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RESEARCH ARTICLE

Coordinate rings of regular nilpotent Hessenberg varieties in the open opposite schubert cell

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Abstract
Dale Peterson has discovered a surprising result that the quantum cohomology ring of the flag variety $\mathrm{GL}_n(\mathbb{C})/B$ is isomorphic to the coordinate ring of the intersection of the Peterson variety Pet_n and the opposite Schubert cell associated with the identity element Ω_e° in $\mathrm{GL}_n(\mathbb{C})/B$. This is an unpublished result, so papers of Kostant and Rietsch are referred for this result. An explicit presentation of the quantum cohomology ring of $\mathrm{GL}_n(\mathbb{C})/B$ is given by Ciocan–Fontanine and Givental–Kim. In this paper, we introduce further quantizations of their presentation so that they reflect the coordinate rings of the intersections of regular nilpotent Hessenberg varieties $\mathrm{Hess}(N,h)$ and Ω_e° in $\mathrm{GL}_n(\mathbb{C})/B$. In other words, we generalize the Peterson's statement to regular nilpotent Hessenberg varieties via the presentation given by Ciocan–Fontanine and Givental–Kim. As an application of our theorem, we show that the singular locus of the intersection of some regular nilpotent Hessenberg variety $\mathrm{Hess}(N,h_m)$ and Ω_e° is the intersection of certain Schubert variety and Ω_e°, where $h_m = (m,n,\ldots,n)$ for $1 < m < n$. We also see that $\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ$ is related with the cyclic quotient singularity.

Contents

1. Introduction

Hessenberg varieties are subvarieties of the full flag variety introduced by De Mari and Shayman and studied by De Mari, Procesi and Shayman [11, 12]. These varieties lie in a fruitful intersection of algebraic combinatorics and representation theory, such as hyperplane arrangements ([4, 14, 34]), Stanley's chromatic symmetric functions ([9, 23, 27, 33]), Postnikov's mixed Eulerian numbers ([8, 24, 30]) and toric orbifolds associated with partitioned weight polytopes ([7, 25]) in a recent development. In this paper, we generalize a result discovered by Dale Peterson to regular nilpotent Hessenberg varieties in type A via the explicit presentation of the quantum cohomology ring of flag varieties given by Ciocan–Fontanine and Givental–Kim.

Let n be a positive integer. The (full) flag variety $Fl(\mathbb{C}^n)$ is the collection of nested sequences of linear subspaces $V_\bullet := (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ in $\mathbb{C}^n$, where each V_i denotes an i-dimensional subspace of $\mathbb{C}^n$. Let N be the regular nilpotent matrix in Jordan canonical form, that is, the nilpotent matrix in Jordan form with exactly one Jordan block. The *Peterson variety* Pet_n is defined to be the subvariety of the flag variety $Fl(\mathbb{C}^n)$ as follows:

$$\mathrm{Pet}_n := \{V_\bullet \in Fl(\mathbb{C}^n) \mid NV_i \subset V_{i+1} \text{ for all } i = 1, 2, \ldots, n-1\}.$$

Dale Peterson has discovered a surprising connection between a geometry of the Peterson variety Pet_n and the quantum cohomology[1] of the flag variety $Fl(\mathbb{C}^n)$, as explained below. Let B^- be the set of lower triangular matrices in the general linear group $\mathrm{GL}_n(\mathbb{C})$. Let Ω_e° be the opposite Schubert cell associated with the identity element e, which is the B^--orbit of the standard flag $F_\bullet = (F_i)_i$, where $F_i = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_i\}$ and $e_1, \ldots, e_n$ are the standard basis of $\mathbb{C}^n$. Note that Ω_e° is the open chart around the standard flag $F_\bullet = (F_i)_i$ in $Fl(\mathbb{C}^n)$. Due to Peterson's statements in [31], the coordinate ring of the intersection $\mathbb{C}[\mathrm{Pet}_n \cap \Omega_e^\circ]$ is isomorphic to the quantum cohomology of the flag variety $QH^*(Fl(\mathbb{C}^n))$ as $\mathbb{C}$-algebras:

$$\mathbb{C}[\mathrm{Pet}_n \cap \Omega_e^\circ] \cong QH^*(Fl(\mathbb{C}^n)).$$

This incredible result discovered by Peterson is unpublished, so we also refer the reader to [29, 32] for the result above. As $\mathbb{C}$-vector spaces, the quantum cohomology $QH^*(Fl(\mathbb{C}^n))$ is $\mathbb{C}1[q_1, \ldots, q_{n-1}] \otimes_{\mathbb{C}} H^*(Fl(\mathbb{C}^n))$ where we call $q_1, \ldots, q_{n-1}$ *quantum parameters*. The product structure on $QH^*(Fl(\mathbb{C}^n))$ is a certain deformation by quantum parameters of the ordinary cup product on $H^*(Fl(\mathbb{C}^n))$. More explicitly, Ciocan–Fontanine and Givental–Kim gave in [16, 22] an efficient presentation of the quantum cohomology ring $QH^*(Fl(\mathbb{C}^n))$ in terms of generators and relations as follows. Let $\check{M}_n$ be the following matrix

$$\check{M}_n := \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & x_{n-1} & q_{n-1} \\ 0 & \cdots & 0 & -1 & x_n \end{pmatrix}. \tag{1.1}$$

The *quantized elementary symmetric polynomial* $\check{E}_i^{(n)}$ $(1 \leq i \leq n)$ in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]$ is defined by the coefficient of λ^{n-i} for the characteristic polynomial of $\check{M}_n$ multiplied by $(-1)^i$, that is,

$$\det(\lambda I_n - \check{M}_n) = \lambda^n - \check{E}_1^{(n)}\lambda^{n-1} + \check{E}_2^{(n)}\lambda^{n-2} + \cdots + (-1)^n\check{E}_n^{(n)}.$$

[1]We work with quantum (and ordinary) cohomology with coefficients in $\mathbb{C}$ throughout this paper.

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Note that by setting $q_s = 0$ for all $1 \leq s \leq n-1$ we have that $\check{E}_i^{(n)}$ is the i-th elementary symmetric polynomial in the variables $x_1, \ldots, x_n$. Then it is known from [16, 22] that there is an isomorphism of $\mathbb{C}$-algebras

$$QH^*(Fl(\mathbb{C}^n)) \cong \mathbb{C}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]/(\check{E}_1^{(n)}, \ldots, \check{E}_n^{(n)}).$$

Combining Peterson's statement and the presentation above, we obtain the isomorphism

$$\mathbb{C}[\mathrm{Pet}_n \cap \Omega_e^\circ] \cong \mathbb{C}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]/(\check{E}_1^{(n)}, \ldots, \check{E}_n^{(n)}) \tag{1.2}$$

as $\mathbb{C}$-algebras. In this paper, we generalize this isomorphism to regular nilpotent Hessenberg varieties by further quantizing the right-hand side above.

Consider a nondecreasing function $h : [n] \to [n]$ such that $h(j) \geq j$ for all $j = 1, \ldots, n$ where $[n] := \{1, 2, \ldots, n\}$, which is called a *Hessenberg function*. We frequently write a Hessenberg function h as $h = (h(1), h(2), \ldots, h(n))$. The *regular nilpotent Hessenberg variety* $\mathrm{Hess}(N, h)$ associated with a Hessenberg function h is defined as

$$\mathrm{Hess}(N, h) := \{V_\bullet \in Fl(\mathbb{C}^n) \mid NV_i \subset V_{h(i)} \text{ for all } i = 1, 2, \ldots, n\}.$$

This object is a generalization of the Peterson variety Pet_n since $\mathrm{Hess}(N, h)$ is equal to Pet_n whenever $h = (2, 3, 4, \ldots, n, n)$. We also note that if $h = (n, n, \ldots, n)$, then $\mathrm{Hess}(N, h) = Fl(\mathbb{C}^n)$ by definition. Recall that the flag variety $Fl(\mathbb{C}^n)$ can be identified with $\mathrm{GL}_n(\mathbb{C})/B$ where B is the set of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$ so that the first j column vectors of a matrix $g \in \mathrm{GL}_n(\mathbb{C})$ generate the j-th vector space V_j for $j \in [n]$. Under the identification $Fl(\mathbb{C}^n) \cong \mathrm{GL}_n(\mathbb{C})/B$, one can write

$$\mathrm{Hess}(N, h) = \{gB \in \mathrm{GL}_n(\mathbb{C})/B \mid g^{-1}Ng \in H(h)\},$$

where $H(h)$ is the set of matrices $(a_{ij})_{i,j \in [n]}$ such that $a_{ij} = 0$ if $i > h(j)$. (Note that a matrix $(a_{ij})_{i,j \in [n]}$ in $H(h)$ is not necessarily invertible.) Since Ω_e° is an affine open set in $Fl(\mathbb{C}^n)$ which is naturally identified with the set of lower triangular unipotent matrices, the intersection $\mathrm{Hess}(N, h) \cap \Omega_e^\circ$ is described as

$$\mathrm{Hess}(N, h) \cap \Omega_e^\circ = \left\{ g = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \cdots & x_{n\,n-1} & 1 \end{pmatrix} \middle| \begin{array}{c} (g^{-1}Ng)_{ij} = 0 \text{ for all} \\ j \in [n-1] \text{ and } h(j) < i \leq n \end{array} \right\}. \tag{1.3}$$

Motivated by this, we set

$$\mathcal{Z}(N, h)_e := \mathrm{Spec}\, \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/\big((g^{-1}Ng)_{ij} \mid j \in [n-1] \text{ and } h(j) < i \leq n\big).$$

We remark that this scheme can be regarded as a zero scheme of some section of certain vector bundle over $\mathrm{GL}_n(\mathbb{C})/B$, which is introduced in [1]. See Section 3 for the details.

We now generalize the matrix $\check{M}_n$ in equation (1.1) to the following matrix

$$M_n := \begin{pmatrix} x_1 & q_{12} & q_{13} & \cdots & q_{1n} \\ -1 & x_2 & q_{23} & \cdots & q_{2n} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & x_{n-1} & q_{n-1\,n} \\ 0 & \cdots & 0 & -1 & x_n \end{pmatrix}. \tag{1.4}$$

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Then we define the q_{rs}*-quantized elementary symmetric polynomial* $E_i^{(n)}$ $(1 \leq i \leq n)$ in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 1 \leq r < s \leq n]$ by the coefficient of λ^{n-i} for the characteristic polynomial of M_n multiplied by $(-1)^i$, namely

$$\det(\lambda I_n - M_n) = \lambda^n - E_1^{(n)} \lambda^{n-1} + E_2^{(n)} \lambda^{n-2} + \cdots + (-1)^n E_n^{(n)}.$$

Note that by setting $q_{rs} = 0$ for $s - r > 1$ and $q_{s\,s+1} = q_s$, our polynomial $E_i^{(n)}$ is the (classical) quantized elementary symmetric polynomial $\check{E}_i^{(n)}$ in $\mathbb{C}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]$. For a Hessenberg function $h : [n] \to [n]$, we define ${}^hE_i^{(n)}$ as the polynomial $E_i^{(n)}$ by setting $q_{rs} = 0$ for all $2 \leq s \leq n$ and $1 \leq r \leq n - h(n+1-s)$:

$${}^hE_i^{(n)} := E_i^{(n)}|_{q_{rs}=0 \ (2 \leq s \leq n \text{ and } 1 \leq r \leq n-h(n+1-s))}.$$

We will pictorially explain which variables q_{rs} are set to 0 in the definition of ${}^hE_i^{(n)}$ in Example 4.11 and surrounding discussion. The main theorem of this paper is as follows:

Theorem 1.1. *Let $h : [n] \to [n]$ be a Hessenberg function. Then there is an isomorphism of $\mathbb{C}$-algebras*

$$\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e}) \cong \frac{\mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 2 \leq s \leq n, n - h(n+1-s) < r < s]}{({}^hE_1^{(n)}, \ldots, {}^hE_n^{(n)})}, \tag{1.5}$$

where $\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e})$ is the set of global sections.

Note that we explicitly construct the isomorphism above. See Theorem 4.13 and Proposition 7.3 for the correspondence. In particular, we see that our quantum parameters q_{rs}'s correspond to polynomials which define regular nilpotent Hessenberg varieties (up to signs). See Corollary 7.4.

As is well known, the cohomology ring $H^*(Fl(\mathbb{C}^n))$ is isomorphic to the quotient of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ by the ideal generated by elementary symmetric polynomials, so the following is a consequence of Theorem 1.1.

Corollary 1.2. *There is an isomorphism of $\mathbb{C}$-algebras*

$$\Gamma(\mathcal{Z}(N,id)_e, \mathcal{O}_{\mathcal{Z}(N,id)_e}) \cong H^*(Fl(\mathbb{C}^n)).$$

We say that a Hessenberg function $h : [n] \to [n]$ is *indecomposable* if it satisfies $h(j) > j$ for all $j \in [n-1]$. It is known from [1, Proposition 3.6] that if h is indecomposable, then the affine scheme $\mathcal{Z}(N,h)_e$ is reduced. Therefore, we can conclude the following result from Theorem 1.1.

Corollary 1.3. *If $h : [n] \to [n]$ is an indecomposable Hessenberg function, then there is an isomorphism of $\mathbb{C}$-algebras*

$$\mathbb{C}[\mathrm{Hess}(N,h) \cap \Omega_e^\circ] \cong \frac{\mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 2 \leq s \leq n, n - h(n+1-s) < r < s]}{({}^hE_1^{(n)}, \ldots, {}^hE_n^{(n)})},$$

where $\mathbb{C}[\mathrm{Hess}(N,h) \cap \Omega_e^\circ]$ is the coordinate ring of the open set $\mathrm{Hess}(N,h) \cap \Omega_e^\circ$ *in* $\mathrm{Hess}(N,h)$.

The isomorphism in Corollary 1.3 is a natural generalization of equation (1.2) since the Hessenberg function $h = (2, 3, 4, \ldots, n, n)$ which defines the Peterson variety Pet_n is indecomposable. In particular, our proof gives an elementary proof for Peterson's statement via the presentation for the quantum cohomology $QH^*(Fl(\mathbb{C}^n))$ given by [16, 22].

We next apply Corollary 1.3 to the study of the singular locus of the open set $\mathrm{Hess}(N,h) \cap \Omega_e^\circ$ in $\mathrm{Hess}(N,h)$ for some Hessenberg functions h. There are partial results for studying singularities of Hessenberg varieties in [3, 15, 26]. Indeed, an explicit presentation of the singular locus for the Peteson variety Pet_n is given by [26]. The singular locus for *nilpotent* Hessenberg varieties of codimension one in the flag variety $Fl(\mathbb{C}^n)$ is explicitly described in [15]. Also, a recent paper [3] combinatorially determines which permutation flags in arbitrary regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ are singular points. We focus on the following Hessenberg function

$$h_m = (m, n, \ldots, n) \quad \text{for } 2 \leq m \leq n-1. \tag{1.6}$$

We derive an explicit presentation of the singular locus of $\mathrm{Hess}(N, h_m)$ from Corollary 1.3. For this purpose, we first study the singular locus of $\mathrm{Hess}(N, h_2)$. More precisely, we show that the singularity of $\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ$ is related with a cyclic quotient singularity. We briefly explain the cyclic quotient singularity. Let $\mathfrak{C}_n$ be the cyclic group of order n generated by ζ, where ζ is a primitive n-th root of unity. Define the action of $\mathfrak{C}_n$ on $\mathbb{C}^2$ by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$ for $\zeta \in \mathfrak{C}_n$ and $(x, y) \in \mathbb{C}^2$. Then, the quotient space $\mathbb{C}^2/\mathfrak{C}_n$ is called the *cyclic quotient singularity* or the *type A_{n-1}-singularity*.

Theorem 1.4. *There is an isomorphism*

$$\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ \cong \mathbb{C}^2/\mathfrak{C}_n \times \mathbb{C}^{\frac{1}{2}(n-1)(n-2)-1},$$

where $h_2 = (2, n, \ldots, n)$.

Recall that $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ is given in equation (1.3). As a corollary of Theorem 1.4, one can give the singular locus of $\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ$ as the solution set of the equations $x_{i1} = 0$ for $2 \leq i \leq n$ and $x_{n2} = 0$. Combining this description and Corollary 1.3, we can explicitly describe the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ as follows.

Theorem 1.5. *Let h_m be the Hessenberg function defined in equation (1.6) for $2 \leq m \leq n-1$. Then, the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ is described as*

$$\left\{ g = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \cdots & x_{n\,n-1} & 1 \end{pmatrix} \;\middle|\; \begin{array}{c} x_{i1} = 0 \textit{ for all } 2 \leq i \leq n \\ \textit{and } x_{nj} = 0 \textit{ for all } 2 \leq j \leq m \end{array} \right\}.$$

The explicit presentation in Theorem 1.5 is certain Schubert variety in the open set Ω_e°, as explained below. Let $\mathfrak{S}_n$ be the permutation group on $[n]$. For $w \in \mathfrak{S}_n$, the Schubert variety X_w is defined by

$$X_w = \{V_\bullet \in Fl(\mathbb{C}^n) \mid \dim(V_p \cap F_q) \geq |\{i \in [p] \mid w(i) \leq q\}| \text{ for all } p, q \in [n]\},$$

where $F_\bullet = (F_i)_i$ is the standard flag, that is, $F_i = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_i\}$ and $e_1, \ldots, e_n$ are the standard basis of $\mathbb{C}^n$. For $2 \leq m \leq n-1$, we define the permutation $w_m \in \mathfrak{S}_n$ by

$$w_m := 1 \;\; n-1 \;\; n-2 \;\; \cdots \;\; n-m+1 \;\; n \;\; n-m \;\; n-m-1 \;\; \cdots \;\; 2$$

in one-line notation. (See also equation (10.6).) Then one can see that

$$X_{w_m} = \{V_\bullet \in Fl(\mathbb{C}^n) \mid V_1 = F_1 \text{ and } V_m \subset F_{n-1}\},$$

so we obtain the following result from Theorem 1.5.

Corollary 1.6. *Let $2 \leq m \leq n-1$. Then, the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ is equal to*

$$\mathrm{Sing}(\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ) = X_{w_m} \cap \Omega_e^\circ.$$

The paper is organized as follows. After reviewing the definition of Hessenberg varieties and their defining equations in Section 2, we focus on regular nilpotent Hessenberg varieties in Section 3. Then we

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state the main theorem (Theorem 4.13) in Section 4. In Section 5, we show that the homomorphism (1.5) is well defined and surjective. In order to prove that it is in fact an isomorphism, we use the commutative algebra's tool of Hilbert series and regular sequences. More specifically, we define certain degrees for the variables $\{x_{ij} \mid 1 \leq j < i \leq n\}$ appeared in equation (1.3) and $\{x_1, \ldots, x_n, q_{rs} \mid 1 \leq r < s \leq n\}$ introduced in equation (1.4) so that the two sides of equation (1.5) are graded $\mathbb{C}$-algebras in Section 6. We give a proof of our main theorem in Section 7. Next, turning our attention to the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$, where h_m is defined in equation (1.6), we give an explicit formula for partial derivatives $\frac{\partial}{\partial x_s} {}^hE_i^{(n)}$ and $\frac{\partial}{\partial q_{rs}} {}^hE_i^{(n)}$ in Section 8. Then we relate the singularity of $\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ$ to the cyclic quotient singularity (Theorem 9.7) in Section 9. One can see that this fact yields an explicit description for the singular locus of $\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ$ (Corollary 9.9). In Section 10, we generalize this result to the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ (Theorem 10.1) by using our main theorem together with the computations for partial derivatives $\frac{\partial}{\partial x_s} {}^hE_i^{(n)}$ and $\frac{\partial}{\partial q_{rs}} {}^hE_i^{(n)}$. We also see that the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ is equal to the intersection of the Schubert variety X_{w_m} and Ω_e° (Corollary 10.2).

2. Hessenberg varieties

In this section, we recall the definitions of Hessenberg varieties in type A_{n-1} and their defining equations. We use the notation $[n] := \{1, 2, \ldots, n\}$ throughout this paper.

