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ON OPERATOR FRACTIONAL LÉVY MOTION: INTEGRAL REPRESENTATIONS AND TIME-REVERSIBILITY

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Abstract

In this paper, we construct operator fractional Lévy motion (ofLm), a broad class of infinitely divisible stochastic processes that are covariance operator self-similar and have wide-sense stationary increments. The ofLm class generalizes the univariate fractional Lévy motion as well as the multivariate operator fractional Brownian motion (ofBm). OfLm can be divided into two types, namely, moving average (maofLm) and real harmonizable (rhofLm), both of which share the covariance structure of ofBm under assumptions. We show that maofLm and rhofLm admit stochastic integral representations in the time and Fourier domains, and establish their distinct small- and large-scale limiting behavior. We also characterize time-reversibility for ofLm through parametric conditions related to its Lévy measure. In particular, we show that, under non-Gaussianity, the parametric conditions for time-reversibility are generally more restrictive than those for the Gaussian case (ofBm).

Keywords: Infinite divisibility; Lévy processes; operator self-similarity

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1. Introduction

Let $X = \{X(t)\}_{t\in\mathbb{R}}$ be an $\mathbb{R}^p$-valued stochastic process with finite second moments. A process X is called *proper* if its distribution at time t is not concentrated on any proper subspace of $\mathbb{R}^p$ for each $t \in \mathbb{R} \setminus \{0\}$. We say X is *covariance operator self-similar* (cov.o.s.s.) if its distribution is proper and its covariance function satisfies

$$\mathrm{Cov}(X(cs), X(ct)) = c^H \mathrm{Cov}(X(s), X(t)) c^{H^*}, \quad s, t \in \mathbb{R}, \quad c > 0, \tag{1}$$

for some (Hurst) matrix H whose eigenvalues have real parts lying in the interval $(0, 1]$. In (1),

$$c^H := \exp\{H \log c\} = \sum_{k\in\mathbb{N}} \frac{(H \log c)^k}{k!},$$

and * denotes the (conjugate) transpose. In this paper, we construct *operator fractional Lévy motion* (ofLm), a broad class of generally non-Gaussian stochastic processes that are cov.o.s.s., have wide-sense stationary increments (namely, the mean and covariance of the increments do not change with time), and display infinitely divisible (ID) marginal distributions. The ofLm

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class subsumes, among others, the univariate fractional Brownian and Lévy motions (fBm and fLm, respectively), as well as the multivariate operator fractional Brownian motion (ofBm). The ofLm class can be divided into two types, namely, moving average (maofLm) and real harmonizable (rhofLm), both of which share the covariance structure of ofBm, under assumptions. We show that both maofLm and rhofLm admit stochastic integral representations in the time and Fourier domains, and establish their distinct small- and large-scale limiting behaviors. We characterize time-reversibility for ofLm, starting from a framework for the uniqueness of finite-second-moment, multivariate stochastic integral representations with respect to ID random measures. In particular, we show that, under non-Gaussianity, the parametric conditions for time-reversibility are more restrictive than those arising in the Gaussian case (ofBm) when the models are comparable.

The concept of self-similarity provides a mathematical underpinning for the modeling of *scale-invariance* in a wide range of natural and social systems, such as in critical phenomena (Sornette [64]), dendrochronology (Bai and Taqqu [4]), stock market prices (Willinger *et al.* [70]), and turbulence (Kolmogorov [34]). A univariate stochastic process X is called *self-similar* if it exhibits the scaling property

$$\{X(ct)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \left\{c^H X(t)\right\}_{t\in\mathbb{R}}, \quad c>0, \tag{2}$$

for some scalar parameter $H \in (0, 1]$, where $\overset{\text{f.d.d.}}{=}$ denotes the equality of finite-dimensional distributions. An example of a self-similar process is the celebrated fBm (Mandelbrot and Van Ness [43], Embrechts and Maejima [26], Pipiras and Taqqu [53]).

On the other hand, new technological developments have ushered in the modern era of 'big data' (Briody [16]). Many systems nowadays are monitored by several low-cost sensors and recording devices, leading to the storage of hundreds to several tens of thousands of time series. In *multivariate* or *high-dimensional* data, scaling behavior does not always appear along standard coordinate axes, and often involves multiple scaling relations. This situation is encountered in many applications, for example in climate studies (Isotta *et al.* [30]), hydrology (Benson *et al.* [11]), finance (Meerschaert and Scalas [49]), neuroscience (Ciuciu *et al.* [18]), and network traffic (Abry and Didier [1]).

A multivariate stochastic process X is called *operator self-similar* (o.s.s.) if it satisfies the relation (2) for some Hurst matrix H whose eigenvalues have real parts lying in the interval (0, 1] (Laha and Rohatgi [39], Hudson and Mason [29]). A canonical model for multivariate fractional systems is ofBm, namely, a Gaussian, o.s.s., stationary-increment stochastic process (Maejima and Mason [41], Mason and Xiao [48], Didier and Pipiras [24]). However, *non-Gaussian* behavior is pervasive in a myriad of natural phenomena and artificial systems. This includes features such as burstiness or heavy tails (Leland *et al.* [40], Paxson and Floyd [52], Willinger *et al.* [69], Boniece *et al.* [12]). Among non-Gaussian scale-invariant constructs, the mathematical generality and richness of fractional Lévy-type processes such as fLm have inspired a large body of work (Brockwell and Marquardt [15], Marquardt [44], Lacaux and Loubes [38], Bender and Marquardt [10], Basse and Pedersen [7], Tikanmäki and Mishura [67]). Fractional Lévy-type processes have also become popular in physical applications since they provide a broad family of second-order models displaying fractional covariance structure (Barndorff-Nielsen and Schmiegel [6], Suciu [65], Magdziarz and Weron [42], Zhang *et al.* [72], Xu *et al.* [71]). While of great importance in applications, the theory of their *multivariate* counterparts is a topic that has been relatively little explored in the literature (e.g., Marquardt [45], Barndorff-Nielsen and Stelzer [5], Moser and Stelzer [51]).

In this paper, we mathematically construct a broad class of (multivariate) cov.o.s.s., wide-sense stationary-increment stochastic processes with ID marginal distributions called *operator fractional Lévy motion* (ofLm). It comprises two subclasses of stochastic processes, framed in the time and frequency (Fourier) domains. In the latter, real harmonizable ofLm (rhofLm) is defined by means of a stochastic integral of the form

$$\big\{\widetilde{X}_H(t)\big\}_{t\in\mathbb{R}} \stackrel{\text{f.d.d.}}{=} \left\{\int_{\mathbb{R}} \left(\frac{e^{\mathbf{i}tx}-1}{\mathbf{i}x}\right)\left\{x_+^{H-(1/2)I}A + x_-^{H-(1/2)I}\overline{A}\right\}\widetilde{\mathcal{M}}(dx)\right\}_{t\in\mathbb{R}} \tag{3}$$

for some complex matrix A, where $\widetilde{\mathcal{M}}(dx)$ is a $\mathbb{C}^p$-valued ID random measure. In the time domain, under mild constraints, moving average ofLm (maofLm) admits the stochastic integral representation

$$\begin{aligned}\big\{X_H(t)\big\}_{t\in\mathbb{R}} \stackrel{\text{f.d.d.}}{=} \Bigg\{\int_{\mathbb{R}} \Big[\Big\{(t-s)_+^{H-(1/2)I} - (-s)_+^{H-(1/2)I}\Big\}M_+ \\ +\Big\{(t-s)_-^{H-(1/2)I} - (-s)_-^{H-(1/2)I}\Big\}M_-\Big]\mathcal{M}(ds)\Bigg\}_{t\in\mathbb{R}}\end{aligned} \tag{4}$$

for real matrices M_+, M_-, where $\mathcal{M}(ds)$ is an $\mathbb{R}^p$-valued ID random measure. In particular, when the random measures are Gaussian, (3) and (4) provide representations of the same stochastic process, namely, ofBm (Didier and Pipiras [23]). The random measures can be induced by multivariate Lévy processes (independent and stationary-increment processes), in which case they generalize Cramér–Wold representations based on Brownian noise (e.g., Doob [25], Rozanov [59]).

OfLm was first considered as a model, without proofs, in Boniece, Didier *et al.* [13] and Boniece, Wendt *et al.* [14]. In this paper, we broadly define ofLm and mathematically establish its fundamental properties such as finite-dimensional distributions and sample path behavior (Theorem 3.1). In particular, ofLm provides a flexible theoretical framework for the study of the effects of departures from non-Gaussianity in multivariate fractional constructs while keeping finite second moments. This can be seen, for instance, in natural alternative stochastic integral representations in the time and Fourier domains (Proposition 3.1; cf. Marquardt and Stelzer [46] on CARMA processes). Moreover, the study of scaling behavior lays bare some of the striking differences from the Gaussian case (cf. Benassi *et al.* [8, 9] on scalar random fields). On the one hand, non-Gaussian ofLm is shown to never be o.s.s. On the other hand, rhofLm and maofLm approach ofBm at short and long time scales, respectively (see Proposition 3.2). In addition, for certain choices of ID random measure (Lévy noise), rhofLm and maofLm approach o.s.s., operator-stable processes at long and short time scales, respectively (see Proposition 3.3).

Recall that a stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ is said to be *time-reversible* if

$$\{X(t)\}_{t\in\mathbb{R}} \stackrel{\text{f.d.d.}}{=} \{X(-t)\}_{t\in\mathbb{R}}. \tag{5}$$

Equivalently, $\{\pm 1\}$ are domain symmetries of X (Didier *et al.* [22]). All univariate, Gaussian stationary or stationary-increment stochastic processes are time-reversible. More generally, in the univariate context, confirmation of time irreversibility is relevant in both theory and modeling because it can be viewed, for example, as evidence of either non-Gaussianity or non-linearity (see Weiss [68], Cox [19, Section 3], Cheng [17], and De Gooijer [21, p. 315]; see also Jacod and Protter [31], Cox [20], and Rosenblatt [55, Chapter 1]). In particular, time-reversibility is well known to be a topic of central importance in physics (e.g., Kuśmierz

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et al. [37]). For an ofBm B_H—a multivariate, stationary-increment Gaussian process—time-reversibility is equivalent to the availability of the classical and convenient fBm-like covariance formula

$$\mathbb{E}B_H(s)B_H(t)^* = \frac{1}{2}\Big\{|s|^H\Sigma|s|^{H^*} + |t|^H\Sigma|t|^{H^*} - |t-s|^H\Sigma|t-s|^{H^*}\Big\}, \quad s, t \in \mathbb{R}, \tag{6}$$

where $\Sigma = \mathbb{E}B_H(1)B_H(1)^*$ (Didier and Pipiras [23], Proposition 5.2). In this paper, we provide parametric characterizations of time-reversibility for maofLm and rhofLm (Theorems 4.1 and 4.2). In particular, the results show that, under regularity assumptions, time-reversibility for ofLm requires parametric conditions that are strictly stronger than those for ofBm (see Examples 4.3 and 4.6). Characterizing time-reversibility involves starting from expressions of the form (3) and (4) and arriving at statements about integrands. In turn, this calls for results on the uniqueness of ID stochastic integrals that replace classical covariance Fourier-inversion-type results for the Gaussian case (as in Didier and Pipiras [23]). For this purpose, we draw upon the seminal work of Kabluchko and Stoev [32] (see also Maruyama [47], Rajput and Rosiński [54], Rosiński [56]) to analyze the uniqueness of finite-second-moment multivariate stochastic integral representations with respect to compensated Poisson random measures.

The paper is organized as follows. In Section 2, we lay out a mathematical setting for multivariate stochastic integrals with respect to finite-second-moment compensated Poisson random measures in both time and Fourier domains. In Section 3, we use the framework of Section 2 to construct rhofLm and maofLm, and establish their essential distributional, sample path, and scaling properties. In Section 4, we characterize time-reversibility for maofLm and rhofLm. All proofs, as well as auxiliary concepts and results, can be found in the appendix.

2. Preliminaries

Let $M(p, q, \mathbb{R})$ and $M(p, q, \mathbb{C})$ be, respectively, the spaces of $\mathbb{R}$- and $\mathbb{C}$-valued $p \times q$ matrices, $p, q \in \mathbb{N}$, and let $M(p, \mathbb{R}) = M(p, p, \mathbb{R})$ and $M(p, \mathbb{C}) = M(p, p, \mathbb{C})$. Also, let $GL(p, \mathbb{R})$ and $GL(p, \mathbb{C})$ denote the corresponding groups of nonsingular matrices on the fields $\mathbb{R}$ and $\mathbb{C}$, respectively. For $M \in M(p, \mathbb{C})$, $\text{eig}(M)$ denotes the set of possibly repeated eigenvalues (characteristic roots) of M, and $\Re\text{eig}(M)$ denotes the set of their (possibly repeated) real parts. Whenever convenient, given $M \in M(p, \mathbb{C})$, we write $\lambda_i(M)$, $i = 1, \dots, p$, for the (possibly repeated) eigenvalues of M, indexed by the ordering $\Re\lambda_1(M) \leq \dots \leq \Re\lambda_p(M)$. The symbol I denotes the identity matrix, and $\text{diag}(d_1, \dots, d_p)$ represents a diagonal matrix with main diagonal entries $d_1, \dots, d_p \in \mathbb{C}$. The symbol $\|\cdot\|$ denotes the Euclidean norm of a vector or the corresponding operator norm for a matrix. In the latter case, for a square matrix M, $\|M\|^2$ is given by the largest eigenvalue of M^*M or MM^*.

2.1. Stochastic integrals

In this section, we use compensated Poisson random measures associated with finite-second-moment Lévy measures to describe a framework for stochastic integration. This framework provides a multivariate generalization of the ones in Benassi *et al.* [8, 9] and Marquardt [44] (see also Marquardt [45]). The ultimate goal is to construct moving average and harmonizable classes of fractional stochastic processes (Section 3), so we consider stochastic integration in both frequency (Fourier) and time domains. In this section, we provide the definitions and expressions that are essential in the construction of ofLm. The Poisson random measures considered herein can be viewed as stemming from the jump measure of a

Lévy process (see, e.g., Sato [62], Chapter 4). More properties of stochastic integrals can be found in Section A.

In the proposed framework, the differences between integration in the Fourier and time domains lie in the Poisson random measure domain ($\mathbb{C}^p$ or $\mathbb{R}^p$, respectively) and in the classes of integrands considered. In the former case, we mainly consider Hermitian integrands, so as to ensure $\mathbb{R}^p$-valued stochastic integrals.

We first consider the Fourier domain. So, let $\mu_{\mathbb{C}^p}(d\mathbf{z}) \equiv \mu(d\mathbf{z})$ be a Lévy measure on $\mathcal{B}(\mathbb{C}^p)$ satisfying

$$\int_{\mathbb{C}^p} \mathbf{z}^*\mathbf{z}\mu(d\mathbf{z}) < \infty, \qquad \mu(\{\mathbf{0}\}) = 0. \tag{7}$$

Example 2.1. Let $\mu(d\mathbf{z}) = c\nu(d\mathbf{z})$, where $c > 0$ and $\nu(d\mathbf{z})$ is any probability measure on $\mathbb{C}^p$ with finite second moments satisfying $\nu(\{\mathbf{0}\}) = 0$. Then (7) is satisfied.

Example 2.2. Let $\alpha \in (0, 2)$. For some $c > 0$, define $\mu(d\mathbf{z}) = e^{-c\|\mathbf{z}\|} \big/ \|\mathbf{z}\|^{1+\alpha} d\mathbf{z}$. Then (7) is satisfied (the measure μ is an instance of a *tempered stable distribution*; see Rosiński [57] or Grabchak [27]).

