and $X \in V$, then forcing with $\mathbb{P}$ adds no new countable subsets of X. In particular, forcing with a countably closed $\mathbb{P}$ adds no new reals.*

§2. Reverse mathematics beyond type 1.

We begin this section by developing a framework for reverse mathematics in higher types; we then define the various higher-type versions of ATR_0 we will consider in this paper, and prove some basic separations and equivalences.

2.1. The base theory.

We begin by making precise the notion of a finite type.[3]

[3]The one oddity of working with types is that the natural formalization is via many-sorted first-order logic, as opposed to ordinary first-order logic. In many-sorted logic, each element of the model and each variable symbol is labelled by one of a fixed collection of sorts; similarly, function, constant, and relation symbols in the signature must be appropriately labelled with sorts. When there are infinitely many sorts — as is the case with Kohlenbach's RCA_0^ω, but not our RCA_0^3 — the resulting logic is subtlely different from single-sorted first-order logic; however, these differences shall not be relevant here. For a careful introduction to many-sorted logic, see Chapter VI of [15].

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Definition 2.1. The *finite types* are defined as follows:

- 0 is a finite type;
- if σ, τ are finite types, then so is $\sigma \to \tau$; and
- only something required to be a finite type by the above rules is a finite type.

We denote the set of all finite types by FT.

The intended interpretation of finite types is as a hierarchy of functionals, with type 0 representing the "atomic" objects — here, natural numbers, or more generally elements of some first-order model of an appropriate theory of arithmetic — and type $\sigma \to \tau$ representing the set of maps from the set of objects of type σ to the set of objects of type τ.

Within the finite types is the special subclass ST of *standard finite types*, defined inductively as follows: 0 is a standard type, and if σ is a standard type, then so is $\sigma \to 0$. The standard types are for simplicity identified with natural numbers: $0 \to 0$ is denoted by "1," $(0 \to 0) \to 0$ by "2," etc.

The appeal of the finite-type framework to reverse mathematics is extremely compelling: the use of finite types lets us talk directly about objects that previously required extensive coding to treat in reverse mathematics, or could not be treated at all. For example, a topological space with cardinality $\leq \beth_i$ (where $\beth_0 = \aleph_0$ and $\beth_{i+1} = 2^{\beth_i}$) can be directly represented as a pair of functionals (F^i, G^{i+1}) corresponding to the characteristic functions of the underlying set and the collection of open subsets. Usually, this representation is even natural. In [11], Kohlenbach developed a base theory for reverse mathematics in all the finite types at once, RCA_0^ω.

However, working with all finite types at once is cumbersome. First, morally speaking, all finite-type functionals are equivalent to functionals of finite standard type via appropriate pairing functions; second, arbitrarily high types are rarely directly relevant. For that reason, we will use a base theory RCA_0^3, defined below, which only treats functionals of types 0, 1, and 2. In a subsequent paper, we will show that our theory is essentially equivalent to Kohlenbach's; specifically, RCA_0^ω is a conservative extension of RCA_0^3.

Definition 2.2. L^3 is the many-sorted first order language, consisting of the following:

- Sorts s_0, s_1, s_2, with corresponding equality predicates $=_0, =_1, =_2$. We will identify sort s_i with type i; recall that the objects of type 0, 1, and 2 are intended to be natural numbers, reals, and maps from reals to naturals, respectively.
- On the sort s_0, the usual signature of arithmetic: two binary functions
$$+, \times : s_0 \times s_0 \to s_0,$$
a binary relation
$$< \subseteq s_0 \times s_0,$$
and two constants
$$0, 1 \in s_0.$$
- Application operators $\cdot_0, \cdot_1$ with
$$\cdot_0 : s_1 \times s_0 \to s_0, \quad \cdot_1 : s_2 \times s_1 \to s_0.$$
These operators will generally be omitted; e.g., Fx or $F(x)$ instead of $F \cdot_1 x$ or $\cdot_1(F, x)$.

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- A binary operation
$$*\colon s_2 \times s_1 \to s_1$$
and a binary operation
$$^\frown\colon s_0 \times s_1 \to s_1.$$

The additional operations $*$ and $^\frown$ allow coding which in Kohlenbach's setting is handled through functionals of nonstandard type. Axioms which completely determine $*$ and $^\frown$ are given in Definition 2.3, below. We will abuse notation slightly and use $^\frown$ to denote both the concatenation of strings, and the specific L^3-symbol, as no confusion will arise. Throughout this paper, "L^3-term" will mean "L^3-term with parameters."

Finally, the syntactic classes Σ^0_i and Π^0_i are defined for L^3 as follows:

- A formula φ is in Σ^0_0 if and only if it has only bounded quantifiers over type 0 objects and no occurrences of $=_1$ or $=_2$. (Note that arbitrary parameters, however, *are* allowed.)
- A formula φ is in Π^0_{i+1} if
$$\varphi \equiv \forall x^0 \theta(x),$$
where $\theta \in \Sigma^0_i$.
- A formula φ is in Σ^0_{i+1} if
$$\varphi \equiv \exists x^0 \theta(x),$$
where $\theta \in \Pi^0_i$.

The higher syntactic classes Σ^1_i, Σ^2_j, etc. are defined in the analogous way, with lower-type quantifiers being "for free" as usual.

The base theory for third-order reverse mathematics which we will use in this paper, RCA^3_0, is then defined as follows:

Definition 2.3. RCA^3_0 is the L^3-theory consisting of the following axioms:

(1) Σ^0_1-induction and the ordered semiring axioms, P^-, for the type 0 objects.
(2) Extensionality axioms for the type 1 and 2 objects:
$$\forall F^1, G^1(\forall x^0(Fx = Gx) \iff F =_1 G) \quad \text{and}$$
$$\forall F^2, G^2(\forall x^1(Fx = Gx) \iff F =_2 G).$$
(3) The Δ^0_1 comprehension[4] schemes for type 1 and 2 objects:
$$\{\forall x^0 \exists! y^0 \varphi(x, y) \implies \exists f^1 \forall x^0(\varphi(x, f(x)))\colon \varphi \in \Sigma^0_1\}$$
and
$$\{\forall x^1 \exists! y^0 \varphi(x, y) \implies \exists F^2 \forall x^1(\varphi(x, F(x)))\colon \varphi \in \Sigma^0_1\}.$$

[4]There are several equivalent formulations of these, including as choice principles; we choose the following presentation, since it seems the most natural. Since we work with functionals which take values in ω instead of with sets (= functionals with values in $\{0, 1\}$), these schemes do look more complicated than the usual Δ^0_1-comprehension scheme in RCA_0; however, the intuition behind them — that if exactly one of an effective collection of existential sentences holds, then we can effectively find which one holds — is the same, and it is straightforward to show that our schemes are equivalent to their "dual" versions in terms of Π^0_1 formulas. For this reason, we slightly abuse terminology and call these schemes "Δ^0_1-comprehension."

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(The notation "∃!" is shorthand for "there exists exactly one.") Recall that Σ^0_1 formulas may have arbitrary parameters.

(4) Finally, the following axioms defining $^\frown$ and $*$:

$$\forall k^0, r^1, n^0[(k^\frown r)(n+1) = r(n) \wedge (k^\frown r)(0) = k],$$

and

$$\forall F^2, r^1, k^0[(F * r)(k) = F(k^\frown r)].$$

Before continuing further, it is worth explaining the definitions of $^\frown$ and $*$. The first axiom just says that $^\frown$ is the usual concatenation operation, appending a natural number to the beginning of a string of natural numbers. The second describes a way to turn type-2 functionals into type-$(1 \to 1)$ functionals, and is slightly more complicated. In order to vew a functional F as a map from reals to reals, we first replace an input real r by the infinite sequence of reals $(0^\frown r, 1^\frown r, \dots)$, and then apply F to each of the reals in this sequence in turn; this yields a sequence of natural numbers — that is, a real — $(F(0^\frown r), F(1^\frown r), \dots)$. This real is $F * r$.

This particular definition of $*$ is merely a technical device, and could be replaced with any of a number of similar constructions; the important point is that we have a way of interpreting a single real r as a sequence of reals, and that by applying a type-2 functional to each real in that sequence we can view the functional as a map from $\mathbb{R}$ to $\mathbb{R}$ instead of a map from $\mathbb{R}$ to ω.

Convention 2.4. Throughout this paper, if $M \models \mathsf{RCA}^3_0$ we will write M_0, M_1, M_2 for the type-0, -1, and -2 parts of M, respectively.

Note that if $(M_0, M_1, M_2; *_0, ^\frown_0), (M_0, M_1, M_2; *_1, ^\frown_1) \models \mathsf{RCA}^3_0$, then in fact

$$(M_0, M_1, M_2; *_0, ^\frown_0) = (M_0, M_1, M_2; *_1, ^\frown_1);$$

that is, models of RCA^3_0 are determined by their 0-, 1-, and 2-type objects, and it is enough to specify these types to specify the full model. Despite this, the symbols $^\frown$ and $*$ are necessary for RCA^3_0 in order for the comprehension schemes to have full force (given that we avoid objects of nonstandard type). As evidence of this, the following two facts are easy to prove, yet crucially rely on comprehension over Δ^0_1 formulas involving $^\frown$ and $*$:

Fact 2.5. RCA^3_0 *proves each of the following statements*:

(1) *For each type-2 functional F, there is a real r such that*

$$\forall s^1, n^0[\forall k^0(s(k) = n) \implies r(n) = F(s)].$$

(2) *For each type-2 functional F, there is a type-2 functional G such that*

$$G(\langle a_0, a_1, a_2, \dots, a_n, \dots\rangle) = F(\langle a_0, a_2, a_4, \dots, a_{2n}, \dots\rangle)$$

Proof. For (1), first note that the type-2 comprehension scheme gives us a functional I such that $\forall r^1[I(r) = r(1)]$, and hence

$$\forall r^1, k^0[\forall i^0(I * (k^\frown r)(i) = k)].$$

Now our desired real r can be defined by

$$r(k) = F(I * (k^\frown \underline{0})),$$

which exists by the type-1 comprehension scheme.

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For (2), let H be the type-2 functional defined by the quantifier-free formula $H(r) = r(2r(0)+1)$; then the desired G is defined by the quantifier-free formula

$$G(r) = k \iff F(H * r) = k,$$

and so again is guaranteed to exist by the type-1 comprehension scheme. ⊣

It can be shown that neither (1) nor (2) is provable if we restrict the Δ^0_1 comprehension schemes to formulas not involving $*$ and $\frown$. Essentially, $*$ and $\frown$ are the price we pay for a base theory which closely resembles RCA_0 and has reasonable models.

To drive this last point home, we end this section by presenting some natural models of RCA^3_0:

Example 2.6. Let $\mathcal{I}$ be a Turing ideal; that is, $\mathcal{I}$ is closed under the join $\oplus$ and is closed downwards under Turing reducibility. Then the smallest ω-model of RCA^3_0 containing precisely the reals in $\mathcal{I}$ is

$$\mathcal{S}_{\mathcal{I}} = (\omega, \mathcal{I}, \{r \mapsto \varphi^{r\oplus s}_e(0) : s \in \mathcal{I} \text{ and } \varphi^{r\oplus s}_e(0)\downarrow \text{ for every } r \in \mathcal{I}\}).$$

We will call such a pair (e, s) a *Turing code* for the map $r \mapsto \varphi^{r\oplus s}_e(0)$.

Proof. Any ω-model of RCA^3_0 whose real part is $\mathcal{I}$ must be at least as large as $\mathcal{S}_{\mathcal{I}}$, so it is enough to show that $\mathcal{S}_{\mathcal{I}} \models \mathsf{RCA}^3_0$. Axioms (1), (2), and (4) are immediate; it only remains to show that the comprehension schema are satisfied.