Let n be a positive integer. A *Hessenberg function* is a function $h : [n] \to [n]$ satisfying the following two conditions

1. $h(1) \leq h(2) \leq \cdots \leq h(n)$;
2. $h(j) \geq j$ for all $j \in [n]$.

Note that $h(n) = n$ by definition. We frequently denote a Hessenberg function by listing its values in sequence, namely $h = (h(1), h(2), \ldots, h(n))$. It is useful to see a Hessenberg function h pictorially by drawing a configuration of boxes on a square grid of size $n \times n$ whose shaded boxes consist of boxes in the i-th row and the j-th column such that $i \leq h(j)$ for $i, j \in [n]$.

Example 2.1. Let $n = 5$. Then, $h = (3, 3, 4, 5, 5)$ is a Hessenberg function and the configuration of the shaded boxes is shown in Figure 1.

Let $\mathfrak{gl}_n(\mathbb{C})$ be the set of $n \times n$ matrices. For a Hessenberg function h, we define

$$H(h) := \{(a_{ij})_{i,j \in [n]} \in \mathfrak{gl}_n(\mathbb{C}) \mid a_{ij} = 0 \text{ if } i > h(j)\}, \tag{2.1}$$

which is called the *Hessenberg space associated to h*.

The (full) flag variety $Fl(\mathbb{C}^n)$ is the set of nested complex linear subspaces of $\mathbb{C}^n$:

$$Fl(\mathbb{C}^n) := \{V_\bullet := (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \text{ for all } i \in [n]\}.$$

Let B be the set of upper triangular matrices in the general linear group $\mathrm{GL}_n(\mathbb{C})$. As is well known, the flag variety $Fl(\mathbb{C}^n)$ can be identified with $\mathrm{GL}_n(\mathbb{C})/B$. Indeed, each flag $V_\bullet \in Fl(\mathbb{C}^n)$ is determined by a matrix g whose first j column vectors generate the j-th vector space V_j for $j \in [n]$. The correspondence

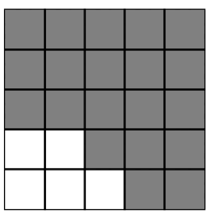

Figure 1. *The configuration corresponding to $h = (3, 3, 4, 5, 5)$*

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$gB \mapsto V_\bullet$ above gives the identification $Fl(\mathbb{C}^n) \cong \mathrm{GL}_n(\mathbb{C})/B$. For a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ and a Hessenberg function $h : [n] \to [n]$, the Hessenberg variety $\mathrm{Hess}(A, h)$ is defined to be the following subvariety of the flag variety $Fl(\mathbb{C}^n)$:

$$\mathrm{Hess}(A, h) := \{V_\bullet \in Fl(\mathbb{C}^n) \mid AV_i \subset V_{h(i)} \text{ for all } i \in [n]\}. \tag{2.2}$$

Hessenberg varieties are introduced in [11, 12]. Note that if $h = (n, n, \ldots, n)$, then $\mathrm{Hess}(A, h) = Fl(\mathbb{C}^n)$. By identifying $Fl(\mathbb{C}^n)$ with $\mathrm{GL}_n(\mathbb{C})/B$, we can write the definition above as

$$\mathrm{Hess}(A, h) = \{gB \in \mathrm{GL}_n(\mathbb{C})/B \mid g^{-1}Ag \in H(h)\}, \tag{2.3}$$

where $H(h)$ is the Hessenberg space defined in equation (2.1).

Definition 2.2. Let $A \in \mathfrak{gl}_n(\mathbb{C})$ and $i, j \in [n]$. We define a polynomial $F^A_{i,j}$ on $\mathrm{GL}_n(\mathbb{C})$ by

$$F^A_{i,j}(g) := \det(v_1 \ \ldots \ v_{i-1} \ Av_j \ v_{i+1} \ \ldots \ v_n),$$

where $g = (v_1 \ \ldots \ v_n) \in \mathrm{GL}_n(\mathbb{C})$ is the decomposition into column vectors. In other words, the polynomial $F^A_{i,j}(g)$ is the determinant of the matrix obtained from g by replacing the i-th column vector of g to the j-th column vector of Ag.

Lemma 2.3. *Let $A \in \mathfrak{gl}_n(\mathbb{C})$. For $g \in \mathrm{GL}_n(\mathbb{C})$ and $i, j \in [n]$, we have*

$$\frac{1}{\det(g)} F^A_{i,j}(g) = (g^{-1}Ag)_{ij},$$

where $(g^{-1}Ag)_{ij}$ denotes the (i, j)-th entry of the matrix $g^{-1}Ag$. In particular, we have

$$\mathrm{Hess}(A, h) = \{gB \in \mathrm{GL}_n(\mathbb{C})/B \mid F^A_{i,j}(g) = 0 \textit{ for all } 1 \leq j \leq n-1 \textit{ and } h(j) < i \leq n\}.$$

Proof. Let c_{ij} be the (i, j) cofactor of g, namely c_{ij} is obtained by multiplying the (i, j) minor of g by $(-1)^{i+j}$. Set $\tilde{g} = (c_{ij})^t_{i,j\in[n]} = (c_{ji})_{i,j\in[n]}$. Since $g^{-1} = \frac{1}{\det(g)} \tilde{g}$, it suffices to show that

$$F^A_{i,j}(g) = (\tilde{g}Ag)_{ij}. \tag{2.4}$$

We write $g = (v_1 \ \ldots \ v_n)$ and $Av_j = (b_{1j}, \ldots, b_{nj})^t$. By the definition of $\tilde{g}$, we have

$$(\tilde{g} \cdot Ag)_{ij} = \sum_{k=1}^{n} c_{ki} b_{kj} = \det(v_1 \ \ldots \ v_{i-1} \ Av_j \ v_{i+1} \ \ldots \ v_n),$$

where we used the cofactor expansion along the i-th column for the last equality. This yields equation (2.4) as desired. The latter statement follows from the former statement and equation (2.3). □

3. Regular nilpotent Hessenberg varieties

In this section, we review geometric properties for regular nilpotent Hessenberg varieties.

Let N be a regular nilpotent element in $\mathfrak{gl}_n(\mathbb{C})$, that is, a matrix whose Jordan canonical form consists of exactly one Jordan block with corresponding eigenvalue equal to 0. For a Hessenberg function h, $\mathrm{Hess}(N, h)$ is called the *regular nilpotent Hessenberg variety*. If $h = (2, 3, 4, \ldots, n, n)$, then $\mathrm{Pet}_n := \mathrm{Hess}(N, h = (2, 3, 4, \ldots, n, n))$ is called the Peterson variety, which is an object of an intense study by Dale Peterson [31]. Surprisingly, the Peterson variety is related with the quantum cohomology of the flag variety ([29, 32]). We will explain the relation in Section 4. For any $p \in \mathrm{GL}_n(\mathbb{C})$, one can

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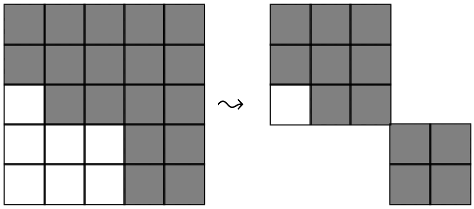

Figure 2. *The decomposition of* $h = (2, 3, 3, 5, 5)$ *into* $h_1 = (2, 3, 3)$ *and* $h_2 = (2, 2)$

see that $\mathrm{Hess}(N, h) \cong \mathrm{Hess}(p^{-1}Np, h)$ which sends gB to $p^{-1}gB$. Thus, we may assume that N is in Jordan form:

$$N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}. \tag{3.1}$$

We summarize geometric properties of $\mathrm{Hess}(N, h)$ as follows.

Theorem 3.1 [5, 26, 29, 34]. *Let* Pet_n *be the Peterson variety. Let* $\mathrm{Hess}(N, h)$ *be the regular nilpotent Hessenberg variety associated with a Hessenberg function h.*

(i) *The Peterson variety* Pet_n *is singular for* $n \geq 3$*. Also,* Pet_n *is normal if and only if* $n \leq 3$.
(ii) *The complex dimension of* $\mathrm{Hess}(N, h)$ *is given by* $\sum_{j=1}^{n}(h(j) - j)$*. In particular, we have* $\dim_{\mathbb{C}} \mathrm{Pet}_n = n - 1$.
(iii) $\mathrm{Hess}(N, h)$ *is irreducible.*

As a property of regular nilpotent Hessenberg varieties, a special case of $\mathrm{Hess}(N, h)$ can be decomposed into a product of smaller regular nilpotent Hessenberg varieties, as explained below. To explain this, we first recall the following terminology from [13, Definition 4.4].

Definition 3.2. A Hessenberg function h is *decomposable* if $h(j) = j$ for some $j \in [n-1]$. A Hessenberg function h is *indecomposable* if $h(j) > j$ for all $j \in [n-1]$. Note that an indecomposable Hessenberg function h satisfies $h(n-1) = h(n) = n$.

If h is a decomposable Hessenberg function, that is, $h(j) = j$ for some $j \in [n-1]$, then the Hessenberg function h can be decomposed into two smaller Hessenberg functions h_1 and h_2 defined by $h_1 = (h(1), \ldots, h(j))$ and $h_2 = (h(j+1) - j, \ldots, h(n) - j)$ as shown in Figure 2.

Then, every $V_\bullet \in \mathrm{Hess}(N, h)$ has $V_j = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_j\}$, where $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{C}^n$ and hence we have

$$\mathrm{Hess}(N, h) \cong \mathrm{Hess}(N_1, h_1) \times \mathrm{Hess}(N_2, h_2), \tag{3.2}$$

where N_1 and N_2 are the regular nilpotent matrices in Jordan canonical form of size j and $n - j$, respectively ([13, Theorem 4.5]).

In a recent paper, Abe–Insko gave the condition that $\mathrm{Hess}(N, h)$ is normal as follows.

Theorem 3.3. ([3, Theorem 1.3]) *Let h be an indecomposable Hessenberg function. Then the regular nilpotent Hessenberg variety* $\mathrm{Hess}(N, h)$ *is normal if and only if h satisfies the condition that* $h(i-1) > i$ *or* $h(i) > i + 1$ *for all* $1 < i < n - 1$.

There is a natural partial order on Hessenberg functions. For two Hessenberg functions $h : [n] \to [n]$ and $h' : [n] \to [n]$, we say $h' \subset h$ if $h'(j) \leq h(j)$ for all $j \in [n]$. Note that if $h' \subset h$, then $\mathrm{Hess}(N, h')$ is a closed subvariety of $\mathrm{Hess}(N, h)$ by the definition (2.2). Since $h = (2, 3, 4, \ldots, n, n)$ is the minimal Hessenberg function among indecomposable Hessenberg functions with respect to the partial order $\subset$,

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the Peterson variety Pet_n is the minimal Hessenberg variety among indecomposable regular nilpotent Hessenberg varieties with respect to the inclusion.

Let h be a Hessenberg function. One can see that the Hessenberg space $H(h)$ in equation (2.1) is stable under the adjoint action of B, so this induces a B-action on the quotient space $\mathfrak{gl}_n(\mathbb{C})/H(h)$. We denote by $\overline{x}$ the image of $x \in \mathfrak{gl}_n(\mathbb{C})$ under the surjection $\mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})/H(h)$. In [1], we consider the vector bundle $\mathrm{GL}_n(\mathbb{C}) \times_B (\mathfrak{gl}_n(\mathbb{C})/H(h))$ over the flag variety $\mathrm{GL}_n(\mathbb{C})/B$ and its section s_A for $A \in \mathfrak{gl}_n(\mathbb{C})$ defined by

$$s_A : \mathrm{GL}_n(\mathbb{C})/B \to \mathrm{GL}_n(\mathbb{C}) \times_B (\mathfrak{gl}_n(\mathbb{C})/H(h));\ gB \mapsto [g, \overline{g^{-1}Ag}],$$

where we denote by $[g,\overline{x}]$ the image of $(g,\overline{x}) \in \mathrm{GL}_n(\mathbb{C}) \times (\mathfrak{gl}_n(\mathbb{C})/H(h))$ under the surjection $\mathrm{GL}_n(\mathbb{C}) \times (\mathfrak{gl}_n(\mathbb{C})/H(h)) \to \mathrm{GL}_n(\mathbb{C}) \times_B (\mathfrak{gl}_n(\mathbb{C})/H(h))$ such that $[g,\overline{x}] = [gb, \overline{b^{-1}xb}]$ for all $b \in B$. By the definition (2.3), one can see that the zero set of s_A is the Hessenberg variety $\mathrm{Hess}(A,h)$.

In general, let $\pi : E \to X$ be a vector bundle of rank r over a scheme X. If s is a section of E, then the *zero scheme* $\mathcal{Z}(s)$ of s is defined as follows (cf. [20]). Let $(U_i,\varphi_i)_i$ be a local trivialization of E, that is, an open covering $\{U_i\}_i$ of X and isomorphisms φ_i of $\pi^{-1}(U_i)$ with $U_i \times \mathbb{C}^r$ over U_i such that the composites $\varphi_i \circ \varphi_j^{-1}$ are linear. Let $s_i : U_i \to \mathbb{C}^r$ determine the section s on U_i, $s_i = (s_{i_1},\dots,s_{i_r})$, s_{i_m} in the coordinate ring of U_i; then $\mathcal{Z}(s)$ is defined in U_i by the ideal generated by $s_{i_1},\dots,s_{i_r}$.

Motivated by the discussion above, we use the following definition introduced in [1].

Definition 3.4. ([1, Definition 3.1]) Let $N \in \mathfrak{gl}_n(\mathbb{C})$ be the regular nilpotent element in equation (3.1). For a Hessenberg function h, let $\mathcal{Z}(N,h)$ denote the zero scheme in $\mathrm{GL}_n(\mathbb{C})/B$ of the section s_N.

Locally around $gB \in \mathrm{GL}_n(\mathbb{C})/B$, the section s_N is represented by a collection of regular functions, and the scheme $\mathcal{Z}(N,h)$ is locally the zero scheme of those functions. (See [1, Lemma 3.4] for the detail.)

Theorem 3.5. *([1, Proposition 3.6]) If h is an indecomposable Hessenberg function, then the zero scheme $\mathcal{Z}(N,h)$ of the section s_N is reduced.*

We analyze the intersection of $\mathrm{Hess}(N,h)$ and the opposite Schubert cell Ω_e° associated with the identity element e. Recall that Ω_e° is the B^--orbit of the identity flag eB/B in $\mathrm{GL}_n(\mathbb{C})/B$, where B^- denotes the set of lower triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. In particular, each flag $V_\bullet \in \Omega_e^\circ$ has V_j spanned by the first j columns of a matrix with 1's in the diagonal positions and 0's to the right of these 1's (cf. [19, Section 10.2]). Thus, one can see that the opposite Schubert cell Ω_e° can be regarded as the following affine space:

$$\Omega_e^\circ \cong \left\{ \left. \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \cdots & x_{n\,n-1} & 1 \end{pmatrix} \right| x_{ij} \in \mathbb{C}\ (1 \le j < i \le n) \right\} \cong \mathbb{C}^{\frac{1}{2}n(n-1)}.$$

Note that Ω_e° is an affine open set of the flag variety $\mathrm{GL}_n(\mathbb{C})/B$ and its coordinate ring is isomorphic to the polynomial ring $\mathbb{C}[x_{ij} \mid 1 \le j < i \le n]$. Thus, we may identify Ω_e° as $\mathrm{Spec}\,\mathbb{C}[x_{ij} \mid 1 \le j < i \le n]$. From now on, we write

$$F_{i,j} \coloneqq F^N_{i,j}(g) \text{ for } 1 \le j < i \le n \text{ where } g = \begin{pmatrix} 1 & & & & \\ x_{21} & 1 & & & \\ x_{31} & x_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n1} & x_{n2} & \cdots & x_{n\,n-1} & 1 \end{pmatrix}$$

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for simplicity. Since the first column vector of Ng is $(x_{21}, x_{31}, x_{41}, \dots, x_{n1}, 0)^t$ and the j-th column vector of Ng is $(\underbrace{0, \dots, 0}_{j-2}, \underbrace{1, x_{j+1\,j}, \dots, x_{nj}, 0}_{n-j+2})^t$ for $j \geq 2$, we can explicitly write

$$F_{i,1} = \begin{vmatrix} 1 & 0 & \cdots & 0 & x_{21} \\ x_{21} & 1 & \ddots & \vdots & x_{31} \\ \vdots & x_{32} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots \\ x_{i1} & x_{i2} & \cdots & x_{i\,i-1} & x_{i+1\,1} \end{vmatrix} \quad \text{for } j = 1; \tag{3.3}$$

$$F_{i,j} = \begin{vmatrix} 1 & 0 & \cdots & 0 & 1 \\ x_{j\,j-1} & 1 & \ddots & \vdots & x_{j+1\,j} \\ x_{j+1\,j-1} & x_{j+1\,j} & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots \\ x_{i\,j-1} & x_{ij} & \cdots & x_{i\,i-1} & x_{i+1\,j} \end{vmatrix} \quad \text{for } j \geq 2 \tag{3.4}$$

from Definition 2.2. Here, we take the convention that $x_{n+1\,j} = 0$. The determinant of $g \in \Omega_e^\circ$ is 1, so we have

$$F_{i,j} = F_{i,j}^N(g) = (g^{-1}Ng)_{ij} \quad \text{for } g \in \Omega_e^\circ \tag{3.5}$$

by Lemma 2.3. We set

$$\begin{aligned} \mathcal{Z}(N,h)_e &:= \mathcal{Z}(N,h) \cap \operatorname{Spec} \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n] \\ &= \operatorname{Spec} \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/((g^{-1}Ng)_{ij} \mid j \in [n-1] \text{ and } h(j) < i \leq n). \end{aligned} \tag{3.6}$$

By the discussion above, we have the following.

Proposition 3.6. *Let h be a Hessenberg function. Then*

$$\mathcal{Z}(N,h)_e = \operatorname{Spec} \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n).$$

In other words, the set of global sections $\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e})$ is given by

$$\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e}) \cong \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n). \tag{3.7}$$

In particular, if h is indecomposable, then the coordinate ring $\mathbb{C}[\operatorname{Hess}(N,h) \cap \Omega_e^\circ]$ of the open set $\operatorname{Hess}(N,h) \cap \Omega_e^\circ$ in $\operatorname{Hess}(N,h)$ is

$$\mathbb{C}[\operatorname{Hess}(N,h) \cap \Omega_e^\circ] = \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-2] \text{ and } h(j) < i \leq n) \tag{3.8}$$

by Theorem 3.5.

4. The main theorem

It can be extracted from Peterson's statements in [31] that the coordinate ring $\mathbb{C}[\operatorname{Pet}_n \cap \Omega_e^\circ]$ is isomorphic to the quantum cohomology ring of the flag variety $QH^*(Fl(\mathbb{C}^n))$. This result is unpublished, so we also refer the reader to [29, 32] for the result. In this section, we first review an explicit presentation for the quantum cohomology ring $QH^*(Fl(\mathbb{C}^n))$ given by [16, 22]. Then we introduce a further quantization of the ring presentation for $QH^*(Fl(\mathbb{C}^n))$ in natural way. Our main theorem relates $\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e})$ and our quantized ring presentation.