Now, for $\mu(d\mathbf{z})$ as in (7), let

$$\widetilde{N}(dx, d\mathbf{z}) = N(dx, d\mathbf{z}) - \mathbb{E}N(dx, d\mathbf{z}) = N(dx, d\mathbf{z}) - dx\mu(d\mathbf{z}) \in \mathbb{R} \tag{8}$$

be a compensated Poisson random measure on $\mathcal{B}(\mathbb{R} \times \mathbb{C}^p)$ (see, for example, Sato [62], Section 19, or Applebaum [3], Section 2.3). We define the space of integration kernels

$$\begin{aligned}\mathcal{L}^2_{dx\otimes\mu(d\mathbf{z})} &\equiv \mathcal{L}^2_{dx\otimes\mu_{\mathbb{C}^p}(d\mathbf{z})} \equiv \mathcal{L}^2\big(\mathbb{R} \times \mathbb{C}^p, \mathcal{B}\big(\mathbb{R} \times \mathbb{C}^p\big), dx \otimes \mu_{\mathbb{C}^p}(d\mathbf{z})\big) \\ &= \Big\{\text{measurable } \varphi : \mathbb{R} \times \mathbb{C}^p \to \mathbb{C}^p : \|\varphi\|_{\mathcal{L}^2_{dx\otimes\mu_{\mathbb{C}^p}(d\mathbf{z})}} < \infty\Big\},\end{aligned}$$

where

$$\|\varphi\|^2_{\mathcal{L}^2_{dx\otimes\mu_{\mathbb{C}^p}(d\mathbf{z})}} \equiv \|\varphi\|^2_{\mathcal{L}^2_{dx\otimes\mu(d\mathbf{z})}} := \int_{\mathbb{R}} \int_{\mathbb{C}^p} \varphi(x, \mathbf{z})^*\varphi(x, \mathbf{z})\mu(d\mathbf{z})dx. \tag{9}$$

Fix the sets $B_{1,i} \times B_{2,i} \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{C}^p)$, as well as the vectors $\varphi_i \in \mathbb{C}^p$, $i = 1, \ldots, I$. Consider the elementary function $\varphi(x, \mathbf{z}) = \sum_{i=1}^{I} \varphi_i 1_{B_{1,i}\times B_{2,i}}(x, \mathbf{z})$. We define the stochastic integral of the elementary function φ with respect to the random measure $\widetilde{N}$ by means of the expression

$$\int_{\mathbb{R}\times\mathbb{C}^p} \varphi(x, \mathbf{z})\widetilde{N}(dx, d\mathbf{z}) := \sum_{i=1}^{I} \varphi_i\widetilde{N}\big(B_{1,i}, B_{2,i}\big).$$

Next, fix $\varphi \in \mathcal{L}^2_{dx\otimes\mu(d\mathbf{z})}$, and let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}^2_{dx\otimes\mu(d\mathbf{z})}$ be elementary functions converging to φ in the norm $\|\cdot\|_{\mathcal{L}^2_{dx\otimes\mu(d\mathbf{z})}}$. Then

$$\mathbb{C}^p \ni \int_{\mathbb{R}\times\mathbb{C}^p} \varphi(x, \mathbf{z})\widetilde{N}(dx, d\mathbf{z}) = L^2(\mathbb{P})\text{–}\lim_{n\to\infty} \int_{\mathbb{R}\times\mathbb{C}^p} \varphi_n(x, \mathbf{z})\widetilde{N}(dx, d\mathbf{z}) \tag{10}$$

is well defined as the *stochastic integral of the function φ with respect to the compensated Poisson random measure $\widetilde{N}$* (see Section A for details). In particular, the limit random vector does not depend on the chosen sequence of elementary functions. Now let

$$\widetilde{g} \in L^2\big(\mathbb{R}, M(p, \mathbb{C})\big) = \Big\{\text{measurable } \mathfrak{g} : \mathbb{R} \to M(p, \mathbb{C}): \int_{\mathbb{R}} \operatorname{tr}\big(\mathfrak{g}(x)\mathfrak{g}(x)^*\big)dx < \infty\Big\}. \tag{11}$$

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We also define the random measure $\widetilde{\mathcal{M}}(dx)$ on $\mathcal{B}(\mathbb{R})$ by means of the relation

$$\int_{\mathbb{R}} \widetilde{g}(x)\widetilde{\mathcal{M}}(dx) := \int_{\mathbb{R}\times\mathbb{C}^p} \{\widetilde{g}(x)\mathbf{z} + \widetilde{g}(-x)\overline{\mathbf{z}}\}\widetilde{N}(dx, d\mathbf{z}) \in \mathbb{C}^p. \tag{12}$$

In particular, for

$$\widetilde{g} \in L^2_{\text{Herm}}(\mathbb{R}) = \left\{\mathfrak{g} \in L^2\big(\mathbb{R}, M(p, \mathbb{C})\big) : \mathfrak{g}(-x) = \overline{\mathfrak{g}(x)}\right\}, \tag{13}$$

the expression (12) reduces to

$$\int_{\mathbb{R}} \widetilde{g}(x)\widetilde{\mathcal{M}}(dx) = \int_{\mathbb{R}\times\mathbb{C}^p} 2\Re\big(\widetilde{g}(x)\mathbf{z}\big)\widetilde{N}(dx, d\mathbf{z}) = 2\Re\bigg(\int_{\mathbb{R}\times\mathbb{C}^p} \widetilde{g}(x)\mathbf{z}\widetilde{N}(dx, d\mathbf{z})\bigg) \in \mathbb{R}^p. \tag{14}$$

So, let $\{\widetilde{g}_t\}_{t\in\mathbb{R}} \subseteq L^2_{\text{Herm}}(\mathbb{R})$. We can define the stochastic process $\widetilde{X} = \{\widetilde{X}(t)\}_{t\in\mathbb{R}}$ by means of the stochastic integral

$$\widetilde{X}(t) = \int_{\mathbb{R}} \widetilde{g}_t(x)\widetilde{\mathcal{M}}(dx), \qquad t \in \mathbb{R}. \tag{15}$$

Equivalently, based on the relation (14), we can re-express $\widetilde{X}$ as

$$\widetilde{X}(t) = \int_{\mathbb{R}\times\mathbb{C}^p} 2\Re\big(\widetilde{g}_t(x)\mathbf{z}\big)\widetilde{N}(dx, d\mathbf{z}) = 2\Re\bigg(\int_{\mathbb{R}\times\mathbb{C}^p} \widetilde{g}_t(x)\mathbf{z}\widetilde{N}(dx, d\mathbf{z})\bigg). \tag{16}$$

To construct the analogous time-domain framework, we start with the following definition. As in (8), we consider the compensated Poisson random measure

$$\widetilde{N}(ds, d\mathbf{z}) = N(ds, d\mathbf{z}) - \mathbb{E}N(ds, d\mathbf{z}) = N(ds, d\mathbf{z}) - ds\mu(d\mathbf{z}) \in \mathbb{R} \tag{17}$$

on $\mathcal{B}\big(\mathbb{R}^{p+1}\big)$, where $\mu(d\mathbf{z})$ is a Lévy measure on $\mathcal{B}\big(\mathbb{R}^p\big)$ satisfying

$$\int_{\mathbb{R}^p} \mathbf{z}^*\mathbf{z}\,\mu(d\mathbf{z}) < \infty, \qquad \mu(\{\mathbf{0}\}) = 0. \tag{18}$$

We naturally define the space of integration kernels $\mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}$ as in (9), where $\mathbb{R}^{p+1}$ replaces $\mathbb{R}\times\mathbb{C}^p$, $\mu(d\mathbf{z})$ is a Lévy measure on $\mathcal{B}(\mathbb{R}^p)$, and

$$\|\varphi\|^2_{\mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}}$$

is defined as in (9) with $\mathbb{R}^{p+1}$ replacing $\mathbb{R}\times\mathbb{C}^p$. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}$ be elementary functions converging to φ in the norm $\|\cdot\|_{\mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}}$. The stochastic integral

$$\int_{\mathbb{R}^{p+1}} \varphi(s, \mathbf{z})\widetilde{N}(ds, d\mathbf{z}) \in \mathbb{R}^p, \qquad \varphi \in \mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}, \tag{19}$$

is then naturally defined as in (10).

We further define the random measure $\mathcal{M}(ds)$ by means of the relation

$$\int_{\mathbb{R}} g(s)\mathcal{M}(ds) := \int_{\mathbb{R}\times\mathbb{R}^p} g(s)\mathbf{z}\widetilde{N}(ds, d\mathbf{z}) \in \mathbb{R}^p, \tag{20}$$

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where

$$g \in L^2\big(\mathbb{R}, M(p, \mathbb{R})\big) = \left\{\text{measurable } \mathfrak{g} : \mathbb{R} \to M(p, \mathbb{R}) \colon \int_{\mathbb{R}} \operatorname{tr}\big(\mathfrak{g}(s)\mathfrak{g}(s)^*\big)ds < \infty\right\} \tag{21}$$

$\big($in particular, $g(s)\mathbf{z} \in \mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}\big)$. Note that the space of integrands for the random measure $\mathcal{M}(ds)$ (i.e., (21)) is different from that for $\widetilde{\mathcal{M}}(dx)$ (i.e., (11)).

Analogously to the expression (15), for $\{g_t(s)\}_{t\in\mathbb{R}} \subseteq L^2\big(\mathbb{R}, M(p, \mathbb{R})\big)$, we can define the stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ by means of the stochastic integral

$$X(t) = \int_{\mathbb{R}} g_t(s)\mathcal{M}(ds).$$

Equivalently, based on the relation (20), we can re-express X as

$$\{X(t)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \left\{\int_{\mathbb{R}\times\mathbb{R}^p} g_t(s)\mathbf{z}\widetilde{N}(ds, d\mathbf{z})\right\}_{t\in\mathbb{R}}. \tag{22}$$

2.2. On integral representations of operator fractional Brownian motion

Recall that an ofBm is a Gaussian, o.s.s., stationary-increment stochastic process. Harmonizable representations are the natural starting point for the study of ofBm. This is so because, as briefly recalled in the introduction, almost every instance of ofBm admits the representation (3), where $\widetilde{\mathcal{M}}(dx) = \widetilde{B}(dx)$ is a p-variate complex Gaussian random measure satisfying $\widetilde{B}(-dx) = \overline{\widetilde{B}(dx)}$ almost surely (a.s.) and $\mathbb{E}\widetilde{B}(dx)\widetilde{B}(dx)^* = dx \times I$ (see Example A.1). So, for notational simplicity, define

$$D = H - (1/2)I. \tag{23}$$

Let

$$L^2_{\text{Herm}}(\mathbb{R}) \ni \widetilde{g}_t(x) = \frac{e^{\mathbf{i}tx} - 1}{\mathbf{i}x}\left\{x_+^{-D}A + x_-^{-D}\overline{A}\right\}, \quad x \neq 0, \tag{24}$$

be the integrand of the harmonizable representation of ofBm. Defining Fourier transforms entrywise, for

$$g_t(s) := \mathcal{F}^{-1}(\widetilde{g}_t)(s) \in L^2\big(\mathbb{R}, M(p, \mathbb{R})\big), \tag{25}$$

ofBm also admits a moving average representation of the form (4), where the Gaussian random measure $\mathcal{M}(ds) = B(ds)$ satisfies $\mathbb{E}B(ds)B(ds)^* = ds \times I$. For most cases of interest, we can explicitly recast the moving average representation of ofBm. In fact, we can set

$$g_t(s) = \begin{cases} \left\{(t-s)_+^D - (-s)_+^D\right\}M_+ + \left\{(t-s)_-^D - (-s)_-^D\right\}M_-, & \text{if } \Re\text{eig}(H) \subseteq (0, 1)\setminus\{1/2\}; \\ \{\operatorname{sign}(t-s) - \operatorname{sign}(-s)\}M + \log\left(\frac{|t-s|}{|s|}\right)N, & \text{if } H = (1/2)I, \end{cases} \tag{26}$$

for some matrix constants $M_+, M_- \in M(p, \mathbb{R})$ or $M, N \in M(p, \mathbb{R})$ (Didier and Pipiras [23], Theorem 3.2). For the instances

$$\Re\text{eig}(H) \subseteq (0, 1)\setminus\{1/2\}, \tag{27}$$

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the expression (26) can be extracted based on the fact that

$$\mathcal{F}\Big((t-s)_{\pm}^{D}-(-s)_{\pm}^{D}\Big)(x):=\int_{\mathbb{R}}e^{\mathbf{i}sx}\Big\{(t-s)_{\pm}^{D}-(-s)_{\pm}^{D}\Big\}ds$$
$$=\frac{e^{\mathbf{i}tx}-1}{\mathbf{i}x}|x|^{-D}\Gamma\big(D+I\big)e^{\mp\mathrm{sign}(x)\mathbf{i}\pi D/2} \qquad (28)$$

(see Proposition 3.1 and Theorem 3.2 in Didier and Pipiras [23], in particular the expressions (3.20), (3.24), and (3.25)). In (28), $\Gamma\big(D+I\big)$ is interpreted as a primary matrix function (Horn and Johnson [28], Sections 6.1 and 6.2). Further note that, when

$$\mathrm{eig}(H)\cap\big\{z\in\mathbb{C}:\Re z=1/2\big\}\neq\emptyset \quad \text{and} \quad H\neq(1/2)I,$$

moving average representations can be quite intricate (see Example 3.1 in Didier and Pipiras [23]).

3. Operator fractional Lévy motion

We are now in a position to define the ofLm class. For the sake of simplicity, hereinafter we focus on purely non-Gaussian constructs. We first define ofLm in the Fourier and time domains, and then establish its fundamental properties.

Definition 3.1. Let $H\in M(p,\mathbb{R})$ be a (Hurst) matrix whose eigenvalues satisfy

$$\Re\mathrm{eig}(H)\subseteq(0,1). \qquad (29)$$

Let $\widetilde{\mathcal{M}}(ds)$ be a $\mathbb{C}^p$-valued random measure as in (14) whose Lévy measure $\mu(d\mathbf{z})\equiv\mu_{\mathbb{C}^p}(d\mathbf{z})$ satisfies the condition (7). A *real harmonizable operator fractional Lévy motion* (rhofLm) without Gaussian component $\widetilde{X}_H=\big\{\widetilde{X}_H(t)\big\}_{t\in\mathbb{R}}$ is a stochastic process such that

(i)
$$\text{the distribution of } \widetilde{X}_H(t) \text{ is proper}, \quad t\neq 0; \qquad (30)$$

(ii) it satisfies the relation

$$\big\{\widetilde{X}_H(t)\big\}_{t\in\mathbb{R}}=\bigg\{\int_{\mathbb{R}}\widetilde{g}_t(x)\widetilde{\mathcal{M}}(dx)\bigg\}_{t\in\mathbb{R}}. \qquad (31)$$

In (31), the integrand is given by $\widetilde{g}_t(x)$ as in (24) for some matrix constant $A\in M(p,\mathbb{C})$.

We can now turn to the time domain.

Definition 3.2. Let $H\in M(p,\mathbb{R})$ be a (Hurst) matrix whose eigenvalues satisfy (29). Also let $\mathcal{M}(ds)$ be an $\mathbb{R}^p$-valued random measure as in (20) whose Lévy measure $\mu(d\mathbf{z})$ satisfies (18). A *moving average operator fractional Lévy motion* (maofLm) without Gaussian component $X_H=\{X_H(t)\}_{t\in\mathbb{R}}$ is an $\mathbb{R}^p$-valued stochastic process such that

(i)
$$\text{the distribution of } X_H(t) \text{ is proper}, \quad t\neq 0; \qquad (32)$$

(ii) it satisfies the relation

$$\big\{X_H(t)\big\}_{t\in\mathbb{R}}\overset{\text{f.d.d.}}{=}\bigg\{\int_{\mathbb{R}}g_t(s)\mathcal{M}(ds)\bigg\}_{t\in\mathbb{R}}. \qquad (33)$$

In (33), the integrand is given by $g_t(s)$ as in (25).

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Remark 3.1. When $p=1$, maofLm reduces to the classical fLm (e.g., Marquardt [44]).

Example 3.1. From (33) and (26), when $\Re\mathrm{eig}(H)\subseteq(0,1)\setminus\{1/2\}$, and $M_+=M_-=:M$, maofLm admits the well-balanced representation

$$\{X_H(t)\}_{t\in\mathbb{R}}=\left\{\int_{\mathbb{R}}\{|t-s|^D-|-s|^D\}M\mathcal{M}(ds)\right\}_{t\in\mathbb{R}} \tag{34}$$

(cf. Benassi *et al.* [9], Definition 2.1).

In the following proposition, we establish fundamental properties of both rhofLm and maofLm. Statement (vii) pertains to sample path properties, whereas all remaining statements pertain to existence, continuity, and distributional properties.

