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THREE SOLUTIONS FOR A SINGULAR QUASILINEAR ELLIPTIC PROBLEM

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Abstract In the present paper we deal with a quasilinear problem involving a singular term. By combining truncation techniques with variational methods, we prove the existence of three weak solutions. As far as we know, this is the first contribution in this direction in the high-dimensional case.

Keywords: singular quasilinear elliptic problem; variational methods; three smooth solutions

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1. Introduction

In the present paper we consider the following singular quasilinear problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \mu u^{s-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_{\lambda,\mu}$}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$ of class C^2, $1 < p < N$; Δ_p is the p-Laplacian operator, i.e. $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$; $f : \Omega \times [0, +\infty[\to [0, +\infty[$ is a Carathéodory function, $0 < s < 1$; and λ, μ are positive parameters. Throughout the following, we assume that for almost all (a.a.) $x \in \Omega$, $f(x, 0) = 0$, and that $f(x, t) > 0$ for $t > 0$. Moreover, we assume that

$$f(x,t) \leq c(1 + t^{q-1}) \quad \text{for a.a. } x \in \Omega \text{ and all } t \geq 0, \tag{H}$$

where $c > 0, 1 < q < p^*$ and p^* is the critical Sobolev exponent. Let $F : \Omega \times [0, +\infty[\to \mathbb{R}$ also be the primitive of f, i.e.

$$F(x,t) = \int_0^t f(x,z)\,\mathrm{d}z.$$

By a weak solution of $(\mathcal{P}_{\lambda,\mu})$ we mean a function $u \in W_0^{1,p}(\Omega)$ such that $u > 0$ almost everywhere (a.e.) in Ω and

$$u^{s-1}\varphi \in L^1(\Omega), \qquad \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi = \int_\Omega [\lambda f(x,u) + \mu u^{s-1}]\varphi$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Owing to the presence of the singular term, the energy functional associated with $(\mathcal{P}_{\lambda,\mu})$ is not differentiable on the whole space $W_0^{1,p}(\Omega)$, even in the sense of Gâteaux. Nevertheless, it is continuous for $0 < s < 1$ and of class C^1 on certain closed convex subsets of $W_0^{1,p}(\Omega)$ (see [4, Corollary A.1]). Moreover, some of its truncations are of class C^1 on the whole space (see [4, Lemma A.3]).

Problems with singular terms were studied primarily in the context of semilinear equations (i.e. $p = 2$). In this regard, we mention the works of Coclite and Palmieri [3]; Lazer and McKenna [8]; Hirano *et al.* [6]; Lair and Shaker [7]; Sun *et al.* [15]; Zhang [17] and the references therein.

Coclite and Palmieri [3] obtained a bifurcation type result for the parametric problem

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{s-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2 < q < 2^*,\ \lambda > 0,\ s < 1$.

More precisely, they showed that if $\partial\Omega$ is of class C^3, then there exists a positive real number λ^* such that the problem has at least one positive solution belonging to $C^2(\Omega) \cap C(\overline{\Omega})$ for $0 < \lambda < \lambda^*$ and no positive solutions for $\lambda > \lambda^*$. The result was improved by Hirano *et al.* [6] who proved, via non-smooth critical point theory, the existence of two smooth positive solutions for $0 < \lambda < \lambda^*$.

Inspired by the work of Lair and Shaker [7], Sun *et al.* [15] and Zhang [17] considered a parametric problem with a singularity of the type $\beta(x)u^{s-1}$ with $0 < s < 1,\ \beta \in L^2(\Omega)_+$.

They produced two positive weak solutions for small $\lambda > 0$ and for suitable functions β. In the framework of the multiplicity of solutions for a semilinear elliptic problem with singular term, let us mention also the recent contribution of Arcoya and Moreno-Mérida [2], in which two solutions are obtained as the limit of two different sequences of solutions of suitable approximated problems.

Recently, several authors have focused on singular equations driven by the p-Laplacian. In this context, we mention the papers of Perera and Zhang [10]; Perera and Silva [9]; Giacomoni *et al.* [5]; and Giacomoni and Saoudi [4].

Both Perera and Silva [9] and Perera and Zhang [10] studied the parametric singular problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \beta(x)u^{s-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,\cdot)$ is $(p-1)$-superlinear near ∞.

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Under certain hypotheses on β, the authors ensured two positive weak solutions for small λ.

Giacomoni *et al.* [5] proved a bifurcation-type result when the reaction term is of the form $\lambda u^{s-1} + u^{q-1}, 0 < s < 1, p < q < p^*$ (see also [4] for more general singularities and superlinear perturbations).

In all the aforementioned works, the existence of at most two positive solutions is proved. Higher multiplicity results seem to be less investigated. As far as we know, the only result ensuring the existence of three weak solutions for parametric singular problems can be found in the paper by Zhao *et al.* [18], in which the authors study the problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \lambda\beta(x)u^{s-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,\cdot)$ is $(p-1)$-sublinear near ∞ and verifies the inequality $f(x,t) \geq \beta(x)$ a.e. in Ω, for t close to zero, where $\beta \geq 0$, $\beta \not\equiv 0$. In this paper, it is crucial that $p > N$. Indeed it turns out to be essential in the proofs, the compactness of the embedding of $W_0^{1,p}(\Omega)$ in $C^0(\overline{\Omega})$.

In the present paper, we prove two multiplicity results. The first ensures the existence of two solutions via an application of the Ambrosetti and Rabinowitz Mountain Pass theorem [1] when $\lambda = \mu$ and f satisfies the so-called Ambrosetti Rabinowitz assumption. In this case, no assumptions on zero are required on f, and direct minimization procedures as well as the relation between C^1 and $W_0^{1,p}$ local minimizers are involved. Thus, we provide here a different proof to that given in [9], where the same result has been obtained by perturbation methods.

The main contribution of the manuscript is the existence of three weak positive solutions for the problem $(\mathcal{P}_{\lambda,\mu})$, provided that the positive parameters λ, μ take values in certain intervals under the assumption that f is $(p-1)$-superlinear at zero and $(p-1)$-sublinear at ∞. We emphasize that, unlike in [18], we consider the higher-dimensional case (i.e. $N > p$). All three obtained solutions belong to a fixed ball in $W_0^{1,p}(\Omega)$. Our approach does not make use of the critical point theory on convex sets or of the more sophisticated theory of weak slope for semicontinuous functionals. This result is based on a very careful application of an abstract result of Ricceri [13] ensuring the existence of two local minimizers, which turn out to be weak solutions of the problem according to the very general definition given here. The existence of the third solution is obtained by applying the well-known Mountain Pass theorem of Pucci and Serrin [11] to an appropriate truncation of the energy functional and combining in a suitable way some arguments of [4], such as regularity theory, a strong comparison principle and the relation between C^1 and $W_0^{1,p}$ local minimizers (all in the framework of the singular case).