The quantum cohomology ring of the flag variety $QH^*(Fl(\mathbb{C}^n))$ is given by

$$QH^*(Fl(\mathbb{C}^n)) = \mathbb{C}[q_1,\dots,q_{n-1}] \otimes_{\mathbb{C}} H^*(Fl(\mathbb{C}^n))$$

as $\mathbb{C}$-vector spaces. Here, $q_1,\dots,q_{n-1}$ are called the *quantum parameters*. The product structure of $QH^*(Fl(\mathbb{C}^n))$ is a certain deformation of the cup product in the ordinary cohomology $H^*(Fl(\mathbb{C}^n))$. In order to describe an explicit presentation for $QH^*(Fl(\mathbb{C}^n))$, we need quantized elementary symmetric polynomials. Consider the matrix

$$\check{M}_n := \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & x_{n-1} & q_{n-1} \\ 0 & \cdots & 0 & -1 & x_n \end{pmatrix},$$

and define *quantized elementary symmetric polynomials* $\check{E}_1^{(n)},\dots,\check{E}_n^{(n)}$ in the polynomial ring $\mathbb{C}[x_1,\dots,x_n,q_1,\dots,q_{n-1}]$ by the following equation

$$\det(\lambda I_n - \check{M}_n) = \lambda^n - \check{E}_1^{(n)}\lambda^{n-1} + \check{E}_2^{(n)}\lambda^{n-2} + \cdots + (-1)^n \check{E}_n^{(n)},$$

where I_n is the identity matrix of size n. By using the cofactor expansion along the s-th column for $\det(\lambda I_s - \check{M}_s)$, we have

$$\det(\lambda I_s - \check{M}_s) = (\lambda - x_s)\det(\lambda I_{s-1} - \check{M}_{s-1}) + q_{s-1}\det(\lambda I_{s-2} - \check{M}_{s-2}).$$

This implies the following recursive formula

$$\check{E}_r^{(s)} = \check{E}_r^{(s-1)} + \check{E}_{r-1}^{(s-1)} x_s + \check{E}_{r-2}^{(s-2)} q_{s-1} \quad \text{for } 1 \le r \le s \le n,$$

where we take the convention that $\check{E}_0^{(s-1)} = 1, \check{E}_{-1}^{(s-2)} = 0$ whenever $r = 1$ and $\check{E}_s^{(s-1)} = 0$ whenever $r = s$. By setting $q_s = 0$ for all $s \in [n-1]$, $\check{E}_i^{(n)}$ is the (ordinary) i-th elementary symmetric polynomial in the variables $x_1,\dots,x_n$.

Theorem 4.1 [16, 22]. *There is an isomorphism of $\mathbb{C}$-algebras*

$$QH^*(Fl(\mathbb{C}^n)) \cong \mathbb{C}[x_1,\dots,x_n,q_1,\dots,q_{n-1}]/(\check{E}_1^{(n)},\dots,\check{E}_n^{(n)}). \tag{4.1}$$

Remark 4.2. The isomorphism above is stated in [22, Theorem 1] for integral coefficients. An explicit presentation of the quantum cohomology ring of partial flag varieties is given by [6, 17, 28].

Remark 4.3. By setting $q_s = 0$ for all $s \in [n-1]$, the isomorphism (4.1) yields the well-known presentation for the (ordinary) cohomology ring of $Fl(\mathbb{C}^n)$. Note that x_i is geometrically the first Chern class of the dual of the i-th tautological line bundle over $Fl(\mathbb{C}^n)$. See, for example, [19, Section 10.2, Proposition 3].

Theorem 4.4 (D. Peterson and [29, 32]). *There is an isomorphism of $\mathbb{C}$-algebras*

$$\mathbb{C}[\mathrm{Pet}_n \cap \Omega_e^\circ] \cong QH^*(Fl(\mathbb{C}^n)).$$

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Remark 4.5. Although we restricted the above discussion to the case of the full flag variety, the isomorphism above is in fact stated in [32, Theorem 4.2] for partial flag varieties. We also refer the reader to a recent paper [10] for general Lie types.

By Theorems 4.1 and 4.4, we obtain an isomorphism

$$\mathbb{C}[\mathrm{Pet}_n \cap \Omega_e^\circ] \cong \mathbb{C}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]/(\check{E}_1^{(n)}, \ldots, \check{E}_n^{(n)}) \tag{4.2}$$

as $\mathbb{C}$-algebras. One can see from [32, Theorems 3.3 and 4.2] that the isomorphism above sends x_{ij} to $\check{E}_{i-j}^{(n-j)}$ under the presentation (3.8) for $h = (2, 3, 4, \ldots, n, n)$. We generalize equation (4.2) to arbitrary Hessenberg function h.

Definition 4.6. Consider the matrix

$$M_n := \begin{pmatrix} x_1 & q_{12} & q_{13} & \cdots & q_{1n} \\ -1 & x_2 & q_{23} & \cdots & q_{2n} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & x_{n-1} & q_{n-1\,n} \\ 0 & \cdots & 0 & -1 & x_n \end{pmatrix},$$

and define q_{rs}*-quantized elementary symmetric polynomials* $E_1^{(n)}, \ldots, E_n^{(n)}$ in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 1 \le r < s \le n]$ by the following equation

$$\det(\lambda I_n - M_n) = \lambda^n - E_1^{(n)}\lambda^{n-1} + E_2^{(n)}\lambda^{n-2} + \cdots + (-1)^n E_n^{(n)}.$$

In other words, $E_i^{(n)}$ is the coefficient of λ^{n-i} for $\det(\lambda I_n - M_n)$ multiplied by $(-1)^i$.

Remark 4.7. If $q_{rs} = 0$ for $s - r > 1$ and $q_{s\,s+1} = q_s$, then the polynomial $E_i^{(n)}$ is equal to the (classical) quantized elementary symmetric polynomial $\check{E}_i^{(n)}$ in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]$.

Lemma 4.8. *For* $1 \le r \le s \le n$, *we have*

$$E_r^{(s)} = E_r^{(s-1)} + E_{r-1}^{(s-1)} x_s + \sum_{k=1}^{r-1} E_{r-1-k}^{(s-1-k)} q_{s-k\,s}$$

with the convention that $E_0^{(*)} = 1$ *for arbitrary* $*$, $\sum_{k=1}^{r-1} E_{r-1-k}^{(s-1-k)} q_{s-k\,s} = 0$ *whenever* $r = 1$, *and* $E_s^{(s-1)} = 0$ *whenever* $r = s$.

Proof. It follows from the cofactor expansion along the s-th column for $\det(\lambda I_s - M_s)$ that $\det(\lambda I_s - M_s)$ is equal to

$$\begin{aligned}(-1)^{s+2} q_{1s} + (-1)^{s+3} q_{2s} \det(\lambda I_1 - M_1) + \cdots + (-1)^{s+s} q_{s-1\,s} \det(\lambda I_{s-2} - M_{s-2}) \\ + (-1)^{s+s}(\lambda - x_s) \det(\lambda I_{s-1} - M_{s-1}).\end{aligned} \tag{4.3}$$

Since we have

$$\det(\lambda I_k - M_k) = \lambda^k - E_1^{(k)}\lambda^{k-1} + E_2^{(k)}\lambda^{k-2} + \cdots + (-1)^k E_k^{(k)}$$

for each $1 \le k \le s - 1$ by the definition, the coefficient of λ^{s-r} for equation (4.3) is given by

$$\begin{aligned}(-1)^r q_{s-r+1\,s} + (-1)^r q_{s-r+2\,s} E_1^{(s-r+1)} + \cdots + (-1)^r q_{s-1\,s} E_{r-2}^{(s-2)} \\ + (-x_s)(-1)^{r-1} E_{r-1}^{(s-1)} + (-1)^r E_r^{(s-1)}.\end{aligned}$$

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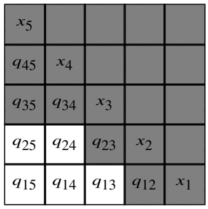

Figure 3. *The polynomial* ${}^hE_i^{(j)} \in \mathbb{C}[x_1, \ldots, x_5, q_{12}, q_{23}, q_{34}, q_{35}, q_{45}]$ *for* $h = (3, 3, 4, 5, 5)$

Namely, the coefficient of λ^{s-r} for $\det(\lambda I_s - M_s)$ is

$$(-1)^r \left(q_{s-r+1\,s} + q_{s-r+2\,s} E_1^{(s-r+1)} + \cdots + q_{s-1\,s} E_{r-2}^{(s-2)} + x_s E_{r-1}^{(s-1)} + E_r^{(s-1)}\right),$$

as desired. □

Example 4.9. Let $n = 3$. Then the $E_r^{(s)}$ have the following form.

$$\begin{aligned}
&E_0^{(1)} = 1, \quad E_1^{(1)} = E_0^{(0)} x_1 = x_1, \\
&E_0^{(2)} = 1, \quad E_1^{(2)} = E_1^{(1)} + E_0^{(1)} x_2 = x_1 + x_2, \quad E_2^{(2)} = E_1^{(1)} x_2 + E_0^{(0)} q_{12} = x_1 x_2 + q_{12}, \\
&E_0^{(3)} = 1, \quad E_1^{(3)} = E_1^{(2)} + E_0^{(2)} x_3 = x_1 + x_2 + x_3, \\
&E_2^{(3)} = E_2^{(2)} + E_1^{(2)} x_3 + E_0^{(1)} q_{23} = x_1 x_2 + x_1 x_3 + x_2 x_3 + q_{12} + q_{23}, \\
&E_3^{(3)} = E_2^{(2)} x_3 + E_1^{(1)} q_{23} + E_0^{(0)} q_{13} = x_1 x_2 x_3 + x_1 q_{23} + x_3 q_{12} + q_{13}.
\end{aligned}$$

Definition 4.10. Let $h : [n] \to [n]$ be a Hessenberg function. For each $1 \leq i \leq j \leq n$, we define ${}^hE_i^{(j)}$ as the polynomial $E_i^{(j)}$ by setting $q_{rs} = 0$ for all $2 \leq s \leq n$ and $1 \leq r \leq n - h(n+1-s)$:

$${}^hE_i^{(j)} := E_i^{(j)}|_{q_{rs}=0 \ (2 \leq s \leq n \text{ and } 1 \leq r \leq n-h(n+1-s))}.$$

The surviving variables in the polynomial ${}^hE_i^{(j)}$ are pictorially shown as follows. Let w_0 be the longest element of the symmetric group $\mathfrak{S}_n$ on n letters $[n]$. Consider the matrix

$$w_0 M_n w_0 = \begin{pmatrix} x_n & -1 & 0 & \cdots & 0 \\ q_{n-1\,n} & x_{n-1} & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ q_{2n} & \cdots & q_{23} & x_2 & -1 \\ q_{1n} & \cdots & q_{13} & q_{12} & x_1 \end{pmatrix}, \tag{4.4}$$

where w_0 is regarded as the permutation matrix. Then $w_0 M_n w_0 \in H(h)$ if and only if $q_{rs} = 0$ for all $2 \leq s \leq n$ and $1 \leq r \leq n - h(n+1-s)$ where $H(h)$ is the Hessenberg space defined in equation (2.1). In other words, the surviving variables in the polynomial ${}^hE_i^{(j)}$ are pictorially the variables q_{rs} arranged in $w_0 M_n w_0$ such that q_{rs} belongs to the configuration of the shaded boxes for the Hessenberg function h.

Example 4.11. Let $n = 5$ and $h = (3, 3, 4, 5, 5)$, which is depicted in Example 2.1. Then ${}^hE_i^{(j)}$ is a polynomial in the variables $x_1, \ldots, x_5, q_{12}, q_{23}, q_{34}, q_{35}, q_{45}$ for $1 \leq i \leq j \leq 5$, as shown in Figure 3.

Remark 4.12. In our setting, the flag variety $Fl(\mathbb{C}^n)$ is identified with $\mathrm{GL}_n(\mathbb{C})/B$, while $Fl(\mathbb{C}^n)$ is regarded as $\mathrm{GL}_n(\mathbb{C})/B^-$ in [32]. Recall that the conjugation by w_0 gives an isomorphism $\mathrm{GL}_n(\mathbb{C})/B \cong \mathrm{GL}_n(\mathbb{C})/B^-$ since $B^- = w_0 B w_0$. This relation might affect the reason why we take the conjugation by w_0 in equation (4.4).

We now state the main theorem of this paper.

Theorem 4.13. *Let $h : [n] \to [n]$ be a Hessenberg function and $\mathcal{Z}(N,h)_e$ the intersection defined in equation (3.6). Then there is an isomorphism of $\mathbb{C}$-algebras*

$$\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e}) \cong \frac{\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 2 \le s \le n, n-h(n+1-s) < r < s]}{({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})}, \tag{4.5}$$

which sends x_{ij} to ${}^hE_{i-j}^{(n-j)}$ under the presentation (3.7). In particular, if h is indecomposable, then there is an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[\mathrm{Hess}(N,h) \cap \Omega_e^\circ] \cong \frac{\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 2 \le s \le n, n-h(n+1-s) < r < s]}{({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})}, \tag{4.6}$$

which sends x_{ij} to ${}^hE_{i-j}^{(n-j)}$ under the presentation (3.8).

We will prove Theorem 4.13 in Section 7. For this purpose, one first see that the homomorphism in equation (4.5) is well defined and surjective in the next section.

Remark 4.14. We will introduce certain degrees for the variables $\{x_{ij} \mid 1 \le j < i \le n\}$ and $\{x_1,\dots,x_n,q_{rs} \mid 1 \le r < s \le n\}$ so that the two sides of equation (4.5) are graded $\mathbb{C}$-algebras (see Section 6 for the detail). We will prove that equation (4.5) is in fact an isomorphism as graded $\mathbb{C}$-algebras in Section 7.

Remark 4.15. The isomorphism (4.6) in the case of $h = (2,3,4,\dots,n,n)$ is exactly equation (4.2).

5. Properties of $E_r^{(s)}$

In this section, we see relations between x_s's, q_{rs}'s, and $E_r^{(s)}$'s. Then we construct an explicit map from $\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e})$ to our quotient ring $\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 2 \le s \le n, n-h(n+1-s) < r < s]/({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})$.

Lemma 5.1. *For $1 \le r < s \le n$, we have*

$$q_{rs} = \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & E_1^{(s)} - E_1^{(s-1)} \\ E_1^{(s-1)} & 1 & 0 & & \vdots & E_2^{(s)} - E_2^{(s-1)} \\ E_2^{(s-1)} & E_1^{(s-2)} & 1 & \ddots & \vdots & E_3^{(s)} - E_3^{(s-1)} \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & E_1^{(r+1)} & 1 & E_{s-r}^{(s)} - E_{s-r}^{(s-1)} \\ E_{s-r}^{(s-1)} & \cdots & \cdots & E_2^{(r+1)} & E_1^{(r)} & E_{s-r+1}^{(s)} - E_{s-r+1}^{(s-1)} \end{vmatrix}$$

in the polynomial ring $\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 1 \le r < s \le n]$.

Proof. For $1 \le s \le n$, it follows from Lemma 4.8 that

$$\begin{cases} E_1^{(s)} &= E_1^{(s-1)} + x_s \\ E_2^{(s)} &= E_2^{(s-1)} + E_1^{(s-1)} x_s + q_{s-1\,s} \\ E_3^{(s)} &= E_3^{(s-1)} + E_2^{(s-1)} x_s + E_1^{(s-2)} q_{s-1\,s} + q_{s-2\,s} \\ & \vdots \\ E_{s-r+1}^{(s)} &= E_{s-r+1}^{(s-1)} + E_{s-r}^{(s-1)} x_s + E_{s-r-1}^{(s-2)} q_{s-1\,s} + E_{s-r-2}^{(s-3)} q_{s-2\,s} + \cdots + E_1^{(r)} q_{r+1\,s} + q_{rs}. \end{cases}$$

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In other words, we have

$$\begin{pmatrix} 1 & & & & & \\ E_1^{(s-1)} & 1 & & & & \\ E_2^{(s-1)} & E_1^{(s-2)} & 1 & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ \vdots & & \ddots & E_1^{(r+1)} & 1 & \\ E_{s-r}^{(s-1)} & \cdots & \cdots & E_2^{(r+1)} & E_1^{(r)} & 1 \end{pmatrix} \begin{pmatrix} x_s \\ q_{s-1\,s} \\ q_{s-2\,s} \\ \vdots \\ q_{r+1\,s} \\ q_{rs} \end{pmatrix} = \begin{pmatrix} E_1^{(s)} - E_1^{(s-1)} \\ E_2^{(s)} - E_2^{(s-1)} \\ E_3^{(s)} - E_3^{(s-1)} \\ \vdots \\ E_{s-r}^{(s)} - E_{s-r}^{(s-1)} \\ E_{s-r+1}^{(s)} - E_{s-r+1}^{(s-1)} \end{pmatrix}.$$

By Cramer's rule, we obtain the desired result. □

Set

$$Q_n := \mathbb{C}[x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n]/(E_1^{(n)}, \dots, E_n^{(n)}), \tag{5.1}$$

and define a map φ from $\mathbb{C}[x_{ij} \mid 1 \le j < i \le n]$ to the ring Q_n by

$$\varphi : \mathbb{C}[x_{ij} \mid 1 \le j < i \le n] \to Q_n;\ x_{ij} \mapsto E_{i-j}^{(n-j)}. \tag{5.2}$$

Proposition 5.2. *The map φ above sends*

$$\varphi(x_{n-s+1\,n-s} - x_{n-s+2\,n-s+1}) = x_s \quad \textit{for } s \in [n]$$
$$\varphi(-F_{n+1-r,n+1-s}) = q_{rs} \quad \textit{for } 1 \le r < s \le n.$$

Here, we take the convention that $x_{n-s+2\,n-s+1} = 0$ whenever $s = 1$ and $x_{n-s+1\,n-s} = 0$ whenever $s = n$. In particular, φ is surjective.

Proof. First, one can see from Definition 4.6 that

$$E_1^{(s)} = x_1 + \cdots + x_s$$

for any $s \in [n]$, which implies that $x_s = E_1^{(s)} - E_1^{(s-1)}$. By the definition of φ we have $\varphi(x_{n-s+1\,n-s} - x_{n-s+2\,n-s+1}) = E_1^{(s)} - E_1^{(s-1)} = x_s$ for $s \in [n]$. (Note that if $s = n$, then we used the relation $E_1^{(n)} = 0$ in the quotient ring $Q_n = \mathbb{C}[x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n]/(E_1^{(n)}, \dots, E_n^{(n)})$.)

Next, it follows from equation (3.3) that

$$\varphi(F_{n+1-r,1}) = \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & E_1^{(n-1)} \\ E_1^{(n-1)} & 1 & 0 & & \vdots & E_2^{(n-1)} \\ E_2^{(n-1)} & E_1^{(n-2)} & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & E_1^{(r+1)} & 1 & E_{n-r}^{(n-1)} \\ E_{n-r}^{(n-1)} & \cdots & \cdots & E_2^{(r+1)} & E_1^{(r)} & E_{n-r+1}^{(n-1)} \end{vmatrix} \tag{5.3}$$

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for $1 \leq r \leq n-1$. On the other hand, by Lemma 5.1 we have

$$q_{rn} = \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & -E_1^{(n-1)} \\ E_1^{(n-1)} & 1 & 0 & & \vdots & -E_2^{(n-1)} \\ E_2^{(n-1)} & E_1^{(n-2)} & 1 & \ddots & \vdots & -E_3^{(n-1)} \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & E_1^{(r+1)} & 1 & -E_{n-r}^{(n-1)} \\ E_{n-r}^{(n-1)} & \cdots & \cdots & E_2^{(r+1)} & E_1^{(r)} & -E_{n-r+1}^{(n-1)} \end{vmatrix} \tag{5.4}$$

since $E_1^{(n)} = 0, \ldots, E_{n-r+1}^{(n)} = 0$ in the quotient ring $Q_n = \mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 1 \leq r < s \leq n]/(E_1^{(n)}, \ldots, E_n^{(n)})$. By equations (5.3) and (5.4), we have $\varphi(F_{n-r+1,1}) = -q_{rn}$ for $1 \leq r \leq n-1$ as desired.