Theorem 3.1. *Let $H\in M(p,\mathbb{R})$ be a (Hurst) matrix satisfying (29). Also let $\widetilde{X}_H=\{\widetilde{X}_H(t)\}_{t\in\mathbb{R}}$ be a rhofLm as in (31) and let $X_H=\{X_H(t)\}_{t\in\mathbb{R}}$ be a maofLm as in (33). Then the following hold:*

(i) *For any $t\in\mathbb{R}$, $\widetilde{X}_H(t)$ and $X_H(t)$ are well defined.*

(ii) *$\widetilde{X}_H$ and X_H are stochastically continuous; namely, $\widetilde{X}_H(t)\overset{\mathbb{P}}{\to}\widetilde{X}_H(t_0)$ and $X_H(t)\overset{\mathbb{P}}{\to}X_H(t_0)$ whenever $t\to t_0\in\mathbb{R}$. In particular, they have measurable modifications.*

(iii) *For any $m\in\mathbb{N}$, any $\mathbf{u}_1,\ldots,\mathbf{u}_m\in\mathbb{R}^p$, and any $t_1<\ldots<t_m$, the characteristic function of the finite-dimensional distributions of $\widetilde{X}_H$ is given by*

$$\mathbb{E}\exp\left\{\mathbf{i}\sum_{j=1}^{m}\langle\mathbf{u}_j,\widetilde{X}_H(t_j)\rangle\right\}$$

$$=\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{C}^p}\left(e^{\mathbf{i}2\sum_{j=1}^m\mathbf{u}_j^*\Re(\widetilde{g}_{t_j}(x)\mathbf{z})}-1-\mathbf{i}2\sum_{j=1}^{m}\mathbf{u}_j^*\Re(\widetilde{g}_{t_j}(x)\mathbf{z})\right)\mu_{\mathbb{C}^p}(d\mathbf{z})dx\right\} \tag{35}$$

$$=\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^{2p}}\left[e^{\mathbf{i}2\sum_{j=1}^m\mathbf{u}_j^*\left(\Re\widetilde{g}_{t_j}(x)\mathbf{z}_1-\Im\widetilde{g}_{t_j}(x)\mathbf{z}_2\right)}-1\right.\right.$$

$$\left.\left.-\mathbf{i}2\sum_{j=1}^{m}\mathbf{u}_j^*\left(\Re\widetilde{g}_{t_j}(x)\mathbf{z}_1-\Im\widetilde{g}_{t_j}(x)\mathbf{z}_2\right)\right]\mu_{\mathbb{R}^{2p}}(d\mathbf{z})dx\right\}, \tag{36}$$

where

$$\mu_{\mathbb{C}^p}\equiv\mu_{\mathbb{R}^{2p}}\text{ and }\mathbf{z}\equiv\begin{pmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{pmatrix}\in\mathbb{R}^p\times\mathbb{R}^p. \tag{37}$$

Moreover, the characteristic function of the finite-dimensional distributions of X_H is given by

$$\mathbb{E}\exp\left\{\mathbf{i}\sum_{j=1}^{m}\langle\mathbf{u}_j,X_H(t_j)\rangle\right\}$$

$$=\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^p}\left(e^{\mathbf{i}\sum_{j=1}^m\mathbf{u}_j^*g_{t_j}(s)\mathbf{z}}-1-\mathbf{i}\sum_{j=1}^{m}\mathbf{u}_j^*g_{t_j}(s)\mathbf{z}\right)\mu(d\mathbf{z})ds\right\}. \tag{38}$$

In particular, $\widetilde{X}_H$ and X_H have mean zero and are cov.o.s.s.

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(iv) *If* $\int_{\mathbb{R}} \mathbf{z}\mathbf{z}^* \mu(d\mathbf{z})$ *has full rank,* X_H *has the same covariance function as an ofBm with Hurst matrix* QHQ^{-1} *and parameter* QA *in its harmonizable representation, where* $Q = \big(\int_{\mathbb{R}} \mathbf{z}\mathbf{z}^* \mu(d\mathbf{z})\big)^{1/2}$*. In particular, if*

$$\int_{\mathbb{R}^p} \mathbf{z}\mathbf{z}^* \mu(d\mathbf{z}) = I \tag{39}$$

and the condition (27) *holds, then* $\mathbb{E}X_H(s)X_H(t)^*$*,* $s, t, \in \mathbb{R}$*, is the covariance function of an ofBm whose time-domain representation has parameters* H*,* M_+*, and* M_- *(see* (26)*). Also, if* $\int_{\mathbb{C}^p} \Re\mathbf{z}\Re\mathbf{z}^* \mu(d\mathbf{z}) = \int_{\mathbb{C}^p} \Im\mathbf{z}\Im\mathbf{z}^* \mu(d\mathbf{z})$ *has full rank, then* $\widetilde{X}_H$ *has the same covariance function as an ofBm with Hurst matrix* $\widetilde{Q}H\widetilde{Q}^{-1}$ *and parameter* $\widetilde{Q}A$ *in its harmonizable representation, where* $\widetilde{Q} = \big(4\int_{\mathbb{C}^p} \Re\mathbf{z}\Re\mathbf{z}^* \mu(d\mathbf{z})\big)^{1/2}$*. In particular, if*

$$4\int_{\mathbb{C}^p} (\Re\mathbf{z})(\Re\mathbf{z})^* \mu(d\mathbf{z}) = I = 4\int_{\mathbb{C}^p} (\Im\mathbf{z})(\Im\mathbf{z})^* \mu(d\mathbf{z}), \tag{40}$$

then $\mathbb{E}\widetilde{X}_H(s)\widetilde{X}_H(t)^*$*,* $s, t, \in \mathbb{R}$*, is the covariance function of an ofBm whose harmonizable representation has parameters* H *and* A *(see* (24)*).*

(v) X_H *has strict-sense stationary increments and* $\widetilde{X}_H$ *has wide-sense stationary increments. If*

$$\mu_{\mathbb{C}^p}(d\mathbf{z}) = \mu_{\mathbb{C}^p}\big(e^{\mathbf{i}\theta} d\mathbf{z}\big), \quad \theta \in [-\pi, \pi) \tag{41}$$

(i.e., $\widetilde{\mathcal{M}}(dx) \stackrel{d}{=} e^{\mathbf{i}hx}\widetilde{\mathcal{M}}(dx)$*,* $h \in \mathbb{R}$*), then* $\widetilde{X}_H$ *also has strict-sense stationary increments.*

(vi) *Let* X_H *be a maofLm whose Hurst matrix* H *satisfies the condition* (27)*. Then* X_H *is not o.s.s. Also, let* $\widetilde{X}_H$ *be a rhofLm. Then* $\widetilde{X}_H$ *is not o.s.s.*

(vii) *Suppose the additional constraint* $\Re\mathrm{eig}(H) \subseteq (1/2, 1)$ *is in place. Then, for every* $\gamma \in (0, \min \Re\mathrm{eig}(H) - 1/2)$*, there exists a modification of maofLm/rhofLm that is a.s.* γ*-Hölder continuous.*

Example 3.2. A simple example of a Lévy measure satisfying (40) is given by

$$\mu_{\mathbb{C}^p}(d\mathbf{z}) = \sum_{k=1}^{p} \delta_{(1+\mathbf{i})e_k}(d\mathbf{z}),$$

where $e_k \in \mathbb{R}^p$, $k = 1, \ldots, p$, are the first p canonical vectors.

Remark 3.2. The properness condition notwithstanding, the integrands in the stochastic integral representations of ofLm can be rank-deficient almost everywhere (a.e.) (see Lemma E.1).

Remark 3.3. Under conditions, the second-order structures of ofLm and ofBm are identical. Therefore, the parametrization of the second-order structure of ofLm is not identifiable (Didier and Pipiras [24]). Characterizing the (non-)identifiability of the parametrization of ofLm—namely, in regard to its finite-dimensional distributions—is a topic for future work.

Recall that, in the Gaussian case (ofBm), harmonizable and moving average stochastic integrals are representations of the same stochastic process (see Section 2.2). Equivalently, they have the same covariance structure. As established in Theorem 3.1, under assumptions on the Lévy measure, rhofLm and maofLm share the covariance structure of ofBm. Nevertheless,

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they are rather distinct from ofBm. We shed light on such differences in the next three propositions. In Proposition 3.1, we provide natural alternative stochastic integral representations of rhofLm and maofLm in the time and Fourier domains, respectively. The representations (i.e., (42) and (43)) are formally similar to (31) and (33), respectively. However, the random measures involved in each expression do *not* satisfy the conditions stated in Definitions 3.1 and 3.2. In particular, the random measures generally display *orthogonal* but *dependent* increments. On the other hand, even though ofLm is never o.s.s., in Proposition 3.2 we establish that (rescaled) maofLm and rhofLm converge to an ofBm over different time ranges, i.e., in the large and small scaling limits, respectively. Remarkably, in Proposition 3.3 we further show that maofLm and rhofLm may display operator self-similarity in the other limit directions; namely, maofLm can be o.s.s. in the *small*-scale limit, whereas rhofLm can be o.s.s. in the *large*-scale limit. However, such limits may display heavy-tailed marginal distributions; i.e., they are not ofBms.

We begin by establishing natural alternative stochastic integral representations of rhofLm and maofLm. These representations are based on random measures with uncorrelated increments; for the reader's convenience they are given explicitly in Proposition E.1.

Proposition 3.1. *Fix a matrix $H \in M(p, \mathbb{R})$ whose eigenvalues satisfy (27). Let $\widetilde{X}_H = \{\widetilde{X}_H(t)\}_{t\in\mathbb{R}}$ be a rhofLm as in (31) under the conditions (40) and (41) on the associated random measure. Also let $X_H = \{X_H(t)\}_{t\in\mathbb{R}}$ be a maofLm as in (33) under the condition (39) on the associated random measure.*

(*i*) *Then, for $\widetilde{g}_t$ as in (24), X_H admits the representation*

$$\{X_H(t)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \left\{\int_{\mathbb{R}} \widetilde{g}_t(x)\Phi_{\mathcal{M}}(dx)\right\}_{t\in\mathbb{R}}, \tag{42}$$

where $\Phi_{\mathcal{M}}(dx)$ is a $\mathbb{C}^p$-valued, zero-mean orthogonal-increment random measure such that $\mathbb{E}\Phi_{\mathcal{M}}(dx)\Phi_{\mathcal{M}}(dx)^* = dx \times I$.

(*ii*) *Furthermore, for g_t as in (25), $\widetilde{X}_H$ admits the representation*

$$\{\widetilde{X}_H(t)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \left\{\int_{\mathbb{R}} g_t(s)\Phi_{\widetilde{\mathcal{M}}}(ds)\right\}_{t\in\mathbb{R}}, \tag{43}$$

where $\Phi_{\widetilde{\mathcal{M}}}(ds)$ is an $\mathbb{R}^p$-valued, zero-mean orthogonal-increment random measure such that $\mathbb{E}\Phi_{\widetilde{\mathcal{M}}}(ds)\Phi_{\widetilde{\mathcal{M}}}(ds)^* = ds \times I$.

Example 3.3. Suppose $M_- = 0$ and

$$\Re\text{eig}(H) \cap \{1/2\} = \emptyset. \tag{44}$$

Let $A = \Gamma(D+I)e^{-i\pi D/2}$. Then, by the expression (28), we can recast the representation (42) in the form

$$\{X_H(t)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \left\{\int_{\mathbb{R}} \frac{e^{\mathbf{i}tx}-1}{\mathbf{i}x}\left\{x_+^{-D}A + x_-^{-D}\overline{A}\right\}\Phi_{\mathcal{M}}(dx)\right\}_{t\in\mathbb{R}}.$$

On the other hand, for this same choice of the parameter A and still assuming the condition (44) holds, again by the expression (28) we can rewrite (43) as

$$\{\widetilde{X}_H(t)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \left\{\int_{\mathbb{R}} \left\{(t-s)_+^D - (-s)_+^D\right\}\Phi_{\widetilde{\mathcal{M}}}(ds)\right\}_{t\in\mathbb{R}}.$$

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In Proposition 3.2, we establish the large- and small-scale behaviors of maofLm and rhofLm, respectively. In the statement of the proposition, $\stackrel{\text{f.d.d.}}{\to}$ denotes the convergence of finite-dimensional distributions.

Proposition 3.2. *Let H be a (Hurst) matrix whose eigenvalues satisfy the condition (29).*

(*i*) *Let $X = \{X_H(t)\}_{t\in\mathbb{R}}$ be a maofLm with Hurst exponent H. Suppose its associated Lévy measure satisfies the condition (39). Then*

$$\left\{c^{-H}X_H(ct)\right\}_{t\in\mathbb{R}} \stackrel{\text{f.d.d.}}{\to} \{B_H(t)\}_{t\in\mathbb{R}}, \quad c\to\infty,$$

where B_H is an ofBm with Hurst exponent H.

(*ii*) *Let $\widetilde{X}_H = \left\{\widetilde{X}_H(t)\right\}_{t\in\mathbb{R}}$ be a rhofLm with exponent H. Suppose its associated Lévy measure satisfies the condition (40). Then, for every fixed $s\in\mathbb{R}$,*

$$\left\{\varepsilon^{-H}\left(\widetilde{X}_H(s+\varepsilon t)-\widetilde{X}_H(s)\right)\right\}_{t\in\mathbb{R}} \stackrel{\text{f.d.d.}}{\to} \{B_H(t)\}_{t\in\mathbb{R}}, \quad \varepsilon\to 0^+,$$

where B_H is an ofBm with Hurst exponent H.

In Proposition 3.3, we show that some maofLm and rhofLm instances are o.s.s. in the small- and large-scale limits, respectively—in both cases, with a different matrix scaling exponent. This occurs when the associated random measures $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are chosen to be 'locally' operator-stable, in the sense that their Lévy measures around $\mathbf{0}$ behave like that of an operator-stable Lévy process. These limiting processes, in turn, are instances of operator-stable o.s.s. processes recently studied in Kremer and Scheffler [36]. For the reader's convenience, the precise definition and more details about such measures and associated independently scattered random measures are provided in Section F.

Recall that for any $M\in M(p,\mathbb{R})$, $\lambda_i(M)$ denotes the ith eigenvalue of M in the ordering $\Re\lambda_1(M)\leq\ldots\leq\Re\lambda_p(M)$, where an arbitrary ordering of eigenvalues is adopted in case real parts are equal.

Proposition 3.3. *Let $B\in M(p,\mathbb{R})$ be such that $\Re\text{eig}(B)\subseteq(1/2,1)$.*

(*i*) *Let $X_H = \{X_H(t)\}_{t\in\mathbb{R}}$ be a maofLm under (27), and suppose its associated Lévy measure is given by $\mu_{B,q}$ as in (148). Further suppose that $HB = BH$, and that $\Re\lambda_p(H-(1/2)I)+\Re\lambda_p(B)<1$.*

$$\widetilde{H}_1 = H + \left(B-(1/2)I\right) \tag{45}$$

in the sense that, for every fixed $s\in\mathbb{R}$,

$$\left\{\varepsilon^{-\widetilde{H}_1}\left(X_H(s+\varepsilon t)-X_H(s)\right)\right\}_{t\in\mathbb{R}} \stackrel{\text{f.d.d.}}{\to} \left\{\Theta_{\widetilde{H}_1,B}(t)\right\}_{t\in\mathbb{R}}, \quad \varepsilon\to 0^+. \tag{46}$$

In (46), $\Theta_{\widetilde{H}_1,B}(t)$ is an $\widetilde{H}_1$-o.s.s. process with representation

$$\Theta_{\widetilde{H}_1,B}(t) = \int_{\mathbb{R}} g_t(s)L_B(ds), \tag{47}$$

where $g_t(s)$ is as in (25), and L_B is an $\mathbb{R}^p$-valued independently scattered ID random measure generated by a full operator-stable random measure with exponent B as in (146).