Our multiplicity results are the following.

Theorem 1.1. *In addition to* (H), *suppose that:*

$H(j)$ *there exist constants* $\eta > p, M > 0$ *such that*

$$0 < \eta F(x,t) \leq f(x,t)t \quad \textit{for a.a. } x \in \Omega \textit{ and all } t \geq M.$$

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Then, there exists $\bar{\lambda} > 0$ *such that, for every* $\lambda \in]0, \bar{\lambda}[$, *problem* $(\mathcal{P}_{\lambda,\lambda})$ *has at least two weak solutions belonging to* $\operatorname{int}(C_0^1(\overline{\Omega})_+)$.

Theorem 1.2. *In addition to* (H), *suppose that*

$H(i)$ $\displaystyle\lim_{t\to 0^+} \frac{\sup_{x\in\Omega} F(x,t)}{t^p} = 0;$

$H(ii)$ $\displaystyle\lim_{t\to +\infty} \frac{\sup_{x\in\Omega} F(x,t)}{t^p} = 0.$

Set

$$\lambda^* = \frac{1}{p} \inf \left\{ \frac{\int_\Omega |\nabla u(x)|^p \, \mathrm{d}x}{\int_\Omega F(x,u(x)) \, \mathrm{d}x} : \int_\Omega F(x,u(x)) \, \mathrm{d}x > 0 \right\}. \tag{1}$$

Then, for each compact interval $[a,b] \subset]\lambda^*, +\infty[$, *there exists* $r > 0$ *with the following property: for every* $\lambda \in [a,b]$, *there exists* $\mu^* > 0$ *such that for each* $\mu \in [0, \mu^*]$, *the problem* $(\mathcal{P}_{\lambda,\mu})$ *has at least three weak solutions belonging to* $\operatorname{int}(C_0^1(\overline{\Omega})_+)$ *whose norms are less than* r.

2. Preliminaries and tools

The present section contains our main tools and some auxiliary results which turn out to be essential for our proof. Let us recall that in the ordered Banach space $C_0^1(\overline{\Omega})$ the positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \ \forall\, x \in \Omega\}$$

has a non-empty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(x) > 0 \ \forall\, x \in \Omega, \ \frac{\partial u}{\partial n}(x) < 0 \ \forall\, x \in \partial\Omega \right\}$$

(n being the outward unit normal to $\partial\Omega$).

Moreover, on the Sobolev space $W_0^{1,p}(\Omega)(1 < p < \infty)$, we consider the norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

Since we are interested in positive solutions and the hypotheses in our theorems concern the positive semiaxis, we may (and will) assume that

$$f(x,t) = 0 \quad \text{for a.a. } x \in \Omega \text{ and all } t \leq 0.$$

Denote by $\Phi, J, \Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ the functionals defined by

$$\Phi(u) = \frac{1}{p}\|u\|^p,$$

$$J(u) = \int_\Omega F(x,u) \, \mathrm{d}x,$$

$$\Psi(u) = \frac{1}{s} \int_\Omega u_+^s \, \mathrm{d}x,$$

where, as usual, $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$.

Define also the energy functional associated to the problem $(\mathcal{P}_{\lambda,\mu})$, i.e. the functional $\mathscr{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\mathscr{E}(u) = \Phi(u) - \lambda J(u) - \mu \Psi(u).$$

Throughout the following, we will adopt the following notations: B_r denotes the open ball in $W_0^{1,p}(\Omega)$ centred at zero of radius r, and c is a constant whose value (unless specified) may vary from line to line. In addition, we denote respectively by φ_1 and λ_1 the $||\cdot||_p$-normalized positive principal eigenfunction and the principal eigenvalue associated with the operator $(-\Delta_p, W_0^{1,p}(\Omega))$. It is well known that $\varphi_1 \in \operatorname{int} C_+$.

We introduce now some key tools which resemble classical results from critical point theory. However, because of the peculiar approach we deal with, we can not make use of the classical theory.

Proposition 2.1. *Assume* (H) *and let* $\lambda, \mu > 0$. *Then, if* u *is a local minimizer of* $\mathscr{E}$, *it is a weak solution of problem* $(\mathcal{P}_{\lambda,\mu})$.

Proof. Let $\rho > 0$ such that $\mathscr{E}(u) \le \mathscr{E}(v)$ for every $v \in B_\rho(u) = u + B_p$. We claim that $u > 0$ a.e. in Ω.

For $t \in (0,1)$ small enough, one has $u + tu_- \in B_\rho(u)$ and $(u + tu_-)_+ = u_+$. So,

$$\begin{aligned}
0 &\le \frac{\mathscr{E}(u + tu_-) - \mathscr{E}(u)}{t} \\
&= \frac{1}{p}\left(\frac{\|u + tu_-\|^p - \|u\|^p}{t}\right) - \lambda \int_\Omega \frac{F(x, u + tu_-) - F(x,u)}{t} \\
&\quad - \frac{\mu}{s}\int_\Omega \frac{(u + tu_-)_+^s - u_+^s}{t} \\
&= \frac{1}{p}\left(\frac{\|u + tu_-\|^p - \|u\|^p}{t}\right) \quad \to \int_\Omega |\nabla u|^{p-2} \nabla u \nabla u_- = -\|u_-\|^p \quad \text{as } t \to 0^+.
\end{aligned}$$

(Recall that for a.a. $x \in \Omega$, $f(x, z) = 0$, for all $z \le 0$.)

From the above computation, it follows that $u_- = 0$, so $u \ge 0$ a.e. in Ω.

Assume that there exists a set of positive measure A such that $u = 0$ in A. Let $\varphi : \Omega \to \mathbb{R}$ be a function in $W_0^{1,p}(\Omega)$, positive in Ω. For $t > 0$ small enough, the function $u + t\varphi \in B_\rho(u)$ and $(u + t\varphi)^s > u^s$ a.e. in Ω, so

$$\begin{aligned}
0 &\le \frac{\mathscr{E}(u + t\varphi) - \mathscr{E}(u)}{t} \\
&= \frac{1}{p}\left(\frac{\|u + t\varphi\|^p - \|u\|^p}{t}\right) - \lambda \int_\Omega \frac{F(x, u + t\varphi) - F(x,u)}{t}
\end{aligned}$$

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$$
\begin{aligned}
&-\frac{\mu}{st^{1-s}}\int_A \varphi^s - \frac{\mu}{s}\int_{\Omega\setminus A}\frac{(u+t\varphi)^s-u^s}{t}\\
&< \frac{1}{p}\left(\frac{\|u+t\varphi\|^p-\|u\|^p}{t}\right) - \lambda\int_\Omega \frac{F(x,u+t\varphi)-F(x,u)}{t}\\
&-\frac{\mu}{st^{1-s}}\int_A \varphi^s \to -\infty \text{ as } t\to 0^+.
\end{aligned}
$$