If $s < n$, then by equation (3.4) we have

$$\varphi(F_{n+1-r,n+1-s}) = \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ E_1^{(s)} & 1 & 0 & & \vdots & E_1^{(s-1)} \\ E_2^{(s)} & E_1^{(s-1)} & 1 & \ddots & \vdots & E_2^{(s-1)} \\ \vdots & E_2^{(s-1)} & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & E_1^{(r+1)} & 1 & \vdots \\ E_{s-r+1}^{(s)} & E_{s-r}^{(s-1)} & \cdots & E_2^{(r+1)} & E_1^{(r)} & E_{s-r+1}^{(s-1)} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \cdots & \cdots & 0 & 1 \\ E_1^{(s)} - E_1^{(s-1)} & 1 & 0 & & \vdots & E_1^{(s-1)} \\ E_2^{(s)} - E_2^{(s-1)} & E_1^{(s-1)} & 1 & \ddots & \vdots & E_2^{(s-1)} \\ \vdots & E_2^{(s-1)} & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & E_1^{(r+1)} & 1 & \vdots \\ E_{s-r+1}^{(s)} - E_{s-r+1}^{(s-1)} & E_{s-r}^{(s-1)} & \cdots & E_2^{(r+1)} & E_1^{(r)} & E_{s-r+1}^{(s-1)} \end{vmatrix}$$

(by subtracting the last column from the first column)

$$= -\begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ E_1^{(s-1)} & 1 & 0 & & \vdots & E_1^{(s)} - E_1^{(s-1)} \\ E_2^{(s-1)} & E_1^{(s-1)} & 1 & \ddots & \vdots & E_2^{(s)} - E_2^{(s-1)} \\ \vdots & E_2^{(s-1)} & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & E_1^{(r+1)} & 1 & \vdots \\ E_{s-r+1}^{(s-1)} & E_{s-r}^{(s-1)} & \cdots & E_2^{(r+1)} & E_1^{(r)} & E_{s-r+1}^{(s)} - E_{s-r+1}^{(s-1)} \end{vmatrix}$$

(by changing the first column and the last column)

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$$= -\begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & E_1^{(s)} - E_1^{(s-1)} \\ E_1^{(s-1)} & 1 & 0 & & \vdots & E_2^{(s)} - E_2^{(s-1)} \\ E_2^{(s-1)} & E_1^{(s-2)} & 1 & \ddots & \vdots & E_3^{(s)} - E_3^{(s-1)} \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & E_1^{(r+1)} & 1 & E_{s-r}^{(s)} - E_{s-r}^{(s-1)} \\ E_{s-r}^{(s-1)} & \cdots & \cdots & E_2^{(r+1)} & E_1^{(r)} & E_{s-r+1}^{(s)} - E_{s-r+1}^{(s-1)} \end{vmatrix}$$

for $1 \le r < s < n$, which is $-q_{rs}$ from Lemma 5.1. This completes the proof. □

It follows from Proposition 5.2 that the image of $\{-F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \le n\}$ under the map φ in equation (5.2) is $\{q_{rs} \mid 2 \le s \le n \text{ and } 1 \le r \le n - h(n+1-s)\}$. Thus, the surjective map φ induces the surjective homomorphism

$$\begin{aligned} \varphi_h : \mathbb{C}[x_{ij} \mid 1 \le j < i \le n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \le n) \\ \twoheadrightarrow Q_n/(q_{rs} \mid 2 \le s \le n \text{ and } 1 \le r \le n-h(n+1-s));\ x_{ij} \mapsto {}^hE_{i-j}^{(n-j)}. \end{aligned} \tag{5.5}$$

6. Hilbert series

In order to prove that φ_h in equation (5.5) is an isomorphism, we introduce certain degrees for the variables $\{x_{ij} \mid 1 \le j < i \le n\}$ and $\{x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n\}$ so that the two sides of equation (5.5) are graded $\mathbb{C}$-algebras. We then show that the two sides have identical Hilbert series.

Let $\mathbb{C}[x_{ij} \mid 1 \le j < i \le n]$ and $\mathbb{C}[x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n]$ be the polynomial rings equipped with a grading defined by

$$\deg x_{ij} = 2(i-j) \quad \text{for } 1 \le j < i \le n; \tag{6.1}$$

$$\deg x_s = 2 \quad \text{for } s \in [n]; \tag{6.2}$$

$$\deg q_{rs} = 2(s-r+1) \quad \text{for } 1 \le r < s \le n. \tag{6.3}$$

Remark 6.1. As mentioned in Remark 4.3, x_s's are degree 2 elements in the cohomology ring of the flag variety by forgetting quantum parameters. Motivated by this fact, our definition for degrees are concentrated in even degrees.

Lemma 6.2. *For $1 \le r \le s \le n$, the polynomial $E_r^{(s)}$ is homogeneous of degree $2r$ in the polynomial ring $\mathbb{C}[x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n]$.*

Proof. We prove this by induction on s. The base case is $s = 1$, which is clear since $E_1^{(1)} = x_1$. Now, suppose that $s > 1$ and assume by induction that the claim is true for arbitrary s' with $s' \le s-1$. From Lemma 4.8, we have

$$E_r^{(s)} = E_r^{(s-1)} + E_{r-1}^{(s-1)} x_s + \sum_{k=1}^{r-1} E_{r-1-k}^{(s-1-k)} q_{s-k\,s}.$$

By the inductive hypothesis with equations (6.2) and (6.3), one can see that $E_r^{(s)}$ is homogeneous of degree $2r$. □

In order to see that the polynomial $F_{i,j}$ is homogeneous, we introduce the following polynomials.

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Definition 6.3. Let $1 \le j \le n-1$ and $j \le m_j < n$. For $i > m_j$, we define polynomials $\tilde{F}_{i,j}^{\langle m_j \rangle}$ by

$$\tilde{F}_{i,1}^{\langle m_1 \rangle} := \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & x_{21} \\ x_{21} & 1 & \ddots & & \vdots & x_{31} \\ x_{31} & x_{32} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ x_{m_1\,1} & x_{m_1\,2} & \cdots & x_{m_1\,m_1-1} & 1 & x_{m_1+1\,1} \\ x_{i1} & x_{i2} & \cdots & x_{i\,m_1-1} & x_{i\,m_1} & x_{i+1\,1} \end{vmatrix};$$

$$\tilde{F}_{i,j}^{\langle m_j \rangle} := \begin{vmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ x_{j\,j-1} & 1 & \ddots & & \vdots & x_{j+1\,j} \\ x_{j+1\,j-1} & x_{j+1\,j} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ x_{m_j\,j-1} & x_{m_j\,j} & \cdots & x_{m_j\,m_j-1} & 1 & x_{m_j+1\,j} \\ x_{i\,j-1} & x_{i\,j} & \cdots & x_{i\,m_j-1} & x_{i\,m_j} & x_{i+1\,j} \end{vmatrix} \quad \text{for } j \ge 2.$$

Here, we take the convention that $x_{n+1\,j} = 0$ for $j \in [n-1]$.

By equations (3.3) and (3.4) one has

$$F_{i,j} = \tilde{F}_{i,j}^{\langle i-1 \rangle} \tag{6.4}$$

for $1 \le j < i \le n$.

Lemma 6.4. *For $1 \le j < i \le n$, the polynomial $F_{i,j}$ is homogeneous of degree $2(i-j+1)$ in the polynomial ring $\mathbb{C}[x_{ij} \mid 1 \le j < i \le n]$.*

Proof. It suffices to show that $\tilde{F}_{i,j}^{\langle m_j \rangle}$ is homogeneous of degree $2(i-j+1)$ for $j \le m_j < i$ by equation (6.4). At first, we fix j. We prove this statement by induction on m_j with this fixed j. The base case is $m_j = j$. For arbitrary $i > j$, since we have

$$\tilde{F}_{i,1}^{\langle 1 \rangle} = \begin{vmatrix} 1 & x_{21} \\ x_{i1} & x_{i+1\,1} \end{vmatrix} = x_{i+1\,1} - x_{i1}x_{21} \text{ for } j = 1;$$

$$\tilde{F}_{i,j}^{\langle j \rangle} = \begin{vmatrix} 1 & 0 & 1 \\ x_{j\,j-1} & 1 & x_{j+1\,j} \\ x_{i\,j-1} & x_{ij} & x_{i+1\,j} \end{vmatrix} = x_{i+1\,j} + x_{j\,j-1}x_{ij} - x_{i\,j-1} - x_{j+1\,j}x_{ij} \text{ for } j > 1,$$

one can easily see from equation (6.1) that $\tilde{F}_{i,j}^{\langle j \rangle}$ is homogeneous of degree $2(i-j+1)$. This shows the base case.

We proceed to the inductive step. Suppose now that $m_j > j$ and that the claim holds for $m_j - 1$ with any allowable choices of the first subscript i' in $\tilde{F}_{i',j}^{\langle m_j - 1 \rangle}$. By the cofactor expansion along the second-to-last column, we have

$$\tilde{F}_{i,j}^{\langle m_j \rangle} = \tilde{F}_{i,j}^{\langle m_j - 1 \rangle} - x_{i\,m_j}\tilde{F}_{m_j,j}^{\langle m_j - 1 \rangle}.$$

By the inductive hypothesis and equation (6.1), the right-hand side above is homogeneous of degree $2(i-j+1)$, as desired. This completes the proof. □

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Recall from the end of Section 5 that we constructed the map φ_h in equation (5.5) from $\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n)$ to $Q_n/(q_{rs} \mid 2 \leq s \leq n \text{ and } 1 \leq r \leq n-h(n+1-s))$. One can see from Lemmas 6.2 and 6.4 that these are graded $\mathbb{C}$-algebras. (Note that Q_n is also a graded $\mathbb{C}$-algebra.) For the rest of this section, we prove that these graded $\mathbb{C}$-algebras have the same Hilbert series.

Definition 6.5. Let $R = \oplus_{i=0}^{\infty} R_i$ be a graded $\mathbb{C}$-algebra where each homogeneous component R_i of degree i is a finite-dimensional vector space over $\mathbb{C}$. Then its *Hilbert series* is defined to be

$$\mathrm{Hilb}(R,t) \coloneqq \sum_{i=0}^{\infty} \dim_{\mathbb{C}} R_i\, t^i.$$

A sequence of homogeneous polynomials $\theta_1, \ldots, \theta_p \in R$ of positive degrees is a *regular sequence* in R if θ_k is a nonzero divisor of $R/(\theta_1, \ldots, \theta_{k-1})$ for all $1 \leq k \leq p$.

The following facts are well known in commutative algebra. See [35, Chapter I, Section 5]. (See also [18, Proposition 5.1].)

Lemma 6.6. *Let $R = \oplus_{i=0}^{\infty} R_i$ be a graded $\mathbb{C}$-algebra with $\dim_{\mathbb{C}} R_i < \infty$ for each i. A sequence of homogeneous polynomials $\theta_1, \ldots, \theta_p \in R$ of positive degrees is a regular sequence in R if and only if the Hilbert series of $R/(\theta_1, \ldots, \theta_p)$ is given by*

$$\mathrm{Hilb}(R/(\theta_1, \ldots, \theta_p), t) = \mathrm{Hilb}(R,t) \cdot \prod_{k=1}^{p} (1 - t^{\deg \theta_k}).$$

Lemma 6.7. *Let R be a polynomial ring $\mathbb{C}[y_1, \ldots, y_n]$. A sequence of homogeneous polynomials $\theta_1, \ldots, \theta_n \in R$ of positive degrees is a regular sequence in R if and only if the solution set of the equations $\theta_1 = 0, \ldots, \theta_n = 0$ in $\mathbb{C}^n$ consists only of the origin $\{0\}$.*

We remark that the number of the homogeneous polynomials $\theta_1, \ldots, \theta_n$ is equal to the number of the variables $y_1, \ldots, y_n$ in the polynomial ring $\mathbb{C}[y_1, \ldots, y_n]$ in Lemma 6.7. By using two lemmas above, we compute the Hilbert series of $\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n)$ and $Q_n/(q_{rs} \mid 2 \leq s \leq n \text{ and } 1 \leq r \leq n - h(n+1-s))$.

Lemma 6.8. *The polynomials $F_{i,j}$ $(1 \leq j < i \leq n)$ form a regular sequence in the polynomial ring $\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]$.*

Proof. By Lemma 6.7, it is enough to show that the solution set of the equations $F_{i,j} = 0$ $(1 \leq j < i \leq n)$ in $\mathbb{C}^{\frac{1}{2}n(n-1)}$ (with the variables x_{ij} $(1 \leq j < i \leq n)$) consists only of the origin $\{0\}$. The intersection of the zero set of $F_{i,j}$ for all $1 \leq j < i \leq n$ is $\mathrm{Hess}(N, id) \cap \Omega_e^{\circ}$ by Lemma 2.3. However, since $\mathrm{Hess}(N, id)$ consists only of the point $\{eB\}$, this means that the equations $F_{i,j} = 0$ $(1 \leq j < i \leq n)$ implies that $x_{ij} = 0$ for all $1 \leq j < i \leq n$, as desired. □

Proposition 6.9. *The Hilbert series of $\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n)$ equipped with a grading in equation (6.1) is equal to*

$$\mathrm{Hilb}(\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n), t)$$
$$= \prod_{\substack{1 \leq j \leq n-1 \\ j < i \leq h(j)}} \frac{1}{1 - t^{2(i-j+1)}} \cdot \prod_{k=1}^{n-1} (1 + t^2 + t^4 + \cdots + t^{2k}).$$

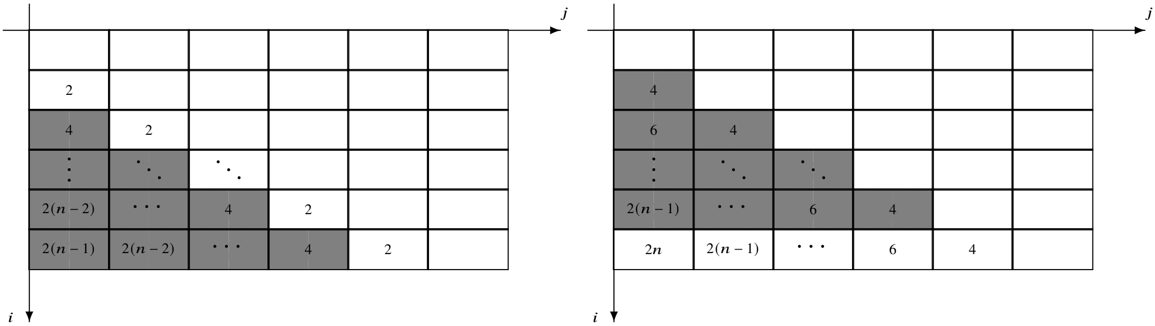

Figure 4. *The values* $2(i-j)$ *and* $2(i-j+1)$ *for* $1 \leq j < i \leq n$

Proof. Since a subsequence of a regular sequence is also a regular sequence from the definition of a regular sequence, by Lemma 6.8, the polynomials $F_{i,j}$ ($j \in [n-1], h(j) < i \leq n$) form a regular sequence in the polynomial ring $\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]$. Thus, it follows from Lemma 6.6 that

$$\begin{aligned}
&\mathrm{Hilb}(\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n), t) \qquad (6.5)\\
&= \mathrm{Hilb}(\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n], t) \cdot \prod_{\substack{1 \leq j \leq n-1 \\ h(j) < i \leq n}} (1 - t^{\deg F_{i,j}})\\
&= \prod_{1 \leq j < i \leq n} \frac{1}{1 - t^{2(i-j)}} \cdot \prod_{\substack{1 \leq j \leq n-1 \\ h(j) < i \leq n}} (1 - t^{2(i-j+1)}) \qquad \text{(by Lemma 6.4).}
\end{aligned}$$

Here, we note that

$$\prod_{1 \leq j < i \leq n} \frac{1}{1 - t^{2(i-j)}} \cdot (1 - t^2)^{n-1} = \prod_{1 \leq j < i \leq n} \frac{1}{1 - t^{2(i-j+1)}} \cdot (1 - t^4)(1 - t^6) \cdots (1 - t^{2n}).$$

In fact, exponents appeared on the left-hand side and exponents on the right-hand side are described as numbers in shaded boxes of the left figure and the right figure in Figure 4, respectively.

This equality leads us to the equality

$$\prod_{1 \leq j < i \leq n} \frac{1}{1 - t^{2(i-j)}} = \prod_{1 \leq j < i \leq n} \frac{1}{1 - t^{2(i-j+1)}} \cdot \prod_{k=1}^{n-1} (1 + t^2 + t^4 + \cdots + t^{2k}),$$

so by equation (6.5), one has

$$\begin{aligned}
&\mathrm{Hilb}(\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(F_{i,j} \mid j \in [n-1] \text{ and } h(j) < i \leq n), t)\\
&= \prod_{1 \leq j < i \leq n} \frac{1}{1 - t^{2(i-j+1)}} \cdot \prod_{k=1}^{n-1} (1 + t^2 + t^4 + \cdots + t^{2k}) \cdot \prod_{\substack{1 \leq j \leq n-1 \\ h(j) < i \leq n}} (1 - t^{2(i-j+1)})\\
&= \prod_{\substack{1 \leq j \leq n-1 \\ j < i \leq h(j)}} \frac{1}{1 - t^{2(i-j+1)}} \cdot \prod_{k=1}^{n-1} (1 + t^2 + t^4 + \cdots + t^{2k}),
\end{aligned}$$

as desired. □

Lemma 6.10. *The polynomials* $E_1^{(n)}, \ldots, E_n^{(n)}, q_{rs}$ $(1 \leq r < s \leq n)$ *form a regular sequence in the polynomial ring* $\mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 1 \leq r < s \leq n]$.