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(*ii*) *Let* $\widetilde{X}_H = \{\widetilde{X}_H(t)\}_{t\in\mathbb{R}}$ *be a rhofLm, and suppose its associated Lévy measure* $\mu_{\mathbb{R}^{2p}}$ *in the identification* (37) *is given by* $\mu_{\widetilde{B},q}$ *as in* (148)*, where* $\widetilde{B} = B \oplus B$*. Further suppose that* H *and* A *commute with* B*, and that* $\Re\lambda_1(H) + \big(\frac{1}{2} - \Re\lambda_p(B)\big) > 0$ *and* $\Re\lambda_p(H) + \big(\frac{1}{2} - \Re\lambda_1(B)\big) < 1$*. Then* $\widetilde{X}_H$ *is asymptotically o.s.s. with exponent*

$$\widetilde{H}_2 = H + ((1/2)I - B) \tag{48}$$

in the sense that

$$\Big\{c^{-\widetilde{H}_2}\widetilde{X}_H(ct)\Big\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{\to} \Big\{\Theta'_{\widetilde{H}_2,B}(t)\Big\}_{t\in\mathbb{R}}, \quad c\to\infty. \tag{49}$$

In (49)*,* $\Theta'_{\widetilde{H}_2,B}$ *is an* $\widetilde{H}_2$*-o.s.s. process with representation*

$$\Theta'_{\widetilde{H}_2,B}(t) = 2\Re\bigg(\int_{\mathbb{R}} \widetilde{g}_t(x)\widetilde{L}_B(dx)\bigg), \tag{50}$$

where $\widetilde{g}_t(x)$ *is as in (24), and* $\widetilde{L}_B(dx)$ *is a* $\mathbb{C}^p$*-valued ID independently scattered random measure generated by a full operator-stable random measure with exponent* $\widetilde{B}$ *as in* (146).

4. Time-reversibility

Recall that a stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ is said to be time-reversible if $\{X(-t)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \{X(t)\}_{t\in\mathbb{R}}$. In this section, we provide characterizations of time-reversibility for maofLm and rhofLm under mild assumptions. In the characterizations, the true difficulty lies in establishing *necessary* conditions, i.e., in what the assumption of time-reversibility implies about the parametric representations of maofLm and rhofLm. The proofs require results on the *uniqueness* of multivariate stochastic integral representations, which are developed in Section D. To provide these uniqueness results, we adapt the fundamental framework constructed in Sections 2.1 and 2.2 of Kabluchko and Stoev [32] (see also Maruyama [47], Chapter 3 of Samorodnitsky [60], and Rosiński [58]).

Example 4.1. If a maofLm X_H is time-reversible and satisfies (39), then its covariance function is given by the fBm-like formula (6) with $\Sigma = \mathbb{E}X_H(1)X_H(1)^*$. If a rhofLm $\widetilde{X}_H$ is time-reversible and satisfies (40), then its covariance function is also given by the formula (6) with $\Sigma = \mathbb{E}\widetilde{X}_H(1)\widetilde{X}_H(1)^*$. In general, an explicit formula for the covariance function of ofLm is not available. In fact, in the Gaussian case, the expression (40) is equivalent to time-reversibility (see Didier and Pipiras [23], Proposition 5.2).

To investigate time-reversibility in the framework of ofLm, it is convenient to slightly generalize the notation. Simply put, the new argument ω stands for either the Fourier or the time argument x or s. In turn, the vector $\boldsymbol{\varpi} = (\omega, \mathbf{z})$ includes both $\omega\in\mathbb{R}$ and the Lévy measure argument $\mathbf{z}\in\mathbb{R}^q$, where either $q = p$ or $q = 2p$. So, more precisely, let $\overline{\Omega} = \mathbb{R}\times\mathbb{R}^q$, $\mathcal{B} = \mathcal{B}(\overline{\Omega})$ (cf. the expression (136)). Let

$$\kappa(d\boldsymbol{\varpi}) = d\omega\otimes\mu(d\mathbf{z}), \tag{51}$$

where $\mu(d\mathbf{z})$ is a Lévy measure satisfying (18). Whenever convenient, we write $\eta(d\omega)\equiv d\omega$. Also define

$$\mathcal{L}^2_{\kappa(d\boldsymbol{\varpi})}(\overline{\Omega}) = \mathcal{L}^2_{d\omega\otimes\mu(d\mathbf{z})}(\mathbb{R}\times\mathbb{R}^q)$$
$$= \bigg\{\varphi:\mathbb{R}\times\mathbb{R}^q\to\mathbb{R}^p : \int_{\mathbb{R}}\int_{\mathbb{R}^q}\varphi(\omega,\mathbf{z})^*\varphi(\omega,\mathbf{z})\mu(d\mathbf{z})d\omega < \infty\bigg\} \tag{52}$$

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(cf. the expression (9)). We then express the compensated Poisson random measure on $\mathcal{B}(\overline{\Omega})$ as

$$\mathbb{R} \ni \widetilde{N}(d\boldsymbol{\varpi}) \equiv N(d\boldsymbol{\varpi}) - \kappa(d\boldsymbol{\varpi}) \equiv \widetilde{N}(d\omega, d\mathbf{z}), \tag{53}$$

where $N(d\boldsymbol{\varpi}) \equiv N(d\omega, d\mathbf{z})$ is a Poisson random measure (cf. (8) and (17)). Let $f_t(\boldsymbol{\varpi})$ and $\mathfrak{g}_t(\omega)$, $t \in \mathbb{R}$, be two families of $\mathbb{R}^p$- and $M(p, q, \mathbb{R})$-valued functions, respectively, where

$$\{f_t(\boldsymbol{\varpi})\}_{t\in\mathbb{R}} := \{\mathfrak{g}_t(\omega)\mathbf{z}\}_{t\in\mathbb{R}} \subseteq \mathcal{L}^2_{\kappa(d\boldsymbol{\varpi})}(\overline{\Omega}). \tag{54}$$

Example 4.2. For (38), we can write (54) with

$$\omega = s, \quad \mathfrak{g}_t(s) = g_t(s) \in M(p, \mathbb{R}), \quad q = p, \quad \text{and} \quad \mu(d\mathbf{z}) \text{ as in (38)}.$$

For (36), we can re-express (54) with

$$\omega = x, \quad \mathfrak{g}_t(x) = \big(\Re\widetilde{g}_t(x), \Im\widetilde{g}_t(x)\big) \in M(p, 2p, \mathbb{R}), \quad q = 2p, \quad \text{and} \quad \mu(d\mathbf{z}) = \big(\mu_{\mathbb{R}^{2p}} \circ \varsigma^{-1}\big)(d\mathbf{z}),$$

where $\varsigma(\mathbf{z}) = (2\Re\mathbf{z}, -2\Im\mathbf{z})$.

The main results in this section require some notion of *minimal* (stochastic integral) representation. In the following definition, we revisit the notion of minimality as put forward in Kabluchko and Stoev [32].

Definition 4.1. Let $T \subseteq \mathbb{R}$, and consider the $\mathbb{R}^p$-valued stochastic process $X = \{X(t)\}_{t\in T}$ given by the stochastic integral representation

$$X(t) = \int_{\overline{\Omega}} f_t(\boldsymbol{\varpi})\widetilde{N}(d\boldsymbol{\varpi}) = \int_{\mathbb{R}\times\mathbb{R}^q} \mathfrak{g}_t(\omega)\mathbf{z}\widetilde{N}(d\omega, d\mathbf{z}), \quad t \in T. \tag{55}$$

We say $\{f_t\}_{t\in T}$ is a *minimal representation* of the ID stochastic process X *with respect to* $\mathcal{B}$ mod κ if the following two conditions hold:

(i) $\sigma(\{f_t\}_{t\in T}) = \mathcal{B} \mod \kappa$, i.e., for every $B \in \mathcal{B}$, there exists $A \in \sigma(\{f_t\}_{t\in T})$ such that $\kappa(A \Delta B) = 0$; and

(ii) there is no $B \in \mathcal{B}$ such that $\kappa(B) > 0$ and, for every $t \in \mathbb{R}$, $f_t \equiv 0$ a.e. on B.

In the following theorem, we characterize time-reversibility for maofLm. For comments on the minimality assumption, see Remark 4.1.

Theorem 4.1. *Let H be a (Hurst) matrix whose eigenvalues satisfy* (27). *Let $X_H = \{X_H(t)\}_{t\in\mathbb{R}}$ be a maofLm with Hurst matrix H. Further assume that*

$$M_+, M_- \in GL(p, \mathbb{R}) \tag{56}$$

and $\{f_t(\boldsymbol{\varpi}), t \in \mathbb{R}\} = \{g_t(s)\mathbf{z}, t \in \mathbb{R}\}$ is a minimal representation of X_H with respect to $\mathcal{B}\big(\mathbb{R}^{p+1}\big)$ mod κ, *where $\kappa(d\boldsymbol{\varpi}) = ds \otimes \mu(d\mathbf{z})$ and $\mu(d\mathbf{z})$ is as in* (38). *Then the following conditions are equivalent:*

(*i*) *X_H is time-reversible.*

(*ii*) *The following two conditions hold:*

(*a*) *$(M_-^{-1}M_+)|_{\text{supp}(\mu)}$ is an involution; i.e.,*

$$M_-^{-1}M_+\mathbf{z} = M_+^{-1}M_-\mathbf{z} \quad \mu(d\mathbf{z})\text{-a.e.} \tag{57}$$

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(*b*) *The map* $\mathbf{z} \mapsto M_+^{-1} M_- \mathbf{z}$ *preserves the measure* μ*; i.e.,*

$$\mu\big((M_-^{-1} M_+) d\mathbf{z}\big) = \mu(d\mathbf{z}). \tag{58}$$

In (ii), the condition (a) can be replaced by the following:

(a')

$$g_{-t}(s)\mathbf{z} = g_t(-s) M_-^{-1} M_+ \mathbf{z} = g_t(-s) M_+^{-1} M_- \mathbf{z} \quad \kappa(ds, d\mathbf{z})\text{-a.e.}$$

Example 4.3. It is illustrative to compare the conditions for time-reversibility for ofBm and maofLm. In the case of the former, time-reversibility (i.e., the expression (6)) is equivalent, in the time domain, to the parametric condition

$$\cos\left(\frac{\pi D}{2}\right)\big(M_+ + M_-\big)\big(M_+^* - M_-^*\big) \sin\left(\frac{\pi D^*}{2}\right)$$
$$= \sin\left(\frac{\pi D}{2}\right)\big(M_+ - M_-\big)\big(M_+^* + M_-^*\big) \cos\left(\frac{\pi D^*}{2}\right) \tag{59}$$

(Didier and Pipiras [23], Corollary 5.1), where $D = H - (1/2)I$ and the cosine and sine of matrices are interpreted in the sense of primary matrix functions (Horn and Johnson [28]).

So, suppose the conditions used in Theorem 4.1 hold; namely, suppose that (27) and (56) are satisfied for a minimal representation, and also that X_H is time-reversible. In addition, assume the Lévy measure μ satisfies the second moment condition (39)—otherwise, ofBm and maofLm have incompatible parametrizations (cf. Theorem 3(iv)). Based on a change of variable $\mathbf{w} = M_+^{-1} M_- \mathbf{z}$, using (58), we have

$$\big(M_+^{-1} M_-\big)\left(\int_{\mathbb{R}^p} \mathbf{z}\mathbf{z}^* \mu(d\mathbf{z})\right)\left(M_-^* \big(M_+^*\big)^{-1}\right) = \int_{\mathbb{R}^p} \mathbf{w}\mathbf{w}^* \mu(d\mathbf{w}) = I.$$

Hence, $M_+ M_+^* = M_- M_-^*$. Also, under (39), (57) implies $M_-^{-1} M_+ = M_+^{-1} M_-$. As a consequence,

$$\big(M_+ + M_-\big)\big(M_+^* - M_-^*\big) = \mathbf{0} = -\big(M_- + M_+\big)\big(M_+^* - M_-^*\big),$$

which in turn implies the condition (59). Conversely, we may pick M_+, M_- satisfying (59) but for which $\big(M_+ + M_-\big)\big(M_+^* - M_-^*\big) \neq \mathbf{0}$. In other words, among the instances of maofLm satisfying (18), the conditions for time-reversibility of maofLm as established in Theorem 4.1 are more stringent than those for the time-reversibility of ofBm.

Example 4.4. Let X_H be a time-reversible maofLm. Assume $\Sigma := \int_{\mathbb{R}^p} \mathbf{z}\mathbf{z}^* \mu(d\mathbf{z})$ has full rank. A simple calculation shows that there exists a symmetric orthogonal matrix O such that

$$M_+^{-1} M_- = M_-^{-1} M_+ = \Sigma^{1/2} O \Sigma^{-1/2} \tag{60}$$

(Lemma E.2). In other words, (60) is a necessary condition for time-reversibility. In light of Theorem 4.1, this implies the following:

(i) If $p = 1$, X_H is time-reversible if and only if either (a) $M_+ = M_-$; or (b) $M_+ = -M_-$ and μ is symmetric (i.e., $\mu(-dz) = \mu(dz)$). (Indeed, the relation (60) implies necessity, whereas sufficiency follows from both conditions (58) and (57)).

(ii) If μ is a full, zero-mean Gaussian measure on $\mathbb{R}^p$, then it is characterized by the matrix Σ. Therefore, X_H is time-reversible if and only if (60) holds.

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Turning to the Fourier domain, let

$$\widetilde{X}_H = \{\widetilde{X}_H(t)\}_{t\in\mathbb{R}} \tag{61}$$

be a rhofLm with kernel $f_t(\boldsymbol{\varpi}) = \widetilde{g}_t(x)\mathbf{z}$ and measure (51) given by $\kappa(d\boldsymbol{\varpi}) = dx \otimes \mu(d\mathbf{z}) \equiv \mu(d\mathbf{z}) \otimes dx$. Note that $\widetilde{X}_H$ can also be represented based on the measure

$$\widetilde{\kappa}(d\boldsymbol{\varpi}) = dx \otimes \widetilde{\mu}(d\mathbf{z}), \tag{62}$$

where

$$\widetilde{\mu}(d\mathbf{z}) = \frac{\mu(d\mathbf{z}) + \mu(\overline{d\mathbf{z}})}{2} \tag{63}$$

(see Lemma E.3). This fact is used in the following theorem, where we characterize time-reversibility for rhofLm. For comments on the minimality assumption, see Remark 4.1.

Theorem 4.2. *Let H be a (Hurst) matrix satisfying (29). Let $\widetilde{X}_H = \{\widetilde{X}_H(t)\}_{t\in\mathbb{R}}$ be a rhofLm with Hurst matrix H. Assume*

$$\{f_t(\boldsymbol{\varpi})\}_{t\in\mathbb{R}} = \{\widetilde{g}_t(x)\mathbf{z}\}_{t\in\mathbb{R}} \text{ is a minimal representation}$$
$$\text{of } \{\widetilde{X}_H(t)\}_{t\in\mathbb{R}} \text{ with respect to } \mathcal{B}(\mathbb{R}\times\mathbb{R}^{2p}) \quad \text{mod } \widetilde{\kappa}, \tag{64}$$

where $\widetilde{\kappa}$ is a measure of the form (62) and we identify $\mu_{\mathbb{C}^p} \equiv \mu_{\mathbb{R}^{2p}}$ (see (37)). Further assume

$$A, \overline{A} \in GL(p, \mathbb{C}). \tag{65}$$

Then $\widetilde{X}_H$ is time-reversible if and only if the map $\mathbf{z} \mapsto -A^{-1}\overline{A}\mathbf{z}$ preserves the measure $\widetilde{\mu}$ as in (62), i.e.,

$$\widetilde{\mu}\Big(-\overline{A}^{-1}Ad\mathbf{z}\Big) = \widetilde{\mu}(d\mathbf{z}). \tag{66}$$

Example 4.5. Under the assumptions of Theorem 4.2, a sufficient condition for $\widetilde{X}_H$ to be time-reversible is that $\Re(A) = \mathbf{0}$. In fact, in this case, $\overline{A}^{-1}A = -I$; hence, the condition (66) holds.

Example 4.6. As in Example 4.3, we now compare the conditions for time-reversibility of ofBm to those for rhofLm. For the former, time-reversibility (i.e., the expression (6)) is equivalent, in the Fourier domain, to the parametric condition

$$AA^* = \overline{AA^*} \tag{67}$$

(Didier and Pipiras [23], Theorem 5.1).