The contradiction ensures that $u>0$. Let us prove now that

$$
u^{s-1}\varphi \in L^1(\Omega) \quad \text{for all } \varphi\in W_0^{1,p}(\Omega) \tag{2}
$$

and

$$
\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi - \lambda\int_\Omega f(x,u)\varphi - \mu\int_\Omega u^{s-1}\varphi \geq 0 \quad \text{for all } \varphi\in W_0^{1,p}(\Omega),\ \varphi\geq 0. \tag{3}
$$

Choose $\varphi\in W_0^{1,p}(\Omega), \varphi\geq 0$. Fix a decreasing sequence $\{t_n\}\subseteq]0,1]$ with $\lim_n t_n=0$. The functions

$$
h_n(x) = \frac{(u(x)+t_n\varphi(x))^s - u(x)^s}{t_n}
$$

are measurable and non-negative, and $\lim_n h_n(x) = su(x)^{s-1}\varphi(x)$ for a.a. $x\in\Omega$. From Fatou's lemma, we deduce

$$
\int_\Omega u^{s-1}\varphi \leq \frac{1}{s}\liminf_n \int_\Omega h_n. \tag{4}
$$

As above, for n large enough,

$$
0\leq \frac{\mathscr{E}(u+t_n\varphi)-\mathscr{E}(u)}{t_n} = \frac{1}{p}\frac{\|u+t_n\varphi\|^p-\|u\|^p}{t_n} - \lambda\int_\Omega\frac{F(x,u+t_n\varphi)-F(x,u)}{t_n} - \frac{\mu}{s}\int_\Omega h_n
$$

so, from (4), passing to the lim inf in the above inequality, we deduce at once condition (2) (it is enough to prove the integrability for a non-negative test function) and

$$
\mu\int_\Omega u^{s-1}\varphi \leq \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi - \lambda\int_\Omega f(x,u)\varphi,
$$

which is claim (3).

In what follows, we adapt the argument of [**15**] (see the proof of Theorem 1). Let $\varepsilon\in]0,1[$ such that $(1+t)u\in B_\rho(u)$ for all $t\in[-\varepsilon,\varepsilon]$. The function $\tilde{\xi}(t)=\mathscr{E}((1+t)u)$

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has a local minimum at zero and

$$
\begin{aligned}
0 = \tilde{\xi}'(0) &= \lim_{t\to 0} \frac{\mathscr{E}((1+t)u) - \mathscr{E}(u)}{t} \\
&= \int_\Omega |\nabla u|^p - \mu \int_\Omega u^s - \lambda \int_\Omega f(x,u)u.
\end{aligned}
$$

So,

$$
\int_\Omega |\nabla u|^p = \mu \int_\Omega u^s + \lambda \int_\Omega f(x,u)u. \tag{5}
$$

Let $\varphi \in W_0^{1,p}(\Omega)$ and plug into (3) the test function $v = (u+\varepsilon\varphi)_+$. Hence, using (5), we have

$$
\begin{aligned}
0 &\le \int_{\{u+\varepsilon\varphi\ge 0\}} |\nabla u|^{p-2}\nabla u \nabla(u+\varepsilon\varphi) - \mu \int_{\{u+\varepsilon\varphi\ge 0\}} u^{s-1}(u+\varepsilon\varphi) \\
&\quad - \lambda \int_{\{u+\varepsilon\varphi\ge 0\}} f(x,u)(u+\varepsilon\varphi) \\
&= \int_\Omega |\nabla u|^p + \varepsilon \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi - \mu \int_\Omega u^s - \varepsilon\mu \int_\Omega u^{s-1}\varphi \\
&\quad - \lambda \int_\Omega f(x,u)u - \varepsilon\lambda \int_\Omega f(x,u)\varphi \\
&\quad - \int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^p - \varepsilon \int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi + \mu \int_{\{u+\varepsilon\varphi<0\}} u^{s-1}(u+\varepsilon\varphi) \\
&\quad + \lambda \int_{\{u+\varepsilon\varphi<0\}} f(x,u)(u+\varepsilon\varphi) \\
&\le \varepsilon \left[\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi - \mu \int_\Omega u^{s-1}\varphi - \lambda \int_\Omega f(x,u)\varphi \right] \\
&\quad - \varepsilon \int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi.
\end{aligned}
$$

Notice that as $\varepsilon \to 0$, the measure of the set $\{u+\varepsilon\varphi<0\} \to 0$, so

$$
\int_{\{u+\varepsilon\varphi<0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi \to 0.
$$

Hence, dividing by ε, and passing to the limit as $\varepsilon \to 0$, we get that

$$
\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi - \mu \int_\Omega u^{s-1}\varphi - \lambda \int_\Omega f(x,u)\varphi \ge 0.
$$

From the arbitrariness of φ, we get at once that u is a weak solution of $(\mathcal{P}_{\lambda,\mu})$. □

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For $\mu > 0$, denote by u_μ the unique global minimizer of the functional

$$u \to \frac{1}{p}\|u\|^p - \frac{\mu}{s}\int_\Omega u_+^s \, dx.$$

It is well known that there exists $\varepsilon_\mu > 0$ with $u_\mu \geq \varepsilon_\mu \varphi_1$ a.e. in Ω (see [4, Lemma A.4]) and that $u_\mu \in \operatorname{int} C_+$ (see [4, Lemmas A.6, A.7, B.1]).

Proposition 2.2. *Assume* (H) *and let* $\lambda, \mu > 0$. *Then every weak solution of problem* $(\mathcal{P}_{\lambda,\mu})$ *belongs to* $C^{1,\beta}(\overline{\Omega}) \cap \operatorname{int} C_+$, *for some* $\beta \in (0, 1)$.

Proof. Let u be a weak solution of $(\mathcal{P}_{\lambda,\mu})$. Since $f \geq 0$, we have that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi - \mu\int_\Omega u^{s-1}\varphi \geq 0 \quad \text{for every } \varphi \in W_0^{1,p}(\Omega) \text{ with } \varphi \geq 0.$$

Also

$$\int_\Omega |\nabla u_\mu|^{p-2}\nabla u_\mu \nabla\varphi - \mu\int_\Omega u_\mu^{s-1}\varphi = 0 \quad \text{for every } \varphi \in W_0^{1,p}(\Omega),$$

and the weak comparison principle ensures that $u \geq u_\mu$ which, from the properties of u_μ, implies that $\operatorname{essinf}_{\mathcal{K}} u > 0$ for every compact set $\mathcal{K} \subseteq \Omega$. Thus, u is a weak solution of $(\mathcal{P}_{\lambda,\mu})$ in the sense of [4]. Now, by Lemmas A.6, A.7 of [4], we have that

$$u \in L^\infty(\Omega), \quad C_1 d(x, \partial\Omega) \leq u(x) \leq C_2 d(x, \partial\Omega) \quad \text{a.e. in } \Omega,$$

for some positive constants C_1, C_2.