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Proof. From Lemma 6.7, it suffices to show that the solution set of the equations $E_1^{(n)} = 0, \dots, E_n^{(n)} = 0, q_{rs} = 0$ $(1 \le r < s \le n)$ in $\mathbb{C}^{\frac{1}{2}n(n+1)}$ (with the variables $x_1, \dots, x_n, q_{rs}$ $(1 \le r < s \le n)$) consists only of the origin $\{0\}$. Since $q_{rs} = 0$ for all $1 \le r < s \le n$, $E_1^{(n)} = 0, \dots, E_n^{(n)} = 0$ implies that $e_1(x_1, \dots, x_n) = 0, \dots, e_n(x_1, \dots, x_n) = 0$, where $e_i(x_1, \dots, x_n)$ is the (ordinary) i-th elementary symmetric polynomial in the variables $x_1, \dots, x_n$. Then one can easily see that $x_i = 0$ for all $i \in [n]$. In fact, $x_1 x_2 \cdots x_n = e_n(x_1, \dots, x_n) = 0$ implies that some x_i must be equal to 0. Without loss of generality, we may assume that $x_n = 0$. This implies that $e_i(x_1, \dots, x_{n-1}) = 0$ for all $i \in [n-1]$. Proceeding in this manner, we conclude that $x_i = 0$ for all $i \in [n]$. Thus, the equations $E_1^{(n)} = 0, \dots, E_n^{(n)} = 0, q_{rs} = 0$ $(1 \le r < s \le n)$ implies that $x_i = 0$ for all $i \in [n]$ and $q_{rs} = 0$ for all $1 \le r < s \le n$, as desired. □

Proposition 6.11. *The Hilbert series of $Q_n/(q_{rs} \mid 2 \le s \le n$ and $1 \le r \le n - h(n+1-s))$ equipped with a grading in equations (6.2) and (6.3) is equal to*

$$\begin{aligned}
&\mathrm{Hilb}(Q_n/(q_{rs} \mid 2 \le s \le n \textit{ and } 1 \le r \le n - h(n+1-s)), t) \\
&= \prod_{\substack{1 \le j \le n-1 \\ j < i \le h(j)}} \frac{1}{1 - t^{2(i-j+1)}} \cdot \prod_{k=1}^{n-1} (1 + t^2 + t^4 + \cdots + t^{2k}).
\end{aligned}$$

Proof. Recall from the definition (5.1) that

$$Q_n = \mathbb{C}[x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n]/(E_1^{(n)}, \dots, E_n^{(n)}).$$

Lemma 6.10 implies that the monomials q_{rs} $(2 \le s \le n, 1 \le r \le n - h(n+1-s))$ form a regular sequence in Q_n by the definition of a regular sequence. Hence, by Lemma 6.6, we have

$$\begin{aligned}
&\mathrm{Hilb}(Q_n/(q_{rs} \mid 2 \le s \le n \text{ and } 1 \le r \le n - h(n+1-s)), t) \qquad (6.6)\\
&= \mathrm{Hilb}(Q_n, t) \cdot \prod_{\substack{2 \le s \le n \\ 1 \le r \le n-h(n+1-s)}} (1 - t^{\deg q_{rs}}) \\
&= \mathrm{Hilb}(Q_n, t) \cdot \prod_{\substack{2 \le s \le n \\ 1 \le r \le n-h(n+1-s)}} (1 - t^{2(s-r+1)}).
\end{aligned}$$

Since a subsequence of a regular sequence is again a regular sequence from the definition of a regular sequence, the polynomials $E_1^{(n)}, \dots, E_n^{(n)}$ form a regular sequence in the polynomial ring $\mathbb{C}[x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n]$ by Lemma 6.10. By using Lemma 6.6 again, one has

$$\begin{aligned}
\mathrm{Hilb}(Q_n, t) &= \mathrm{Hilb}(\mathbb{C}[x_1, \dots, x_n, q_{rs} \mid 1 \le r < s \le n], t) \cdot \prod_{k=1}^{n} (1 - t^{\deg E_k^{(n)}}) \qquad (6.7)\\
&= \frac{1}{(1-t^2)^n} \cdot \prod_{1 \le r < s \le n} \frac{1}{1 - t^{2(s-r+1)}} \cdot \prod_{k=1}^{n} (1 - t^{2k}) \qquad \text{(by Lemma 6.2)} \\
&= \prod_{1 \le r < s \le n} \frac{1}{1 - t^{2(s-r+1)}} \cdot \prod_{k=1}^{n-1} (1 + t^2 + t^4 + \cdots + t^{2k}).
\end{aligned}$$

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By equations (6.6) and (6.7), we obtain

$$\begin{aligned}
&\mathrm{Hilb}(Q_n/(q_{rs}\mid 2\le s\le n \text{ and } 1\le r\le n-h(n+1-s)),t)\\
&=\prod_{1\le r<s\le n}\frac{1}{1-t^{2(s-r+1)}}\cdot\prod_{k=1}^{n-1}(1+t^2+t^4+\cdots+t^{2k})\cdot\prod_{\substack{2\le s\le n\\ 1\le r\le n-h(n+1-s)}}(1-t^{2(s-r+1)})\\
&=\prod_{1\le j<i\le n}\frac{1}{1-t^{2(i-j+1)}}\cdot\prod_{k=1}^{n-1}(1+t^2+t^4+\cdots+t^{2k})\cdot\prod_{\substack{1\le j\le n-1\\ h(j)+1\le i\le n}}(1-t^{2(i-j+1)})\\
&\qquad\qquad\text{(by setting } i=n+1-r \text{ and } j=n+1-s \text{ in the third product)}\\
&=\prod_{\substack{1\le j\le n-1\\ j<i\le h(j)}}\frac{1}{1-t^{2(i-j+1)}}\cdot\prod_{k=1}^{n-1}(1+t^2+t^4+\cdots+t^{2k}),
\end{aligned}$$

as desired. □

7. Proof of Theorem 4.13

We now prove Theorem 4.13.

Proof of Theorem 4.13. We first note that there exists a canonical isomorphism

$$\begin{aligned}
&Q_n/(q_{rs}\mid 2\le s\le n \text{ and } 1\le r\le n-h(n+1-s))\\
\cong&\mathbb{C}[x_1,\dots,x_n,q_{rs}\mid 2\le s\le n, n-h(n+1-s)<r<s]/({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})
\end{aligned}$$

by Definition 4.10 and equation (5.1). Under the identification above and the presentation in equation (3.7), we can rewrite φ_h in equation (5.5) as

$$\varphi_h:\Gamma(\mathcal{Z}(N,h)_e,\mathcal{O}_{\mathcal{Z}(N,h)_e})\twoheadrightarrow\frac{\mathbb{C}[x_1,\dots,x_n,q_{rs}\mid 2\le s\le n, n-h(n+1-s)<r<s]}{({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})},$$

which is defined by

$$\varphi_h(x_{ij})={}^hE_{i-j}^{(n-j)}.$$

Here, by slightly abuse of notation we used the same symbol φ_h for the map above. The map φ_h is surjective and this preserves the gradings on both graded $\mathbb{C}$-algebras from equation (6.1) and Lemma 6.2. It follows from Propositions 6.9 and 6.11 that the two sides of φ_h have identical Hilbert series. Therefore, we conclude that φ_h is an isomorphism. □

Remark 7.1. Our proof gives an alternative proof of the isomorphism (4.2).

Corollary 7.2. *There is an isomorphism of $\mathbb{C}$-algebras*

$$\Gamma(\mathcal{Z}(N,id)_e,\mathcal{O}_{\mathcal{Z}(N,id)_e})\cong H^*(Fl(\mathbb{C}^n)).$$

Proof. Applying the isomorphism (4.5) to the case when $h=id$, we obtain

$$\Gamma(\mathcal{Z}(N,id)_e,\mathcal{O}_{\mathcal{Z}(N,id)_e})\cong\mathbb{C}[x_1,\dots,x_n]/(e_1(x_1,\dots,x_n),\dots,e_n(x_1,\dots,x_n)),$$

where $e_i(x_1,\dots,x_n)$ denotes the i-th elementary symmetric polynomial in the variables $x_1,\dots,x_n$. As is well known, the right hand side above is a presentation for the cohomology ring $H^*(Fl(\mathbb{C}^n))$ (e.g., [19, Section 10.2, Proposition 3]). □

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We constructed the isomorphism of graded $\mathbb{C}$-algebras

$$\varphi_h : \Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e}) \xrightarrow{\cong} \frac{\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 2 \le s \le n, n-h(n+1-s) < r < s]}{({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})}, \tag{7.1}$$

which sends x_{ij} to ${}^hE_{i-j}^{(n-j)}$ for all $1 \le j < i \le n$ from the homomorphism φ in equation (5.2). The following result follows from Proposition 5.2.

Proposition 7.3. *The inverse map of φ_h in equation (7.1) is given by*

$$\varphi_h^{-1}(x_s) = x_{n-s+1\,n-s} - x_{n-s+2\,n-s+1} \quad \text{for } s \in [n]$$
$$\varphi_h^{-1}(q_{rs}) = -F_{n+1-r,n+1-s} \quad \text{for } 2 \le s \le n \text{ and } n-h(n+1-s) < r < s$$

with the convention that $x_{n-s+2\,n-s+1} = 0$ whenever $s = 1$ and $x_{n-s+1\,n-s} = 0$ whenever $s = n$.

We conclude the following result from the discussion above.

Corollary 7.4. *Let $h : [n] \to [n]$ be a Hessenberg function. Then the following commutative diagram holds*

$$\begin{array}{ccc}
\mathbb{C}[x_{ij} \mid 1 \le j < i \le n] & \xrightarrow[\cong]{\varphi} & \dfrac{\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 1 \le r < s \le n]}{(E_1^{(n)},\dots,E_n^{(n)})} \\
\Big\downarrow {\scriptstyle F_{i,j}=0\ (1\le j\le n-1 \text{ and } h(j)<i\le n)} & & \Big\downarrow {\scriptstyle q_{rs}=0\ (2\le s\le n \text{ and } 1\le r\le n-h(n+1-s))} \\
\Gamma(\mathcal{Z}(N,h)_e, \mathcal{O}_{\mathcal{Z}(N,h)_e}) & \xrightarrow[\cong]{\varphi_h} & \dfrac{\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 2 \le s \le n, n-h(n+1-s) < r < s]}{({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})},
\end{array}$$

where both vertical arrows denote the canonical surjective maps under the presentation (3.7). In particular, if h is indecomposable, then we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathbb{C}[Fl(\mathbb{C}^n) \cap \Omega_e^\circ] & \xrightarrow[\cong]{\varphi} & \dfrac{\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 1 \le r < s \le n]}{(E_1^{(n)},\dots,E_n^{(n)})} \\
\Big\downarrow {\scriptstyle F_{i,j}=0\ (1\le j\le n-2 \text{ and } h(j)<i\le n)} & & \Big\downarrow {\scriptstyle q_{rs}=0\ (3\le s\le n \text{ and } 1\le r\le n-h(n+1-s))} \\
\mathbb{C}[\mathrm{Hess}(N,h) \cap \Omega_e^\circ] & \xrightarrow[\cong]{\varphi_h} & \dfrac{\mathbb{C}[x_1,\dots,x_n,q_{rs} \mid 2 \le s \le n, n-h(n+1-s) < r < s]}{({}^hE_1^{(n)},\dots,{}^hE_n^{(n)})},
\end{array}$$

where the left vertical arrow is induced from the inclusion $\mathrm{Hess}(N,h) \cap \Omega_e^\circ \hookrightarrow Fl(\mathbb{C}^n) \cap \Omega_e^\circ$. Note that both vertical arrows are surjective.

8. Jacobian matrix

It is an interesting and challenging problem to find an explicit description of the singular locus of (regular nilpotent) Hessenberg varieties $\mathrm{Hess}(N,h)$. There are already partial results for the problem stated above in [3, 15, 26]. For the rest of the paper, we will analyze the singular locus of $\mathrm{Hess}(N,h) \cap \Omega_e^\circ$ for some Hessenberg function h as an application of our result. The isomorphism (4.6) in Theorem 4.13 yields that if h is indecomposable, then the singular locus of the open set $\mathrm{Hess}(N,h) \cap \Omega_e^\circ$ in $\mathrm{Hess}(N,h)$ is isomorphic to the singular locus of the zero set defined by n polynomials ${}^hE_1^{(n)},\dots,{}^hE_n^{(n)}$ in $\mathbb{C}^{n+\sum_{j=1}^n (h(j)-j))}$ with the variables $x_1,\dots,x_n,q_{rs}$ $(2 \le s \le n, n-h(n+1-s) < r < s)$. In this section, we give an explicit formula for partial derivatives $\partial E_i^{(n)}/\partial x_s$ $(1 \le s \le n)$ and $\partial E_i^{(n)}/\partial q_{rs}$ $(1 \le r < s \le n)$ for each $i \in [n]$.

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For positive integers a and b with $1 \leq a \leq b \leq n$, we set

$$[a, b] := \{a, a+1, \ldots, b\}.$$

As in Definition 4.6, we introduce the following polynomials.

Definition 8.1. Let $M_{[a,b]}$ be the matrix of size $(b-a+1) \times (b-a+1)$ defined as

$$M_{[a,b]} := \begin{pmatrix} x_a & q_{a\,a+1} & q_{a\,a+2} & \cdots & q_{a\,b} \\ -1 & x_{a+1} & q_{a+1\,a+2} & \cdots & q_{a+1\,b} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & x_{b-1} & q_{b-1\,b} \\ 0 & \cdots & 0 & -1 & x_b \end{pmatrix}.$$

We define polynomials $E_1^{[a,b]}, \ldots, E_{b-a+1}^{[a,b]} \in \mathbb{C}[x_a, \ldots, x_b, q_{rs} \mid a \leq r < s \leq b]$ by

$$\det(\lambda I_{b-a+1} - M_{[a,b]}) = \lambda^{b-a+1} - E_1^{[a,b]}\lambda^{b-a} + E_2^{[a,b]}\lambda^{b-a-1} + \cdots + (-1)^{b-a+1}E_{b-a+1}^{[a,b]}.$$

Note that if $a = 1$, then we obtain

$$E_i^{[1,b]} = E_i^{(b)} \quad \text{for each } i \in [b] \tag{8.1}$$

by the definition. In what follows, we use the symbol

$$[a, a-1] := \emptyset \quad \text{for each } a \in [n+1]$$

and we take the following convention

$$E_i^{[a,b]} := \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i > b-a+1 \end{cases} \tag{8.2}$$

for $b \geq a-1$. By the same argument for the proof of Lemma 4.8, one can prove the lemma below. We leave the detail to the reader.

Lemma 8.2. *Let $1 \leq a \leq b \leq n$. For $1 \leq i \leq b-a+1$, we have*

$$E_i^{[a,b]} = E_i^{[a,b-1]} + E_{i-1}^{[a,b-1]}x_b + \sum_{k=1}^{i-1} E_{i-1-k}^{[a,b-1-k]} q_{b-k\,b}$$

with the convention that $\sum_{k=1}^{i-1} E_{i-1-k}^{[a,b-1-k]} q_{b-k\,b} = 0$ whenever $i = 1$.

Lemma 8.3. 1. *Let $s \in [n]$. For $i \in [n]$, we have*

$$\frac{\partial}{\partial x_s} E_i^{(n)} = \sum_{k=0}^{i-1} E_{i-1-k}^{[1,s-1]} E_k^{[s+1,n]}.$$

2. *Let $1 \leq r < s \leq n$. Then*

$$\frac{\partial}{\partial q_{rs}} E_{i+s-r}^{(n)} = \begin{cases} 0 & \text{if } 1-(s-r) \leq i \leq 0, \\ \sum_{k=0}^{i-1} E_{i-1-k}^{[1,r-1]} E_k^{[s+1,n]} & \text{if } 1 \leq i \leq n-(s-r). \end{cases}$$

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Proof. (1) By Definition 4.6, one has

$$\det(\lambda I_n - M_n) = \lambda^n - E_1^{(n)}\lambda^{n-1} + E_2^{(n)}\lambda^{n-2} + \cdots + (-1)^n E_n^{(n)}. \tag{8.3}$$

We think of λ as a variable in the equality above, and we partial differentiate the both sides with respect to x_s. Then $\frac{\partial}{\partial x_s}E_i^{(n)}$ is equal to the coefficient of λ^{n-i} for $\frac{\partial}{\partial x_s}\det(\lambda I_n - M_n)$ multiplied by $(-1)^i$. Since the variable x_s appears in only the (s,s)-th entry of the matrix $(\lambda I_n - M_n)$, from the cofactor expansion along the s-th column for $\det(\lambda I_n - M_n)$, we can write

$$\det(\lambda I_n - M_n) = (\lambda - x_s)\det(\lambda I_{s-1} - M_{[1,s-1]})\det(\lambda I_{n-s} - M_{[s+1,n]}) + F$$

for some polynomial $F \in \mathbb{C}[x_1,\dots,\widehat{x_s},\dots,x_n,q_{ij} \mid 1 \le i < j \le n]$. Here, the caret sign $\widehat{\ }$ over x_s means that the entry is omitted. Hence, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x_s}\det(\lambda I_n - M_n) &= -\det(\lambda I_{s-1} - M_{[1,s-1]})\det(\lambda I_{n-s} - M_{[s+1,n]}) \\
&= -\left(\sum_{u=0}^{s-1}(-1)^u E_u^{[1,s-1]}\lambda^{s-1-u}\right)\left(\sum_{v=0}^{n-s}(-1)^v E_v^{[s+1,n]}\lambda^{n-s-v}\right) \\
&\qquad \text{(by Definition 8.1)} \\
&= -\sum_{u=0}^{s-1}\sum_{v=0}^{n-s}(-1)^{u+v}E_u^{[1,s-1]}E_v^{[s+1,n]}\lambda^{n-1-(u+v)} \\
&= -\sum_{\ell=0}^{n-1}\left(\sum_{k=0}^{\ell}(-1)^\ell E_{\ell-k}^{[1,s-1]}E_k^{[s+1,n]}\right)\lambda^{n-1-\ell}.
\end{aligned}$$

Therefore, we conclude that the coefficient of λ^{n-i} for $\frac{\partial}{\partial x_s}\det(\lambda I_n - M_n)$ multiplied by $(-1)^i$ is equal to

$$\frac{\partial}{\partial x_s}E_i^{(n)} = \sum_{k=0}^{i-1}E_{i-1-k}^{[1,s-1]}E_k^{[s+1,n]},$$

as desired.

(2) We partial differentiate the both sides of equation (8.3) with respect to q_{rs}, then $\frac{\partial}{\partial q_{rs}}E_j^{(n)}$ is equal to the coefficient of λ^{n-j} for $\frac{\partial}{\partial q_{rs}}\det(\lambda I_n - M_n)$ multiplied by $(-1)^j$. Since the variable q_{rs} appears in only the (r,s)-th entry of the matrix $(\lambda I_n - M_n)$, one can see from similar arguments as in the previous case that

$$\begin{aligned}
\frac{\partial}{\partial q_{rs}}\det(\lambda I_n - M_n) &= (-1)^{r+s}\frac{\partial}{\partial q_{rs}}(-q_{rs})\det(\lambda I_{r-1} - M_{[1,r-1]})\det(\lambda I_{n-s} - M_{[s+1,n]}) \\
&= (-1)^{r+s+1}\left(\sum_{u=0}^{r-1}(-1)^u E_u^{[1,r-1]}\lambda^{r-1-u}\right)\left(\sum_{v=0}^{n-s}(-1)^v E_v^{[s+1,n]}\lambda^{n-s-v}\right) \\
&= (-1)^{r+s+1}\sum_{u=0}^{r-1}\sum_{v=0}^{n-s}(-1)^{u+v}E_u^{[1,r-1]}E_v^{[s+1,n]}\lambda^{n-1-(u+v+s-r)} \\
&= (-1)^{r+s+1}\sum_{\ell=s-r}^{n-1}\left(\sum_{k=0}^{\ell-s+r}(-1)^{\ell-s+r}E_{\ell-s+r-k}^{[1,r-1]}E_k^{[s+1,n]}\right)\lambda^{n-1-\ell}.
\end{aligned}$$

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Thus, the coefficient of λ^{n-j} for $\frac{\partial}{\partial q_{rs}} \det(\lambda I_n - M_n)$ multiplied by $(-1)^j$ is

$$\frac{\partial}{\partial q_{rs}} E_j^{(n)} = \begin{cases} 0 & \text{if } 1 \leq j \leq s - r, \\ \sum_{k=0}^{j-1-s+r} E_{j-1-s+r-k}^{[1,r-1]} E_k^{[s+1,n]} & \text{if } s - r + 1 \leq j \leq n, \end{cases}$$

as desired. This completes the proof. □

In what follows, we set

$$q_{ss} := x_s \quad \text{for } s \in [n], \tag{8.4}$$

and we see the Jacobian matrix $\left(\frac{\partial}{\partial q_{rs}} E_i^{(n)}\right)_{i,(r,s)}$ below.

Example 8.4. Let $n = 3$. Then the Jacobian matrix $\left(\frac{\partial}{\partial q_{rs}} E_i^{(3)}\right)_{i,(r,s)}$ is described as

$$\begin{array}{c} \\ E_1^{(3)} \\ E_2^{(3)} \\ E_3^{(3)} \end{array} \begin{array}{c} \begin{array}{cccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial q_{12}} & \frac{\partial}{\partial q_{23}} & \frac{\partial}{\partial q_{13}} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ E_1^{[2,3]} & E_1^{[1,1]} + E_1^{[3,3]} & E_1^{[1,2]} & 1 & 1 & 0 \\ E_2^{[2,3]} & E_1^{[1,1]} E_1^{[3,3]} & E_2^{[1,2]} & E_1^{[3,3]} & E_1^{[1,1]} & 1 \end{pmatrix} \end{array}$$

by Lemma 8.3. Also, the Jacobian matrix $\left(\frac{\partial}{\partial q_{rs}} E_i^{(4)}\right)_{i,(r,s)}$ for $n = 4$ is given by

$$\begin{array}{c} \\ E_1^{(4)} \\ E_2^{(4)} \\ E_3^{(4)} \\ E_4^{(4)} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial q_{12}} & \frac{\partial}{\partial q_{23}} & \frac{\partial}{\partial q_{34}} & \frac{\partial}{\partial q_{13}} & \frac{\partial}{\partial q_{24}} & \frac{\partial}{\partial q_{14}} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_1^{[2,4]} & E_1^{[1,1]} + E_1^{[3,4]} & E_1^{[1,2]} + E_1^{[4,4]} & E_1^{[1,3]} & 1 & 1 & 1 & 0 & 0 & 0 \\ E_2^{[2,4]} & E_1^{[1,1]} E_1^{[3,4]} + E_2^{[3,4]} & E_2^{[1,2]} + E_1^{[1,2]} E_1^{[4,4]} & E_2^{[1,3]} & E_1^{[3,4]} & E_1^{[1,1]} + E_1^{[4,4]} & E_1^{[1,2]} & 1 & 1 & 0 \\ E_3^{[2,4]} & E_1^{[1,1]} E_2^{[3,4]} & E_2^{[1,2]} E_1^{[4,4]} & E_3^{[1,3]} & E_2^{[3,4]} & E_1^{[1,1]} E_1^{[4,4]} & E_2^{[1,2]} & E_1^{[4,4]} & E_1^{[1,1]} & 1 \end{pmatrix} \end{array}.$$

The Jacobian matrices above have a full rank. In general, one can verify from Lemma 8.3 that the rank of the Jacobian matrix $\left(\frac{\partial}{\partial q_{rs}} E_i^{(n)}\right)_{i,(r,s)}$ is full for arbitrary n. This fact also follows from the well-known fact that $Fl(\mathbb{C}^n) \cap \Omega_e^\circ$ is smooth and Theorem 4.13 for the case when $h = (n, \ldots, n)$.