So, suppose the conditions used in Theorem 4.2 hold; namely, suppose the conditions (29), (65), and (64) are satisfied. In addition, assume the Lévy measure μ satisfies the second moment condition (40), so as to ensure that ofBm and rhofLm have the same covariance structure and compatible parametrizations (cf. Theorem 3.1(iv)). Then, for $\mathbf{z} = \mathbf{z}_1 + \mathbf{i}\mathbf{z}_2$,

$$\int_{\mathbb{C}^p} \mathbf{z}\mathbf{z}^*\widetilde{\mu}(d\mathbf{z}) = (1/2)I + \mathbf{i}\int_{\mathbb{C}^p} (\mathbf{z}_2\mathbf{z}_1^* - \mathbf{z}_1\mathbf{z}_2^*)\widetilde{\mu}(d\mathbf{z}) = (1/2)I, \tag{68}$$

where the last equality is a consequence of the general property $\widetilde{\mu}(d\mathbf{z}) = \widetilde{\mu}(\overline{d\mathbf{z}})$. Thus, assuming time-reversibility, based on a change of variable $\mathbf{w} = -\overline{A}^{-1}A\mathbf{z}$, the condition (66) implies that

$$\Big(\overline{A}^{-1}A\Big)\int_{\mathbb{C}^p} \mathbf{z}\mathbf{z}^*\widetilde{\mu}(d\mathbf{z})\Big(\overline{A}^{-1}A\Big)^* = \int_{\mathbb{C}^p} \mathbf{w}\mathbf{w}^*\widetilde{\mu}(d\mathbf{w}) = (1/2)I. \tag{69}$$

Hence, $\left(\overline{A}^{-1}A\right)\left(\overline{A}^{-1}A\right)^* = I$, which implies the condition (67). In regard to the converse, however, by choosing A, A^* satisfying (67), we may easily find a Lévy measure μ under the condition (40) such that (66) is not satisfied. In other words, among the instances of rhofLm satisfying (40), the conditions for time-reversibility of rhofLm as established in Theorem 4.2 are stronger than those for time-reversibility of ofBm.

Example 4.7. Write $\Sigma = \int_{\mathbb{C}^p} \mathbf{z}\mathbf{z}^* \widetilde{\mu}(d\mathbf{z})$, and suppose Σ has full rank. By reasoning similar to that of Example 4.4, a necessary condition for the time-reversibility of $\widetilde{X}_H$ is that

$$A^{-1}\overline{A} = \Sigma^{1/2} U \Sigma^{-1/2}, \tag{70}$$

where $U \in M(p, \mathbb{C})$ is some unitary matrix. This implies the following:

(i) If $p = 1$, $\widetilde{X}_H$ is time-reversible if and only if $\widetilde{\mu}(-dz) = \widetilde{\mu}\left(e^{\mathbf{i}2\theta} dz\right)$, where $\theta = \arg A$.

(ii) If μ is a zero-mean Gaussian measure on $\mathbb{C}^p$ satisfying $\mu(d\mathbf{z}) = \mu(e^{\mathbf{i}\theta} d\mathbf{z})$, $\theta \in (-\pi, \pi]$, then $\widetilde{X}_H$ is time-reversible if and only if (70) holds. (Indeed, in this case $\widetilde{\mu} = \mu$, and μ is completely determined by Σ, showing that (70) holds if and only if (66) holds by taking second moments.)

Remark 4.1. The minimality of a representation $\{f_t\}_{t \in T}$ can always be enforced by replacing Ω with $\operatorname{supp}\{f_t, t \in T\}$ and by choosing $\mathcal{B} = \sigma\{f_t, t \in T\}$. However, establishing the minimality of a representation based on a given Borel space such as $\left(\mathbb{R}^{q+1}, \mathcal{B}\left(\mathbb{R}^{q+1}\right)\right)$ is not, in general, straightforward (see Definition D.1; cf. Kabluchko and Stoev [32], Remark 2.18). It can be shown by elementary—though long and tedious—arguments that minimality over $\left(\mathbb{R}^{q+1}, \mathcal{B}\left(\mathbb{R}^{q+1}\right)\right)$ naturally holds for some simple cases, such as when μ is a point mass. The question of when it holds in general remains open.

5. Conclusion

In this paper, we construct ofLm, a broad class of generally non-Gaussian stochastic processes that are covariance operator self-similar, have wide-sense stationary increments, and display infinitely divisible marginal distributions. The ofLm class generalizes the univariate fractional Lévy motion as well as the multivariate ofBm. The ofLm class can be divided into two types, namely, maofLm (moving average) and rhofLm (real harmonizable), both of which share the covariance structure of ofBm, under assumptions. We show that both maofLm and rhofLm admit stochastic integral representations in the time and Fourier domains. Though never o.s.s., the small- and large-scale limiting behaviors of maofLm and rhofLm are generally distinct. This stands in sharp contrast with the Gaussian case, where moving average and harmonizable stochastic integrals are representations of the same stochastic process. We characterize time-reversibility for ofLm in terms of its parameters and Lévy measure, starting from a framework for the uniqueness of finite-second-moment multivariate stochastic integral representations. In particular, we show that, under non-Gaussianity, the parametric conditions for time-reversibility are generally more restrictive those in the Gaussian case (ofBm).

This work leaves a number of issues to be explored and open research questions. These include the following: (i) efficient simulation schemes for ofLm, in particular with regard to the effect of the dimension p; (ii) the construction of statistical methodology that accounts for the impact of the tails of Lévy noise as a measure of non-Gaussianity; (iii) applications in fields such as in physics or signal processing, where the presence of fractal second-order behavior is well established, but where the modeling of non-Gaussian features is still a wide-open area of research.

Appendix A. Properties of stochastic integrals

Before establishing the results in Section 3, we lay out a few basic facts about the general stochastic integrals defined in Section 2.1. We start off with the Fourier domain. By construction, for $\varphi_1, \varphi_2 \in \mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}$, the $\mathbb{C}^p$-valued stochastic integrals of the form (10) satisfies the isometry-type property

$$\mathbb{E}\Big(\int_{\mathbb{R}\times\mathbb{C}^p} \varphi_1(x,\mathbf{z})\widetilde{N}(dx,d\mathbf{z})\Big)\Big(\int_{\mathbb{R}\times\mathbb{C}^p} \varphi_2(x',\mathbf{z}')\widetilde{N}(dx',d\mathbf{z}')\Big)^* = \int_{\mathbb{R}}\int_{\mathbb{C}^p} \varphi_1(x,\mathbf{z})\varphi_2(x,\mathbf{z})^*\mu(d\mathbf{z})dx.$$

In particular, consider the functions

$$\varphi_i(x,\mathbf{z}) = 2\Re\big(\widetilde{g}_i(x)\mathbf{z}\big), \quad \widetilde{g}_i \in L^2_{\mathrm{Herm}}(\mathbb{R}), \quad i=1,2. \tag{71}$$

Then, since $\widetilde{g}_i$, $i=1,2$, is Hermitian,

$$\mathbb{E}\Big(\int_{\mathbb{R}} \widetilde{g}_1(x)\widetilde{\mathcal{M}}(dx)\Big)\Big(\int_{\mathbb{R}} \widetilde{g}_2(x')\widetilde{\mathcal{M}}(dx')\Big)^* = 4\int_{\mathbb{R}} \Re\widetilde{g}_1(x)\Big(\int_{\mathbb{C}^p}\Re\mathbf{z}\Re\mathbf{z}^*\mu(d\mathbf{z})\Big)\Re\widetilde{g}_2(x)^*dx + 4\int_{\mathbb{R}} \Im\widetilde{g}_1(x)\Big(\int_{\mathbb{C}^p}\Im\mathbf{z}\Im\mathbf{z}^*\mu(d\mathbf{z})\Big)\Im\widetilde{g}_2(x)^*dx. \tag{72}$$

Moreover, the joint characteristic function of the real and imaginary parts of the $\mathbb{C}^p$-valued stochastic integral $\int \varphi d\widetilde{N}$, $\varphi \in \mathcal{L}^2_{dx\otimes\mu(d\mathbf{z})}$, is given by

$$\mathbb{E}\exp\Big\{\mathbf{i}\Big(\mathbf{u}_1^*\int \Re(\varphi)d\widetilde{N} + \mathbf{u}_2^*\int\Im(\varphi)d\widetilde{N}\Big)\Big\} = \exp\Big\{\int_{\mathbb{R}}\int_{\mathbb{C}^p}\Big[e^{\mathbf{i}\left(\mathbf{u}_1^*\Re(\varphi)+\mathbf{u}_2^*\Im(\varphi)\right)} - 1 - \mathbf{i}\big(\mathbf{u}_1^*\Re(\varphi)+\mathbf{u}_2^*\Im(\varphi)\big)\Big]\mu(d\mathbf{z})dx\Big\}, \tag{73}$$

for $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^p$ (Sato [63]). Note that, under the condition (7), the integral on the right-hand side of (73) is finite in view of the inequality

$$\big|e^{\mathbf{i}y} - 1 - \mathbf{i}y\big| \le |y|^2, \quad y \in \mathbb{R}. \tag{74}$$

In particular, for a function $\varphi(x,\mathbf{z})$ of the form (71),

$$\mathbb{E}\exp\Big\{\mathbf{i}\Big(\mathbf{u}^*\int_{\mathbb{R}\times\mathbb{C}^p}\varphi(x,\mathbf{z})\widetilde{N}(dx,d\mathbf{z})\Big)\Big\} = \exp\Big\{\int_{\mathbb{R}}\widetilde{\psi}(\widetilde{g}(x)^*\mathbf{u})dx\Big\}, \quad \mathbf{u}\in\mathbb{R}^p, \tag{75}$$

where

$$\widetilde{\psi}(\mathbf{v}) = \int_{\mathbb{C}^p}\Big(e^{\mathbf{i}2\Re\langle\mathbf{v},\mathbf{z}\rangle} - 1 - \mathbf{i}2\Re\langle\mathbf{v},\mathbf{z}\rangle\Big)\mu(d\mathbf{z}), \quad \mathbf{v}\in\mathbb{C}^p, \tag{76}$$

and $\langle\mathbf{v},\mathbf{z}\rangle := \mathbf{v}^*\mathbf{z}$. Equivalently, if we regard $\mu(d\mathbf{z}) = \mu_{\mathbb{R}^{2p}}(d\mathbf{z})$ as a measure on $\mathcal{B}(\mathbb{R}^{2p})$, identifying each $\mathbf{z}_1 + \mathbf{i}\mathbf{z}_2 \in \mathbb{C}^p$ with $(\mathbf{z}_1,\mathbf{z}_2)\in\mathbb{R}^{2p}$, then for $\mathbf{z},\mathbf{v}\in\mathbb{R}^{2p}$, we may write

$$\widetilde{\psi}(\mathbf{v}) = \int_{\mathbb{R}^{2p}}\Big(e^{\mathbf{i}2\langle\mathbf{v},\mathbf{z}\rangle} - 1 - \mathbf{i}2\langle\mathbf{v},\mathbf{z}\rangle\Big)\mu_{\mathbb{R}^{2p}}(d\mathbf{z}), \quad \mathbf{v}\in\mathbb{R}^{2p}. \tag{77}$$

Given the stochastic integral (16), for every n, any $(t_1, \ldots, t_n) \in \mathbb{R}^n$, and any $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^p$, the characteristic function of the finite-dimensional distributions of the $\mathbb{R}^p$-valued stochastic process $\widetilde{X}$ is given by

$$\begin{aligned}
&\mathbb{E} \exp\left\{\mathbf{i} \sum_{k=1}^{n} \langle \mathbf{u}_k, \widetilde{X}(t_k)\rangle\right\} \\
&= \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{C}^p} \left[e^{\mathbf{i}2\Re\left(\sum_{k=1}^{n} \mathbf{u}_k^* \widetilde{g}_{t_k}(x)\mathbf{z}\right)} - 1 - \mathbf{i}2\Re\left(\sum_{k=1}^{n} \mathbf{u}_k^* \widetilde{g}_{t_k}(x)\mathbf{z}\right)\right]\mu(d\mathbf{z})dx \right\}. \quad (78)
\end{aligned}$$

In particular, the random vectors $\widetilde{X}(t)$, $t \in \mathbb{R}$, are ID (cf. Samorodnitsky [60], Theorem 3.3.2(ii)).

Turning to the time domain, let $\varphi_i(s, \mathbf{z}) = g_i(s)\mathbf{z}$, $i = 1, 2$. By construction, the stochastic integral (20) satisfies the isometry property

$$\mathbb{E}\left(\int_{\mathbb{R}} g_1(s)\mathcal{M}(ds)\right)\left(\int_{\mathbb{R}} g_2(s')\mathcal{M}(ds')\right)^* = \int_{\mathbb{R}} g_1(s)\left(\int_{\mathbb{R}^p} \mathbf{z}\mathbf{z}^* \mu(d\mathbf{z})\right) g_2(s)^* ds, \quad (79)$$

where $\mathrm{tr}\left(\int_{\mathbb{R}^p} \mathbf{z}\mathbf{z}^*\mu(d\mathbf{z})\right) < \infty$. Moreover, for $\varphi(s, \mathbf{z}) \in \mathcal{L}^2_{ds\otimes\mu(d\mathbf{z})}$, the characteristic function of the $\mathbb{R}^p$-valued stochastic integral $\int \varphi(s, \mathbf{z})\widetilde{N}(ds, d\mathbf{z})$ is given by

$$\begin{aligned}
&\mathbb{E}e^{\mathbf{i}\mathbf{u}^* \int_{\mathbb{R}^{p+1}} \varphi(s,\mathbf{z})\widetilde{N}(ds,d\mathbf{z})} \\
&= \exp\left\{\int_{\mathbb{R}} \int_{\mathbb{R}^p} \left(e^{\mathbf{i}\langle \mathbf{u}, \varphi(s,\mathbf{z})\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \varphi(s, \mathbf{z})\rangle\right)\mu(d\mathbf{z})ds\right\}, \quad \mathbf{u} \in \mathbb{R}^p. \quad (80)
\end{aligned}$$

Under the condition (7) (restricted to $\mathbb{R}^p$), the integral on the right-hand side of (80) is convergent in view of the inequality (74).

Given the expression (22), for any n, the joint characteristic function of the stochastic process $X = \{X(t)\}_{t\in\mathbb{R}}$ at the time points $t_1, \ldots, t_n$ is given by

$$\begin{aligned}
\mathbb{E}e^{\mathbf{i}\sum_{k=1}^{n}\langle \mathbf{u}_k, X(t_k)\rangle} &= \exp\left\{\int_{\mathbb{R}} \int_{\mathbb{R}^p} \left(e^{\mathbf{i}\sum_{k=1}^{n}\langle \mathbf{u}_k, g_{t_k}(s)\mathbf{z}\rangle} - 1 - \mathbf{i}\sum_{k=1}^{n}\langle \mathbf{u}_k, g_{t_k}(s)\mathbf{z}\rangle\right)\mu(d\mathbf{z})ds\right\} \\
&= \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{k=1}^{n} g_{t_k}(s)^* \mathbf{u}_k\right) ds\right\}, \quad \mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^p. \quad (81)
\end{aligned}$$

In (81), the Lévy symbol ψ can be expressed as

$$\psi(\mathbf{u}) = \int_{\mathbb{R}^p} \left(e^{\mathbf{i}\langle \mathbf{u}, \mathbf{z}\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \mathbf{z}\rangle\right)\mu(d\mathbf{z}), \quad \mathbf{u} \in \mathbb{R}^p. \quad (82)$$

In particular, the random vectors $X(t)$, $t \in \mathbb{R}$, are ID (cf. Samorodnitsky [60], Theorem 3.3.2(ii)).