If we put

$$\psi(x) = \lambda f(x, u(x)) + \mu u(x)^{s-1},$$

then

$$0 < \psi(x) \leq \lambda c(1 + ||u||_\infty^{q-1}) + \mu C_1^{s-1} d(x, \partial\Omega)^{s-1}$$

$$\leq C_3 d(x, \partial\Omega)^{s-1} \quad \text{a.e. in } \Omega,$$

where

$$C_3 = \lambda c(1 + ||u||_\infty^{q-1}) \cdot \operatorname{diam}(\overline{\Omega})^{1-s} + \mu C_1^{s-1}.$$

Moreover, $\psi \in L^\infty_{loc}(\Omega)$, since for each compact set $\mathcal{K} \subseteq \Omega$ we have $d(\mathcal{K}, \partial\Omega) > 0$ and

$$0 < \psi(x) \leq C_3 d(x, \partial\Omega)^{s-1} \leq C_3 d(\mathcal{K}, \partial\Omega)^{s-1} \quad \text{a.e. in } \mathcal{K}.$$

The assumptions of Theorem B.1 of [4] are fulfilled, and so we deduce $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$.

Also,

$$-\Delta_p u(x) = \lambda f(x, u(x)) + \mu u(x)^{s-1} \geq 0 \quad \text{a.e. in } \Omega,$$

so, $u \in \operatorname{int} C_+$, thanks to the strong maximum principle of Vazquez [16]. □

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The following proposition is an easy consequence of the results proved in [4]. It will be a crucial tool in our proof.

Proposition 2.3. *Assume* (H) *and let* $\lambda, \mu > 0$. *Define* $g : \Omega \times \mathbb{R} \to [0, +\infty)$, $\tilde{\Psi}$ *and* $\mathscr{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$ *by*

$$g(x,t) = \begin{cases} t^{s-1} & \text{if } x \in \Omega \text{ and } t \geq u_\mu(x), \\ u_\mu(x)^{s-1} & \text{if } x \in \Omega \text{ and } t \leq u_\mu(x), \end{cases}$$

$$\tilde{\Psi}(u) = \int_\Omega \int_0^{u_+} g(x,t)\,\mathrm{d}t\,\mathrm{d}x,$$

and

$$\mathscr{F}(u) = \frac{1}{p}\|u\|^p - \lambda J(u) - \mu\tilde{\Psi}(u),$$

respectively. Then, $\mathscr{F} \in C^1(W_0^{1,p}(\Omega))$ *and the following hold:*

(a) *if* u_0 *is a critical point of* $\mathscr{F}$, *then* $u_0 \geq u_\mu$ *a.e. in* Ω;

(b) *if* u_0 *is a critical point of* $\mathscr{F}$, *then it is a weak solution of* $(\mathcal{P}_{\lambda,\mu})$;

(c) *if* $u_0 \in \operatorname{int} C_+$ *is a local minimizer of* $\mathscr{F}$ *in the* $C_0^1(\overline{\Omega})$*-topology, then* u_0 *is also a local minimizer of* $\mathscr{F}$ *in the* $W_0^{1,p}(\Omega)$*-topology.*

Proof. The fact that $\mathscr{F} \in C^1(W_0^{1,p}(\Omega))$ follows from the proof of Lemma A.3 of [4], and its derivative at u is given by

$$\langle \mathscr{F}'(u), \varphi \rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi - \lambda \int_\Omega f(x,u)\varphi - \mu \int_\Omega g(x,u)\varphi$$

for every $\varphi \in W_0^{1,p}(\Omega)$.

(a) Let u_0 be a critical point of $\mathscr{F}$. Choosing $(u_0 - u_\mu)_-$ as a test function, one has

$$-\int_{\{u_0<u_\mu\}} |\nabla u_0|^{p-2}\nabla u_0 \cdot (\nabla u_0 - \nabla u_\mu) + \int_{\{u_0<u_\mu\}} [\lambda f(x,u_0) + \mu u_\mu^{s-1}](u_0 - u_\mu) = 0.$$

Bearing in mind that u_μ is a global minimum of $u \to (1/p)\|u\|^p - (\mu/s)\int_\Omega u_+^s$, we also obtain that

$$-\int_{\{u_0<u_\mu\}} |\nabla u_\mu|^{p-2}\nabla u_\mu \cdot (\nabla u_0 - \nabla u_\mu) + \int_{\{u_0<u_\mu\}} \mu u_\mu^{s-1}(u_0 - u_\mu) = 0.$$

Hence, subtracting the two equalities,

$$\begin{aligned} &\int_{\{u_0<u_\mu\}} (|\nabla u_0|^{p-2}\nabla u_0 - |\nabla u_\mu|^{p-2}\nabla u_\mu) \cdot (\nabla u_0 - \nabla u_\mu) \\ &\quad = \int_{\{u_0<u_\mu\}} \lambda f(x,u_0)(u_0 - u_\mu) \leq 0. \end{aligned}$$

By the strong monotonicity of the p-Laplacian operator, we deduce that $\|(u_0 - u_\mu)_-\| = 0$, that is, $u_0 \geq u_\mu$ a.e. in Ω.

(b) This follows from (a).

(c) Assume that $u_0 \in \operatorname{int} C_+$ is a local minimizer of $\mathscr{F}$ in the $C_0^1(\overline{\Omega})$-topology. Then, for $\varphi \in C_0^1(\overline{\Omega})$ and t small, one has

$$0 \leq \lim_{t \to 0} \frac{\mathscr{F}(u_0 + t\varphi) - \mathscr{F}(u_0)}{t}$$
$$= \int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \nabla \varphi - \lambda \int_\Omega f(x, u_0)\varphi - \mu \int_\Omega g(x, u_0)\varphi.$$

Rewriting the above inequality, replacing φ with $-\varphi$, we obtain

$$\int_\Omega |\nabla u_0|^{p-2} \nabla u_0 \nabla \varphi - \lambda \int_\Omega f(x, u_0)\varphi - \mu \int_\Omega g(x, u_0)\varphi = 0.$$

By density, u_0 is a critical point of $\mathscr{F}$ in $W_0^{1,p}(\Omega)$. Thus $u_0 \geq u_\mu$ (see (a)).