We focus on the type-2 case; the type-1 case is identical. Intuitively, we should be able to compute the value of any functional defined according to the Δ^0_1-comprehension scheme effectively from the real parameters and Turing codes for the type-2 parameters involved, since the value of the functional is determined by an effective collection of Σ^0_1 sentences. The only possible difficulty could arise from the new symbols, $\frown$ and $*$. To ensure that these pose no problems, we first observe that we can — uniformly in a Turing code for a functional F and a natural number $c \in \omega$ — find a Turing code for the map $r \mapsto F(c\frown r)$. The case of $*$ is slightly more interesting, but still poses no problems. By a straightforward induction on n, if $F_1, \cdots, F_n \in \mathcal{S}_{\mathcal{I}}$, then there is some $e \in \omega$ and $s \in \mathcal{I}$ such that, for every $r \in \mathcal{I}$ and $i \in \omega$, we have

$$\Phi^{r\oplus s}_e(i) = F_1 * (F_2 * (\cdots * (F_n * r)))(i).$$

It now follows by a tedious but straightforward induction on formula complexity that we can effectively compute the values of any type-2 functional defined in a Δ^0_1 fashion from parameters in $\mathcal{S}_{\mathcal{I}}$, uniformly in the real parameters and in Turing codes for the type-2 parameters. But this yields a Turing code for the functional so defined, which is therefore already in $\mathcal{S}_{\mathcal{I}}$. ⊣

Corollary 2.7. *The structure* $\mathcal{C} = (\omega, \mathbb{R}, \{f : \mathbb{R} \to \omega : f \text{ is continuous}\})$ *— when interpreted as an* L^3*-structure in the natural way — is the smallest model of* RCA^3_0 *containing all the reals.*

Example 2.8. Recall that the class of *Borel* sets is the smallest class of subsets of $\mathbb{R}$ containing the open sets which is closed under complementation and countable unions. Note that if a set is Borel, then this is witnessed by a well-founded tree whose terminal nodes are open intervals with rational endpoints, and whose other

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nodes correspond either to complementation or countable union; we will call the smallest rank of such a witnessing tree the *Borel rank* of the set. Then:

- a map $f : \mathbb{R} \to \omega$ is *Borel measurable* if $f^{-1}(i)$ is Borel for every $i \in \omega$, and
- a map $f : \mathbb{R} \to \mathbb{R}$ is *Borel measurable* if $f^{-1}(X)$ is Borel for every Borel $X \subseteq \mathbb{R}$.

We let $\mathcal{B}$ be the three-sorted structure $(\omega, \mathbb{R}, \{F : \mathbb{R} \to \omega : F \text{ is Borel measurable}\})$. Since $\mathcal{B}$ contains all the reals, each of the symbols in the language of RCA_0^3 has a natural interpretation in $\mathcal{B}$; in particular, $\mathcal{B}$ is closed under the operations $^\frown$ and $*$. Then $\mathcal{B} \models \mathsf{RCA}_0^3$.

We will use this model in a separation result later (2.23).

Proof. Since the first- and second-order parts of $\mathcal{B}$ are ω and $\mathbb{R}$, $\mathcal{B}$ is clearly closed under $^\frown$ and $*$, and satisfies parts (1), (2), and (4) of the axioms of RCA_0^3, as well as the Δ_1^0-comprehension scheme for type-1 objects. So it only remains to show that $\mathcal{B}$ satisfies the comprehension scheme for type-2 functionals.

There are multiple ways to show that $\mathcal{B}$ satisfies the Δ_1^0-comprehension scheme for type-2 functionals. First, we observe that if $X \subseteq \mathbb{R}$ is Borel and $k \in \omega$, then $X_k = \{r : k^\frown r \in X\}$ is also Borel; this is proved by a straightforward induction on the Borel rank of X, which we omit.

Now suppose $Y : \mathbb{R} \to \omega$ is Δ_1^0 relative to some type-1 parameters $r_0, \cdots, r_m$ and some type-2 parameters $F_0, \cdots, F_n$. That is, there is a Σ_0^0-formula $\psi(x^0, y^1, z^0)$ with only the displayed variables, which does not involve equality between type-1 or type-2 terms, with some type-1 parameters $r_0, \cdots, r_m$ and some type-2 parameters $F_0, \cdots, F_n$, such that

$$Y(r) = k \iff \exists x^0 \psi(x, r, k);$$

we will show that Y is Borel-measurable, and hence in $\mathcal{B}$.

To see this, fix $i \in \omega$ and consider the set of reals $X = Y^{-1}(i)$; we must show that X is Borel. Note that since

$$X = \bigcup_{j \in \omega} \{s : \psi(j, s, i)\},$$

so it is enough to show that the sets $X_j = \{s : \psi(j, s, i)\}$ are each Borel. So fix $j \in \omega$. The set X_j is a Boolean combination of sets of the form $X_{j,\alpha} = \{s : \alpha(j, r, i)\}$ for α atomic; so it is enough to show that each such set is Borel. So fix such an atomic α. Since ψ cannot involve any instances of equality of type-1 or -2 objects, α must have the form $\mathsf{t}_0 = \mathsf{t}_1$, for terms $\mathsf{t}_0, \mathsf{t}_1$ of type 0. It is easy to see that $X_{j,\alpha}$ is Borel if for each $k \in \{0, 1\}$ and $c \in \omega$, $\{s : \mathsf{t}_k(s) = c\}$ is Borel.

We will now be finished if we can show that, if $c \in \omega$ and t is a term with one free type-1 variable y^1, real parameters $r_0, \cdots, r_m$, and type-2 parameters $F_0, \cdots, F_n$, then $\{s : \mathsf{t}(s) = c\}$ is Borel. This is proved by induction on the complexity of the term. We omit most of the induction, since it is straightforward, and prove the only difficult part:

Claim. *if $F_0, \cdots, F_n$ are Borel, then the map $r \mapsto F_0 * (F_1 * (\cdots * (F_n * r)))$ is Borel.*

Proof of claim. It suffices to show that if F is Borel-measurable and $X \subseteq \mathbb{R}$ is Borel, then the set $P_X = \{r : F * r \in X\}$ is also Borel; that is, $r \mapsto F * r$ is Borel-measurable. This is proved by induction on the Borel rank of X. If X is open, then if $r \in P_X$ there is some finite initial segment σ of $F * r$ such that if s is any

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real extending σ, we have $r \in X$. So P_X contains the set $C_\sigma = \{s : \sigma \prec F * s\}$. But that set is the intersection of finitely many sets of the form $\{s : F(c^\frown s) = i\}$, which are Borel since F is Borel-measurable; so C_σ is Borel. Moreover, every element of P_X is contained in some C_σ, of which there are only countably many; so P_X is Borel.

Since taking preimages commutes with unions and complementation, the rest of the induction is immediate. ⊣

The remaining induction is uneventful. ⊣

2.2. Higher-type analogues of ATR_0. In what follows, we treat higher-type determinacy principles, and towards that end some definitions are necessary. We study games of length ω on $\mathbb{R}$ — that is, players I and II alternate playing real numbers, building an ω-sequence of reals. We identify both clopen games and open games with their underlying *game trees*, which are subtrees of $\mathbb{R}^{<\omega}$: thus, we identify clopen games with well-founded trees — players alternate playing reals, moving further along the tree, and the first player to be unable to play and stay on the tree loses — and we identify open games with trees — players alternate playing reals, and player I (Open) wins if and only if the play ever leaves the tree. There are several reasonable ways to encode game trees $\subseteq \mathbb{R}^{<\omega}$ as type-2 functionals and finite strings of reals as individual reals, and the specific choice of coding is unimportant. We will assume such a coding method in the background, so that we may for instance apply a type-2 functional to a node on a subtree of $\mathbb{R}^{<\omega}$; there will be no subtleties in this regard.

When discussing plays, however, things become more complicated. If Σ, Π are strategies, then the kth stage in the play $\Sigma \otimes \Pi$, $(\Sigma \otimes \Pi)_k$ — or rather, a real coding $(\Sigma \otimes \Pi)_k$ — is defined as follows. There is a functional F, whose existence is guaranteed by the comprehension scheme, such that $F * (k^\frown r)$ is the kth "row" of r; specifically, F is defined by

$$s \mapsto s(2 + \langle s(0), s(1)\rangle).$$

We say that a real r *codes* $(\Sigma \otimes \Pi)_k$ if

- $F * (0^\frown r) = \underline{0}$,
- $\forall 0 < 2j + 1 \leq k[F * ((2j + 1)^\frown r) = \Sigma * (F * ((2j)^\frown r))]$, and
- $\forall 0 < 2j + 2 \leq k[F * ((2j + 2)^\frown r) = \Pi * (F * ((2j + 1)^\frown r))]$;

similarly, we say that r codes the whole play $\Sigma \otimes \Pi$ if r codes $(\Sigma \otimes \Pi)_k$ for all k. This definition lets us refer to the play $\Sigma \otimes \Pi$ inside the language of RCA_0^3; and we use, e.g., "$(\Sigma \otimes \Pi)_k \notin T$" as shorthand for "there is a real r coding $(\Sigma \otimes \Pi)_k$, and $r \notin T$."

There is a subtlety here, which arises due to a particular weakness in the base theory RCA_0^3. (The end of this paper addresses the foundational aspects of this; for now, we simply treat it as it affects us.) RCA_0^3 is too weak to guarantee the existence of a real coding the whole play $\Sigma \otimes \Pi$. This is a consequence of Hunter's proof[5] ([8], Theorem 2.5) that the theory

[5]Originally formulated for Kohlenbach's theory RCA_0^ω, but immediately adaptable to RCA_0^3.

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$$\mathsf{RCA}_0^3{+}\mathcal{E}_1 := \mathsf{RCA}_0^3{+}\text{“}\exists J^2\forall x^1(J * x = x')\text{”}$$

is conservative over ACA_0: if Σ and Π are each the operator J described above, then "$\Sigma \otimes \Pi$ exists" implies "$\mathbf{0}^{(\omega)}$ exists," so that sentence cannot be a consequence of $\mathsf{RCA}_0^3{+}\mathcal{E}_1$, let alone RCA_0^3 itself.

This can be salvaged in general by altering the base theory; and in fact, since this same subtlety arises in other ways, this is a reasonable course of action — see the end of Section 4 of this paper. In our case, however, all potential difficulties are handled by the strength of the principles we consider. For example, in the definition of clopen and open determinacy, we use a strong definition of "winning strategy:" e.g., a strategy Σ for Open in an open game is winning if for every strategy Π for Closed, there is a real coding some stage $(\Sigma \otimes \Pi)_k$ of the game by which Σ has won. This builds into the statements of the theorems we examine all the strength we need to perform the intuitively natural calculations involving stages of games.

The end result is that, although we cannot meaningfully talk about the play of a game $\Sigma \otimes \Pi$ directly within RCA_0^3, the principles we study in this paper happen to have enough power to allow us to do so. As an example of this, it is easy to see that each of the principles introduced in Definition 2.9 below imply that at most one player has a winning strategy in an open or clopen game; however, this is not provable in the base theory RCA_0^3 alone.