Example A.1. For the reader's convenience, we explicitly show how $\widetilde{\mathcal{M}}(dx)$ and $\mathcal{M}(ds)$ can be constructed to encompass both Gaussian and purely non-Gaussian instances of ofLm. For rhofLm, consider a $\mathbb{C}^p$-valued independently scattered ID random measure on $\mathbb{R}$ given by

$$\mathfrak{M}(dx) := \mathfrak{B}(dx) + \int_{\mathbb{C}^p} \mathbf{z}\widetilde{N}(dx, d\mathbf{z}) = \Re(\mathfrak{M}(dx)) + \mathbf{i}\Im(\mathfrak{M}(dx)) =: \mathcal{M}^{(1)}(dx) + \mathbf{i}\mathcal{M}^{(2)}(dx),$$

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where $\mathfrak{B}(dx) := \mathfrak{B}^{(1)}(dx) + \mathbf{i}\mathfrak{B}^{(2)}(dx)$ is independent of $\widetilde{N}(dx, d\mathbf{z})$ and $\mathfrak{B}^{(\ell)}(dx)$, $\ell = 1, 2$, are $\mathbb{R}^p$-valued independently scattered Gaussian random measures on $\mathbb{R}$. We may then define

$$\widetilde{\mathcal{M}}(dx) = \Big(\widetilde{\mathcal{M}}^{(1)}(dx) + \widetilde{\mathcal{M}}^{(1)}(-dx)\Big) + \mathbf{i}\Big(\widetilde{\mathcal{M}}^{(2)}(dx) - \widetilde{\mathcal{M}}^{(2)}(-dx)\Big)$$

(n.b.: $\widetilde{\mathcal{M}}(dx)$ is not independently scattered). When $\widetilde{N}(ds, d\mathbf{z})$ is absent ($\mathfrak{M}(dx) \equiv \mathfrak{B}(dx)$), the resulting measure $\widetilde{\mathcal{M}}(dx) =: \widetilde{B}(dx)$ is Gaussian. Assuming properness, the corresponding process (31) is an ofBm. Without loss of generality, we may further assume $\mathfrak{B}^{(1)}(dx)$, $\mathfrak{B}^{(2)}(dx)$ are independent and take $\mathbb{E}\widetilde{B}(dx)\widetilde{B}(dx)^* = dx \times I$ (note that a slightly different construction of $\widetilde{B}(dx)$—but one that is equivalent for representing ofBm—is given in Didier and Pipiras [23]; see also Samorodnitsky and Taqqu [61], Section 7.2.2). When $\mathfrak{M}(dx)$ is purely non-Gaussian ($\mathfrak{B}(dx) \equiv \mathbf{0}$), one recovers rhofLm as in Definition 3.1.

To encompass both Gaussian and non-Gaussian instances of maofLm, one can take $\mathcal{M}(ds) = B(ds) + \int_{\mathbb{R}^p} \mathbf{z}\widetilde{N}(ds, d\mathbf{z})$, where $B(ds)$ is an $\mathbb{R}^p$-valued independently scattered Gaussian random measure on $\mathbb{R}$ independent of $N(ds, d\mathbf{z})$, without loss of generality satisfying $\mathbb{E}B(ds)B(ds)^* = ds \times I$. In this case, when $\widetilde{N}$ is absent from $\mathcal{M}$, assuming properness, the corresponding process (33) is an ofBm.

Appendix B. Proofs: Section 3

Proof of Theorem 3.1: Statement (i) is a consequence of the facts that $\widetilde{g}_t \in L^2_{\mathrm{Herm}}(\mathbb{R})$, $g_t \in L^2(\mathbb{R}, M(p, \mathbb{R}))$, $t \in \mathbb{R}$ (cf. Didier and Pipiras [23]).

In regard to (ii), it follows from the dominated convergence theorem that the covariance functions of both $\widetilde{X}_H$ and X_H are continuous. Therefore, both processes are stochastically continuous.

As for (iii), the relation (35) is a consequence of the expression (78) for the characteristic function of stochastic integrals of the form (15). The expression (36) now follows from the fact that, for $\mathbf{z} = \mathbf{z}_1 + \mathbf{i}\mathbf{z}_2 \in \mathbb{C}^p$, and $j = 1, \dots, m$,

$$\begin{aligned}\Re\Big\{\Big(\Re g_{t_j}(x) + \mathbf{i}\Im g_{t_j}(x)\Big)\mathbf{z}\Big\} &= \Re\Big\{\Big[\Re g_{t_j}(x)\mathbf{z}_1 - \Im g_{t_j}(x)\mathbf{z}_2 + \mathbf{i}\Big(\Re g_{t_j}(x)\mathbf{z}_2 + \Im g_{t_j}(x)\mathbf{z}_1\Big)\Big]\Big\}\\ &= \Re g_{t_j}(x)\mathbf{z}_1 - \Im g_{t_j}(x)\mathbf{z}_2.\end{aligned}$$

In turn, the relation (38) is a consequence of (81). Moreover, $\widetilde{X}_H$ and X_H are cov.o.s.s. as a consequence of the isometry relations (72) and (79), as well as of the scaling properties

$$\widetilde{g}_{ct}(x) = c^{D+I}\widetilde{g}_t(cx) \quad \text{a.e.}, \tag{83}$$

$$g_{ct}(cs) = \mathcal{F}^{-1}\Big(c^{-1}\widetilde{g}_{ct}\big(c^{-1}\,\cdot\big)\Big)(s) = c^D\mathcal{F}^{-1}\big(\widetilde{g}_t\big)(s) = c^D g_t(s) \quad \text{a.e.}, \tag{84}$$

for any $c > 0$. This establishes (iii).

We now turn to (iv). First consider X_H, and let $H = PJ_HP^{-1}$ be the Jordan decomposition of H. Define $H' = QHQ^{-1}$, and observe that $\Re\mathrm{eig}(H') \in (0, 1)$. The expression (79) and Parseval–Plancherel imply $\mathbb{E}X_H(s)X_H(t)^* = \int_{\mathbb{R}} g_s(u)QQ^*g_t(u)^*du = \int_{\mathbb{R}} \widetilde{g}_s(x)QQ^*\widetilde{g}_t(x)^*dx$, which is the covariance function of an ofBm with parameters H' and QA in its harmonizable representation. This establishes the claim. The claim under (27) follows, since in this case $Q = I$. The statements for $\widetilde{X}_H$ follow similarly based on the expression (72).

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In regard to (v), for any $t, h \in \mathbb{R}$,

$$\begin{aligned} X_H(t+h) - X_H(h) &= \int_{\mathbb{R}} \big(g_{t+h}(s) - g_h(s)\big)\mathcal{M}(ds) \\ &= \int_{\mathbb{R}} \Big(\Big\{(t+h-s)_+^D - (h-s)_+^D\Big\}M_+ + \Big\{(t+h-s)_-^D - (h-s)_-^D\Big\}M_-\Big)\mathcal{M}(ds) \\ &\overset{\text{f.d.d.}}{=} \int_{\mathbb{R}} \Big(\Big\{\big(t-s'\big)_+^D - \big(-s'\big)_+^D\Big\}M_+ + \Big\{\big(t-s'\big)_-^D - \big(-s'\big)_-^D\Big\}M_-\Big)\mathcal{M}(ds). \end{aligned} \tag{85}$$

In (85), the equality of finite-dimensional distributions follows from a change of variable $h - s = s'$ in the characteristic function for

$$\big(X_H(t+h_1) - X_H(h_1), \ldots, X_H(t+h_m) - X_H(h_m)\big), \quad h_1, \ldots, h_m \in \mathbb{R},$$

which in turn stems from the expression (81). This shows that the maofLm X_H has strict-sense stationary increments. That $\widetilde{X}_H$ has wide-sense stationary increments is a direct consequence of (72) and (79). Moreover, under the condition (41), for any $t, h \in \mathbb{R}$,

$$\begin{aligned} \widetilde{X}_H(t+h) - \widetilde{X}_H(h) &= \int_{\mathbb{R}} \big(\widetilde{g}_{t+h}(x) - \widetilde{g}_h(x)\big)\widetilde{\mathcal{M}}(dx) \\ &= \int_{\mathbb{R}} \left(\frac{e^{\mathbf{i}tx} - 1}{\mathbf{i}x}\right)\{x_+^{-D}A + x_-^{-D}\overline{A}\}e^{\mathbf{i}hx}\widetilde{\mathcal{M}}(dx) \\ &\overset{\text{f.d.d.}}{=} \int_{\mathbb{R}} \left(\frac{e^{\mathbf{i}tx} - 1}{\mathbf{i}x}\right)\{x_+^{-D}A + x_-^{-D}\overline{A}\}\widetilde{\mathcal{M}}(dx) = \widetilde{X}_H(t). \end{aligned}$$

This shows that the rhofLm $\widetilde{X}_H$ has strict-sense stationary increments. This establishes (v).

We now show (vi). For the sake of contradiction, suppose X_H is o.s.s. Then the relation (2) holds for X_H and some matrix $\mathcal{H}$. So, fix $c > 0$ and $t \neq 0$. Then

$$\begin{aligned} &\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^p} \left(e^{\mathbf{i}\mathbf{u}^* g_{ct}(s)\mathbf{z}} - 1 - \mathbf{i}\mathbf{u}^* g_{ct}(s)\mathbf{z}\right)\mu(d\mathbf{z})ds\right\} \\ &= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^p} \left(e^{\mathbf{i}\mathbf{u}^* c^{\mathcal{H}} g_t(s)\mathbf{z}} - 1 - \mathbf{i}\mathbf{u}^* c^{\mathcal{H}} g_t(s)\mathbf{z}\right)\mu(d\mathbf{z})ds\right\}, \quad \mathbf{u} \in \mathbb{R}^p. \end{aligned} \tag{86}$$

Define the measurable functions $H_1(s, \mathbf{z}) = g_{ct}(s)\mathbf{z}$ and $H_2(s, \mathbf{z}) = c^{\mathcal{H}} g_t(s)\mathbf{z}$, and consider the product measure $\mu \otimes \eta$. Now define the induced measures on $\mathbb{R}^p$ via

$$\nu_i(d\mathbf{y}) = [H_i(\mu \otimes \eta)](d\mathbf{y}) = (\mu \otimes \eta)\big[H_i^{-1}(d\mathbf{y})\big], \quad i = 1, 2.$$

By a change of measures, we can rewrite (86) as

$$\exp\left\{\int_{\mathbb{R}^p} \left(e^{\mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle\right)\nu_1(d\mathbf{y})\right\} = \exp\left\{\int_{\mathbb{R}^p} \left(e^{\mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle\right)\nu_2(d\mathbf{y})\right\}, \quad \mathbf{u} \in \mathbb{R}^p.$$

By the measure-theoretic convention $0 \times \infty = 0$, we arrive at

$$\begin{aligned} &\exp\left\{\int_{\mathbb{R}^p \backslash \{\mathbf{0}\}} \left(e^{\mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle\right)\nu_1(d\mathbf{y})\right\} \\ &= \exp\left\{\int_{\mathbb{R}^p \backslash \{\mathbf{0}\}} \left(e^{\mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle\right)\nu_2(d\mathbf{y})\right\}, \quad \mathbf{u} \in \mathbb{R}^p. \end{aligned}$$

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By the uniqueness of the Lévy measure, $\nu_1(B) = \nu_2(B)$ for all $B \in \mathcal{B}(\mathbb{R}^p \setminus \{\mathbf{0}\})$. Equivalently,

$$\int_{\mathbb{R}} \int_{\mathbb{R}^p} 1_B\big(g_{ct}(s)\mathbf{z}\big)\mu(d\mathbf{z})ds = \int_{\mathbb{R}} \int_{\mathbb{R}^p} 1_B\big(c^{\mathcal{H}} g_t(s)\mathbf{z}\big)\mu(d\mathbf{z})ds, \quad B \in \mathcal{B}(\mathbb{R}^p \setminus \{\mathbf{0}\}). \tag{87}$$

Note that the kernel g_t satisfies the scaling relation (84). By a change of variable $s = cw$ and (84) applied to the integral $\int_{\mathbb{R}} \int_{\mathbb{R}^p} 1_B\big(g_{ct}(s)\mathbf{z}\big)\mu(d\mathbf{z})ds$, we can rewrite the relation (87) as

$$c \int_{\mathbb{R}} \int_{\mathbb{R}^p} 1_{c^{(1/2)I-H}B}\big(g_t(w)\mathbf{z}\big)\mu(d\mathbf{z})dw = \int_{\mathbb{R}} \int_{\mathbb{R}^p} 1_{c^{-\mathcal{H}}B}\big(g_t(s)\mathbf{z}\big)\mu(d\mathbf{z})ds. \tag{88}$$

Let

$$\nu_*(B) = \int_{\mathbb{R}} \int_{\mathbb{R}^p} 1_B\Big(g_t(s)\mathbf{z}\Big)\mu(d\mathbf{z})ds, \quad B \in \mathcal{B}(\mathbb{R}^p \setminus \{\mathbf{0}\}), \tag{89}$$

and set $\nu_*(\{\mathbf{0}\}) := 0$. Note that

$$\int_{\mathbb{R}^p} \|\mathbf{y}\|^2 \nu_*(d\mathbf{y}) = \int_{\mathbb{R}} \int_{\mathbb{R}^p} \|g_t(s)\mathbf{z}\|^2 \mu(d\mathbf{z})ds \le \int_{\mathbb{R}} \|g_t(s)\|^2 ds \cdot \int_{\mathbb{R}^p} \|\mathbf{z}\|^2 \mu(d\mathbf{z}) < \infty.$$

Based on ν_*, we can rewrite (88) as

$$c\nu_*\big(c^{(1/2)I-H}B\big) = \nu_*\big(c^{-\mathcal{H}}B\big). \tag{90}$$

Fix $B_0 \in \mathcal{B}(\mathbb{R}^p)$. Then, for $B(c) := c^{H-(1/2)I}B_0 \in \mathcal{B}(\mathbb{R}^p)$, (90) implies that $c\nu_*(B_0) = \nu_*\big(c^{-\mathcal{H}}c^{H-(1/2)I}B_0\big)$, i.e.,

$$c\nu_*(d\mathbf{y}) = \nu_*\big(c^{-\mathcal{H}}c^{H-(1/2)I}d\mathbf{y}\big). \tag{91}$$

Starting from the Lévy measure ν_*, we can define a Lévy process L_* by means of the characteristic function

$$\mathbb{E}e^{\mathbf{i}\langle \mathbf{u}, L_*(t)\rangle} = \exp\left\{t \int_{\mathbb{R}^p} \Big(e^{\mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \mathbf{y}\rangle\Big)\nu_*(d\mathbf{y})\right\}. \tag{92}$$

In particular, $L_*(0) = 0$, L_* has finite second moment, and $L_*(t)$ is proper for $t \neq 0$. By (91),

$$\mathbb{E}e^{\mathbf{i}\langle \mathbf{u}, L_*(ct)\rangle} = \mathbb{E}e^{\mathbf{i}\left\langle \mathbf{u}, c^{\mathcal{H}}c^{(1/2)I-H}L_*(t)\right\rangle}, \quad c > 0. \tag{93}$$

Since L_* is a Lévy process, the relation (93) implies that, for every $c > 0$, there exist a linear operator $B(c)$ and a vector $b(c)$—namely, $B(c) = c^{\mathcal{H}}c^{(1/2)I-H}$ and $b(c) = \mathbf{0}$—such that $\{L_*(ct)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \{B(c)L_*(t) + b(c)\}_{t\in\mathbb{R}}$. Therefore, since L_* is proper, Theorem 1 in Hudson and Mason [29] shows that there exist a matrix H' and a nonrandom function $d : [0, \infty) \to \mathbb{R}^p$ such that $\{L_*(ct)\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \big\{c^{H'}L_*(t) + d(c)\big\}_{t\in\mathbb{R}}$. Furthermore, since L_* is stochastically continuous, Theorem 7 in Hudson and Mason [29] implies that $d \equiv 0$, and that $L_*(1)$ is operator-stable (with exponent H'). However, L_* has finite second moment, which in turn implies L_* must be Gaussian. This contradicts the fact that, by (92), L_* has no Gaussian component. Hence, maofLm X_H is not o.s.s., as claimed. The case for rhofLm $\widetilde{X}_H$ is analogous.