Suppose on the contrary that u_0 is *not* a local minimizer of $\mathscr{F}$ in the $W_0^{1,p}(\Omega)$-topology. Choose $r \in (q, p^*)$ and consider the closed convex sets

$$S_n = \left\{ u \in W_0^{1,p}(\Omega) : \frac{1}{r} ||u - u_0||_r^r \leq \frac{1}{n} \right\}, \quad n \geq 1$$

(here, $|| \cdot ||_r$ stands for the $L^r(\Omega)$-norm). Since $\mathscr{F}$ is sequentially weakly lower semicontinuous and coercive on S_n, we may find $v_n, n \geq 1$, such that

$$v_n \in S_n, \qquad \mathscr{F}(v_n) = \min_{u \in S_n} \mathscr{F}(u), \qquad \mathscr{F}(v_n) < \mathscr{F}(u_0), \quad n \geq 1. \tag{6}$$

Claim. *$v_n \geq u_\mu$, for all $n \geq 1$.*

Arguing indirectly, suppose that for some $n \geq 1$, we have $(u_\mu - v_n)_+ \not\equiv 0$. Set

$$w_t = v_n + t(u_\mu - v_n)_+, \qquad \xi(t) = \mathscr{F}(w_t), \quad t \in [0, 1].$$

Then, on $\{u_\mu > v_n\}$, we have

$$w_t - u_\mu = (1 - t)(v_n - u_\mu) < 0 \quad \text{for all } t \in (0, 1).$$

Therefore, for $t \in (0, 1)$,

$$\begin{aligned}
\xi'(t) &= \langle \mathscr{F}'(w_t), (u_\mu - v_n)_+ \rangle \\
&= \int_{\{u_\mu > v_n\}} |\nabla w_t|^{p-2} \nabla w_t \cdot (\nabla u_\mu - \nabla v_n) - \lambda \int_{\{u_\mu > v_n\}} f(x, w_t)(u_\mu - v_n) \\
&\quad - \mu \int_{\{u_\mu > v_n\}} g(x, w_t)(u_\mu - v_n) \\
&\leq \int_{\{u_\mu > v_n\}} |\nabla w_t|^{p-2} \nabla w_t \cdot (\nabla u_\mu - \nabla v_n) - \mu \int_{\{u_\mu > v_n\}} u_\mu^{s-1}(u_\mu - v_n) \\
&= - \int_{\{u_\mu > v_n\}} (|\nabla w_t|^{p-2} \nabla w_t - |\nabla u_\mu|^{p-2} \nabla u_\mu) \cdot (\nabla v_n - \nabla u_\mu)
\end{aligned}$$

(owing to the choice of u_μ),

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so,

$$(1-t)\xi'(t) \le -\int_{\{u_\mu > v_n\}} (|\nabla w_t|^{p-2}\nabla w_t - |\nabla u_\mu|^{p-2}\nabla u_\mu)\cdot(\nabla w_t - \nabla u_\mu)$$
$$< 0 \quad \text{(by the strong monotonicity of the } p\text{-Laplacian operator).}$$

Consequently, ξ is strictly decreasing on $[0,1]$. In particular, we have

$$\xi(1) < \xi(0) \Rightarrow \mathscr{F}(w_1) < \mathscr{F}(v_n).$$

However, since $u_0 \ge u_\mu$, we may check that $|w_1 - u_0| \le |v_n - u_0|$. Thus, $w_1 \in S_n$, which contradicts the fact that v_n is a global minimizer of $\mathscr{F}$ on S_n and finishes the proof of the claim.

Then the Lagrange multiplier rule gives rise to a sequence $k_n, n \ge 1$ such that

$$\mathscr{F}'(v_n) = k_n E'(v_n), \quad n \ge 1,$$

where $E(u) = ||u - u_0||_r^r/r, u \in W_0^{1,p}(\Omega)$.

Now, the above claim combined with the definition of g yields that for all $n \ge 1$,

$$\begin{cases} -\Delta_p v_n(x) = \lambda f(x, v_n(x)) + \mu v_n(x)^{s-1} + k_n|v_n(x) - u_0(x)|^{r-2}(v_n(x) - u_0(x)), \\ \quad \text{a.e. in } \Omega, \\ v_n \mid_{\partial\Omega} = 0. \end{cases}$$

We also remark that $k_n \le 0, n \ge 1$. Indeed, for each $n \ge 1$, the function

$$\zeta_n(t) = \mathscr{F}((1-t)v_n + tu_0), \quad t \in [0,1]$$

attains its minimum at $t_0 = 0$, so $\zeta_n'(0) \ge 0 \Rightarrow \langle \mathscr{F}'(v_n), u_0 - v_n\rangle \ge 0$, which implies $k_n||v_n - u_0||_r^r \le 0$ and thus $k_n \le 0$.

Then we proceed as in the proof of Theorem 1.1 of [4, p. 701] to reach a contradiction. □

We recall now some abstract results of Ricceri that we will use in the following.

Theorem A (see the proof of Theorem 2.5 in [12]). *Let X be a reflexive real Banach space, and let $\Gamma, \Upsilon : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals. Assume also that Γ is (strongly) continuous and satisfies $\lim_{\|x\|\to+\infty} \Gamma(x) = +\infty$. For each $\rho > \inf_X \Gamma$, put*

$$\vartheta(\rho) = \inf_{x\in\Gamma^{-1}(]-\infty,\rho[)} \frac{\Upsilon(x) - \inf_{\overline{(\Gamma^{-1}(]-\infty,\rho[))}^w} \Upsilon}{\rho - \Gamma(x)},$$

where $\overline{(\Gamma^{-1}(]-\infty,\rho[))}^w$ is the closure of $\Gamma^{-1}(]-\infty,\rho[)$ in the weak topology. Then, for each $\rho > \inf_X \Gamma$ and each $\mu > \vartheta(\rho)$, the restriction of the functional $\Upsilon + \mu\Gamma$ to $\Gamma^{-1}(]-\infty,\rho[)$ has a global minimum.

Theorem B (see [13, Theorem 4]). *Let (X, τ) be a Hausdorff topological space, and let $P, Q : X \to \mathbb{R}$ be two sequentially lower semicontinuous functions. Assume that*

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there is $\sigma > \inf_X P$ such that the set $\overline{P^{-1}(]-\infty, \sigma[)}$ is compact and first countable. Moreover, assume that there is a strict local minimizer of P, say x_0, such that $\inf_X P < P(x_0) < \sigma$. Then, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the function $P + \mu Q$ has at least two τ_P local minimizers lying in $P^{-1}(]-\infty, \sigma[)$, where τ_P denotes the smallest topology on X which contains both τ and the family of sets $\{P^{-1}(]-\infty, \rho[)\}_{\rho\in\mathbb{R}}$.

We will also need the following theorem, which we state here in a convenient form for our purposes.

Theorem C (see [14, Theorem C]). *Let X be a reflexive and separable real Banach space, $I : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous functional and $p > 0$. Denote by $\mathcal{I} : X \to \mathbb{R}$ the functional*

$$\mathcal{I}(u) = \frac{1}{p}\|u\|^p + I(u),$$

and assume that $\mathcal{I}$ is coercive. Then, any strict local minimizer of $\mathcal{I}$ in the strong topology is so in the weak topology.