Consider the following four theorems, all equivalent to ATR_0 over RCA_0:

- *Comparability of well-orderings*: If X, Y are well-orders with domain $\subseteq \mathbb{N}$, then there is an embedding from one into the other.
- *Clopen determinacy*: Every well-founded subtree of $\omega^{<\omega}$, viewed as a clopen game, is determined.
- *Open determinacy*: Every subtree of $\omega^{<\omega}$, viewed as an open game, is determined.
- Σ_1^1 *separation*: If $\varphi(A)$ is a Σ_1^1 sentence (possibly with parameters) with a single free set variable, and $X = (X_i)_{i\in\omega}$ is an array of sets such that
$$\forall k \in \omega \exists j \in 2(\neg\varphi(X_{\langle k,j\rangle})),$$
then there is some set Y such that
$$\forall k \in \omega(\neg\varphi(X_{\langle k,Y(k)\rangle})).$$

These each have reasonable higher-type analogues, each of which is a theorem of ZFC:

Definition 2.9. Over RCA_0^3, we define the following principles:

- The comparability of well-orderings of reals, $\mathsf{CWO}^{\mathbb{R}}$: If X, Y are well-orderings with domain $\subseteq \mathbb{R}$, then there is an embedding from one into the other.
- Clopen determinacy for reals, $\Delta_1^{\mathbb{R}}$-Det: for every tree $T \subseteq \mathbb{R}^{<\omega}$ which is well-founded, viewed as a clopen game, either there is a winning strategy for player I:
$$\exists\Sigma\colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall\Pi\colon \mathbb{R}^{<\omega} \to \mathbb{R}[\exists k \in \omega((\Sigma\otimes\Pi)_{2k+1} \in T \wedge (\Sigma\otimes\Pi)_{2k+2} \notin T)];$$
or there is a winning strategy for player II:
$$\exists\Pi\colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall\Sigma\colon \mathbb{R}^{<\omega} \to \mathbb{R}[\exists k \in \omega((\Sigma\otimes\Pi)_{2k} \in T \wedge (\Sigma\otimes\Pi)_{2k+1} \notin T)].$$

- Open determinacy for reals, $\Sigma_1^{\mathbb{R}}$-Det: for every tree $T \subseteq \mathbb{R}^{<\omega}$, viewed as an open game, either there is a winning strategy for player I (Open):
$$\exists \Sigma\colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall \Pi\colon \mathbb{R}^{<\omega} \to \mathbb{R}[\exists k \in \omega((\Sigma \otimes \Pi)_k \notin T)];$$
or there is a winning strategy for player II (Closed):
$$\exists \Pi\colon \mathbb{R}^{<\omega} \to \mathbb{R}, \forall \Sigma\colon \mathbb{R}^{<\omega} \to \mathbb{R}[\forall k \in \omega((\Sigma \otimes \Pi)_k \in T)].$$
- The Σ_1^2-separation principle, Σ_1^2-$\mathsf{Sep}^{\mathbb{R}}$: If $\varphi(f^2)$ is a Σ_1^2-formula with a single type-2 free variable, and $X = (X_\eta)_{\eta \in \mathbb{R}}$, $Y = (Y_\eta)_{\eta \in \mathbb{R}}$ are real-indexed collections of type-2 functionals[6] such that
$$\neg \exists x^1 (\varphi(X_x) \wedge \varphi(Y_x)),$$
then there is some type-2 object F such that
$$\forall x^1 [\varphi(X_x) \implies F(x) = 1 \quad \text{and} \quad \varphi(Y_x) \implies F(x) = 0].$$

(Note that, strictly speaking, Σ_1^2-$\mathsf{Sep}^{\mathbb{R}}$ is an infinite scheme, as opposed to a single sentence.) It is these principles which we choose to study in this paper. The remainder of this section is devoted to the simpler parts of their analysis; the separation of clopen and open determinacy for reals is the subject of the following section.

Note that the determinacy principles above are not provable in ZF alone, whereas $\mathsf{CWO}^{\mathbb{R}}$ and Σ_1^2-$\mathsf{Sep}^{\mathbb{R}}$ are, so in order to analyze these principles properly we need some version of the axiom of choice:

Definition 2.10. Let $\langle \cdot, \cdot \rangle$ be an appropriate pairing function on $\mathbb{R}$. The *selection principle* for $\mathbb{R}$, $\mathsf{SF}(\mathbb{R})$, is the assertion that for every $\mathbb{R}$-indexed set of nonempty sets of reals has a selection functional; that is, for every type-2 functional F — interpreted as the $\mathbb{R}$-indexed set of reals
$$\{\{s \in \mathbb{R} : F(\langle r, s \rangle) = 1\} : r \in \mathbb{R}\}$$
— there is a type-2 functional G satisfying
$$\forall r^1 (F(\langle r, G * r \rangle) = 1).$$

Now we turn to the implications. Clearly $\Sigma_1^{\mathbb{R}}$-Det implies $\Delta_1^{\mathbb{R}}$-Det. A more interesting implication is the following:

Fact 2.11. *Over* RCA_0^3, *we have*
$$\Sigma_1^2\text{-}\mathsf{Sep}^{\mathbb{R}} + \mathsf{SF}(\mathbb{R}) \implies \Delta_1^{\mathbb{R}}\text{-Det}.$$

Proof. This is somewhat involved. We begin with three technical results, which are of independent interest:

Fact 2.12 (Comprehension). $\mathsf{RCA}_0 + \Sigma_1^2$-$\mathsf{Sep}^{\mathbb{R}}$ *implies* Δ_1^2*-comprehension for type-2 functionals for each n: given* $n \in \omega$ *and any* Σ_1^2 *formula* φ *with one type-1 variable which is equivalent to a* Π_1^2 *formula, there is a functional F such that*
$$F(r) = 1 \iff \varphi(r), \quad F(r) = 0 \iff \neg\varphi(r)$$

[6]A real-indexed set of type-2 functionals $(Z_s)_{s \in \mathbb{R}}$ is coded by the type-2 functional
$$\hat{Z}\colon r \mapsto Z_{P_0 * r}(P_1 * r),$$
where P_0, P_1 correspond to the left and right projections of a reasonable pairing function $\mathbb{R}^2 \cong \mathbb{R}$.

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Proof. Apply Σ^2_1-Sep$^{\mathbb{R}}$ to the pair $(\varphi, \neg\varphi)$. ⊣

Fact 2.13 (Iteration). RCA^3_0+Σ^2_1-Sep$^{\mathbb{R}}$ *proves that given a functional F and a real r, we can form the iteration sequence $(r, F * r, F * (F * r), \cdots)$. Formally, for every type-2 functional F there is a type-2 functional G such that — for every r — we have*

$$G * (n^\frown r) = F * (F * (\cdots * (F * r)))$$

(*where the right hand side contains n applications of "$F*$"*).

In particular, note that this implies that given any strategies Σ_0 and Σ_1, the real coding their entire play $\Sigma_0 \otimes \Sigma_1$ exists.

Proof. Fix a real r. Let $\chi(x^1, y^0, z^1)$ be the formula asserting that x is a real whose first y rows are of the form $z, F * z, F * (F * z), \cdots$ Then let $\varphi(w^1)$ be the formula

$$\exists s(\chi(s, w(1), w^{--}), s(w(0) = 0))$$

and let ψ be the formula

$$\forall s(\chi(s, w(1), w^{--}) \implies s(w(0) = 1)).$$

Clearly φ and ψ are Σ^2_1 and $\varphi \wedge \psi$ is inconsistent, so we may apply Σ^2_1-Sep$^{\mathbb{R}}$. This yields the desired G. ⊣

Proposition 2.14 (Paths from subtrees). RCA^3_0+Σ^2_1-Sep$^{\mathbb{R}}$+$\mathsf{SF}(\mathbb{R})$ *proves that a tree $T \subseteq \mathbb{R}^{<\omega}$ is well-founded if and only if it has no nonempty subtrees with no terminal nodes.*

Proof. Clearly a witness to T being ill-founded yields a subtree with no terminal nodes. In the other direction, suppose T is well-founded but has a nonempty subtree S with no terminal nodes. By $\mathsf{SF}(\mathbb{R})$ and 2.12, there is a functional F such that if σ is a predecessor of an element of S, then $F(\sigma) \in S$ is an extension of σ. Fix $\sigma_0 \in S$ and let $\sigma_{i+1} = F(\sigma_i)$ for $i > 0$; by 2.13, the sequence $(\sigma_0, \sigma_1, \cdots)$ exists. ⊣

We now return to the proof of 2.11. Let $T \subseteq \mathbb{R}^{<\omega}$ be a well-founded tree, viewed as a clopen game; we will show that T is determined.

Definition 2.15. For $\sigma \in T$, let $T[\sigma] = \{\rho : \sigma \preccurlyeq \rho \in T\}$. A *$U$-tree for σ* is a functional $F : T[\sigma] \to \{Safe, Unsafe\}$ satisfying the following properties:

1. $ev(\sigma) = Unsafe$;
2. $ev(\tau) = Safe \iff$ there is some immediate extension ρ of τ such that $\rho \in T$ and $ev(\rho) = Unsafe$.

An *S-tree for σ* is a pair (ρ, F) such that $\rho \in T$ is an immediate extension of σ and F is a U-tree for ρ. We let $\varphi_U(\sigma)$ and $\varphi_S(\sigma)$ be the sentences, "There is a U-tree for σ" and "There is an S-tree for σ," respectively; note that both φ_U and φ_S are Σ^2_1.

Intuitively, the existence of a U-tree for σ indicates that σ is *unsafe*, that is, the game G_σ is a win for player II. Similarly, the existence of an S-tree for σ provides one bit of information towards a winning strategy for player I in G_σ.

Lemma 2.16. *No $\sigma \in T$ satisfies $\varphi_U(\sigma) \wedge \varphi_S(\sigma)$.*

Proof. Otherwise, let T_0 and T_1 be U- and S-trees for σ; then the set of nodes ρ on which $T_0(\rho) \neq T_1(\rho)$ forms a nonempty subtree of T with no terminal nodes, contradicting the wellfoundedness of T via 2.14. ⊣

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By 2.16 we can apply Σ^2_1-$\mathsf{Sep}^{\mathbb{R}}$to get a functional $ev: T \rightarrow \{Safe, Unsafe\}$ such that

- $\varphi_U(\sigma) \implies ev(\sigma) = Unsafe$,
- $\varphi_S(\sigma) \implies ev(\sigma) = Safe$.

It is now enough to show that either ev is a U-tree for $\langle\rangle$, or there is some length-1 string $\langle a\rangle \in T$ such that the restriction $ev_{\langle a\rangle}$ to $T[\langle a\rangle]$ is a U-tree for $\langle a\rangle$. To see that this is sufficient, suppose ev is a U-tree for $\langle\rangle$ (the other case is identical). Then by $\mathsf{SF}(\mathbb{R})$there is a strategy for player II such that, if $|\sigma|$ is odd and $ev(\sigma) = Safe$, then $ev(\sigma^\frown\Sigma(\sigma))$ is on T and is marked *Unsafe* by ev; this strategy can clearly never lose, so by 2.13 Σ is a winning strategy.

Definition 2.17. Say that a node σ of T is *bad* if one of the following conditions holds:

(1) $ev(\sigma) = Safe$ but for every immediate extension ρ of σ we have $ev(\rho) = Safe$, or
(2) $ev(\sigma) = Unsafe$ but there is some immediate extension ρ of σ such that $ev(\rho) = Unsafe$.

Lemma 2.18. *If σ is bad, then there is some proper extension of σ which is bad.*

Proof. Suppose σ is bad but no proper extension of σ is bad.

If σ is bad via case (1), then the map

$$ev': T[\sigma] \rightarrow \{Safe, Unsafe\}: \rho \mapsto \begin{cases} ev(\rho) & \text{if } \rho \neq \sigma \\ Unsafe & \text{if } \rho = \sigma \end{cases}$$

is a U-tree for σ whose existence follows from 2.12; but this contradicts the definition of ev.

If σ is bad via case (2), then we can similarly construct an S-tree for σ, again contradicting the definition of ev. ⊣

Corollary 2.19. *There are no bad nodes of T.*

Proof. Suppose otherwise. The set of bad nodes exists by 2.12; by 2.18 and 2.14, this contradicts the well-foundedness of T. ⊣

But now we are done: since T has no bad nodes, either ev is a U-tree for $\langle\rangle$, or — letting σ be some length-1 node of T which satisfies $ev(\sigma) = Unsafe$ — ev_σ is a U-tree for σ, and as observed above either possibility allows us to produce a winning strategy for T. ⊣

Note that at the close of the proof, we conclude that in fact for every $\sigma \in T$ we have $\varphi_U(\sigma) \iff \neg\varphi_S(\sigma)$; yet since the proof of this fact itself goes through Σ^2_1-$\mathsf{Sep}^{\mathbb{R}}$, the slightly weaker theory $\mathsf{SF}(\mathbb{R})$+Δ^2_1-comprehension for type-2 functionals does not seem to imply $\Delta^{\mathbb{R}}_1$-Det.