We now turn to (vii). Without loss of generality, we can rewrite $\Re h_1 \le \ldots \le \Re h_p$. Let $d_1 = \Re h_1 - \frac{1}{2}$. Again without loss of generality, let $0 \le s < t \le 1$, and write $r = t - s$. Recall that the Frobenius inner product $\langle\cdot,\cdot\rangle_F$ of two matrices $A, B \in M(p, \mathbb{C})$ is given by $\langle A, B\rangle_F = \mathrm{tr}(A^*B)$,

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and write $\|\cdot\|_F$ for the corresponding norm. Also, write $H = PJ_HP^{-1}$ for the Jordan decomposition of H, and recall that for any Jordan block J_h corresponding to $h \in \mathrm{eig}(H)$ of size k,

$$r^{J_h} = \begin{pmatrix} r^h & 0 & 0 & \dots & 0 \\ (\log r)r^h & r^h & 0 & \dots & 0 \\ \frac{(\log r)^2}{2!}r^h & (\log r)r^h & r^h & & \\ \vdots & \vdots & & \ddots & \\ \frac{(\log r)^{k-1}}{(k-1)!}r^h & \frac{(\log r)^{k-2}}{(k-2)!}r^h & \dots & (\log r)r^h & r^h \end{pmatrix}, \quad r > 0.$$

By stationarity of the increments of maofLm,

$$\begin{aligned} \mathbb{E}\|X_H(t) - X_H(s)\|^2 &= \mathbb{E}\|X_H(r)\|^2 = \mathrm{tr}\big(\mathbb{E}X_H(r)X_H(r)^*\big) \\ &= \mathrm{tr}\big(r^H\mathbb{E}X_H(1)X_H(1)^* r^{H^*}\big) = \mathrm{tr}\big(r^{H^*}r^H\mathbb{E}X_H(1)X_H(1)^*\big) \\ &\le \big\|r^{H^*}r^H\big\|_F \big\|\mathbb{E}X_H(1)X_H(1)^*\big\|_F \le C\big\|r^H\big\|_F^2 \le C'\big\|r^{J_H}\big\|_F^2 \\ &\le C'' \max_{h\in\mathrm{eig}(H)} \big\|r^{J_h}\big\|_F^2 \le C'''\big(|\log r|^{2(p-1)} \vee 1\big) r^{2\min\{\Re h_1,\dots,\Re h_p\}} \\ &\le C'''' r^{2d_1+1-\varepsilon}, \end{aligned}$$

for every $\varepsilon > 0$. Hence, by the Kolmogorov–Čentsov theorem (e.g., Kallenberg [33], Theorem 2.23), X has a modification that is a.s. locally γ-Hölder continuous for each $\gamma \in (0, d_1)$, as claimed. The statement also holds for rhofLm in view of its wide-sense stationary increments (see Statement (v)).

Proof of Proposition 3.1: Both statements are a consequence of the Parseval-type relations (142), (145) and Proposition E.1.

Proof of Proposition 3.2: We begin with (i). For any $t \ne 0$, let g_t be as in (25), and recall the scaling relation (84) that the kernel g_t satisfies. For $n \in \mathbb{N}$, let $t_1, \dots, t_n \in \mathbb{R}$. By Theorem 3.1(iii), the joint characteristic function of $X_H(t_1), \dots, X_H(t_n)$ is given by

$$\mathbb{E}\exp\left\{\mathbf{i}\sum_{j=1}^n \langle \mathbf{u}_j, X_H(t_j)\rangle\right\} = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^n g_{t_j}(s)^*\mathbf{u}_j\right) ds\right\},$$

where the Lévy symbol ψ is as in (82). Now consider the collection of rescaled vectors

$$c^{-H}X_H(ct_1), c^{-H}X_H(ct_2), \dots, c^{-H}X_H(ct_n).$$

Then their joint characteristic function is given by

$$\mathbb{E}e^{\mathbf{i}\sum_{j=1}^n \left\langle \mathbf{u}_j, c^{-H}X_H(ct_j)\right\rangle} = \mathbb{E}e^{\mathbf{i}\sum_{j=1}^n \left\langle c^{-H^*}\mathbf{u}_j, X_H(ct_j)\right\rangle} = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^n g_{ct_j}(s)^* c^{-H^*}\mathbf{u}_j\right) ds\right\}$$

$$\stackrel{cv=s}{=} \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^n g_{ct_j}(cv)^* c^{-H^*}\mathbf{u}_j\right) cdv\right\} = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^n g_{t_j}(v)^* c^{-\frac{1}{2}I}\mathbf{u}_j\right) cdv\right\}$$

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$$= \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^p} \left(\exp\left\{ \mathbf{i} \left\langle \sum_{j=1}^n g_{t_j}(v)^* c^{-\frac{1}{2}I} \mathbf{u}_j, \mathbf{z} \right\rangle \right\} - 1 - \mathbf{i} \left\langle \sum_{j=1}^n g_{t_j}(v)^* c^{-\frac{1}{2}I} \mathbf{u}_j, \mathbf{z} \right\rangle \right) c\mu(d\mathbf{z}) dv \right\}$$
$$= \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}^p} \left(\exp\left\{ c^{-1/2} \mathbf{i} \left\langle \sum_{j=1}^n g_{t_j}(v)^* \mathbf{u}_j, \mathbf{z} \right\rangle \right\} - 1 - \mathbf{i} \left\langle c^{-1/2} \sum_{j=1}^n g_{t_j}(v)^* \mathbf{u}_j, \mathbf{z} \right\rangle \right) c\mu(d\mathbf{z}) dv \right\}. \tag{94}$$

Note that

$$h_c(y) := c\left(e^{\mathbf{i}c^{-1/2}y} - 1 - \mathbf{i}c^{-1/2}y\right) \sim -\frac{1}{2}y^2$$

as $c \to \infty$ and that $|h_c(y)| \leq y^2$ for all $y \in \mathbb{R}$. If we write $\xi(v, \mathbf{z}) = \langle \sum_{j=1}^n g_{t_j}(v)^* \mathbf{u}_j, \mathbf{z} \rangle$, the integrand $h_c(\xi(v, \mathbf{z}))$ in the expression (94) satisfies

$$|h_c(\xi(v, \mathbf{z}))| \leq \xi(v, \mathbf{z})^2 = \sum_{j,k=1}^n \mathbf{u}_j^* g_{t_j}(v) \mathbf{z}\mathbf{z}^* g_{t_k}^*(v) u_k,$$

which is integrable with respect to $\mu(d\mathbf{z})dv$ since each $g_{t_j}(\cdot) \in L^2(\mathbb{R}, M(p, \mathbb{R}))$ and $\left\| \int_{\mathbb{R}^p} \mathbf{z}\mathbf{z}^* \mu(d\mathbf{z}) \right\| = 1$. Thus, by the dominated convergence theorem, as $c \to \infty$, (94) converges to

$$\exp\left\{ -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^p} \left(\sum_{j=1}^n \langle g_{t_j}(v)^* \mathbf{u}_j, \mathbf{z} \rangle \right)^2 \mu(d\mathbf{z}) dv \right\}$$
$$= \exp\left\{ -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^p} \sum_{j,k=1}^n \mathbf{u}_j^* g_{t_j}(v) \mathbf{z}\mathbf{z}^* g_{t_k}^*(v) \mathbf{u}_k \mu(d\mathbf{z}) dv \right\}$$
$$= \exp\left\{ -\frac{1}{2} \int_{\mathbb{R}} \sum_{j,k=1}^n \mathbf{u}_j^* g_{t_j}(v) g_{t_k}^*(v) \mathbf{u}_k dv \right\}, \tag{95}$$

where we use the condition (39). Note that (95) is equal to $\exp\{-\frac{1}{2}\mathbf{u}^* \Sigma_{B_H} \mathbf{u}\}$, where Σ_{B_H} is the $pn \times pn$ block matrix

$$\Sigma_{B_H} = \left(\int_{\mathbb{R}} g_{t_i}(s) g_{t_j}(s)^* ds \right)_{i,j=1,\ldots,n}.$$

Hence, (95) is the characteristic function of an ofBm at times $t_1, \ldots, t_n$.

For (ii), for any $t \neq 0$, let $\widetilde{g}_t$ be as in (24). For $s \in \mathbb{R}$ and $\varepsilon > 0$, Theorem 3.1 (iv) implies that

$$\widetilde{X}_H(s + \varepsilon t) - \widetilde{X}_H(s) \overset{\text{f.d.d.}}{=} \widetilde{X}_H(\varepsilon t).$$

So, fix $n \in \mathbb{N}$ and let $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^p$, $t_1, \ldots, t_n \in \mathbb{R}$. Note that $\widetilde{g}_{\varepsilon t}(\varepsilon^{-1}x) = x^{D+I}\widetilde{g}_t(x)$. By Theorem 3.1(iii),

$$\mathbb{E}e^{\mathbf{i}\sum_{j=1}^N \mathbf{u}_j^* \varepsilon^{-H} \widetilde{X}_H(t_j)} = \exp\left\{ \int_{\mathbb{R}} \widetilde{\psi}\left(\sum_{j=1}^n \widetilde{g}_{\varepsilon t_j}(x)^* \varepsilon^{-H^*} \mathbf{u}_j \right) dx \right\}$$
$$\overset{y=\varepsilon x}{=} \exp\left\{ \int_{\mathbb{R}} \widetilde{\psi}\left(\sum_{j=1}^n \widetilde{g}_{t_j}(y\varepsilon^{-1})^* \varepsilon^{-H^*} \mathbf{u}_j \right) \frac{dy}{\varepsilon} \right\}$$
$$= \exp\left\{ \int_{\mathbb{R}} \widetilde{\psi}\left(\sum_{j=1}^n \widetilde{g}_{t_j}(y)^* \varepsilon^{\frac{1}{2}I} \mathbf{u}_j \right) \frac{dy}{\varepsilon} \right\}, \tag{96}$$

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where $\widetilde{\psi}$ is given by (76). Recast $\mathbf{z}=\mathbf{z}_1+\mathbf{i}\mathbf{z}_2$. By a dominated convergence argument similar to that of (i), and based on the fact that $\widetilde{g}_{t_j}\in L^2_{\mathrm{Herm}}(\mathbb{R})$, $j=1,\ldots,n$, as $\varepsilon\to 0$ the expression (96) converges to

$$\exp\left\{-2\left(\int_{\mathbb{R}\times\mathbb{C}^p}\sum_{j,k=1}^{n}\mathbf{u}_j^*\widetilde{g}_{t_j}(y)\mathbf{z}_1\mathbf{z}_1^*\widetilde{g}_{t_k}^*(y)\mathbf{u}_k+\sum_{j,k=1}^{n}\mathbf{u}_j^*\widetilde{g}_{t_j}(y)\mathbf{z}_2\mathbf{z}_2^*\widetilde{g}_{t_k}^*(y)\mathbf{u}_k\right)\mu(d\mathbf{z})dy\right\}.$$

By using the condition (40), we arrive at

$$\exp\left\{-\frac{1}{2}\int_{\mathbb{R}}\sum_{j,k=1}^{n}\mathbf{u}_j^*\widetilde{g}_{t_j}(y)\widetilde{g}_{t_k}^*(y)\mathbf{u}_k\,dy\right\}.$$

This establishes (ii).

Proof of Proposition 3.3: Whenever convenient, we write $D=H-(1/2)I$ (see (23)). First note that, since $\Re\lambda_p(B)<1$, the expression (147) is well defined and corresponds to the Lévy symbol of a full operator-stable distribution in $\mathbb{R}^p$ (see Section 11). Furthermore, the process (47) is well defined by Theorem 5.4 in Maejima and Mason [41] (see also Theorem 4.2 in Kremer and Scheffler [36]), since

$$\Re\lambda_1\big(\widetilde{H}-B\big)+\Re\lambda_1(B)=\Re\lambda_1(D)+\Re\lambda_1(B)>0$$

and

$$\Re\lambda_p\big(\widetilde{H}-B-I\big)+\Re\lambda_p(B)=\Re\lambda_p(D)-1+\Re\lambda_p(B)<0.$$

(N.b.: there is a typo in the original statement of Theorem 5.4 in Maejima and Mason [41]; in the notation of that paper, their assumption should read $\Lambda_{D-B-I}+\Lambda_B<0$.) We now proceed as in the proof of Proposition 3.2(i). In fact, fix $n\in\mathbb{N}$ and $t_1,\ldots,t_n\in\mathbb{R}$, as well as the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_n\in\mathbb{R}^p$. Let $\varepsilon>0$. Consider g_{t_j} as in (25), $j=1,\ldots,n$, and recall the scaling relation (84). Then, by stationary increments and (45),

$$\mathbb{E}e^{\mathbf{i}\sum_{j=1}^{n}\left\langle\mathbf{u}_j,\varepsilon^{-\widetilde{H}_1}\left(X_H(s+\varepsilon t_j)-X_H(s)\right)\right\rangle}=\mathbb{E}e^{\mathbf{i}\sum_{j=1}^{n}\left\langle\mathbf{u}_j,\varepsilon^{-\widetilde{H}_1}X_H\left(\varepsilon t_j\right)\right\rangle}$$

$$=\exp\left\{\int_{\mathbb{R}}\psi\left(\sum_{j=1}^{n}g_{\varepsilon t_j}(s)^*\varepsilon^{-\widetilde{H}_1^*}\mathbf{u}_j\right)ds\right\}\overset{\varepsilon v=s}{=}\exp\left\{\int_{\mathbb{R}}\psi\left(\sum_{j=1}^{n}g_{\varepsilon t_j}(\varepsilon v)^*\varepsilon^{-\widetilde{H}_1^*}\mathbf{u}_j\right)\varepsilon dv\right\}$$

$$=\exp\left\{\int_{\mathbb{R}}\psi\left(\sum_{j=1}^{n}g_{t_j}(v)^*\varepsilon^{(D-\widetilde{H}_1)^*}\mathbf{u}_j\right)\varepsilon dv\right\}=\exp\left\{\int_{\mathbb{R}}\psi\left(\sum_{j=1}^{n}\varepsilon^{-B^*}g_{t_j}(v)^*\mathbf{u}_j\right)\varepsilon dv\right\}$$

$$=\exp\left\{\int_{\mathbb{R}}\int_{S_0}\int_{\mathbb{R}_+}\left(\exp\left\{\mathbf{i}\left\langle\sum_{j=1}^{n}g_{t_j}(v)^*\mathbf{u}_j,\,(r/\varepsilon)^B\boldsymbol{\theta}\right\rangle\right\}\right.\right.$$

$$\left.\left.-1-\mathbf{i}\left\langle\sum_{j=1}^{n}g_{t_j}(v)^*\mathbf{u}_j,\,(r/\varepsilon)^B\boldsymbol{\theta}\right\rangle\right)\varepsilon q(r,\boldsymbol{\theta})\frac{dr}{r^2}\lambda(d\boldsymbol{\theta})dv\right\}$$

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$$\begin{aligned}&\overset{\xi=r/\varepsilon}{=} \exp\left\{\int_{\mathbb{R}}\int_{S_0}\int_{\mathbb{R}_+}\left(\exp\left\{\mathbf{i}\left\langle\sum_{j=1}^n g_{t_j}(v)^*\mathbf{u}_j,\,\xi^B\boldsymbol{\theta}\right\rangle\right\}\right.\right.\\&\left.\left.-1-\mathbf{i}\left\langle\sum_{j=1}^n g_{t_j}(v)^*\mathbf{u}_j,\,\xi^B\boldsymbol{\theta}\right\rangle\right)q(\varepsilon\xi,\boldsymbol{\theta})\frac{d\xi}{\xi^2}\lambda(d\boldsymbol{\theta})dv\right\}\\&\to \exp\left\{\int_{\mathbb{R}}\int_{S_0}\int_{\mathbb{R}_+}\left(\exp\left\{\mathbf{i}\left\langle\sum_{j=1}^n g_{t_j}(v)^*\mathbf{u}_j,\,\xi^B\boldsymbol{\theta}\right\rangle\right\}\right.\right.\\&\left.\left.-1-\mathbf{i}\left\langle\sum_{j=1}^n g_{t_j}(v)^*\mathbf{u}_j,\,\xi^B\boldsymbol{\theta}\right\rangle\right)\frac{d\xi}{\xi^2}\lambda(d\boldsymbol{\theta})dv\right\},\end{aligned} \tag{97}$$

as $\varepsilon \to 0^+$. The limit in (97) is a consequence of the dominated convergence theorem and of the relation (150), since $|q(r,\boldsymbol{\theta})| \le 1$. By Proposition 2.17 in Sato [63], (97) is the characteristic function of (47). Therefore, the limit (46) of the rescaled finite-dimensional distributions of X_H holds.