Let $h : [n] \to [n]$ be a Hessenberg function. As in Definition 4.10, for $0 \leq i \leq n$ and a, b with $b \geq a - 1$, we define

$${}^h E_i^{[a,b]} := E_i^{[a,b]}|_{q_{rs}=0 \ (2 \leq s \leq n \text{ and } 1 \leq r \leq n - h(n+1-s))}. \tag{8.5}$$

Note that we take the convention in equation (8.2). By the definition (8.5), it is straightforward to see that for arbitrary Hessenberg function $h : [n] \to [n]$, $i \in [n]$, and (r, s) with $2 \leq s \leq n, n - h(n+1-s) < r \leq s$,

$$\frac{\partial}{\partial q_{rs}} {}^h E_i^{(n)} = \left(\frac{\partial}{\partial q_{rs}} E_i^{(n)}\right)\Bigg|_{q_{uv}=0 \ (2 \leq v \leq n \text{ and } 1 \leq u \leq n - h(n+1-v))}.$$

Combining this and Lemma 8.3, we have that for $1 \leq r \leq s \leq n$,

$$\frac{\partial}{\partial q_{rs}} {}^{h}E_{i+s-r}^{(n)} = \begin{cases} 0 & \text{if } 1-(s-r) \leq i \leq 0, \\ 1 & \text{if } i = 1, \\ \sum_{k=0}^{i-1} {}^{h}E_{i-1-k}^{[1,r-1]} \, {}^{h}E_{k}^{[s+1,n]} & \text{if } 2 \leq i \leq n-(s-r). \end{cases} \tag{8.6}$$

For $2 \leq m \leq n-1$, we define a Hessenberg function $h_m : [n] \to [n]$ by

$$h_m = (m, n, \ldots, n). \tag{8.7}$$

The zero set of $\{{}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\}$ in $\mathbb{C}^{\frac{1}{2}n(n+1)-(n-m)}$ with the variables $x_1 \ldots, x_n, q_{rs}$ $(2 \leq s \leq n, n - h_m(n+1-s) < r < s)$ is denoted by $V\big({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\big)$, that is,

$$V\big({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\big) := \{(a,p) \in \mathbb{C}^{\frac{1}{2}n(n+1)-(n-m)} \mid {}^{h_m}E_i^{(n)}(a,p) = 0 \text{ for } i \in [n]\}, \tag{8.8}$$

where $(a,p) := (a_1, \ldots, a_n, p_{rs})_{2 \leq s \leq n, n-h_m(n+1-s)<r<s}$.

Proposition 8.5. *Let $2 \leq m \leq n-1$ and h_m be the Hessenberg function in equation (8.7). Then, the singular locus of $V\big({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\big)$ in equation (8.8) is given by the solution set of the equations*

$$\frac{\partial}{\partial q_{rs}} {}^{h_m}E_n^{(n)} = 0 \quad \textit{for all } 2 \leq s \leq n \textit{ and } n - h_m(n+1-s) < r \leq s.$$

Here, we recall our convention (8.4) that $q_{rs} = x_s$ whenever $r = s$.

Before we prove Proposition 8.5, we give an example of the singular locus of the zero set $V\big({}^{h_m}E_1^{(3)}, {}^{h_m}E_2^{(3)}, {}^{h_m}E_3^{(3)}\big)$ for $m = 2$ by using Proposition 8.5.

Example 8.6. Consider the case when $h = (2,3,3)$ for $n = 3$. The Jacobian matrix $\big(\frac{\partial}{\partial q_{rs}} {}^{h}E_i^{(3)}\big)_{i,(r,s)\neq(1,3)}$ is obtained from $\big(\frac{\partial}{\partial q_{rs}} E_i^{(3)}\big)_{i,(r,s)}$ by forgetting the quantum parameter q_{13}. As seen in Example 8.4, the Jacobian matrix $\big(\frac{\partial}{\partial q_{rs}} {}^{h}E_i^{(3)}\big)_{i,(r,s)\neq(1,3)}$ is shown as

$$\begin{array}{c} \\ {}^{h}E_1^{(3)} \\ {}^{h}E_2^{(3)} \\ {}^{h}E_3^{(3)} \end{array} \begin{array}{c} \begin{array}{ccccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial q_{12}} & \frac{\partial}{\partial q_{23}} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ {}^{h}E_1^{[2,3]} & {}^{h}E_1^{[1,1]} + {}^{h}E_1^{[3,3]} & {}^{h}E_1^{[1,2]} & 1 & 1 \\ {}^{h}E_2^{[2,3]} & {}^{h}E_1^{[1,1]} \, {}^{h}E_1^{[3,3]} & {}^{h}E_2^{[1,2]} & {}^{h}E_1^{[3,3]} & {}^{h}E_1^{[1,1]} \end{pmatrix} \end{array}.$$

It follows from Proposition 8.5 that the singular locus of $V\big({}^{h}E_1^{(3)}, {}^{h}E_2^{(3)}, {}^{h}E_3^{(3)}\big)$ is given by the solution set of the equations ${}^{h}E_2^{[2,3]} = {}^{h}E_1^{[1,1]} \, {}^{h}E_1^{[3,3]} = {}^{h}E_2^{[1,2]} = {}^{h}E_1^{[3,3]} = {}^{h}E_1^{[1,1]} = 0$. The resulting solution is as follows:

$$x_1 = 0, \ x_3 = 0, \ q_{12} = 0, \ q_{23} = 0.$$

Then it follows from equation (4.6) and Proposition 7.3 that the singular locus of $\mathrm{Pet}_3 \cap \Omega_e^\circ$ is given by the solution set of the equations $x_{32} = 0, \ x_{21} = 0, \ F_{3,2} = 0, \ F_{2,1} = 0$. This is equivalent to

$$x_{21} = 0, \ x_{32} = 0, \ x_{31} = 0.$$

Hence, the singular locus of $\mathrm{Pet}_3 \cap \Omega_e^\circ$ is $\{eB\}$. Note that the singular locus of the Peterson variety Pet_n is given by [26]. We will explain this in Appendix A.

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Proof of Proposition 8.5. Recall that a point $(a, p) := (a_1, \dots, a_n, p_{rs})_{2 \le s \le n, n-h_m(n+1-s) < r < s}$ in $\mathbb{C}^{\frac{1}{2}n(n+1)-(n-m)}$ is a singular point of $V({}^{h_m}E_1^{(n)}, \dots, {}^{h_m}E_n^{(n)})$ if and only if the rank of the Jacobian matrix $(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_i^{(n)}(a, p))_{i,(r,s)}$ is not full. It is clear that the Jacobian matrix $(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_i^{(n)}(a, p))_{i,(r,s)}$ is not full if the n-th row vector is zero. Thus, it is enough to prove that if a point (a, p) in $\mathbb{C}^{\frac{1}{2}n(n+1)-(n-m)}$ is a singular point of $V({}^{h_m}E_1^{(n)}, \dots, {}^{h_m}E_n^{(n)})$, then the n-th row vector $(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_n^{(n)}(a, p))_{(r,s)}$ of the Jacobian matrix is zero.

Since the column vector with respect to $\frac{\partial}{\partial q_{rs}}$ of the Jacobian matrix $(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_i^{(n)}(a, p))_{i,(r,s)}$ is of the form $(\underbrace{0, \dots, 0}_{s-r}, 1, *, \dots, *)^t$ by equation (8.6), the first $n-1$ row vectors of the Jacobian matrix $(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_i^{(n)}(a, p))_{i,(r,s)}$ are linearly independent. By the assumption that $(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_i^{(n)}(a, p))_{i,(r,s)}$ does not have full rank, the n-th row vector must be written as a linear combination of the first $n-1$ row vectors, that is,

$$\left(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_n^{(n)}(a, p)\right)_{(r,s)} = \sum_{i=1}^{n-1} c_i \left(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_i^{(n)}(a, p)\right)_{(r,s)} \tag{8.9}$$

for some $c_1, \dots, c_{n-1} \in \mathbb{C}$. We note that the pair (r, s) can be taken as $2 \le s \le n$ and $n - h_m(n+1-s) < r \le s$ in the equality above. Recall that we denote the singular point $(a_1, \dots, a_n, p_{rs})_{2 \le s \le n, n-h_m(n+1-s) < r < s}$ of $V({}^{h_m}E_1^{(n)}, \dots, {}^{h_m}E_n^{(n)})$ by (a, p).

Claim 1. The coefficients c_i of equation (8.9) must be $(-1)^{n-i+1}a_n^{n-i}$ for $i \in [n-1]$.

We prove Claim 1 by descending induction on i. The base case is $i = n-1$. Comparing the $(r, s) = (1, n-1)$-th component of equation (8.9), we have

$${}^{h_m}E_1^{[n,n]}(a, p) = c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_{n-2} \cdot 0 + c_{n-1} \cdot 1$$

by equation (8.6). Since ${}^{h_m}E_1^{[n,n]} = x_n$, the equality above implies that $a_n = c_{n-1}$. This shows the base case.

We now assume that $i < n-1$ and the assertion of the claim holds for arbitrary k with $k \ge i+1$. It follows from equation (8.6) that the $(r, s) = (n-i, n-1)$-th component of equation (8.9) gives

$$\begin{aligned}
&{}^{h_m}E_{n-i-1}^{[1,n-i-1]}(a, p) \cdot {}^{h_m}E_1^{[n,n]}(a, p) \\
=&c_i \cdot 1 + c_{i+1} \cdot \left({}^{h_m}E_1^{[1,n-i-1]}(a, p) + {}^{h_m}E_1^{[n,n]}(a, p)\right) \\
&+ c_{i+2} \cdot \left({}^{h_m}E_2^{[1,n-i-1]}(a, p) + {}^{h_m}E_1^{[1,n-i-1]}(a, p) \cdot {}^{h_m}E_1^{[n,n]}(a, p)\right) \\
&+ c_{i+3} \cdot \left({}^{h_m}E_3^{[1,n-i-1]}(a, p) + {}^{h_m}E_2^{[1,n-i-1]}(a, p) \cdot {}^{h_m}E_1^{[n,n]}(a, p)\right) \\
&+ \cdots + c_{n-1} \cdot \left({}^{h_m}E_{n-i-1}^{[1,n-i-1]}(a, p) + {}^{h_m}E_{n-i-2}^{[1,n-i-1]}(a, p) \cdot {}^{h_m}E_1^{[n,n]}(a, p)\right).
\end{aligned}$$

Since ${}^{h_m}E_1^{[n,n]}(a, p) = a_n$ and $c_k = (-1)^{n-k+1}a_n^{n-k}$ for all $i+1 \le k \le n-1$ by our inductive assumption, ${}^{h_m}E_{n-i-1}^{[1,n-i-1]}(a, p) \cdot a_n$ is equal to

$$\begin{aligned}
&c_i + \sum_{k=1}^{n-i-1} (-1)^{n-i-k+1} a_n^{n-i-k} \left({}^{h_m}E_k^{[1,n-i-1]}(a, p) + {}^{h_m}E_{k-1}^{[1,n-i-1]}(a, p) \cdot a_n\right) \\
=&c_i + (-1)^{n-i} a_n^{n-i-1} \cdot a_n + a_n \cdot {}^{h_m}E_{n-i-1}^{[1,n-i-1]}(a, p).
\end{aligned}$$

This yields that $c_i = (-1)^{n-i+1}a_n^{n-i}$ and we proved Claim 1.

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It follows from Claim 1 and equation (8.9) that

$$\left(\frac{\partial}{\partial q_{rs}}\, {}^{h_m}E_n^{(n)}(a,p)\right)_{(r,s)} = \sum_{i=1}^{n-1}(-1)^{n-i+1}a_n^{n-i}\left(\frac{\partial}{\partial q_{rs}}\, {}^{h_m}E_i^{(n)}(a,p)\right)_{(r,s)} \tag{8.10}$$

for all $2 \le s \le n$ and $n - h_m(n+1-s) < r \le s$.

Claim 2. It holds that $p_{in} = 0$ for all $n - m + 1 \le i \le n - 1$.

We show Claim 2 by descending induction on i. The base case is $i = n - 1$. The $(r, s) = (1, n-2)$-th component of equation (8.10) is equal to

$$ {}^{h_m}E_2^{[n-1,n]}(a,p) = -a_n^2 \cdot 1 + a_n \cdot {}^{h_m}E_1^{[n-1,n]}(a,p)$$

from equation (8.6). Since ${}^{h_m}E_2^{[n-1,n]}(a,p) = a_{n-1}a_n + p_{n-1\,n}$ and ${}^{h_m}E_1^{[n-1,n]}(a,p) = a_{n-1} + a_n$, we have $p_{n-1\,n} = 0$, which proves the base case.

Suppose now that $i < n - 1$ and that the claim holds for any k with $k \ge i + 1$, that is, $p_{kn} = 0$ for all $i + 1 \le k \le n - 1$. By equation (8.6), the $(r, s) = (1, i - 1)$-th component of equation (8.10) is

$$\begin{aligned} {}^{h_m}E_{n-i+1}^{[i,n]}(a,p) =& (-1)^{n-i+2}a_n^{n-i+1}\cdot 1 + (-1)^{n-i+1}a_n^{n-i}\cdot {}^{h_m}E_1^{[i,n]}(a,p) \\ &+ (-1)^{n-i}a_n^{n-i-1}\cdot {}^{h_m}E_2^{[i,n]}(a,p) + \cdots + a_n\cdot {}^{h_m}E_{n-i}^{[i,n]}(a,p). \end{aligned} \tag{8.11}$$

Here, we note that

$$\det(\lambda I_{n-i+1} - M_{[i,n]})|_{q_{kn}=0\ (i+1\le k\le n-1)} = (-1)^{n-i}(-q_{in}) + \det(\lambda I_{n-i} - M_{[i,n-1]})\cdot(\lambda - x_n)$$

by the cofactor expansion along the last column. The left-hand side is written as

$$\sum_{\ell=0}^{n-i+1}\left((-1)^{\ell}E_\ell^{[i,n]}|_{q_{kn}=0\ (i+1\le k\le n-1)}\right)\lambda^{n-i+1-\ell},$$

and the right-hand side is

$$\lambda^{n-i+1} + \sum_{\ell=1}^{n-i}\left((-1)^{\ell}(x_nE_{\ell-1}^{[i,n-1]} + E_\ell^{[i,n-1]})\right)\lambda^{n-i+1-\ell} + (-1)^{n-i+1}(x_nE_{n-i}^{[i,n-1]} + q_{in})$$

by definition. Thus, we obtain that

$$\begin{aligned} E_\ell^{[i,n]}|_{q_{kn}=0\ (i+1\le k\le n-1)} &= x_nE_{\ell-1}^{[i,n-1]} + E_\ell^{[i,n-1]} \quad \text{for } 1 \le \ell \le n-i; \\ E_{n-i+1}^{[i,n]}|_{q_{kn}=0\ (i+1\le k\le n-1)} &= x_nE_{n-i}^{[i,n-1]} + q_{in}. \end{aligned}$$

In particular, by our inductive hypothesis $p_{kn} = 0$ for all $i + 1 \le k \le n - 1$, one has

$$\begin{aligned} {}^{h_m}E_\ell^{[i,n]}(a,p) &= a_n\, {}^{h_m}E_{\ell-1}^{[i,n-1]}(a,p) + {}^{h_m}E_\ell^{[i,n-1]}(a,p) \quad \text{for } 1 \le \ell \le n-i; \\ {}^{h_m}E_{n-i+1}^{[i,n]}(a,p) &= a_n\, {}^{h_m}E_{n-i}^{[i,n-1]}(a,p) + p_{in}. \end{aligned}$$

Substituting these equalities to equation (8.11), the left-hand side of equation (8.11) is

$$a_n\, {}^{h_m}E_{n-i}^{[i,n-1]}(a,p) + p_{in}.$$

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On the other hand, the right-hand side of equation (8.11) is

$$(-1)^{n-i+2}a_n^{n-i+1} + \sum_{k=1}^{n-i}(-1)^{n-i-k+2}a_n^{n-i-k+1} \cdot \Big(a_n\, {}^{h_m}E_{k-1}^{[i,n-1]}(a,p) + {}^{h_m}E_k^{[i,n-1]}(a,p)\Big)$$
$$= a_n \cdot {}^{h_m}E_{n-i}^{[i,n-1]}(a,p).$$

Hence, we obtain that $p_{in} = 0$ as desired. This proves Claim 2.

Claim 3. We have $a_n = 0$.

It follows from Theorem 4.13 that

$$\mathrm{Hess}(N,h_m) \cap \Omega_e^\circ \cong V\big({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\big).$$

We denote by $b = (b_{ij})_{1 \le j < i \le n} \in \mathrm{Hess}(N,h_m) \cap \Omega_e^\circ \subset \mathbb{C}^{\frac{1}{2}n(n-1)}$ the image of the singular point (a,p) of $V\big({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\big)$ under the isomorphism above. One can see from Proposition 7.3 and Claim 2 that

$$F_{i,1}(b) = -q_{n+1-i\,n}(a,p) = -p_{n+1-i\,n} = 0 \quad \text{for } 2 \le i \le m. \tag{8.12}$$

Since $b \in \mathrm{Hess}(N,h_m) \cap \Omega_e^\circ$, we also have

$$F_{i,1}(b) = 0 \quad \text{for } m+1 \le i \le n \tag{8.13}$$

by Lemma 2.3. It follows from equations (8.12) and (8.13) and Lemma 2.3 again that the point $b = (b_{ij})_{1 \le j < i \le n}$ belongs to $\mathrm{Hess}(N,h_1) \cap \Omega_e^\circ$, where h_1 is the decomposable Hessenberg function defined by $h_1 := (1,n,\ldots,n)$. As seen in Definition 3.2 and surrounding discussion, every flag $V_\bullet \in \mathrm{Hess}(N,h_1)$ has $V_1 = \mathbb{C} \cdot (1,0,\ldots,0)^t$ which implies that $b_{i1} = 0$ for all $2 \le i \le n$. It then follows from Proposition 7.3 that

$$a_n = x_n(a,p) = -x_{21}(b) = -b_{21} = 0,$$

as desired. We have proven Claim 3.

Combining Claim 3 and equation (8.10), we conclude that the n-th row vector $\big(\frac{\partial}{\partial q_{rs}}\, {}^{h_m}E_n^{(n)}(a,p)\big)_{(r,s)}$ of the Jacobian matrix is zero. This completes the proof. □

By using Proposition 8.5, we will explicitly describe the singular locus of $\mathrm{Hess}(N,h_m) \cap \Omega_e^\circ$ in Section 10. For this purpose, we will first study the singularity of $\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ$ in next section.

9. Cyclic quotient singularity

In this section, we analyze the singularity of $\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ$, where $h_2 = (2,n,\ldots,n)$. In fact, we see that the singularity is related with a cyclic quotient singularity. Then we can explicitly describe the singular locus of $\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ$.

First, we study relations between $F_{i,j}$'s and $\tilde{F}_{i,j}^{\langle m_j \rangle}$'s, which are defined in equations (3.3) and (3.4), and Definition 6.3, respectively.