To compliment 2.11, we show that the assumption of $\mathsf{SF}(\mathbb{R})$ cannot be removed:

Fact 2.20. *Over* RCA^3_0, $\Delta^{\mathbb{R}}_1$-Det *implies* $\mathsf{SF}(\mathbb{R})$.

Proof. Let F be an instance of $\mathsf{SF}(\mathbb{R})$, viewed as an $\mathbb{R}$-indexed family of sets of reals $\{F_r\}_{r\in\mathbb{R}}$. Consider the game in which player I plays a real r, and player II wins if and only if they immediately play a real $s \in F_r$. A winning strategy cannot exist

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for player I, and any winning strategy for player II immediately yields a selection functional for F. ⊣

2.11 and 2.20 together raise the question of the role that variants of the axiom of choice might play in higher-order versions of ATR_0. This will be treated in more detail in a forthcoming paper; for now, we introduce one final principle, which is close to the classical statement of ATR_0 itself and which captures exactly the choiceless part of $\Delta^{\mathbb{R}}_1$-Det:

Definition 2.21. For a tree $T \subseteq \mathbb{R}^{<\omega}$ and a node $\sigma \in T$ we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$, and if $F : T \to \omega$ we let $F_\sigma : \tau \mapsto F(\sigma^\frown\tau)$. Σ^1_1 *rank-recursion on* $\mathbb{R}$, denoted "$\mathsf{RR}_1(\mathbb{R})$," is then the scheme asserting that for any tree $T \subseteq \mathbb{R}^{<\omega}$ which does not contain a nonempty subtree with no terminal nodes (recall that, absent choice, this is a strengthening of well-foundedness) and every Σ^1_1 formula Σ^1_1-formula $\varphi(Y^2, Z^2)$ with only the displayed free variables, there is a type-2 functional F with range $\subseteq \{0, 1\}$ such that, for $\sigma \in T$,

$$F(\sigma) = 1 \iff \varphi(F_\sigma, T_\sigma).$$

Theorem 2.22. *Over* RCA^3_0, *we have* $\mathsf{RR}_1(\mathbb{R}) + \mathsf{SF}(\mathbb{R}) \iff \Delta^{\mathbb{R}}_1$-Det.

Proof. $\Delta^{\mathbb{R}}_1$-Det $\implies \mathsf{RR}_1(\mathbb{R}) + \mathsf{SF}(\mathbb{R})$: we have already observed (2.20) that $\Delta^{\mathbb{R}}_1$-Detimplies $\mathsf{SF}(\mathbb{R})$. To show that $\Delta^{\mathbb{R}}_1$-Detimplies $\mathsf{RR}_1(\mathbb{R})$, given a well-founded T and an appropriate formula φ, consider the following well-founded game. First, player I chooses some $\sigma \in T$; then, player II responds by playing either "Safe" or "Unsafe." The game then continues by playing the clopen game T_σ, with player II going first if she chose "Safe" and player II going second if she chose "Unsafe." Clearly only player II can have a winning strategy, and any winning strategy computes the desired h by setting $h(\sigma) = 0$ if the winning strategy for II tells II to play "Unsafe" if I plays σ.

In the other direction, given a clopen game G, use $\mathsf{RR}_1(\mathbb{R})$ with the formula

$$\varphi(Y^2, Z^2) \equiv \exists a^1(a \in Z \text{ and } Y(a) = 0)$$

(recall that Y is meant to stand for F_σ and Z for T_σ). The resulting function h then computes a winning quasistrategy for G: if $h(\sigma) = 0$, then σ is a loss for whoever's turn it is, and one player or the other can win by ensuring that their opponent always plays from nodes marked 0 by h. $\mathsf{SF}(\mathbb{R})$ then lets us pass from this winning quasistrategy to a genuine winning strategy for G. ⊣

We end this section by presenting a straightforward separation result — the first instance of divergence from the standard reverse-mathematical picture. Given the low complexity of wellfoundedness at higher types, it is reasonable to expect that $\mathsf{CWO}^{\mathbb{R}}$ is quite weak relative to the higher-type determinacy principles. This is, in fact, true:

Lemma 2.23. *Over* RCA^3_0, $\mathsf{CWO}^{\mathbb{R}}$ *does not imply* $\Delta^{\mathbb{R}}_1$-Det.

Proof. We will show that in fact the model $\mathcal{B}$ generated by the Borel sets, defined in 2.8, satisfies $\mathsf{CWO}^{\mathbb{R}}$ but not $\Delta^{\mathbb{R}}_1$-Det. Showing that $\mathcal{B} \models \mathsf{CWO}^{\mathbb{R}}$ is straightforward: any uncountable Borel set of reals contains a perfect subset, and there is no Borel well-ordering of $\mathbb{R}$. These facts follow from Borel determinacy ([9], Theorem 20.5), and together imply that all Borel well-orderings are countable. It then follows that any two Borel well-orderings are comparable by a boldface $\mathbf{\Sigma^0_2}$ embedding, so $\mathcal{B} \models \mathsf{CWO}^{\mathbb{R}}$.

To show that $\mathcal{B} \models \neg\Delta^{\mathbb{R}}_1$-Det, fix some analytic (that is, boldface $\mathbf{\Sigma^1_1}$) set $X \subseteq \mathbb{R}$ which is not Borel. Let $T \subseteq \omega^{<\omega}$ be a tree such that

$$X = \{a \in \mathbb{R} : \exists b \in \mathbb{R}((\langle a(i), b(i)\rangle)_{i\in\omega} \in [T])\};$$

such a tree is guaranteed to exists since X is Σ^1_1, and since $\mathcal{B}$ contains all reals we have that $T \in \mathcal{B}$. Now consider the game $\mathcal{G}$ which proceeds as follows:

- Player I plays a real a.
- Player II guesses whether $a \in X$ or not.
- If player II guesses "yes," then player II must also play a real b; player I then plays a natural number k; the game is now over, and player I wins if and only if $(\langle a(i), b(i)\rangle)_{i<k} \notin T$.
- If player I guesses "no," then player I plays a real b, player II plays a natural number k; the game is now over, and this time player II wins if and only if $(\langle a(i), b(i)\rangle)_{i<k} \notin T$.

Informally, player I is challenging player II to correctly compute X, and the tree T is used to evaluate whether II's guess was correct. This is a clopen game, and viewed as a subtree of $\mathbb{R}^{<\omega}$ it is clearly Borel, so $\mathcal{G} \in \mathcal{B}$.

However, this game is undetermined in $\mathcal{B}$. To see this, note that since $\mathcal{B}$ contains every real, a strategy in $\mathcal{B}$ is winning in $\mathcal{B}$ if and only if it is actually winning, since otherwise any play defeating it would be coded by a real and hence exist in $\mathcal{B}$. So if $\mathcal{B}$ satisfies $\Delta^{\mathbb{R}}_1$-Det, then $\mathcal{B}$ must contain an actual winning strategy for $\mathcal{G}$; but X is Borel relative to any winning strategy for $\mathcal{G}$ (since such a strategy Σ must be a strategy for player II, and must have the property that $\Sigma(\langle a\rangle) = 1 \iff a \in X$). Since $\mathcal{B}$ consists precisely of the Borel functionals, if $\mathcal{G}$ were determined in $\mathcal{B}$ then X would have to be Borel, which is a contradiction. ⊣

Note that Borel instances of $\Delta^{\mathbb{R}}_1$-Det can be constructed whose winning strategies are much more complex than Σ^1_1; so $\mathsf{CWO}^{\mathbb{R}}$ is in fact *far* weaker than $\Delta^{\mathbb{R}}_1$-Det.

§3. Separating clopen and open determinacy. In this section we construct a model M of $\mathsf{RCA}^3_0+\Delta^{\mathbb{R}}_1$-Det $+\neg\Sigma^{\mathbb{R}}_1$-Det, using a variation of Steel's tagged tree forcing; see [23], and also [16] and [19]. Throughout this section, we work over a transitive ground model V of ZFC+CH.

Remark 3.1. Recently, Sherwood Hachtman [5] has developed an alternate proof of this result; using methods from inner model theory, he shows that the smallest initial segment of Goedel's constructible universe L which is a model of $\mathsf{RCA}^3_0+\Delta^{\mathbb{R}}_1$-Det does not satisfy $\Sigma^{\mathbb{R}}_1$-Det. More precisely, he shows that if θ is the least ordinal such that $(\omega, \mathbb{R}, \omega^{\mathbb{R}})^{L_\theta} \models \mathsf{RCA}^3_0+\Delta^{\mathbb{R}}_1$-Det, then $(\omega, \mathbb{R}, \omega^{\mathbb{R}})^{L_\theta} \models \neg\Sigma^{\mathbb{R}}_1$-Det.

The general picture of classical Steel forcing is as follows. Conditions are well-founded trees, with additional information representing rank, ordered by extension (with certain restrictions). The full generic object is an infinite, ill-founded tree, whose nodes are labelled with their ranks in the tree, together with a collection of distinguished paths. The model built from this generic is gotten by looking at all sets hyperarithmetic relative to the tree and finitely many of the paths; in particular, the ordinal labels are forgotten. This loss of information is crucial.

In our case, our conditions will be countable, ill-founded trees with additional information, ordered appropriately. The generic object will be, as in the classical case, a tree whose nodes are labelled essentially with their rank. This tree can be viewed as the game tree of an open game; this open game, which is classically a win for player II (Closed), will exist but be undetermined in our model. The difficult portion of the proof is ensuring that the model we build satisfies $\Delta^{\mathbb{R}}_1$-Det. Rather than use a higher-order notion of hyperarithmeticity (see below), we construct our model out of those functionals which depend on the generic tree only in a limited way; see Definition 3.4.

The idea behind the game we construct is as follows. Consider the clopen game G_α, for α an ordinal, in which players I and II alternately build decreasing sequences of ordinals less than α, and the first player whose sequence terminates loses. Clearly player II wins this game, since all she has to do is consistently play slightly larger ordinals than what player I plays.

$$G_\alpha : \quad \begin{array}{l|lllll} \text{Player I} & \alpha_0 & & \alpha_1 < \alpha_0 & & \cdots \\ \hline \text{Player II} & & \beta_0 & & \beta_1 < \beta_0 \cdots & \end{array}$$

Now there is a natural open game, $\mathcal{O}_\alpha$, associated to G_α. $\mathcal{O}_\alpha$ has the same rules as G_α, except that on player I's turn, she can give up and start over, playing an arbitrary ordinal below α. If she does this, then player II gets to play an arbitrary ordinal below α as well. After a restart, play then continues as normal, until player II loses or player I restarts again. Player I (Open) wins if player II's sequence ever reaches zero; player II (Closed) wins otherwise.

$$\mathcal{O}_\alpha : \quad \begin{array}{l|lllll} \text{Player I (Open)} & \alpha_0 & & \alpha_1 & & \cdots \\ \hline \text{Player II (Closed)} & & \beta_0 & & \beta_1 \cdots & \end{array} \quad (\forall i, \alpha_{i+1} < \alpha_i \implies \beta_{i+1} < \beta_i).$$

Essentially, $\mathcal{O}_\alpha$ is gotten by "pasting together" ω-many copies of G_α, one after the other, and player II must win *all* of these clopen sub-games in order to win $\mathcal{O}_\alpha$. This is still a win for player II, but in a more complicated fashion. In particular, if player II happened to not be able to directly see the ordinals player I played, but was only able to see the underlying game tree itself, she would need quite a lot of transfinite recursion to be able to figure out what move to play next - seemingly more than she would need to win G_α, since there is much more "noise" in the structure of $\mathcal{O}_\alpha$. This is roughly the situation we create in the construction below. We will define a forcing notion which adds a tree $T_G \subseteq \mathbb{R}^{<\omega}$. This tree can be viewed as an open game on $\mathbb{R}$ of length ω in the usual manner. In the full generic extension, this game will be identical to the game $\mathcal{O}_{\omega_2^V}$ — that is, the game tree of $\mathcal{O}_{\omega_2^V}$ will be isomorphic to T_G in the full generic extension — but the function which assigns to nodes of T_G their ordinal ranks will be extremely complicated.