3. Proof of the theorems

3.1. Proof of Theorem 1.1

Proof. In order to prove our first multiplicity result, we apply Theorem A with $X = W_0^{1,p}(\Omega)$, $\Gamma = \Phi$, $\Upsilon = -(J + \Psi)$. Fix $\rho > 0$ and let $\bar{\lambda} = (1/\vartheta(\rho))$ (if $\vartheta(\rho) = 0$, put $\bar{\lambda} = +\infty$). Thus, for $\lambda \in]0, \bar{\lambda}[$, the energy $\mathscr{E} = \Phi - \lambda(J + \Psi)$ has a local minimizer u_1 in the open ball $B_{(p\rho)^{1/p}}$. From Propositions 2.1 and 2.2, $u_1 \in \operatorname{int} C_+$ is a weak solution of $(\mathcal{P}_{\lambda,\lambda})$.

For such fixed λ, denote by $u_\lambda \in \operatorname{int} C_+$ the global minimizer of the functional $u \to \|u\|^p/P - (\lambda/s)\int_\Omega u_+^s$, and let $g, \tilde{\Psi}$ and $\mathscr{F}$ be as in Proposition 2.3. From the strong comparison principle for singular problems (Theorem 2.3 of [5]), we deduce that $u_1 - u_\lambda \in \operatorname{int} C_+$. (Recall that $f(x, t) > 0$, for $t > 0$.)

Also, as u_1 is a $W_0^{1,p}(\Omega)-$ local minimizer of $\mathscr{E}$, it is a $C_0^1(\overline{\Omega})-$ local minimizer of $\mathscr{E}$. Since $u_1 - u_\lambda \in \operatorname{int} C_+$ and $\operatorname{int} C_+$ is open in the $C_0^1(\overline{\Omega})-$ topology, there exists a neighbourhood V of u_1 in this topology such that $V \subseteq u_\lambda + \operatorname{int} C_+$ and $\mathscr{E}(u) \geq \mathscr{E}(u_1)$ for all $u \in V$.

Notice that for every $u \in u_\lambda + \operatorname{int} C_+$, we have that

$$\begin{aligned}
\mathscr{E}(u) &= \frac{1}{p}\|u\|^p - \lambda J(u) - \lambda \int_\Omega \int_0^{u_\lambda(x)} t^{s-1}\, dt\, dx - \lambda \int_\Omega \int_{u_\lambda(x)}^{u(x)} t^{s-1}\, dt\, dx \\
&\quad - \lambda \int_\Omega \int_0^{u_\lambda(x)} u_\lambda(x)^{s-1}\, dt\, dx + \lambda \int_\Omega \int_0^{u_\lambda(x)} u_\lambda(x)^{s-1}\, dt\, dx \\
&= \mathscr{F}(u) - \lambda \int_\Omega \int_0^{u_\lambda(x)} t^{s-1}\, dt\, dx + \lambda \int_\Omega \int_0^{u_\lambda(x)} u_\lambda(x)^{s-1}\, dt\, dx \\
&= \mathscr{F}(u) + \text{const.}
\end{aligned}$$

By virtue of the above equality, we obtain that u_1 is a $C_0^1(\overline{\Omega})-$ local minimizer of $\mathscr{F}$. But then Proposition 2.3 implies that u_1 is also a $W_0^{1,p}(\Omega)-$ local minimizer of $\mathscr{F}$.

Next, we note that $\mathscr{F}(\zeta\varphi_1) \to -\infty$, as $\zeta \to +\infty$. Indeed, $H(j)$ implies that

$$\lim_{t\to+\infty} \frac{F(x,t)}{t^p} = +\infty,$$

uniformly for a.a. $x \in \Omega$.

This fact combined with (H) enables us to choose $C_4 > \lambda_1, C_5 > 0$ such that for a.a. $x \in \Omega$,

$$F(x,t) \geq C_4 \frac{t^p}{p} - C_5 \quad \text{for all } t \geq 0.$$

Then

$$\mathscr{F}(\zeta\varphi_1) \leq \lambda_1 \frac{\zeta^p}{p} - C_4 \frac{\zeta^p}{p} + C_5|\Omega| - \int_\Omega \int_0^{\zeta\varphi_1(x)} g(x,z)\,\mathrm{d}z\,\mathrm{d}x \to -\infty,$$

as $\zeta \to +\infty$. (Recall that $C_4 > \lambda_1$ and $g(\cdot,\cdot) > 0$.)

Finally, $\mathscr{F}$ satisfies the Palais–Smale condition. The proof of this fact follows from $H(j)$ in a standard way. We sketch the proof for the sake of completeness.

Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence subject to

$$|\mathscr{F}(u_n)| \leq M_1 \quad \text{for some } M_1 > 0 \text{ for all } n \geq 1 \tag{7}$$

and

$$||\mathscr{F}'(u_n)||_* \to 0 \quad \text{as } n \to \infty.$$

We intend to prove that $\{u_n\}_{n\geq 1}$ is bounded.

Choose a sequence $\varepsilon_n \to 0^+$ such that for all $h \in W_0^{1,p}(\Omega)$,

$$\left| \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla h - \lambda \int_\Omega f(x,u_n)h - \mu \int_\Omega g(x,u_n)h \right| \leq \varepsilon_n \|h\|, \quad n \geq 1. \tag{8}$$

Putting into (8) $h = -u_{n-}$ and observing that $f(x,u_n)u_{n-} = 0$, we obtain

$$\|u_{n-}\|^p \leq \|u_{n-}\|^p + \mu \int_\Omega g(x,u_n)u_{n-} \leq \varepsilon_n \|u_{n-}\|, \quad n \geq 1$$

so,

$$\lim_{n\to\infty} \|u_{n-}\| = 0. \tag{9}$$

It remains to prove that $\{u_{n+}\}$ is bounded.

In (8), we choose $h = u_{n+}$. Owing to the definition of g and to hypothesis (H), we get that

$$-\|u_{n+}\|^p + \mu \int_{\{u_n > u_\mu\}} (u_{n+})^s + \lambda \int_{\{u_n > u_\mu\}} f(x,u_{n+})u_{n+} \leq M_2 + \varepsilon_n \|u_{n+}\| \tag{10}$$

for some $M_2 > 0$, all $n \geq 1$.

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On the other hand, from (7) and (9), and by using again the definition of g and (H), we may check that for some $M_3 > 0$,

$$\|u_{n+}\|^p - \frac{p\mu}{s}\int_{\{u_n>u_\mu\}}(u_{n+})^s - \lambda p\int_{\{u_n>u_\mu\}}F(x,u_{n+}) \le M_3, \quad n\ge 1. \tag{11}$$

Adding (10) and (11) and taking into account (H), we obtain

$$\lambda\int_\Omega [f(x,u_{n+})u_{n+} - pF(x,u_{n+})] \le M_4 + \mu\left(\frac{p}{s}-1\right)\|u_{n+}\|_s^s + \varepsilon_n\|u_{n+}\|, \quad n\ge 1, \tag{12}$$

for some $M_4 > 0$.