Lemma 9.1. *Let $1 \le j \le n-1$ and $j \le m_j < n$. For $i > m_j$, we have*

$$F_{i,j} = \tilde{F}_{i,j}^{\langle m_j \rangle} - \sum_{\ell=m_j+1}^{i-1} x_{i\ell} F_{\ell,j},$$

where we take the convention that $\sum_{\ell=m_j+1}^{m_j} x_{i\ell} F_{\ell,j} = 0$.

Proof. It suffices to show that

$$\tilde{F}_{i,j}^{\langle i-k\rangle} = \tilde{F}_{i,j}^{\langle m_j\rangle} - \sum_{\ell=m_j+1}^{i-k} x_{i\ell}F_{\ell,j} \quad \text{for } 1 \leq k \leq i-m_j. \tag{9.1}$$

Indeed, equation (9.1) for $k = 1$ is the desired equality by equation (6.4). We prove equation (9.1) by descending induction on k. The base case is $k = i - m_j$, which is clear. Now, suppose that $k < i - m_j$ and assume by induction that the claim is true for $k + 1$, that is,

$$\tilde{F}_{i,j}^{\langle i-k-1\rangle} = \tilde{F}_{i,j}^{\langle m_j\rangle} - \sum_{\ell=m_j+1}^{i-k-1} x_{i\ell}F_{\ell,j}. \tag{9.2}$$

Then we show equation (9.1). The left-hand side of equation (9.1) is equal to

$$\tilde{F}_{i,j}^{\langle i-k-1\rangle} - x_{i\,i-k}F_{i-k,j}$$

by using the cofactor expansion along the second-to-last column. Combining this with the inductive hypothesis (9.2), we have proven equation (9.1). □

Proposition 9.2. *Let* $1 \leq j \leq n-1$ *and* $j \leq m_j < n$*. For* $i > m_j$*, the ideal*

$$(F_{m_j+1,j}, F_{m_j+2,j}, \ldots, F_{i,j})$$

is equal to the ideal

$$(\tilde{F}_{m_j+1,j}^{\langle m_j\rangle}, \tilde{F}_{m_j+2,j}^{\langle m_j\rangle}, \ldots, \tilde{F}_{i,j}^{\langle m_j\rangle})$$

in the polynomial ring $\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]$*. In particular, if* $h : [n] \to [n]$ *is an indecomposable Hessenberg function such that* $h \neq (n, \ldots, n)$*, then we have*

$$\mathbb{C}[\mathrm{Hess}(N,h) \cap \Omega_e^\circ] \cong \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(\tilde{F}_{i,j}^{\langle h(j)\rangle} \mid j \in J_h \text{ and } h(j) < i \leq n), \tag{9.3}$$

where $J_h \coloneqq \{j \in [n-2] \mid h(j) < n\}$*.*

Proof. We prove the first statement by induction on i. The base case $i = m_j + 1$ is clear since $F_{m_j+1,j} = \tilde{F}_{m_j+1,j}^{\langle m_j\rangle}$ by equation (6.4). We proceed to the inductive step. Suppose that $i > m_j + 1$ and that the claim holds for $i - 1$. Then we have

$$\begin{aligned}(F_{m_j+1,j}, \ldots, F_{i-1,j}, F_{i,j}) &= (F_{m_j+1,j}, \ldots, F_{i-1,j}, \tilde{F}_{i,j}^{\langle m_j\rangle}) \quad \text{(from Lemma 9.1)}\\ &= (\tilde{F}_{m_j+1,j}^{\langle m_j\rangle}, \ldots, \tilde{F}_{i-1,j}^{\langle m_j\rangle}, \tilde{F}_{i,j}^{\langle m_j\rangle}) \quad \text{(by the inductive assumption)}\end{aligned}$$

as desired. The isomorphism (9.3) follows from equation (3.8) and the former statement by setting $m_j = h(j)$. □

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Example 9.3. Let $m_j = j + 1$. Then one has

$$\tilde{F}_{i,1}^{\langle 2\rangle} = \begin{vmatrix} 1 & 0 & x_{21} \\ x_{21} & 1 & x_{31} \\ x_{i1} & x_{i2} & x_{i+1\,1} \end{vmatrix} \text{ for } i > 2; \tag{9.4}$$

$$\tilde{F}_{i,j}^{\langle j+1\rangle} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ x_{j\,j-1} & 1 & 0 & x_{j+1\,j} \\ x_{j+1\,j-1} & x_{j+1\,j} & 1 & x_{j+2\,j} \\ x_{i\,j-1} & x_{ij} & x_{i\,j+1} & x_{i+1\,j} \end{vmatrix} \text{ for } 2 \leq j \leq n-2 \text{ and } i > j+1.$$

By equation (9.3), the coordinate ring $\mathbb{C}[\mathrm{Pet}_n \cap \Omega_e^\circ]$ is

$$\mathbb{C}[\mathrm{Pet}_n \cap \Omega_e^\circ] \cong \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(\tilde{F}_{i,j}^{\langle j+1\rangle} \text{ for all } 1 \leq j \leq n-2 \text{ and } j+1 < i \leq n). \tag{9.5}$$

We now explain the cyclic quotient singularity which is also called the type A-singularity. Let ζ be a primitive n-th root of unity and $\mathfrak{C}_n$ the cyclic group of order n generated by ζ. Consider the action of $\mathfrak{C}_n$ on $\mathbb{C}^2$ defined by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$ for $\zeta \in \mathfrak{C}_n$ and $(x, y) \in \mathbb{C}^2$. This induces the action of $\mathfrak{C}_n$ on the polynomial ring $\mathbb{C}[x, y]$ (which is the coordinate ring of $\mathbb{C}^2$), and it is given by $\zeta \cdot x = \zeta x$ and $\zeta \cdot y = \zeta^{-1} y$. As is well known, the $\mathfrak{C}_n$-invariants in $\mathbb{C}[x, y]$ is isomorphic to

$$\mathbb{C}[\mathbb{C}^2/\mathfrak{C}_n] \cong \mathbb{C}[x, y]^{\mathfrak{C}_n} \cong \mathbb{C}[X, Y, Z]/(XY - Z^n),$$

which sends X to x^n, Y to y^n, and Z to xy. The quotient space $\mathbb{C}^2/\mathfrak{C}_n$ is called the *cyclic quotient singularity* or the *type A_{n-1}-singularity*.

Example 9.4. Let $n = 3$. As seen in Example 9.3, the coordinate ring of $\mathrm{Pet}_3 \cap \Omega_e^\circ$ is given by

$$\mathbb{C}[\mathrm{Pet}_3 \cap \Omega_e^\circ] \cong \mathbb{C}[x_{21}, x_{31}, x_{32}]/(\tilde{F}_{3,1}^{\langle 2\rangle}).$$

Here, one can compute $\tilde{F}_{3,1}^{\langle 2\rangle}$ as

$$\begin{aligned}\tilde{F}_{3,1}^{\langle 2\rangle} = \begin{vmatrix} 1 & 0 & x_{21} \\ x_{21} & 1 & x_{31} \\ x_{31} & x_{32} & 0 \end{vmatrix} &= x_{21}^2 x_{32} - x_{21}x_{31} - x_{31}x_{32} \\ &= -x_{21}^3 + (x_{21}^2 - x_{31})(x_{21} + x_{32}).\end{aligned}$$

Thus, we have

$$\mathbb{C}[\mathrm{Pet}_3 \cap \Omega_e^\circ] \cong \mathbb{C}[X, Y, Z]/(XY - Z^3),$$

which sends X to $x_{21}^2 - x_{31}$, Y to $x_{21} + x_{32}$ and Z to x_{21}. The ring isomorphism above yields the isomorphism $\mathrm{Pet}_3 \cap \Omega_e^\circ \cong \mathbb{C}^2/\mathfrak{C}_3$.

In Example 9.4, we constructed the polynomial $XY - Z^n$ from $\tilde{F}_{3,1}^{\langle 2\rangle}$ for $n = 3$. We now generalize this fact to arbitrary n. More specifically, we construct the polynomial $XY - Z^n$ from $\tilde{F}_{i,1}^{\langle 2\rangle}$ $(2 < i \leq n)$ defined in equation (9.4). By the cofactor expansion along the second column, we have

$$\tilde{F}_{i,1}^{\langle 2\rangle} = \begin{vmatrix} 1 & x_{21} \\ x_{i1} & x_{i+1\,1} \end{vmatrix} - x_{i2} \begin{vmatrix} 1 & x_{21} \\ x_{21} & x_{31} \end{vmatrix} \text{ for } i > 2. \tag{9.6}$$

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Lemma 9.5. *For* $2 < i \leq n$*, we have*

$$\begin{aligned}&\tilde{F}^{\langle 2\rangle}_{n,1} + x_{21}\tilde{F}^{\langle 2\rangle}_{n-1,1} + x_{21}^2\tilde{F}^{\langle 2\rangle}_{n-2,1} + \cdots + x_{21}^{n-i}\tilde{F}^{\langle 2\rangle}_{i,1}\\ =&\begin{vmatrix} 1 & x_{21}^{n-i+1} \\ x_{i1} & 0 \end{vmatrix} - (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-i}x_{i2})\begin{vmatrix} 1 & x_{21} \\ x_{21} & x_{31} \end{vmatrix}.\end{aligned}$$

Proof. We prove this by descending induction on i. The base case $i = n$ is the equality (9.6) for $i = n$. Suppose now that $i < n$ and that the claim holds for $i + 1$. Then we have

$$\begin{aligned}&\tilde{F}^{\langle 2\rangle}_{n,1} + x_{21}\tilde{F}^{\langle 2\rangle}_{n-1,1} + x_{21}^2\tilde{F}^{\langle 2\rangle}_{n-2,1} + \cdots + x_{21}^{n-i-1}\tilde{F}^{\langle 2\rangle}_{i+1,1} + x_{21}^{n-i}\tilde{F}^{\langle 2\rangle}_{i,1}\\ =&\begin{vmatrix} 1 & x_{21}^{n-i} \\ x_{i+1\,1} & 0 \end{vmatrix} - (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-i-1}x_{i+1\,2})\begin{vmatrix} 1 & x_{21} \\ x_{21} & x_{31} \end{vmatrix} + x_{21}^{n-i}\tilde{F}^{\langle 2\rangle}_{i,1}\\ &\qquad\text{(by our descending induction hypothesis)}\\ =&\begin{vmatrix} 1 & x_{21}^{n-i} \\ x_{i+1\,1} & 0 \end{vmatrix} - (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-i-1}x_{i+1\,2})\begin{vmatrix} 1 & x_{21} \\ x_{21} & x_{31} \end{vmatrix}\\ &+ x_{21}^{n-i}\begin{vmatrix} 1 & x_{21} \\ x_{i1} & x_{i+1\,1} \end{vmatrix} - x_{21}^{n-i}x_{i2}\begin{vmatrix} 1 & x_{21} \\ x_{21} & x_{31} \end{vmatrix} \qquad \text{(by equation (9.6))}\\ =&\begin{vmatrix} 1 & x_{21}^{n-i+1} \\ x_{i1} & 0 \end{vmatrix} - (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-i}x_{i2})\begin{vmatrix} 1 & x_{21} \\ x_{21} & x_{31} \end{vmatrix}\end{aligned}$$

as desired. □

Proposition 9.6. *We set*

$$\begin{cases} X = x_{21}^2 - x_{31} \\ Y = x_{21}^{n-2} + x_{32}x_{21}^{n-3} + \cdots + x_{n-1\,2}x_{21} + x_{n2} \\ Z = x_{21}. \end{cases}$$

Then one can write

$$\tilde{F}^{\langle 2\rangle}_{n,1} + x_{21}\tilde{F}^{\langle 2\rangle}_{n-1,1} + x_{21}^2\tilde{F}^{\langle 2\rangle}_{n-2,1} + \cdots + x_{21}^{n-3}\tilde{F}^{\langle 2\rangle}_{3,1} = XY - Z^n.$$

Proof. By using Lemma 9.5 for $i = 3$, we obtain

$$\begin{aligned}&\tilde{F}^{\langle 2\rangle}_{n,1} + x_{21}\tilde{F}^{\langle 2\rangle}_{n-1,1} + x_{21}^2\tilde{F}^{\langle 2\rangle}_{n-2,1} + \cdots + x_{21}^{n-3}\tilde{F}^{\langle 2\rangle}_{3,1}\\ =&\begin{vmatrix} 1 & x_{21}^{n-2} \\ x_{31} & 0 \end{vmatrix} - (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-3}x_{32})\begin{vmatrix} 1 & x_{21} \\ x_{21} & x_{31} \end{vmatrix}\\ =&-x_{21}^{n-2}x_{31} - (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-3}x_{32})(x_{31} - x_{21}^2)\\ =&-x_{21}^{n} + x_{21}^{n-2}(x_{21}^2 - x_{31}) + (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-3}x_{32})(x_{21}^2 - x_{31})\\ =&-x_{21}^{n} + (x_{n2} + x_{21}x_{n-1\,2} + x_{21}^2 x_{n-2\,2} + \cdots + x_{21}^{n-3}x_{32} + x_{21}^{n-2})(x_{21}^2 - x_{31})\\ =&XY - Z^n.\end{aligned}$$

□

Theorem 9.7. *If* $h_2 = (2, n, \ldots, n)$*, then there is an isomorphism of* $\mathbb{C}$*-algebras*

$$\mathbb{C}[\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ] \cong \frac{\mathbb{C}[X, Y, Z]}{(XY - Z^n)} \otimes \mathbb{C}[x_{32}, x_{42}, \ldots, x_{n-1\,2}] \otimes \mathbb{C}[x_{ij} \mid 3 \leq j < i \leq n]$$

which sends

$$\begin{aligned}
&X \mapsto x_{21}^2 - x_{31};\\
&Y \mapsto x_{21}^{n-2} + x_{32}x_{21}^{n-3} + \cdots + x_{n-1\,2}x_{21} + x_{n2};\\
&Z \mapsto x_{21};\\
&x_{i2} \mapsto x_{i2} \text{ for } 3 \leq i \leq n-1;\\
&x_{ij} \mapsto x_{ij} \text{ for } 3 \leq j < i \leq n.
\end{aligned} \tag{9.7}$$

In other words,

$$\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ \cong \mathbb{C}^2/\mathfrak{C}_n \times \mathbb{C}^{\frac{1}{2}(n-1)(n-2)-1}.$$

Remark 9.8. We have seen the case when $n = 3$ in Example 9.4.

Proof of Theorem 9.7. By equation (9.3), we have

$$\mathbb{C}[\mathrm{Hess}(N, h_2) \cap \Omega_e^\circ] \cong \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(\tilde{F}_{3,1}^{\langle 2\rangle}, \tilde{F}_{4,1}^{\langle 2\rangle}, \ldots, \tilde{F}_{n,1}^{\langle 2\rangle}). \tag{9.8}$$

Put

$$P_n := (x_{21}^2 - x_{31})(x_{21}^{n-2} + x_{32}x_{21}^{n-3} + \cdots + x_{n-1\,2}x_{21} + x_{n2}) - x_{21}^n.$$

It then follows from Proposition 9.6 that

$$\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(\tilde{F}_{3,1}^{\langle 2\rangle}, \ldots, \tilde{F}_{n-1,1}^{\langle 2\rangle}, \tilde{F}_{n,1}^{\langle 2\rangle}) \cong \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(\tilde{F}_{3,1}^{\langle 2\rangle}, \ldots, \tilde{F}_{n-1,1}^{\langle 2\rangle}, P_n). \tag{9.9}$$

By the definition (9.4), $\tilde{F}_{i,1}^{\langle 2\rangle} = 0$ if and only if

$$x_{i+1\,1} = x_{21}x_{i1} + x_{31}x_{i2} - x_{21}^2 x_{i2}$$

for $3 \leq i \leq n-1$. These equalities for $i = n-1, n-2, \ldots, 3$ lead us to the isomorphism

$$\mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]/(\tilde{F}_{3,1}^{\langle 2\rangle}, \ldots, \tilde{F}_{n-1,1}^{\langle 2\rangle}, P_n) \cong \mathbb{C}[x_{21}, x_{31}, x_{ij} \mid 2 \leq j < i \leq n]/(P_n). \tag{9.10}$$

It is straightforward to see that

$$\begin{aligned}
&\mathbb{C}[x_{21}, x_{31}, x_{ij} \mid 2 \leq j < i \leq n]/(P_n)\\
\cong &\mathbb{C}[X, Y, Z]/(XY - Z^n) \otimes \mathbb{C}[x_{32}, x_{42}, \ldots, x_{n-1\,2}] \otimes \mathbb{C}[x_{ij} \mid 3 \leq j < i \leq n],
\end{aligned} \tag{9.11}$$

which sends

$$\begin{aligned}
&X \mapsto x_{21}^2 - x_{31};\\
&Y \mapsto x_{21}^{n-2} + x_{32}x_{21}^{n-3} + \cdots + x_{n-1\,2}x_{21} + x_{n2};\\
&Z \mapsto x_{21};\\
&x_{i2} \mapsto x_{i2} \text{ for } 3 \leq i \leq n-1;\\
&x_{ij} \mapsto x_{ij} \text{ for } 3 \leq j < i \leq n.
\end{aligned}$$

In fact, the inverse map is given by

$$
\begin{aligned}
&x_{21} \mapsto Z;\\
&x_{31} \mapsto -X + Z^2;\\
&x_{n2} \mapsto Y - Z^{n-2} - x_{32}Z^{n-3} - \cdots - x_{n-1\,2}Z;\\
&x_{i2} \mapsto x_{i2} \text{ for } 3 \leq i \leq n-1;\\
&x_{ij} \mapsto x_{ij} \text{ for } 3 \leq j < i \leq n.
\end{aligned}
$$

Combining equations (9.8), (9.9), (9.10) and (9.11), we conclude that

$$
\mathbb{C}[\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ] \cong \frac{\mathbb{C}[X,Y,Z]}{(XY - Z^n)} \otimes \mathbb{C}[x_{32}, x_{42}, \ldots, x_{n-1\,2}] \otimes \mathbb{C}[x_{ij} \mid 3 \leq j < i \leq n].
$$

We complete the proof. □

For a polynomial $f \in \mathbb{C}[\Omega_e^\circ] \cong \mathbb{C}[x_{ij} \mid 1 \leq j < i \leq n]$, we denote the zero set of f by

$$
V(f) := \{g \in \Omega_e^\circ \cong \mathbb{C}^{\frac{1}{2}n(n-1)} \mid f(g) = 0\}. \tag{9.12}
$$

Corollary 9.9. *The singular locus of* $\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ$ *is given by*

$$
\mathrm{Sing}(\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ) = \bigcap_{i=2}^{n} V(x_{i1}) \cap V(x_{n2}).
$$

Proof. By Theorem 9.7, we have

$$
\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ \cong \mathbb{C}^2/\mathfrak{C}_n \times \mathbb{C}^{\frac{1}{2}(n-1)(n-2)-1} \cong V(XY - Z^n).
$$

Here, $V(XY - Z^n)$ denotes the hypersurface defined by the equation $XY - Z^n = 0$ in $\mathbb{C}^{\frac{1}{2}(n-1)(n-2)+2}$ with the variables $X, Y, Z, x_{32}, x_{42}, \ldots, x_{n-1\,2}, x_{ij}$ $(3 \leq j < i \leq n)$. The singular locus of $V(XY - Z^n)$ is the solution set of the equations

$$
X = 0,\ Y = 0,\ Z = 0.
$$

It follows from the correspondence (9.7) that the image of the ideal (X, Y, Z) under the isomorphism $\mathbb{C}[\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ] \cong \frac{\mathbb{C}[X,Y,Z]}{(XY-Z^n)} \otimes \mathbb{C}[x_{32}, x_{42}, \ldots, x_{n-1\,2}] \otimes \mathbb{C}[x_{ij} \mid 3 \leq j < i \leq n]$ is

$$
(x_{21}^2 - x_{31}, x_{21}^{n-2} + x_{32}x_{21}^{n-3} + \cdots + x_{n-1\,2}x_{21} + x_{n2}, x_{21}) = (x_{31}, x_{n2}, x_{21})
$$

in $\mathbb{C}[\mathrm{Hess}(N,h_2) \cap \Omega_e^\circ]$. Hence, by the presentation (9.3), the singular locus of $\mathrm{Hess}(N,h_2)$ is the solution set of the equations

$$
\begin{cases}
x_{21} = 0\\
x_{31} = 0\\
x_{n2} = 0\\
\tilde{F}_{i,1}^{\langle 2\rangle} = 0 \quad \text{for } 3 \leq i \leq n-1.
\end{cases} \tag{9.13}
$$

Note that the equation $\tilde{F}_{n,1}^{\langle 2\rangle} = 0$ can be removed above since $\tilde{F}_{n,1}^{\langle 2\rangle} = 0$ is derived from the equations (9.13) by Proposition 9.6. It then follows from the equality $\tilde{F}_{i,1}^{\langle 2\rangle} = x_{i+1\,1} + x_{21}^2 x_{i2} - x_{21}x_{i1} - x_{31}x_{i2}$ for $3 \leq i \leq n-1$ that the solution set of the equations (9.13) is given by $x_{i1} = x_{n2} = 0$ for $2 \leq i \leq n$, as desired. □

10. Singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$

We now give an explicit description for the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ where h_m is defined in equation (8.7).