There are several differences between our construction and Steel's tagged tree forcing, however. Most importantly, our forcing is countably closed. Countable closure is an extremely powerful condition, which we use throughout this argument but especially in Lemma 3.16; at the same time, countable closure also adds a layer of complexity to the proof of the retagging lemma, an important combinatorial property of Steel-type forcings, which usually follows from well-foundedness of the trees underlying the forcing conditions. In our case, the proof of the retagging lemma uses a much weaker "local well-foundedness" property. Additionally, there is an important shift in how we define the desired substructure of the full generic

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extension. In classical Steel forcing, the desired substructure is defined by first picking out specific elements of the generic extension — usually paths through a certain tree — and then closing under hyperarithmetic reducibility; the proof then continues by showing that every element of the resulting model depends only on "bounded" information about the generic. In our case, we start at the end, and simply consider the part of the generic extension depending on the generic in a "bounded" way. This is both clearer and more flexible a method than the standard approach; also, higher-type analogue of the hyperarithmetic sets — the so-called "hyperanalytic" sets — is more complicated to work with. See [17] for a definition of this analogue, as well as an account of some early difficulties faced in its study.

3.1. Constructing the model. The forcing we use in this section is the following:

DEFINITION 3.2. Let $\omega_2^* = \omega_2 \cup \{\infty\}$, ordered by taking the usual order on ω_2 and setting $\infty > x$ for all $x \in \omega_2^*$ (including $\infty > \infty$). $\mathbb{P}$ is the forcing consisting of all partial maps $p\colon \subseteq \mathbb{R}^{<\omega} \to \omega_2^* \times \omega_2^*$ satisfying the following conditions, ordered by reverse inclusion:

- $dom(p)$ is a countable subtree of $\mathbb{R}^{<\omega}$ with $p(\langle\rangle)\downarrow= (\infty, \infty)$ (the game starts with player Open moving, and no meaningful tags);
- $\sigma \in dom(p) \implies [(|\sigma| = 2k + 1 \wedge p(\sigma^-)_1 = p(\sigma)_1) \vee (|\sigma| = 2k \wedge p(\sigma^-)_0 = p(\sigma)_0)]$ (player Open is playing $p(\sigma)_0$, Closed is playing $p(\sigma)_1$, and on a given turn exactly one of these values changes);
- if $p_1(\sigma) = 0$, then no extension of σ is in the domain of p (if Closed ever hits 0, she loses); and
- $\sigma^\frown\langle a, b\rangle \in dom(p)$, $|\sigma| = 2k$, $\infty \neq p(\sigma)_0 > p(\sigma^\frown\langle a, b\rangle)_0 \implies p(\sigma)_1 > p(\sigma^\frown\langle a, b\rangle)_1$ (as long as player Open has not just played an ∞, or failed to play less than her previous play, Closed's next play has to be less than her previous play).

Note that the way this last condition is phrased allows $p(\sigma)_1$ to be anything when $p(\sigma)_0 = \infty$, for $|\sigma| = 2k$, since we have $\infty > \infty$. Also, if $|\sigma| = 2k$ and $p(\sigma^-)_1 = \infty$, then $p(\sigma)_1$ can be anything.

From this point on, we fix a filter $G \subseteq \mathbb{P}$ which is $\mathbb{P}$-generic over V.

The main difference between our forcing $\mathbb{P}$ and Steel forcing is that $\mathbb{P}$ is countably closed (recall 1.2). The immediate use of countable closure is that it lets us completely control the type-1 objects in our model; later, we will use countable closure in a more subtle way, to show that no well-orderings of reals of length $\geq \omega_2^V$ are in our model, even though such well-orderings will exist in the full generic extension (Lemma 3.16).

As with Steel forcing, we have a retagging notion:

DEFINITION 3.3. For $p, q \in \mathbb{P}$ and $\alpha \in \omega_2$, we say that q is an *α-retagging of p*, and write $p \approx_\alpha q$, if

- $dom(p) = dom(q)$;
- for $\sigma \in dom(p), i \in 2$ we have

$$p(\sigma)_i < \alpha \implies q(\sigma)_i = p(\sigma)_i$$

and

$$p(\sigma)_i \geq \alpha \implies q(\sigma)_i \geq \alpha.$$

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These retagging relations let us define the set of names which depend on the generic in a "bounded" way:

DEFINITION 3.4. Let ν be a name for a type-2 functional, that is, a map $\mathbb{R} \to \omega$, and suppose $\alpha \in \omega_2$. Then ν is *α-stable* if for all $a \in \mathbb{R}$, $k \in \omega$, we have

$$\forall p, q \in \mathbb{P}[p \approx_\alpha q, p \Vdash \nu(a) = k \implies q \Vdash \nu(a) = k.]$$

Finally, we can define our desired model:

DEFINITION 3.5. Fix G $\mathbb{P}$-generic over V. M is defined inductively to be the L^3-structure

$$M = (\omega, \mathbb{R}, \{\nu[G] : \exists\alpha < \omega_2(\nu \text{ is } \alpha\text{-stable})\}).$$

The purpose of this section is to prove

THEOREM 3.6. $M \models \mathsf{RCA}_0^3 + \Delta_1^{\mathbb{R}}\text{-}\mathsf{Det} + \neg\Sigma_1^{\mathbb{R}}\text{-}\mathsf{Det}$.

We begin with two simple properties of the model M.

DEFINITION 3.7. T_G is the underlying tree of G; that is,

$$T_G = \{\sigma \in \mathbb{R}^{<\omega} : \exists p \in G(\sigma \in dom(p))\}.$$

FACT 3.8.

(1) $\mathcal{P}(\omega^\omega) \cap V \subset M_2$.
(2) $T_G \in M_2$.

PROOF. (1) follows from the fact that canonical names for sets in V do not depend on the poset $\mathbb{P}$, and are hence 0-stable. For (2), the only way to force $\sigma \notin T_G$ is to have some $p \in G$, $\tau \prec \sigma$ such that $p(\tau)_1 = 0$, so it follows that the canonical name for T_G is 1-stable. ⊣

We can now prove the first nontrivial fact about M: that it does not satisfy open determinacy for reals. Specifically, we will show that T_G, viewed as an open game, is undetermined in M.

The first step is the following:

LEMMA 3.9. $V[G] \models T_G$ *is a win for Closed.*

PROOF. By a straightforward density argument, if G is generic, then whenever $|\sigma| = 2k + 1$, $p \in G$, and $p(\sigma)_1 = \infty$, there is some $q \in G$ and $a \in \mathbb{R}$ such that $q(\sigma^\frown\langle a\rangle)_1 = \infty$. It follows that the strategy

$$\Pi(\sigma) = \text{ the } \leq_W\text{-least } a \text{ such that } \exists p \in G(p(\sigma^\frown\langle a\rangle)_1 = \infty)$$

is winning for Closed. ⊣

The indeterminacy of T_G in M then follows from a two-part argument: strategies for Open can be defeated using 3.9 and the countable closure of $\mathbb{P}$, and stable strategies for Closed can be defeated by pulling the rug out from under her:

LEMMA 3.10. $M \models \neg\Sigma_1^{\mathbb{R}}\text{-}\mathsf{Det}$.

PROOF. Consider the open game corresponding to T_G (in which player I is Open). Recall that T_G is in M and T_G is "really" a win for player Closed by 3.8 and 3.9, respectively; we claim that this game is undetermined in M.

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Suppose Σ is a strategy for player Open in M. Consider the tree of game-states "allowed" by Σ:

$$A_\Sigma = \{\sigma \in T_G : \exists\Pi(\sigma \prec \Sigma \otimes \Pi)\}.$$

Since T_G is actually a win for Closed, the tree A_Σ must be ill-founded. Let $f \in V[G]$ be a path through T_G. Then $f \in V$, since $\mathbb{P}$ is countably closed and f can be coded by a single real. But then within V, we can construct a strategy Π which defeats Σ by playing along f:

$$\tau \prec f \implies \Pi(\tau) = f(|\tau|), \quad \tau \not\prec f \implies \Pi(\tau) = 0.$$

Since Π exists in V, $\Pi \in M_2$; so T_G is not a win for Open in M.

Now suppose Π is a strategy for player Closed in M, and suppose (towards a contradiction) that

$$p \Vdash \nu \text{ is a winning strategy in } T_G,$$

where ν is an α-stable name for Π, $\alpha \in \omega_2$. We can find

- $q \leq p$,
- $a \in \mathbb{R} - \{c : \langle c\rangle \in dom(p)\}$,
- $b \in \mathbb{R}$, and
- $\beta > \alpha$

such that $\langle a, b\rangle \in dom(q)$, $q(\langle a\rangle) = (\beta, \infty)$, and $q \Vdash \nu(\langle a\rangle) = b$. Now since $q \leq p$ and p forces that Π wins, we must have $q(\langle a, b\rangle) = (\beta, \gamma)$ with $\gamma > \beta$; so $\gamma > \alpha$. But then we can find a $\hat{q} \approx_\alpha q$ such that $\hat{q} \leq p$ and $\hat{q}(\langle a, b\rangle) = (\hat{\beta}, \hat{\gamma})$ for some $\hat{\beta} > \hat{\gamma}$. But then $\hat{q}$ forces that there is some finite play extending $\langle a, b\rangle$ which is a win for Open; and since every possible finite play exists in M, this contradicts the assumption that ν was forced to be a name for a winning strategy. ⊣

To analyze M further, we require the analogue of Steel's retagging lemma for our forcing:

LEMMA 3.11 (Retagging). *Suppose $\alpha < \omega_2$ has uncountable cofinality, $p \approx_\alpha q$, $r \leq q$, and $\gamma < \alpha$. Then there is some $\hat{r} \leq p$ with $\hat{r} \approx_\gamma r$.*

PROOF. This is a straightforward combinatorial construction. It is worth noting, however, that Steel's retagging lemma is proved using the fact that conditions in Steel forcing are (essentially) well-founded trees. Of course, are conditions are not well-founded, so we must be slightly more subtle: the heart of this proof is the realization that conditions in $\mathbb{P}$, though not well-founded, are "locally well-founded" in a precise sense. Intuitively, when deciding how to tag a given node of r', we only need to look at a well-founded piece of the domain of r; using the ranks of these well-founded pieces as parameters gives us enough "room" for the natural construction to go through.

Formally, we proceed as follows. Since α has uncountable cofinality, we can find a $\tilde{\gamma}$ such that $\gamma < \tilde{\gamma} < \alpha$ and $\tilde{\gamma}$ is larger than every $r(\sigma)_i$ and $p(\tau)_i$ ($i \in \{0, 1\}, \sigma, \tau \in \mathbb{R}^{<\omega}$) which is less than α.