Now choose $\vartheta\in(1,p)$. Then, for each $n\ge 1$,

$$\|u_{n+}\|_s^s = \int_\Omega (u_{n+})^s \le |\Omega| + \int_{\{u_{n+}>1\}}(u_{n+})^\vartheta \le \text{const.}\,(1+||u_{n+}||^\vartheta) \tag{13}$$

(we have also used Hölder's and Poincaré's inequalities).

Meanwhile, hypothesis $H(j)$ enables us to find $C_6, C_7 > 0$ such that

$$tf(x,t) - pF(x,t) \ge C_6F(x,t) - C_7 \quad \text{for all } t\ge 0. \tag{14}$$

Combining (11)–(14), we infer that $\{u_{n+}\}$ is bounded, thus $\{u_n\}$ is bounded (see (9)). By using standard arguments based on the monotonicity properties of the negative p-Laplacian, we may extract a strongly convergent subsequence of $\{u_n\}$.

Now, from the classical Mountain Pass theorem of Ambrosetti and Rabinowitz (see [**1**, Theorem 1]), we obtain the existence of a second critical point for $\mathscr{F}$, i.e. a second solution of problem $(\mathcal{P}_{\lambda,\lambda})$ as it follows from Proposition 2.3. □

3.2. Proof of Theorem 1.2

Proof. Existence of two local minimizers. We are going to apply Theorem B with $X = W_0^{1,p}(\Omega)$ and τ the weak topology on X. Let us prove that

$$\lim_{u\to 0}\frac{J(u)}{\Phi(u)} = 0, \tag{15}$$

where Φ and J are as in the previous section. Fix $\varepsilon > 0$ and $\theta\in]\max\{p,q\},p^*[$. Hypothesis $H(i)$ and the growth of f (see (H)) imply that for some constant $c_\varepsilon > 0$,

$$0\le F(x,t)\le \frac{\varepsilon}{p}|t|^p + c_\varepsilon|t|^\theta \qquad \text{for a.a. } x\in\Omega \text{ and all } t\in\mathbb{R}.$$

It follows that for some $c'_\varepsilon > 0$,

$$0\le\frac{J(u)}{\Phi(u)}\le\frac{\varepsilon}{\lambda_1} + c'_\varepsilon||u||^{\theta-p} \quad \text{for all } u\in X\setminus\{0\}.$$

Then

$$\limsup_{u\to 0}\frac{J(u)}{\Phi(u)}\le\varepsilon$$

and, since $\varepsilon>0$ is arbitrary, (15) follows.

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From $H(ii)$, we easily deduce that

$$\lim_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)} = 0. \tag{16}$$

For all $\lambda > 0$, set

$$P_\lambda = \Phi - \lambda J.$$

The functional P_λ is sequentially weakly lower semicontinuous and coercive (see (16)), whereas 0 turns out to be a (strong) strict local minimizer of P_λ (see (15)). Applying Theorem C, we get that 0 is a local minimizer of P_λ in the weak topology. Moreover, by the definition of λ^* (see (1)), we obtain that for every $\lambda > \lambda^*$, 0 is not a global minimizer of P_λ. In fact, $\inf_X P_\lambda < P_\lambda(0) = 0$.

We point out that $\lambda^* > 0$. This follows from assumptions (H) and $H(i)$–(ii). Indeed, there exists a constant $c > 0$ such that

$$F(x,t) \le c|t|^p \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Thus,

$$\int_\Omega F(x,u(x))\,\mathrm{d}x \le c\|u\|_p^p \le c'\|u\|^p \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

where $c' > 0$ also involves the Sobolev embedding constant. This implies that $\lambda^* > 0$.

To proceed, fix $[a,b] \subset]\lambda^*, +\infty[$ and choose $\sigma > 0$.

From the coercivity of P_λ, it clearly follows that the sets $\overline{P_\lambda^{-1}(]-\infty,\sigma[)}^w$ are compact and metrizable (thus, first countable) with respect to the weak topology. (Recall that the weak closure of a bounded subset of a separable reflexive Banach space is compact and metrizable with respect to the weak topology.) Notice that

$$\bigcup_{\lambda\in[a,b]} \{u \in X : P_\lambda(u) < \sigma\} \subseteq \{u \in X : \Phi(u) - bJ(u) < \sigma\} \subseteq B_\eta,$$

for some positive radius η (this follows from the fact that $J(u) \ge 0$ for every $u \in W_0^{1,p}(\Omega)$ and from the coercivity of $\Phi - bJ$). Put also $c^* = \sup_{B_\eta}(\Phi - aJ)$ and let $r > \eta$ such that

$$\bigcup_{\lambda\in[a,b]} \{u \in X : P_\lambda(u) \le c^* + 2\} \subseteq B_r. \tag{17}$$

Next, choose $\lambda \in [a,b]$. Note that Ψ is sequentially weakly continuous but not differentiable in X, since $0 < s < 1$.

In order to obtain a uniform estimate of the norm of our solutions, we need to introduce a function $\alpha \in C^1(\mathbb{R})$, bounded, such that $\alpha(t) = t$ for every t such that $|t| \le \sup_{B_{2r}} \Psi$. Therefore,

$$(\alpha \circ \Psi)(u) = \Psi(u) \quad \text{for every } u \in B_{2r}. \tag{18}$$

Now Theorem B guarantees the existence of some $\delta = \delta(\lambda) > 0$ such that for every $\mu \in [0,\delta]$, $P_\lambda - \mu(\alpha \circ \Psi)$ has two local minimizers in the τ_{P_λ} topology, say u_1 and u_2, such that

$$u_1, u_2 \in P_\lambda^{-1}(]-\infty, \sigma[) \subseteq B_\eta \subseteq B_r. \tag{19}$$

Since P_λ is continuous, the topology τ_{P_λ} is weaker than the strong topology, and u_1 and u_2 turn out to be local minimizers of the functional

$$\mathscr{E}_\alpha : X \to \mathbb{R}, \quad \mathscr{E}_\alpha(u) = \frac{1}{p}\|u\|^p - \lambda J(u) - \mu(\alpha \circ \Psi)(u).$$

Notice that if $\|u - u_i\| < r$, then $\|u\| < \|u_i\| + r < 2r$ for $i = 1, 2$. Therefore, since (from (18)) $\mathscr{E}_\alpha = \mathscr{E}$ in B_{2r}, u_1 and u_2 turn out to be local minimizers of $\mathscr{E}$.

Put $\mu^* = \mu^*(\lambda) = \min\{\delta, (\sup_{\mathbb{R}} \alpha)^{-1}\}$ and fix $\mu \in [0, \mu^*]$.