Theorem 10.1. *Let* $2 \leq m \leq n-1$. *Then, the singular locus of* $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ *is described as*

$$\mathrm{Sing}(\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ) = \bigcap_{i=2}^{n} V(x_{i1}) \cap \bigcap_{j=2}^{m} V(x_{nj}),$$

where the notation $V(f)$ *is defined in equation (9.12).*

Proof. We prove this by induction on m. The base case is $m = 2$, which follows from Corollary 9.9. Suppose now that $m > 2$ and that the claim holds for $m-1$. It follows from Proposition 8.5 that the singular locus of $V\left({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\right)$ is given by the solution set of the equations

$$\frac{\partial}{\partial q_{rs}} {}^{h_m}E_n^{(n)} = 0 \quad \text{for all } 2 \leq s \leq n \text{ and } n - h_m(n+1-s) < r \leq s. \tag{10.1}$$

By Lemma 8.2, we have

$$\begin{cases} E_1^{[n,n]} & = x_n \\ E_2^{[n-1,n]} & = E_1^{[n-1,n-1]} x_n + q_{n-1\,n} \\ E_3^{[n-2,n]} & = E_2^{[n-2,n-1]} x_n + E_1^{[n-2,n-2]} q_{n-1\,n} + q_{n-2\,n} \\ & \vdots \\ E_m^{[n-m+1,n]} & = E_{m-1}^{[n-m+1,n-1]} x_n + \cdots + E_1^{[n-m+1,n-m+1]} q_{n-m+2\,n} + q_{n-m+1\,n}. \end{cases} \tag{10.2}$$

Since it holds that

$${}^{h_m}E_i^{[n-i+1,n]} = \frac{\partial}{\partial q_{1\,n-i}} {}^{h_m}E_n^{(n)} = 0 \quad \text{for all } 1 \leq i \leq m$$

by equations (8.6) and (10.1), the equalities (10.2) lead us to the following equation

$$\begin{pmatrix} 1 & & & & \\ {}^{h_m}E_1^{[n-1,n-1]} & 1 & & & \\ {}^{h_m}E_2^{[n-2,n-1]} & {}^{h_m}E_1^{[n-2,n-2]} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ {}^{h_m}E_{m-1}^{[n-m+1,n-1]} & {}^{h_m}E_{m-2}^{[n-m+1,n-2]} & \cdots & {}^{h_m}E_1^{[n-m+1,n-m+1]} & 1 \end{pmatrix} \begin{pmatrix} x_n \\ q_{n-1\,n} \\ q_{n-2\,n} \\ \vdots \\ q_{n-m+1\,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which yields that $x_n = q_{n-1\,n} = q_{n-2\,n} = \cdots = q_{n-m+1\,n} = 0$. Since we have $q_{n-m+1\,n} = 0$ and

$$\left(\frac{\partial}{\partial q_{rs}} {}^{h_m}E_n^{(n)} \right) \Bigg|_{q_{n-m+1\,n}=0} = \frac{\partial}{\partial q_{rs}} {}^{h_{m-1}}E_n^{(n)}$$

for $2 \leq s \leq n$ and $n - h_{m-1}(n+1-s) < r \leq s$, the equations (10.1) imply that

$$\begin{cases} \frac{\partial}{\partial q_{rs}} {}^{h_{m-1}}E_n^{(n)} & = 0 \quad \text{for all } 2 \leq s \leq n \text{ and } n - h_{m-1}(n+1-s) < r \leq s \\ \frac{\partial}{\partial q_{n-m+1\,n}} {}^{h_m}E_n^{(n)} & = 0 \\ q_{n-m+1\,n} & = 0. \end{cases} \tag{10.3}$$

Conversely, the equations (10.3) yield equation (10.1). In fact, since one can write ${}^{h_m}E_n^{(n)} = {}^{h_{m-1}}E_n^{(n)} + q_{n-m+1\,n} \cdot F$ for some polynomial F by Definition 4.10, we have $\frac{\partial}{\partial q_{rs}}\, {}^{h_m}E_n^{(n)} = \frac{\partial}{\partial q_{rs}}\, {}^{h_{m-1}}E_n^{(n)} + q_{n-m+1\,n} \cdot \frac{\partial}{\partial q_{rs}} F$ for all $2 \leq s \leq n$ and $n - h_{m-1}(n+1-s) < r \leq s$. Hence, equation (10.1) is equivalent to equation (10.3). One has

$$\frac{\partial}{\partial q_{n-m+1\,n}}\, {}^{h_m}E_n^{(n)} = {}^{h_m}E_{n-m}^{[1,n-m]} = {}^{h_m}E_{n-m}^{(n-m)}$$

from equations (8.1) and (8.6), so the singular locus of $V\big({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\big)$ is given by the solution set of the equations

$$\begin{cases} \frac{\partial}{\partial q_{rs}}\, {}^{h_{m-1}}E_n^{(n)} & = 0 \quad \text{for all } 2 \leq s \leq n \text{ and } n - h_{m-1}(n+1-s) < r \leq s \\ {}^{h_m}E_{n-m}^{(n-m)} & = 0 \\ q_{n-m+1\,n} & = 0. \end{cases}$$

By Corollary 7.4, the following commutative diagram holds

$$\begin{array}{ccc} \mathbb{C}[Fl(\mathbb{C}^n) \cap \Omega_e^\circ] & \xrightarrow[\cong]{\varphi} & \dfrac{\mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 1 \leq r < s \leq n]}{(E_1^{(n)}, \ldots, E_n^{(n)})} \\ \Big\downarrow {\scriptstyle F_{i,1}=0\ (m+1 \leq i \leq n)} & & \Big\downarrow {\scriptstyle q_{rn}=0\ (1 \leq r \leq n-m)} \\ \mathbb{C}[\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ] & \xrightarrow[\cong]{\varphi_{h_m}} & \dfrac{\mathbb{C}[x_1, \ldots, x_n, q_{rs} \mid 2 \leq s \leq n, n - h_m(n+1-s) < r < s]}{({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)})}, \end{array} \tag{10.4}$$

where both vertical arrows are surjective. The ideal defining the singular locus of the zero set $V\big({}^{h_m}E_1^{(n)}, \ldots, {}^{h_m}E_n^{(n)}\big)$ is

$$\left(\frac{\partial}{\partial q_{rs}}\, {}^{h_{m-1}}E_n^{(n)} \,\middle|\, 2 \leq s \leq n, n - h_{m-1}(n+1-s) < r \leq s \right) + \big({}^{h_m}E_{n-m}^{(n-m)}\big) + (q_{n-m+1\,n}).$$

The image of the ideal above under the isomorphism $\varphi_{h_m}^{-1}$ in equation (10.4) is equal to

$$\left(\varphi^{-1}\left(\frac{\partial}{\partial q_{rs}}\, {}^{h_{m-1}}E_n^{(n)} \right) \,\middle|\, 2 \leq s \leq n, n - h_{m-1}(n+1-s) < r \leq s \right) + (x_{nm}) + (F_{m,1})$$

in $\mathbb{C}[\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ]$ by the definition (7.1) and Proposition 7.3. Hence, the singular locus of $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ is the solution set of the equations

$$\begin{cases} \varphi^{-1}\left(\frac{\partial}{\partial q_{rs}}\, {}^{h_{m-1}}E_n^{(n)} \right) = 0 \quad \text{for all } 2 \leq s \leq n \text{ and } n - h_{m-1}(n+1-s) < r \leq s \\ F_{m,1} = 0 \\ F_{i,1} = 0 \quad \text{for all } m+1 \leq i \leq n \end{cases} \tag{10.5}$$

and

$$x_{nm} = 0$$

by Lemma 2.3. Since the singular locus of $\mathrm{Hess}(N, h_{m-1}) \cap \Omega_e^\circ$ is the solution set of the equations (10.5) from Proposition 8.5, Theorem 4.13 and Lemma 2.3 again, we can describe

$$\mathrm{Sing}(\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ) = \mathrm{Sing}(\mathrm{Hess}(N, h_{m-1}) \cap \Omega_e^\circ) \cap V(x_{nm}).$$

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One can see from our inductive assumption that $\mathrm{Sing}(\mathrm{Hess}(N, h_{m-1}) \cap \Omega_e^\circ) = \bigcap_{i=2}^n V(x_{i1}) \cap \bigcap_{j=2}^{m-1} V(x_{nj})$, so we complete the proof. □

Let $\mathfrak{S}_n$ be the permutation group on $[n]$. For $w \in \mathfrak{S}_n$, the Schubert cell X_w° is defined to be the B-orbit of the permutation flag wB in the flag variety $\mathrm{GL}_n(\mathbb{C})/B$. The Schubert variety X_w is defined by the closure of the Schubert cell X_w°, that is, $X_w = \overline{BwB/B}$. We put $F_i := \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_i\}$ for $i \in [n]$, where $e_1, \dots, e_n$ are the standard basis for $\mathbb{C}^n$. Under the identification $Fl(\mathbb{C}^n) \cong \mathrm{GL}_n(\mathbb{C})/B$, one can describe the Schubert variety X_w as

$$X_w = \{V_\bullet \in Fl(\mathbb{C}^n) \mid \dim(V_p \cap F_q) \geq r_w(p, q) \text{ for all } p, q \in [n]\},$$

where $r_w(p, q) = |\{i \in [p] \mid w(i) \leq q\}|$ (e.g., [19, Section 10.5]). For $2 \leq m \leq n-1$, we define the permutation $w_m \in \mathfrak{S}_n$ as

$$w_m(i) = \begin{cases} 1 & \text{if } i = 1, \\ n+1-i & \text{if } 2 \leq i \leq m, \\ n & \text{if } i = m+1, \\ n+2-i & \text{if } m+2 \leq i \leq n. \end{cases} \tag{10.6}$$

Then one can verify from [21] that the Schubert variety X_{w_m} is described as

$$X_{w_m} = \{V_\bullet \in Fl(\mathbb{C}^n) \mid \dim(V_1 \cap F_1) \geq 1 \text{ and } \dim(V_m \cap F_{n-1}) \geq m\}.$$

In other words, $V_\bullet \in X_{w_m}$ if and only if $V_1 = F_1 = \mathrm{span}_{\mathbb{C}}\{e_1\}$ and $V_m \subset F_{n-1} = \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_{n-1}\}$. In particular, we have

$$X_{w_m} \cap \Omega_e^\circ = \bigcap_{i=2}^{n} V(x_{i1}) \cap \bigcap_{j=2}^{m} V(x_{nj}).$$

Hence, we obtain the following result from Theorem 10.1.

Corollary 10.2. *Let $2 \leq m \leq n-1$. Then, the singular locus of* $\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ$ *is equal to*

$$\mathrm{Sing}(\mathrm{Hess}(N, h_m) \cap \Omega_e^\circ) = X_{w_m} \cap \Omega_e^\circ.$$

Remark 10.3. Let A be a nilpotent matrix (not necessarily regular nilpotent). Then it is known from [15, Theorem 5] that the singular locus of $\mathrm{Hess}(A, h_{n-1})$ is

$$\mathrm{Sing}(\mathrm{Hess}(A, h_{n-1})) = \mathrm{Hess}(A, h = (1, n-1, \dots, n-1, n)).$$

Consider the regular nilpotent case. The Hessenberg function $h = (1, n-1, \dots, n-1, n)$ is decomposable, so every flag $V_\bullet \in \mathrm{Hess}(N, h)$ has $V_1 = \mathrm{span}_{\mathbb{C}}\{e_1\}$ and $V_{n-1} = \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_{n-1}\}$ (see Definition 3.2 and surrounding discussion). Hence, the result of [15, Theorem 5] for $A = N$ gives

$$\mathrm{Sing}(\mathrm{Hess}(N, h_{n-1})) = X_{w_{n-1}}.$$

A. Singular locus of Peterson variety

The singular locus of the Peterson variety Pet_n has been studied in [26]. Combining the result of [26] and some result in [2], we describe the decomposition for the singular locus of the Peterson variety into irreducible components.

It is known that the flag variety $Fl(\mathbb{C}^n)$ has a decomposition into Schubert cells

$$Fl(\mathbb{C}^n) = \bigsqcup_{w \in \mathfrak{S}_n} X_w^\circ, \tag{A.1}$$

where each X_w° is isomorphic to $\mathbb{C}^{\ell(w)}$ and $\ell(w)$ denotes the length of w (e.g., [19]). Tymoczko generalized this fact to the Hessenberg varieties in [36]. In what follows, we explain the work in [36] for the case of the Peterson variety Pet_n.

Let I be a subset of $[n-1]$. We may regard $[n-1]$ as the set of vertices of the Dynkin diagram in type A_{n-1}. Then, $I \subset [n-1]$ can be decompose into the connected components of the Dynkin diagram of type A_{n-1}:

$$I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_m.$$

In other words, each I_j $(1 \leq j \leq m)$ denotes a maximal consecutive subset of $[n-1]$. To each connected component I_j, one can assign the longest element $w_0^{(I_j)}$ of the permutation subgroup $\mathfrak{S}_{I_j}$ on I_j. Then, we define the permutation $w_I \in \mathfrak{S}_n$ by

$$w_I := w_0^{(I_1)} w_0^{(I_2)} \cdots w_0^{(I_m)}.$$

Example A.1. Let $n = 9$ and $I = \{1, 2, 3\} \sqcup \{6, 7\}$. Then, the one-line notation of w_I is

$$w_I = 432158769.$$

The Schubert cell X_v° intersects with the Peterson variety Pet_n if and only if $v = w_I$ for some $I \subset [n-1]$ by [36] (see also [2, Lemma 3.5]). We set

$$X_I^\circ := X_{w_I}^\circ \cap \mathrm{Pet}_n \ \text{ and } X_I := X_{w_I} \cap \mathrm{Pet}_n = \overline{X_{w_I}^\circ} \cap \mathrm{Pet}_n\,.$$

By intersecting with the Peterson variety Pet_n, the decomposition in equation (A.1) yields that

$$\mathrm{Pet}_n = \bigsqcup_{J \subset [n-1]} X_J^\circ.$$

It is known from [36] that $X_J^\circ \cong \mathbb{C}^{|J|}$ for any $J \subset [n-1]$. In general, it follows from [2, Equation (3.7)] that for each $I \subset [n-1]$, we have

$$X_I = \bigsqcup_{J \subset I} X_J^\circ. \tag{A.2}$$

It is known that X_I is a regular nilpotent Hessenberg variety for a certain Hessenberg function h_I as described below. For $I \subset [n-1]$, we define a Hessenberg function $h_I : [n] \to [n]$ by

$$h_I(i) := \begin{cases} i+1 & \text{if } i \in I, \\ i & \text{if } i \notin I. \end{cases}$$

Note that if $I = [n-1]$, then $h_I = (2, 3, 4, \ldots, n, n)$ is the Hessenberg function for the Peterson case. Otherwise, h_I is decomposable (Definition 3.2).

Proposition A.2. ([2, *Proposition 3.4*]) *For a subset I of* $[n-1]$, *we have*

$$X_I = \overline{X_I^\circ} = \mathrm{Hess}(N, h_I),$$

where N is the regular nilpotent element defined in equation (3.1).

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It follows from Theorem 3.1 and Proposition A.2 that X_I is irreducible for any $I \subset [n-1]$. For positive integers a, b with $a \leq b$, we denote by $[a, b]$ the set $\{a, a+1, \ldots, b\}$. The singular locus of the Peterson variety Pet_n is described in [26] as follows.

Theorem A.3. *([26, Theorem 4]) The singular locus of* Pet_n *is given by*

$$\mathrm{Sing}(\mathrm{Pet}_n) = \bigsqcup_{\substack{J \subset [n-1] \\ J \neq [n-1], [2,n-1], [1,n-2]}} X_J^{\circ}.$$

Lemma A.4. *We have*

$$\{J \subset [n-1] \mid J \neq [n-1], [2,n-1], [1,n-2]\} = \bigcup_{j=2}^{n-2} \{J \subset [n-1] \mid J \not\ni j\} \cup \{J \subset [2, n-2]\}.$$

Proof. We first show that the left-hand side is included in the right-hand side. For this, we take a subset J of $[n-1]$ such that $J \neq [n-1], [2,n-1], [1,n-2]$. Note that $|J| \leq n-2$.

Case (i): Suppose that $|J| \leq n-3$. If J contains $[2, n-2]$, then $J = [2, n-2]$ since $|J| \leq n-3$. Otherwise, we have $J \not\ni j$ for some $2 \leq j \leq n-2$. In both cases, J belongs to the right-hand side.

Case (ii): Suppose that $|J| = n-2$. Since $J \neq [2, n-1], [1, n-2]$, we see that $J = [n-1] \setminus \{j\}$ for some $2 \leq j \leq n-2$, which belongs to the right-hand side.

Hence, we proved that the left-hand side is included in the right-hand side.

Conversely, let J be a subset of $[n-1]$ appeared in the right-hand side. If $J \not\ni j$ for some $2 \leq j \leq n-2$, then we have that $J \neq [n-1], [2,n-1], [1,n-2]$. If $J \subset [2, n-2]$, then it is clear that $J \neq [n-1], [2,n-1], [1,n-2]$. Thus, the right-hand side is included in the left-hand side, so the equality holds. □

The following proposition gives the decomposition for the singular locus of the Peterson variety into irreducible components.

Proposition A.5. *The singular locus of* Pet_n *is decomposed into irreducible components as follows:*

$$\begin{aligned}\mathrm{Sing}(\mathrm{Pet}_n) &= \left(\bigcup_{2 \leq j \leq n-2} X_{[n-1]\setminus\{j\}} \right) \cup X_{[2,n-2]} \\ &= \left(\bigcup_{2 \leq j \leq n-2} \mathrm{Hess}(N, h_{[n-1]\setminus\{j\}}) \right) \cup \mathrm{Hess}(N, h_{[2,n-2]}).\end{aligned}$$

Proof. By Theorem A.3 and Lemma A.4, we have

$$\mathrm{Sing}(\mathrm{Pet}_n) = \left(\bigcup_{2 \leq j \leq n-2} \bigsqcup_{J \subset [n-1]\setminus\{j\}} X_J^{\circ} \right) \cup \left(\bigsqcup_{J \subset [2,n-2]} X_J^{\circ} \right).$$

By using the decomposition in equation (A.2), the right-hand side above is equal to

$$\left(\bigcup_{2 \leq j \leq n-2} X_{[n-1]\setminus\{j\}} \right) \cup X_{[2,n-2]},$$

as desired. Also, this coincides with $\left(\bigcup_{2 \leq j \leq n-2} \mathrm{Hess}(N, h_{[n-1]\setminus\{j\}}) \right) \cup \mathrm{Hess}(N, h_{[2,n-2]})$ by Proposition A.2. □

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