For $\sigma \in dom(r) - dom(p)$, let

$$T_\sigma = \{\tau : \sigma^\frown\tau \in dom(r) \wedge \forall\rho \prec \tau(|\sigma^\frown\rho| \text{ odd } \implies \infty \neq r(\sigma^\frown\rho^-)_0 > r(\sigma^\frown\rho)_0)\}$$

be the set of ways to extend σ within $dom(r)$ which according to r don't involve player Open restarting after σ, and note that for each $\sigma \in dom(r) - dom(p)$ the tree T_σ is well-founded. Also, let N be the set of nodes of $dom(r)$ that are new (that is, not in $dom(p)$) but don't follow any new restarts by player Open:

$$\{\sigma \in dom(r) - dom(p) \colon \forall \tau \preccurlyeq \sigma(\tau \in dom(r) - dom(p), |\tau| \text{ odd} \\ \implies r(\tau^-)_0 > r(\tau)_0 \neq \infty)\}.$$

The idea is that we really only need to focus on nodes in N: nodes in $dom(p)$ have already had their tags determined, and nodes not in $N \cup dom(p)$ will have no constraints on their tags coming from p at all, since they must follow a restart by Open. In order to define the value of $\hat{r}$ on some node σ in N, though, we need an upper bound on how large N is above σ to keep from running out of ordinals prematurely; this is provided by taking the rank of T_σ.

Formally, we build the retagged condition as follows. Recalling that $V \models$ZFC, fix in V a well-ordering of $\mathbb{R}^{<\omega}$, and via that ordering let $rk(S)$ be the rank of S for $S \subseteq \mathbb{R}^{<\omega}$ a well-founded tree. Then we define $\hat{r}$ as follows:

$$\hat{r}(\sigma) = \begin{cases} \uparrow, & \text{if } \sigma \notin dom(r), \\ p(\sigma), & \text{if } \sigma \in dom(p), \\ r(\sigma), & \text{if } \sigma \notin (N \cup dom(p)), \\ (\min\{\tilde{\gamma} + rk(T_\sigma), r(\sigma)_0\}, \hat{r}(\sigma^-)_1), & \text{if } \sigma \in N \text{ and } |\sigma| \text{ is odd}, \\ (\hat{r}(\sigma^-)_0, \min\{\tilde{\gamma} + rk(T_\sigma), r(\sigma)_1\}), & \text{if } \sigma \in N \text{ and } |\sigma| \text{ is even}. \end{cases}$$

It is readily checked that $\hat{r} \in \mathbb{P}$ — the assumption on $\tilde{\gamma}$ being used here to show that the coordinates of $\hat{r}$ are decreasing when the corresponding coordinates of r drop from $\geq \alpha$ to $< \alpha$ — and that $\hat{r} \leq p$ and $\hat{r} \approx_\gamma r$ (in fact, $\hat{r} \approx_{\tilde{\gamma}} r$). ⊣

As a straightforward application of the retagging lemma, we can now show that M is a model of RCA_0^3:

Lemma 3.12. $M \models \mathsf{RCA}_0^3$.

Proof. P^-, the extensionality axioms, the axioms defining $*$ and $\frown$, and comprehension for reals are all trivially satisfied, the last of these since M contains precisely the reals in V and $V \models$ ZFC. Only the comprehension scheme for type-2 functionals is nontrivial. We will prove that *arithmetic* comprehension for type-2 functionals holds in M, since this proof is no harder than the proof for Δ^0_1 comprehension.

Intuitively, we will show that functionals defined in an arithmetic way depend, value-by-value, on only countably many bits of information. From this, and the countable closure of our forcing, we will be able to find stable names for such functionals.

Let $\varphi(X^1, y^0)$ be an arithmetic (that is, Σ^0_n for some $n \in \omega$; recall Definition 2.2) formula such that for each $a \in \mathbb{R}$ there is precisely one $k \in \omega$ with

$$M \models \varphi(a, k).$$

Since each natural number is definable, we can assume φ has no type-0 parameters. Let $(F_i)_{i<n}$ be the type-2 parameters used in φ, let $(s_j)_{j<m}$ be the type-1 parameters used in φ, and let v_i be an α-stable name for F_i; since each F_i has a stable name, and there are only finitely many F_i, we can find some large enough $\alpha < \omega_2$ so that such names exist. Note that we can work directly with the s_j, as opposed to just dealing with their names, since our forcing adds no new reals.

For $a \in \mathbb{R}$, let $\mathcal{C}_a$ be any countable set of names for reals such that

- $\mathcal{C}_a$ contains a name for a and each s_j;
- whenever a name μ is in $\mathcal{C}_a$ and $k \in \omega$, $\mathcal{C}_a$ contains a name ν such that $\Vdash \nu = k^\frown \mu$; and
- whenever μ is in $\mathcal{C}_a$ and $i < n$, $\mathcal{C}_a$ contains a name μ' such that $\Vdash \mu' = \nu_i * \mu$.

Although we have not been completely precise in defining the sets $\mathcal{C}_a$, it is clear that the definition above is effective in the sense that a suitable set of sets of names $\{\mathcal{C}_a : a \in \mathbb{R}\}$ exists in the ground model, V.

The key fact about the $\mathcal{C}_a$ is that, by construction, they determine the truth value of the formula φ at a: that is, the truth value of $\varphi(a, k)$ depends only on the values of the F_i on the reals named by elements of $\mathcal{C}_a$. Formally,

$$\forall \mu \in \mathcal{C}_a \exists k \in \mathbb{R}[(p \Vdash \mu = k) \wedge (q \Vdash \mu = k)] \\ \implies \forall l \in \omega[(p \Vdash \varphi(a, l)) \iff (q \Vdash \varphi(a, l))].$$

Now let ν be a name for the functional defined by φ. We will show that ν is $(\alpha + \omega_1)$-stable.

Let $r \in \mathbb{R}$ and $p, q \in \mathbb{P}$ such that $p \approx_{\alpha+\omega_1} q$ and $p \Vdash \nu(r) = k$. Let

$$D_r = \{t \in \mathbb{P} : \forall \mu \in \mathcal{C}_r \exists s \in \mathbb{R}(t \Vdash \mu = s)\}$$

be the set of conditions which decide the value of each name in $\mathcal{C}_r$. Since $\mathcal{C}_r$ is countable, and $\mathbb{P}$ is countably closed, the set D_r is dense. Now suppose towards contradiction that $q \nVdash \nu(r) = k$. Then since D_r is dense, we can find some $q' \leq q$ such that

$$q' \in D_r \quad \text{and} \quad q' \Vdash \nu(r) = l$$

for some natural $l \neq k$. By the retagging lemma, there is some $p' \leq p$ such that $p' \approx_\alpha q'$; but since each of the ν_i are α-stable, we must have

$$\forall i < n, t \in \mathbb{R}, \mu \in \mathcal{C}_r[(q' \Vdash \mu = t) \iff (p' \Vdash \mu = t)].$$

But since the truth value of $\varphi(r, k)$ depends only on the values of the $\mathcal{C}_r$, this contradicts the fact that $p' \leq p$ and $p \Vdash \nu(r) = k$. ⊣

3.2. Clopen determinacy in M. Showing that M satisfies clopen determinacy for reals, however, requires a more delicate proof. Intuitively, given a stable name for a clopen game, we ought to be able to inductively construct a stable name for a winning (quasi)strategy in that game by just iterating the retagging lemma in the right way. However, since the rank of a stable name is required to be $< \omega_2$, we cannot iterate the retagging lemma ω_2-many times, so we need all clopen games in M to have rank $< \omega_2$. This cannot be derived from the retagging lemma alone; instead, we need to look at particular subposets of $\mathbb{P}$:

Definition 3.13. For $\alpha < \omega_2$, $\mathbb{P}_\alpha$ is the subposet of $\mathbb{P}$ defined by

$$\mathbb{P}_\alpha = \{p \in \mathbb{P} \colon \forall \sigma \in dom(p), i \in 2(p(\sigma)_i < \alpha \vee p(\sigma)_i = \infty)\}.$$

Conditions in $\mathbb{P}_\alpha$ will turn out to satisfy a slightly stronger retagging property with respect to $\approx_\alpha$ — the projecting lemma, below — than conditions in general, and this will be used to prove that this forcing adds no stable well-orderings of reals longer

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than any in the ground model. Note that this is false for unstable well-orderings; in particular, forcing with $\mathbb{P}$ collapses ω_2 in the full generic extension.

DEFINITION 3.14. For $p \in \mathbb{P}$, $\alpha < \omega_2$, we let the *α-projection of p*,

$$p^\alpha : dom(p) \to (\alpha \cup \{\infty\}) \times (\alpha \cup \{\infty\}),$$

be the map given by

$$\forall \sigma \in dom(p), i \in 2, \quad p^\alpha(\sigma)_i = \begin{cases} p(\sigma)_i & \text{if } p(\sigma)_i < \alpha \\ \infty & \text{otherwise.} \end{cases}$$

LEMMA 3.15 (Projecting). *For all $p \in \mathbb{P}$, $\alpha < \omega_2$, we have*:

(1) $p^\alpha \in \mathbb{P}_\alpha$;
(2) $p^\alpha \approx_\alpha p$;
(3) $p \leq q \implies p^\alpha \leq q^\alpha$;
(4) $|\mathbb{P}_\alpha|^V = \aleph_1$; *and*
(5) $\mathbb{P}_\alpha$ *is countably closed.*

PROOF. For (1), note that since we set $\infty > \infty$, the map

$$x \mapsto \begin{cases} x & \text{if } x < \alpha \\ \infty & \text{otherwise} \end{cases}$$

satisfies $x < y \iff \pi(x) < \pi(y)$. So as long as p is in $\mathbb{P}$, the projection p^α will not contain any illegal instances of the second coordinate increasing (which is the only possible obstacle to being a condition), and so will also be in $\mathbb{P}$ - and clearly if $p^\alpha \in \mathbb{P}$, then $p^\alpha \in \mathbb{P}_\alpha$.

(2) and (3) are immediate consequences of (1). Property (3) shows that we can allow $\gamma = \alpha$ in the retagging lemma above if p is assumed to be in $\mathbb{P}_\alpha$, and that we can take $\hat{r}$ to be in $\mathbb{P}_\alpha$ as well in that case.

For (4), note that elements of $\mathbb{P}_\alpha$ can be coded by countable subsets of $\mathbb{R} \times \omega_1$; the result then follows since $V \models \mathsf{CH}$.

Finally, for (5), let $(p_i)_{i\in\omega}$ be a sequence of conditions in $\mathbb{P}_\alpha$ with $p_{i+1} \leq p_i$. Then since $\mathbb{P}$ is countably closed, we have some $q \in \mathbb{P}$ with $q \leq p_i$ for all $i \in \omega$; but then $q^\alpha \in \mathbb{P}_\alpha$ by (1), and since each $p_i \in \mathbb{P}_\alpha$, we have $p_i^\alpha = p_i$ and hence $q^\alpha \leq p_i$ by (3). ⊣

This lemma helps provide us with explicit upper bounds on the lengths of type-2 well-orderings in M, via the construction below. We can use this result to provide a bound on the lengths of well-orderings in M, which in turn allows the induction necessary for showing clopen determinacy to go through.

LEMMA 3.16 (Bounding). *Suppose ν is a stable name for a well-ordering of $\mathbb{R}$ (that is, $\Vdash \nu$ is a well-ordering of $\mathbb{R}$). Then there is some ordinal $\lambda < \omega_2$ such that*

$$\Vdash \nu \preccurlyeq \lambda.$$

That is, ω_2 is not collapsed in a stable way by forcing with $\mathbb{P}$.

PROOF. Suppose ν is an α-stable name for a well-ordering of a set of reals. The proof takes place around the subposet $\mathbb{P}_\alpha$. For a sequence of reals $\overline{a} = \langle a_0, \cdots, a_n \rangle$

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and a condition $p \in \mathbb{P}$, say that p is *adequate* for $\overline{a}$, and write $Ad(p, \overline{a})$, if p forces that $\overline{a}$ is a descending sequence through v:

$$p \Vdash a_0 >_v \cdots >_v a_n.$$

Note that since v is α-stable, p is adequate for $\overline{a}$ if and only if p^α is adequate for $\overline{a}$, by (2) of the previous lemma.