From Propositions 2.1 and 2.2, u_1 and u_2 are weak solutions of $(\mathcal{P}_{\lambda,\mu})$ belonging to $\mathrm{int}\, C_+ \cap C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$.

Existence of the third solution. The existence of a third solution is obtained via regularization methods.

For $\mu \in [0, \mu^*]$, let u_μ be the unique global minimizer of the functional

$$u \to \frac{1}{p}\|u\|^p - \frac{\mu}{s}\int_\Omega u_+^s \,\mathrm{d}x,$$

define $g, \tilde{\Psi}$ and $\mathscr{F}$ as in Proposition 2.3, and let

$$\mathscr{F}_\alpha : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad \mathscr{F}_\alpha(u) = \frac{1}{p}\|u\|^p - \lambda J(u) - \mu(\alpha \circ \tilde{\Psi})(u).$$

It is clear that since $g(x, t) \le t^{s-1}$ for every $t > 0$, if $\|u\| \le 2r$, one has

$$\tilde{\Psi}(u) \le \Psi(u) \le \sup_{B_{2r}} \Psi,$$

and $(\alpha \circ \tilde{\Psi})(u) = \tilde{\Psi}(u)$, so that $\mathscr{F}_\alpha$ coincides with $\mathscr{F}$ in B_{2r}.

From the strong comparison principle for singular problems (Theorem 2.3 of [**5**]), we deduce that $u_1 - u_\mu \in \mathrm{int}\, C_+$ and $u_2 - u_\mu \in \mathrm{int}\, C_+$. (Recall that $f(x, t) > 0$, for $t > 0$.)

As in the proof of Theorem 1.1, we notice that u_1 is a $C_0^1(\overline{\Omega})-$ local minimizer of $\mathscr{E}$ and, since $u_1 - u_\mu \in \mathrm{int}\, C_+$ and $\mathrm{int}\, C_+$ is open in the $C_0^1(\overline{\Omega})-$ topology, there exists a neighbourhood V of u_1 in this topology such that $V \subseteq u_\mu + \mathrm{int}\, C_+$ and $\mathscr{E}(u) \ge \mathscr{E}(u_1)$ for all $u \in V$.

Moreover, since (again as in Theorem 1.1)

$$\mathscr{E}(u) = \mathscr{F}(u) + \text{const.} \quad \text{for all } u \in u_\mu + \mathrm{int}\, C_+,$$

u_1 is a $C_0^1(\overline{\Omega})-$ local minimizer of $\mathscr{F}$. But then Proposition 2.3 implies that u_1 is also a $W_0^{1,p}(\Omega)-$ local minimizer of $\mathscr{F}$. Similarly, u_2 turns out to be a $W_0^{1,p}(\Omega)-$ local minimizer of $\mathscr{F}$. Moreover, since for every $\|u\| < 2r$ one has $\mathscr{F}(u) = \mathscr{F}_\alpha(u)$, u_1 and u_2 are actually $W_0^{1,p}(\Omega)-$ local minimizers of $\mathscr{F}_\alpha$.

The functional $\mathscr{F}_\alpha$ is of class C^1 in $W_0^{1,p}(\Omega)$. Indeed, since $u_\mu \ge \varepsilon_\mu \varphi_1$, the functional $\tilde{\Psi}$ is of class C^1 in $W_0^{1,p}(\Omega)$ (see the proof of Lemma A.3 in [**4**]). Therefore, $\alpha \circ \tilde{\Psi} \in C^1(W_0^{1,p}(\Omega))$, and the same is true for $\mathscr{F}_\alpha$. Also, it verifies the well known Palais–Smale

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condition. Indeed, hypothesis $H(ii)$ implies that $\mathscr{F}_\alpha$ is coercive. Then, owing to the strong monotonicity of the negative p-Laplacian operator, the Palais–Smale condition follows in a standard way. (Recall that the operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ defined by

$$\langle Tu, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v$$

is of *type* $(S)_+$, i.e. it satisfies the following condition:

'if $(u_n) \subseteq W_0^{1,p}(\Omega)$ weakly converges to u and $\limsup_n \langle Tu_n, u_n - u\rangle \le 0$, then (u_n) strongly converges to u'.)

By Theorem 1 of [**11**], there exists a critical point for $\mathscr{F}_\alpha$, say u_3, such that

$$\mathscr{F}_\alpha(u_3) = \inf_{\gamma\in S} \sup_{t\in[0,1]} \mathscr{F}_\alpha(\gamma(t)),$$

where

$$S = \{\gamma \in C^0([0,1], W_0^{1,p}(\Omega)) : \ \gamma(0) = u_1, \ \gamma(1) = u_2\}.$$

In particular, if $\tilde\gamma(t) = tu_1 + (1-t)u_2, t \in [0,1]$, then $\tilde\gamma \in S$ and

$$\tilde\gamma(t) \in B_\eta \quad \text{for all } t \in [0,1].$$

(Recall that $u_1, u_2 \in B_\eta$ (see (19)).

So, by the definition of c^* and μ^*, one has

$$\begin{aligned}\mathscr{F}_\alpha(u_3) &\le \sup_{t\in[0,1]} \mathscr{F}_\alpha(\tilde\gamma(t)) \\ &\le \sup_{u\in B_\eta} [\Phi(u) - aJ(u)] + \mu^* \sup_{u\in B_\eta} (\alpha\circ\tilde\Psi)(u) \\ &\le c^* + 1.\end{aligned}$$

Therefore,

$$P(u_3) = \Phi(u_3) - \lambda J(u_3) \le c^* + 1 + \mu(\alpha\circ\tilde\Psi)(u_3) \le c^* + 2$$

and, from (17),

$$u_3 \in B_r.$$

It is clear that u_3 is a critical point of $\mathscr{F}$ and, from Proposition 2.3, $u_3 \ge u_\mu$. Thus, from Propositions 2.3 and 2.2, $u_3 \in \operatorname{int} C_+$ is a positive solution of problem $(\mathcal{P}_{\lambda,\mu})$, and the proof is concluded. □

Remark 3.1. Our main theorem is the result of a very careful application of the abstract result of Ricceri. It turns out to be crucial to apply it to the functional Ψ (which does not depend on μ) in order to individuate the positive numbers δ and μ^* and subsequently, after fixing $\mu \in]0, \mu^*[$, to work with the functional $\tilde\Psi$ (which on the contrary depends on μ). The functional Ψ has enough regularity that it is possible to apply Theorem A. It remains an open question whether the existence of three solutions can also be obtained for problem $(\mathcal{P}_{\lambda,\mu})$ when $s \le 0$. In this case, it is clear that the above method can not be used, as the functional $\Psi(u) = (1/s)\int_\Omega u_+(x)^s \,\mathrm{d}x$ (for $s < 0$) is not well defined even on the Sobolev space $W_0^{1,p}(\Omega)$.

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