In order to bound the size of v in any generic extension, we create in the ground model an approximation to the tree of descending sequences through v, as follows:

$$\mathcal{T}_v = \{\langle (p_i, a_i)\rangle_{i<n} \colon p_i \in \mathbb{P}_\alpha \wedge \forall i < j < n(p_j \leq p_i \wedge Ad(p_j, \langle a_0, \cdots, a_{i-1}\rangle))\}.$$

Elements of $\mathcal{T}_v$ are potential descending sequences, together with witnesses to their possibility. Now since v is a name for a well-ordering, we must have that $\mathcal{T}_v$ is well-founded. Otherwise, we would have a sequence of condition/real pairs, $\langle (p_i, a_i)\rangle_{i\in\omega}$, which build an infinite descending sequence through v, that is,

$$p_{i+1} \leq p_i, \quad p_{i+2} \Vdash a_i >_v a_{i+1}.$$

But then a common strengthening $q \leq p_i$, which exists by the countable closure of $\mathbb{P}_\alpha$, would create an infinite descending chain in v; and this contradicts the assumption that $\Vdash v$ is well-founded.

Additionally, $|\mathcal{T}_v| = \aleph_1$, since $\mathcal{T}_v \subseteq (\mathbb{P}_\alpha \times \mathbb{R})^{<\omega}$ and $|\mathbb{P}_\alpha| = \aleph_1$ by Lemma 3.15(4). Fixing in V a bijection between ω_1 and $\mathcal{T}_v$ we can take the Kleene-Brouwer ordering $\mathcal{L}_v$ of $\mathcal{T}_v$. Since $\mathcal{T}_v$ is well-founded, this is a well-ordering; below, we will show that in fact

$$\Vdash v \preccurlyeq \mathcal{L}_v.$$

Let

$$K_v^G = \{\langle a_0, \cdots, a_n\rangle \colon a_0 >_{v[G]} \cdots >_{v[G]} a_n\}$$

be the tree of descending sequences through $v[G]$ in $V[G]$, and fix a well-ordering $\leq_W$ of $\mathbb{P}_\alpha$ in V. For $\overline{a} \in K_v^G$, we define a condition in $\mathbb{P}_\alpha$ by recursion as follows:

$$h(\overline{a}) = \text{ the } \leq_W\text{-least } p \in \mathbb{P}_\alpha \text{ such that } p \leq h(\overline{b}) \text{ for all } \overline{b} \prec \overline{a} \text{ and } Ad(p, \overline{a}).$$

(Note that by the previous lemma and the fact that v is α-stable, h is defined for all $\overline{a} \in K_v^G$.) An embedding from K_v^G into $\mathcal{T}_v$ can then be defined:

$$e \colon K_v^G \to \mathcal{T}_v \colon \langle a_i\rangle_{i<n} \mapsto \langle (h(\langle a_0, \cdots, a_i\rangle), a_i)\rangle_{i<n}.$$

It follows that $v[G] \preccurlyeq \mathcal{L}_v$, as desired. ⊣

Now we are finally ready to prove that M satisfies clopen determinacy. For simplicity, this proof is broken into three pieces. First, we show that the rank of a node in a clopen game can be determined in an α-stable way, for appropriately large α. Then we define a set which encodes the rank of nodes in a clopen game, as well as which player these nodes are winning for, and show that this set is similarly well-behaved. Finally, we use this to give stable names for winning strategies in clopen games which themselves have stable names — and this will suffice to show that $\Delta_1^{\mathbb{R}}$-Det holds in M. Unfortunately, the first two steps in this proof is exceedingly tedious, as we require more and more room to retag conditions, but the intuition is that of a straightforward induction.

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Fix in V a well-ordering $\leq_W$ of $\mathbb{R}$. Using this well-ordering, we can define the rank $rk(T)$ of a well-founded tree $T \subset \mathbb{R}^{<\omega}$ in the usual way; and for $\sigma \in T$, we let $rk_T(\sigma) = rk(\{\tau : \sigma^\frown\tau \in T\})$. If ν is a name for a well-founded tree, then $rk(\nu)$ and $rk_\nu(\sigma)$ are the standard names for $rk(\nu[G])$ and $rk_{\nu[G]}(\sigma)$.

Lemma 3.17. *Let ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$, $p \in \mathbb{P}$, $\gamma < \omega_2$, and $\sigma \in \mathbb{R}^{<\omega}$ such that*

$$p \Vdash rk_\nu(\sigma) = \gamma,$$

and suppose $q \approx_{\beta+\omega_1(\gamma 2+2)} p$; then

$$q \Vdash rk_\nu(\sigma) = \gamma.$$

Proof. By induction on γ. For $\gamma = 0$, suppose q is a counterexample to the claim; then we can find $r \leq q$ and $a \in \mathbb{R}$ such that

$$r \Vdash \sigma^\frown\langle a\rangle \in \nu.$$

Now by the retagging lemma, we can find some $\hat{r} \leq p$ such that $\hat{r} \approx_\beta r$. Since ν is β-stable, we have

$$r \Vdash \sigma^\frown\langle a\rangle \in \nu,$$

which contradicts the assumption on p.

Now suppose the lemma holds for all $\gamma < \theta$, and let $p \Vdash rk_\nu(\sigma) = \theta$; then

$$p \Vdash \forall a \in \mathbb{R}(\sigma^\frown\langle a\rangle \in \nu \implies rk_\nu(\sigma^\frown\langle a\rangle) < \theta).$$

Suppose towards a contradiction that

$$q \approx_{\beta+\omega_1(\theta 2+2)} p \quad \text{and} \quad q \not\Vdash rk_\nu(\sigma) = \theta;$$

then there is some $r \leq q$, $a \in \mathbb{R}$ such that

$$r \Vdash \sigma^\frown\langle a\rangle \in \nu \wedge rk_\nu(\sigma^\frown\langle a\rangle) \geq \theta.$$

By the retagging lemma we get some $\hat{r} \leq p$ such that $\hat{r} \approx_{\beta+\omega_1(\theta 2+1)} r$, and since ν is β-stable we have $\hat{r} \Vdash \sigma^\frown\langle a\rangle \in \nu$. Since $\hat{r} \leq p$, and $p \Vdash rk_\nu(\sigma) = \theta$, we must be able to find some $\delta < \theta$ and $s \leq \hat{r}$ such that $s \Vdash rk_\nu(\sigma^\frown\langle a\rangle) = \delta$; using the retagging lemma a second time, we can get some $\hat{s} \leq r$ such that $\hat{s} \approx_{\beta+\omega_1(\delta 2+2)} s$. But then by the induction hypothesis $s \Vdash rk_\nu(\sigma^\frown\langle a\rangle) = \delta$, contradiction the assumption on r. ⊣

Definition 3.18. If $T \subset \mathbb{R}^{<\omega}$ is a well-founded tree, thought of as a clopen game, a node σ on T is *safe* if the corresponding clopen game

$$T^\sigma = \{\tau : \sigma^\frown\tau \in T\}$$

is a win for player I. For ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$ with rank $< \alpha$ for some $\alpha < \omega_2$ (see Lemma 3.16), let Δ_ν be a name for the set which encodes rank and safety of nodes on ν:

$$\Delta_\nu[G] := \{(\sigma, \delta, i) : \sigma \in \nu[G] \quad \text{and} \quad rk_{\nu[G]}(\sigma) = \delta \quad \text{and} \quad i = [\![\sigma \text{ is safe in } \nu[G]]\!]\}.$$

We will show that Δ_ν is well-behaved, in the sense of stability, and use this to give a stable name for a winning strategy for ν.

Lemma 3.19. *Let ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$ of rank $<\alpha$; and for simplicty, let $\kappa = \beta + \omega_1(\alpha 2 + 2)$. If $p \Vdash (\sigma, \delta, i) \in \Delta_\nu$, and $q \approx_{\kappa+\omega_1(\delta 2+2)} p$, then $q \Vdash (\sigma, \delta, i) \in \Delta_\nu$.*

Proof. Suppose not. Let δ be the least ordinal such that for some σ, i there are conditions p, q such that

- $q \approx_{\kappa+\omega_1(\delta 2+2)} p$,
- $p \Vdash (\sigma, \delta, i) \in \Delta_\nu$, and
- $q \not\Vdash (\sigma, \delta, i) \in \Delta_\nu$.

There are two cases. If $\delta = 0$, then we must have $i = 0$; since ν is β-stable, there can be no condition below q which adds a child of σ to ν (since then we can use the retagging lemma to force this below p, which already forces that σ is terminal in ν), and so $q \Vdash (\sigma, 0, 0) \in \Delta_\nu$.

So suppose $\delta > 0$. Since $p \Vdash rk_\nu(\sigma) = \delta$, by the previous lemma we have $q \Vdash rk_\nu(\sigma) = \delta$; so q just disagrees on whether σ is safe, which means we must be able to find some $r \leq q$ such that

$$r \Vdash (\sigma, \delta, 1 - i) \in \Delta_\nu.$$

By the retagging lemma we can find an $\hat{r} \leq p$ such that $\hat{r} \approx_{\kappa+\omega_1(\delta 2+1)} r$.

Now the proof breaks into two subcases based on whether $i = 0$ or $i = 1$. We treat the first case; the proofs are essentially identical.

We have $r \approx_{\kappa+\omega_1(\delta 2+1)} \hat{r}$ and $r \Vdash (\sigma, \delta, 1) \in \Delta_\nu$. Since r thinks σ is safe, r must think there is some immediate successor of σ which is unsafe. That is, we can find $s \leq r$, $\theta < \delta$, and $a \in \mathbb{R}$ such that $s \Vdash (\sigma^\frown\langle a\rangle, \theta, 0) \in \Delta_\nu$; by retagging again we can find

$$\hat{s} \leq \hat{r}, \hat{s} \approx_{\kappa+\omega_1(\theta 2+2)} s,$$

which by our assumption on δ means that

$$\hat{s} \Vdash \sigma^\frown\langle a\rangle \in \nu \text{ and is unsafe.}$$

But $\hat{s} \leq \hat{r} \leq p$ and p believes σ is unsafe, which means p believes σ has no safe extensions - a contradiction. ⊣

Finally, we are ready to show that ν is determined in M:

Corollary 3.20. *Let ν be a β-stable name for a well-founded subtree of $\mathbb{R}^{<\omega}$, viewed as a clopen game, with $rk(\nu) < \alpha < \omega_2$ for some limit ordinal α. Then there is an $(\beta + \omega_1(\alpha 4 + 5))$-stable name for a (type-2 functional coding a) winning strategy for ν.*

Proof. (Note that requiring α to be a limit is a benign hypothesis, as we can always make α larger if necessary; this assumption is just made to simplify some ordinal arithmetic below.) Recall that $\leq_W$ is a well-ordering of $\mathbb{R}$ in V. Let μ be a name for the type-2 functional which encodes the strategy picking out the $\leq_W$-least winning move at any given stage:

$$\mu[G](n^\frown\sigma) = \begin{cases} a(n) & \text{if } a \text{ is the } \leq_W\text{-least real such that } \exists\beta < \alpha[(\sigma^\frown\langle a\rangle, \beta, 0) \in \Delta_\nu], \\ 0 & \text{if no such real } s \text{ exists.} \end{cases}$$

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For simplicity, we assume that $\Vdash$"no string containing a '0' is on ν," so that there is no ambiguity in this definition. Clearly μ yields a winning strategy for whichever player wins ν.

All that remains to show is that μ is stable. Let $\lambda = (\beta + \omega_1(\alpha 4 + 5))$, fix σ and a, and let $p \approx_\lambda q$ are conditions in $\mathbb{P}$ such that $p \Vdash \mu(\sigma) = a$. We can find some $r \leq q$ and some b such that $q' \Vdash \mu(\sigma) = b$; we'll show that $b = a$, and so we must have had $q \Vdash \mu(\sigma) = a$ already.

There are two cases:

Case 1. $a = 0$. Suppose towards a contradiction that $b \neq 0$. Since $a = 0$, we have $p \Vdash \forall\delta < \alpha, \forall b \in \mathbb{R}[(\sigma^\frown\langle b\rangle, \delta, 0) \notin \Delta_\nu]$. Let $s \leq r$ and $\delta < \alpha$ be such that $s \Vdash (\sigma^\frown\langle b\rangle, \delta, 0) \in \Delta_\nu$; by the retagging lemma, there is $p' \leq p$ with $p' \approx_{\beta+\omega_1(\alpha 4+4)} s$, which by Lemma 3.19 is impossible.

Case 2. $a \neq 0$. By identical logic as in the previous case, we must have $b \neq 0$; suppose towards contradiction that $b \neq a$. With two applications of the retagging lemma, we can find ordinals $\delta_0, \delta_1 < \alpha$ and conditions $p' \leq p$, $r' \leq r$ such that

- $p' \approx_{\beta+\omega_1(\alpha 4+2)} r'$,
- $p' \Vdash (\sigma^\frown\langle a\rangle, \delta_0, 0) \in \Delta_\nu$, and
- $r' \Vdash (\sigma^\frown\langle b\rangle, \delta_1, 0) \in \Delta_\nu$.

By Lemma 3.19, we have $r' \Vdash (\sigma^\frown\langle a\rangle, \delta_0, 0) \in \Delta_\nu$ and $p' \Vdash (\sigma^\frown\langle b\rangle, \delta_1, 0) \in \Delta_\nu$ as well. Also note that we have $p' \Vdash \mu(\sigma) = a$, $r' \Vdash \mu(\sigma) = b$, since $p' \leq p$ and $r' \leq r \leq q$. Now since $a \neq b$, either $a <_W b$ or $b <_W a$, and so either way we have a contradiction.

This completes the proof. ⊣

Since $M_1 = \mathbb{R}$, M computes well-foundedness of subtrees of $\mathbb{R}^{<\omega}$ correctly; so by Lemma 3.16, it then follows that every clopen game in M has a winning strategy in M. Together with Lemmas 3.10 and 3.12, this completes the proof of Theorem 3.6.

§4. Conclusion.

In this paper, we have sought to understand how the passage to higher types affects mathematical constructions related to the system ATR_0; given both the sheer number of such constructions, and the relative youth of higher-order reverse mathematics, this remains necessarily incomplete. We close by mentioning three particular directions for further research we find most immediately compelling:

- Despite the analysis provided by this paper, there are still basic questions remaining unaddressed. It is unclear what is the relationship between $\Sigma^{\mathbb{R}}_1$-Det and Σ^2_1-Sep$^{\mathbb{R}}$. We suspect that these principles are incomparable, but separations at this level are unclear: for example, it is open even whether $\Sigma^{\mathbb{R}}_1$-Det implies the Π^2_2-comprehension principle for type-2 functionals, although the answer is almost certainly that it does not.

  For that matter, in this paper we have focused entirely on the strengths of third-order theorems relative to other third-order theorems; their strength relative to second-order principles has been completely unexplored. For instance, it is entirely possible, albeit unlikely, that $\Delta^{\mathbb{R}}_1$-Det and $\Sigma^{\mathbb{R}}_1$-Det have the same second-order consequences.

- One interesting aspect of the shift to higher types we have not touched on at all is the extra structure available in higher-type versions of classical theorems. Given a Π^1_2 principle
$$\varphi \equiv \forall X^1 \exists Y^1 \theta(X, Y),$$
we can take its higher-type (so prima facie Π^2_2) analogue
$$\varphi^* \equiv \forall F^2 \exists G^2 \theta^*(F, G).$$
Now, individual reals are topologically uninteresting, but passing to a higher type changes the situation considerably. Specifically, we can consider *topologically restricted* versions of φ^*: given a pointclass Γ, let
$$\varphi^*[\Gamma] \equiv \forall F^2 \in \Gamma \exists G^2 \theta(F, G).$$
The relevant example is restricted forms of determinacy: the principles $\Delta^{\mathbb{R}}_1$-$\mathsf{Det}[\Gamma]$ (resp., $\Sigma^{\mathbb{R}}_1$-$\mathsf{Det}[\Gamma]$) assert determinacy for clopen (resp., open) games whose underlying trees when viewed as sets of reals are in Γ. In particular, the system $\Sigma^{\mathbb{R}}_1$-$\mathsf{Det}[Open]$ is extremely weak, at least by the standards of higher-type determinacy theorems: it is equivalent over RCA^3_0 to the classical system ATR_0.
The techniques used in the proof of Theorem 3.6 are topologically badly behaved. In particular, they tell us nothing about the restricted versions $\Delta^{\mathbb{R}}_1$-$\mathsf{Det}[\Gamma]$ and $\Sigma^{\mathbb{R}}_1$-$\mathsf{Det}[\Gamma]$. With some work the argument of this paper might extend to show that $\Delta^{\mathbb{R}}_1$-$\mathsf{Det}[\Gamma] \not\vdash \Sigma^{\mathbb{R}}_1$-$\mathsf{Det}[\Gamma]$ over RCA^3_0, for reasonably large pointclasses Γ, but not immediately; and certainly a detailed understanding of which restricted forms of open determinacy for reals are implied by which restricted forms of clopen determinacy will require substantially new ideas. This finer structure seems to allow a rich connection between classical descriptive set theory and higher reverse mathematics, and is worth investigating.
- Finally, there is a serious foundational question regarding the base theory for higher-order reverse mathematics. The language of higher types is a natural framework for reverse mathematics, as explained at the beginning of Section 2.1; however, the specific base theory RCA^ω_0 is not entirely justified from a computability-theoretic point of view. While proof-theoretically natural, it does not necessarily capture "computable higher-type mathematics." The most glaring example of this concerns the Turing jump operator. In the theory RCA^ω_0, the existence of a functional corresponding to the jump operator
$$\mathfrak{J}^{1\to 1}: f \mapsto f'$$
is conservative over ACA_0 ([8], Theorem 2.5). However, intuitively we can compute the ωth jump (and much more) of a given real by iterating $\mathfrak{J}$; thus, given a model M of RCA^ω_0, there may be algorithms using only parameters from M and effective operations which compute reals not in M. From a computability theoretic point of view, then, RCA^ω_0 may be an unsatisfactorily weak base theory.
Of course, this discussion hinges on what, precisely, "computability" means for higher types. A convincing approach is given in [10], justified by arguments by Kleene and others (see especially [3]) similar in spirit to Turing's original

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informal argument. It is thus desirable — at least for higher-type reverse mathematics motivated by computability theory, as opposed to proof theory — to have a base theory corresponding to full Kleene recursion.[7] We will address these, and other, aspects of the base theory issue in a future paper. However, the search for the "right" base theory is very fertile mathematical ground, drawing on and responding to foundational ideas from proof theory, generalized recursion theory, and even set theory, and deserves attention from many corners and active debate.

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REFERENCES

[1] Harvey Friedman, *The analysis of mathematical texts, and their calibration in terms of intrinsic strength I*, published on author's website, 1975.

[2] ———, *Some systems of second order arithmetic and their use*, ***Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974)***, vol. 1, pp. 235–242, Canadian Mathematical Congress, Montreal, Quebec, 1975.

[3] R. O. Gandy, *Computable functionals of finite type. I*, ***Sets, Models and Recursion Theory Proceedings of the Summer School in Mathematical Logic and tenth Logic Colloquium (Leicester, 1965)***, pp. 202–242, North-Holland, Amsterdam, 1967.

[4] Noam Greenberg and Julia F. Knight, *Computable structure theory using admissible recursion theory on ω_1*. ***Effective Mathematics of the Uncountable***, Cambridge University Press, New York, 2013.

[5] Sherwood Hachtman, *Determinacy in third order arithmetic*, submitted.

[6] Joseph Harrison, *Recursive pseudo-well-orderings*. ***Transactions of the American Mathematical Society***, vol. 131(1968), pp. 526–543.

[7] Victor Harnik, *Set existence axioms for general (not necessarily countable) stability theory*. ***Annals of Pure and Applied Logic***, vol. 34 (1987), no. 3, pp. 231–243. Stability in model theory (Trento, 1984).

[8] James Hunter, ***Higher-order reverse topology***, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[9] Alexander S. Kechris, ***Classical descriptive set theory***, vol. 156, Graduate Texts in Mathematics, Springer-Verlag, New York, 1995.

[10] S. C. Kleene, *Recursive functionals and quantifiers of finite types. I*. ***Transactions of the American Mathematical Society***, vol. 91 (1959), pp. 1–52.

[11] Ulrich Kohlenbach, *Higher order reverse mathematics*, ***Reverse mathematics 2001***, vol. 21, Lecture Notes in Logic, pp. 281–295, Association of Symbolic Logic, La Jolla, CA, 2005.

[12] Alexander P. Kreuzer, *Non-principal ultrafilters, program extraction and higher-order reverse mathematics*. ***Journal of Mathematical Logic***, vol. 12 (2012), no. 1, 1250002, p. 16.

[7]It should be noted that the separations 2.23 and 3.6 in this paper do not suffer from the choice of base theory. This is because — by a straightforward, albeit tedious, induction — Kleene computability from a type-2 object satisfies the following *countable use condition*: if F is a given type-2 object, and φ_e^F is a type-2 object computed from F, then for each real r there is a countable set of reals C_r such that

$$\forall G^2(G \upharpoonright C_r = F \upharpoonright C_r \implies \varphi_e^F(r) = \varphi_e^G(r)).$$

The models in 2.23 and 3.6 then satisfy this stronger theory by essentially the same argument as in 3.12.

[13] Kenneth Kunen, ***Set theory***, vol. 102, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1983.

[14] John R. Longley, *Notions of computability at higher types. I*, ***Logic Colloquium 2000***, Lecture notes in Logic, vol. 19, pp. 32–142, Association of Symbolic Logic, Urbana, IL, 2005.

[15] María Manzano, ***Extensions of first order logic***, Cambridge Tracts in Theoretical Computer Science, vol. 19, Cambridge University Press, Cambridge, 1996.

[16] Antonio Montalbán, *On the* Π^1_1*-separation principle*. ***MLQ Mathematical Logic Quarterly***, vol. 54 (2008), no. 6, pp. 563–578.

[17] Yiannis N. Moschovakis, *Hyperanalytic predicates*. ***Transactions of the American Mathematical Society***, vol. 129 (1967), pp. 249–282.

[18] ———, ***Descriptive set theory***, second edition, Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009.

[19] Itay Neeman, *Necessary use of* Σ^1_1 *induction in a reversal*, this Journal, vol. 76 (2011), no. 2, pp. 561–574.

[20] Noah Schweber, PhD thesis, University of California, Berkeley.

[21] Richard A. Shore, *Reverse mathematics, countable and uncountable: a computational approach*, ***Effective Mathematics of the Uncountable***, Cambridge University Press, New York, 2013.

[22] Stephen G. Simpson, ***Subsystems of second order arithmetic***, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1999.

[23] John R. Steel, *Forcing with tagged trees*. ***Annals of Mathematical Logic***, vol. 15 (1978), no. 1, pp. 55–74.

[24] Nobuyuki Sakamoto and Takeshi Yamazaki, *Uniform versions of some axioms of second order arithmetic*. ***Mathematical Logic Quarterly***, vol. 50 (2004), no. 6, pp. 587–593.

[25] Henry Towsner, *Ultrafilters in reverse mathematics*, published in the ArXiv, 09 2011.

[26] Bartel L and van der Waerden, *Eine Bemerkung über die Unzerlegbarkeit von Polynomen*. ***Mathematische Annalen***, vol. 102 (1930), no. 1, pp. 738–739.

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