For (ii), we first verify the existence of the limiting process (50) by applying Theorem 2.5 in Kremer and Scheffler [36]. To use the proposition, we view (50) as a process in $\mathbb{R}^{2p}$, where we make the identification

$$\mathbb{R}^{2p\times 2p} \ni \widetilde{g}_t(x) \equiv \begin{pmatrix}\Re\widetilde{g}_t(x) & -\Im\widetilde{g}_t(x)\\ 0 & 0\end{pmatrix}$$

(see Proposition 5.10 in Kremer and Scheffler [35]). Fix $t \neq 0$. To apply Theorem 2.5 in Kremer and Scheffler [36], we need to verify

$$\int_{\{x:\|\widetilde{g}_t(x)\|<R\}}\|\widetilde{g}_t(x)\|^{\frac{1}{\Re\lambda_p(B)}-\delta_1}dx+\int_{\{x:\|\widetilde{g}_t(x)\|>R\}}\|\widetilde{g}_t(x)\|^{\frac{1}{\Re\lambda_1(B)}+\delta_2}dx<\infty \tag{98}$$

for some $R > 0$ and appropriate

$$\delta_1 \in \left(0,\ \frac{1}{\Re\lambda_p(B)}\right), \qquad \delta_2 > 0.$$

Recall that $D = H - (1/2)I$. Since $\|\widetilde{g}_t(x)\| \to 0$ as $|x| \to \infty$, and $\|\widetilde{g}_t(\cdot)\|$ is continuous on $\mathbb{R}\setminus\{0\}$, for R large enough there exists an $\varepsilon > 0$ such that the set $\{x:\|\widetilde{g}_t(x)\| > R\} \subseteq \{x:|x| < \varepsilon\}$. Further note that for each $\delta > 0$ there exists $C > 0$ such that

$$\begin{aligned}\max\left\{\left\|\Re\widetilde{g}_t(x)\right\|,\ \left\|\Im\widetilde{g}_t(x)\right\|\right\}\mathbf{1}_{\{|x|>\varepsilon\}} &\le C|x|^{\delta-\Re\lambda_1(D)-1},\\ \max\left\{\left\|\Re\widetilde{g}_t(x)\right\|,\ \left\|\Im\widetilde{g}_t(x)\right\|\right\}\mathbf{1}_{\{|x|\le\varepsilon\}} &\le C|x|^{-\delta-\Re\lambda_p(D)},\end{aligned} \tag{99}$$

(see Theorem 2.2.4 in Meerschaert and Scheffler [50]), where in the second inequality we used the fact that $\frac{e^{\mathbf{i}tx}-1}{\mathbf{i}x}$ is bounded for all small $|x|$.

So, take $\delta > 0$ small enough so that

$$\Re\lambda_1(D)+1-\Re\lambda_p(B)=\Re\lambda_1(H)+\left(\frac{1}{2}-\Re\lambda_p(B)\right)>\delta \tag{100}$$

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and

$$\Re\lambda_p(D) + 1 - \Re\lambda_1(B) = \Re\lambda_p(H) + \left(\frac{1}{2} - \Re\lambda_1(B)\right) < 1 - \delta. \tag{101}$$

Let $C > 0$ be the constant satisfying both inequalities (99). For notational simplicity, write $\rho_1 = -(\delta - \Re\lambda_1(D) - 1)$ and $\rho_2 = \delta + \Re\lambda_p(D)$. Now, for any $\delta_1 > 0$,

$$\int_{\{|x|>\varepsilon\}} \max\left\{\left\|\Re\widetilde{g}_t(x)\right\|, \left\|\Im\widetilde{g}_t(x)\right\|\right\}^{\frac{1}{\Re\lambda_p(B)}-\delta_1} dx \le C\int_{\{|x|>\varepsilon\}} |x|^{\frac{-\rho_1}{\Re\lambda_p(B)}+\delta_1\rho_1} dx. \tag{102}$$

By (100),

$$\frac{-\rho_1}{\Re\lambda_p(B)} = \frac{\delta - \Re\lambda_1(D) - 1}{\Re\lambda_p(B)} < -1.$$

By choosing δ_1 so $\rho_1\delta_1$ is small enough, we obtain

$$\frac{-\rho_1}{\Re\lambda_p(B)} + \delta_1\rho_1 < -1,$$

implying that the integral (102) is finite. This shows that the first summand in (98) is finite, since

$$\int_{\{x:\|\widetilde{g}_t(x)\|<R\}} \left\|\widetilde{g}_t(x)\right\|^{\frac{1}{\Re\lambda_p(B)}+\delta_1} \left(\mathbf{1}_{\{|x|\le\varepsilon\}} + \mathbf{1}_{\{|x|>\varepsilon\}}\right) dx \le C' + \int_{\{|x|>\varepsilon\}} |x|^{\frac{-\rho_1}{\Re\lambda_p(B)}+\delta_1\rho_1} dx$$

by (102). Now, for any $\delta_2 > 0$,

$$\int_{\{|x|\le\varepsilon\}} \max\left\{\left\|\Re\widetilde{g}_t(x)\right\|, \left\|\Im\widetilde{g}_t(x)\right\|\right\}^{\frac{1}{\Re\lambda_1(B)}+\delta_2} dx \le C\int_{\{|x|\le\varepsilon\}} |x|^{\frac{-\rho_2}{\Re\lambda_1(B)}-\delta_2\rho_2} dx. \tag{103}$$

By (101), we see that

$$\frac{-\rho_2}{\Re\lambda_1(B)} = \frac{-\delta - \Re\lambda_p(D)}{\Re\lambda_1(B)} > -1.$$

Hence, by choosing δ_2 small enough we have

$$\frac{\rho_2}{\Re\lambda_1(B)} + \delta_2\rho_2 > -1,$$

and the integral (103) is also finite. If we write $\|\cdot\|_{q\times q}$ for the operator norm in $\mathbb{R}^{q\times q}$, then clearly $\|\widetilde{g}_t(x)\|_{2p\times 2p} \le 2\max\left\{\left\|\Re\widetilde{g}_t(x)\right\|_{p\times p}, \left\|\Im\widetilde{g}_t(x)\right\|_{p\times p}\right\}$. Hence, the conditions of Theorem 2.5 in Kremer and Scheffler [36] are satisfied, which implies that the process (50) exists by Proposition 5.10 in Kremer and Scheffler [35]. Moreover, by Corollary 5.11(b) in Kremer and Scheffler [35], the characteristic function of the candidate limiting process (50) at times $t_1, \ldots, t_n$ is given by

$$\begin{aligned}
&\exp\int_{\mathbb{R}}\int_{\mathbb{R}^{2p}} W\left(2\sum_{k=1}^{n} \mathbf{u}_k^*\left(\Re\widetilde{g}_{t_k}(y)\mathbf{z}_1 - \Im\widetilde{g}_{t_k}(y)\mathbf{z}_2\right)\right)\mu_{\widetilde{B}}(d\mathbf{z})dy \\
&\quad = \exp\int_{\mathbb{R}}\int_{S_0}\int_{\mathbb{R}_+} W\left(2\sum_{k=1}^{n} \mathbf{u}_k^*\left(\Re\widetilde{g}_{t_k}(y)\xi^B\boldsymbol{\theta} - \Im\widetilde{g}_{t_k}(y)\xi^B\boldsymbol{\theta}\right)\right)\frac{d\xi}{\xi^2}\lambda(d\boldsymbol{\theta})dy
\end{aligned} \tag{104}$$

(see (146)), where for notational simplicity we used the expression $W(y) = e^{\mathbf{i}y} - 1 - \mathbf{i}y$, $y \in \mathbb{R}$. Now, to establish the convergence (49), observe that the scaling relation

$$\widetilde{g}_{ct}\big(xc^{-1}\big)c^{-\widetilde{H}_2} = \widetilde{g}_t(x)c^{H+\frac{1}{2}I-\widetilde{H}_2} = \widetilde{g}_t(x)c^B$$

holds. So the characteristic function of the rescaled vector $\Big(c^{-\widetilde{H}_2}\widetilde{X}_H(ct_1), \dots, c^{-\widetilde{H}_2}\widetilde{X}_H(ct_1)\Big)^*$ is given by

$$\begin{aligned}
&\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{C}^p}\left[e^{\mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* c^{-\widetilde{H}_2}\widetilde{g}_{ct_k}(x)\mathbf{z}\right)} - 1 - \mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* c^{-\widetilde{H}_2}\widetilde{g}_{ct_k}(x)\mathbf{z}\right)\right]\mu(d\mathbf{z})dx\right\}\\
&= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^{2p}} W\left(2\sum_{k=1}^n \mathbf{u}_k^* c^{-\widetilde{H}_2}\big(\Re\widetilde{g}_{ct_k}(x)\mathbf{z}_1 - \Im\widetilde{g}_{ct_k}(x)\mathbf{z}_2\big)\right)\mu_{\mathbb{R}^{2p}}(d\mathbf{z})dx\right\},\\
&\overset{y=cx}{=} \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^{2p}} W\left(2\sum_{k=1}^n \mathbf{u}_k^* c^{-\widetilde{H}_2}\big(\Re\widetilde{g}_{ct_k}(yc^{-1})\mathbf{z}_1 - \Im\widetilde{g}_{ct_k}\big(yc^{-1}\big)\mathbf{z}_2\big)\right)\mu_{\mathbb{R}^{2p}}(d\mathbf{z})\frac{dy}{c}\right\},\\
&= \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^{2p}} W\left(2\sum_{k=1}^n \mathbf{u}_k^* \big(\Re\widetilde{g}_{ct_k}(y)c^B\mathbf{z}_1 - \Im\widetilde{g}_{ct_k}(y)c^B\mathbf{z}_2\big)\right)\mu_{\mathbb{R}^{2p}}(d\mathbf{z})\frac{dy}{c}\right\},\\
&\overset{(148)}{=} \exp\int_{\mathbb{R}}\int_{S_0}\int_{\mathbb{R}_+} W\left(2\sum_{k=1}^n \mathbf{u}_k^* \big(\Re\widetilde{g}_{t_k}(y)(cr)^B\boldsymbol{\theta} - \Im\widetilde{g}_{t_k}(y)(cr)^B\boldsymbol{\theta}\big)\right)q(r,\boldsymbol{\theta})\frac{dr}{r^2}\lambda(d\boldsymbol{\theta})\frac{dy}{c}\\
&\overset{\xi=cr}{=} \exp\int_{\mathbb{R}}\int_{S_0}\int_{\mathbb{R}_+} W\left(2\sum_{k=1}^n \mathbf{u}_k^* \big(\Re\widetilde{g}_{t_k}(y)\xi^B\boldsymbol{\theta} - \Im\widetilde{g}_{t_k}(y)\xi^B\boldsymbol{\theta}\big)\right)q\big(c^{-1}\xi,\boldsymbol{\theta}\big)\frac{d\xi}{\xi^2}\lambda(d\boldsymbol{\theta})dy\\
&\overset{c\to\infty}{\to} \exp\int_{\mathbb{R}}\int_{S_0}\int_{\mathbb{R}_+} W\left(2\sum_{k=1}^n \mathbf{u}_k^* \big(\Re\widetilde{g}_{t_k}(y)\xi^B\boldsymbol{\theta} - \Im\widetilde{g}_{t_k}(y)\xi^B\boldsymbol{\theta}\big)\right)\bigg)\frac{d\xi}{\xi^2}\lambda(d\boldsymbol{\theta})dy,
\end{aligned}$$

where the limit is again a consequence of the dominated convergence theorem and the relation (149). The conclusion follows.

Appendix C. Proofs: Section 4

Proof of Theorem 4.1: Let $\{Y_H(t)\}_{t\in\mathbb{R}} = \{X_H(-t)\}_{t\in\mathbb{R}}$ be the time-reversed process. First note that, if $\{f_t(\boldsymbol{\varpi}), t \in \mathbb{R}\} = \{g_t(s)\mathbf{z}, t \in \mathbb{R}\}$ is a minimal representation of X_H with respect to $\mathcal{B}$ mod κ, then

$$\{f_{-t}(\boldsymbol{\varpi}), t \in \mathbb{R}\} \text{ is a minimal representation of } Y_H \text{ with respect to } \mathcal{B} \mod \kappa. \tag{105}$$

We first show that (ii) ⇒ (i). Note that, by (57),

$$\begin{aligned}
g_{-t}(s)\mathbf{z} &= \Big\{\Big[(-t-s)_+^D - (-s)_+^D\Big]M_+ + \Big[(-t-s)_-^D - (-s)_-^D\Big]M_-\Big\}\mathbf{z}\\
&= \Big[(t+s)_-^D - (s)_-^D\Big]M_-\Big(M_+^{-1}M_-\Big)\mathbf{z} + \Big[(t+s)_+^D - s_+^D\Big]M_+\Big(M_+^{-1}M_-\Big)\mathbf{z} \quad \mu(d\mathbf{z})\text{-a.e.}
\end{aligned} \tag{106}$$

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So, for any $m\in\mathbb{N}$, fix $t_1<\ldots<t_m$, and pick any vectors $\mathbf{u}_1,\ldots,\mathbf{u}_m\in\mathbb{R}^p$. By the expressions (81) and (106), the finite-dimensional distributions of $\{X_H(-t)\}_{t\in\mathbb{R}}$ are given by

$$\mathbb{E}\exp\left\{\sum_{j=1}^{n}\mathbf{u}_j^* X_H(-t_j)\right\}=\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^p}\left(e^{\mathbf{i}\sum_{j=1}^{n}\mathbf{u}_j^* g_{-t_j}(s)\mathbf{z}}-1-\mathbf{i}\sum_{j=1}^{n}\mathbf{u}_j^* g_{-t_j}(s)\mathbf{z}\right)\mu(d\mathbf{z})ds\right\}$$

$$=\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^p}\left(e^{\mathbf{i}\sum_{j=1}^{n}\mathbf{u}_j^* g_{t_j}\left(s'\right)\mathbf{z}'}-1-\mathbf{i}\sum_{j=1}^{n}\mathbf{u}_j^* g_{t_j}\left(s'\right)\mathbf{z}'\right)\mu\left(d\mathbf{z}'\right)ds\right\}=\mathbb{E}\exp\left\{\sum_{j=1}^{n}\mathbf{u}_j^* X_H(t_j)\right\},$$

where we make the change of variable $\left(s',\mathbf{z}'\right)=\left(-s,\left(M_+^{-1}M_-\right)\mathbf{z}\right)$ and apply the conditions (57) and (58). Therefore, X_H is time-reversible. This establishes (i).

Now, we establish (i) $\Rightarrow$ (ii). So, suppose X_H is time-reversible. We first show that (57) holds. In terms of spectral representations, time-reversibility means that, for $f_t(\boldsymbol{\varpi})$ as in (54),

$$\left\{\int_{\mathbb{R}^{p+1}}f_t(\boldsymbol{\varpi})\widetilde{N}(d\boldsymbol{\varpi})\right\}_{t\in\mathbb{R}}\overset{\text{f.d.d.}}{=}\left\{\int_{\mathbb{R}^{p+1}}f_{-t}(\boldsymbol{\varpi})\widetilde{N}(d\boldsymbol{\varpi})\right\}_{t\in\mathbb{R}}.$$

By assumption and by (105), both $\{f_t(\boldsymbol{\varpi})\}_{t\in\mathbb{R}}$ and $\{f_{-t}(\boldsymbol{\varpi})\}_{t\in\mathbb{R}}$ are minimal representations of X_H on the space $\left(\mathbb{R}^{p+1},\mathcal{B}\left(\mathbb{R}^{p+1}\right),\kappa\right)$. Then, Proposition D.1 implies that there is a (unique modulo κ-null sets) mapping

$$\Phi:\mathbb{R}^{p+1}\to\mathbb{R}^{p+1},\quad \boldsymbol{\varpi}\mapsto(\Phi_1(\boldsymbol{\varpi}),\Phi_2(\boldsymbol{\varpi})),\quad \Phi_1(\boldsymbol{\varpi})\in\mathbb{R},\ \Phi_2(\boldsymbol{\varpi})\in\mathbb{R}^p,$$

such that, for all $t\in\mathbb{R}$,

$$f_{-t}(\boldsymbol{\varpi})=f_t(\Phi(\boldsymbol{\varpi}))\quad \kappa(d\boldsymbol{\varpi})\text{-a.e.}\tag{107}$$

So, for each $t\in\mathbb{R}$, let

$$V_t=\left\{\boldsymbol{\varpi}:\text{(107) holds at }\boldsymbol{\varpi}\right\}\in\mathcal{B}(\mathbb{R}\times\mathbb{R}^p).$$

Define $V=\bigcap_{t\in\mathbb{Q}}V_t$. Observe that

$$f_{-t}(\boldsymbol{\varpi})=f_t(\Phi(\boldsymbol{\varpi})),\quad \boldsymbol{\varpi}=(s,\mathbf{z})\in V,\quad t\in\mathbb{Q}.\tag{108}$$

Now, for notational simplicity, consider the integrand g_t with s in place of $-s$. Fix any

$$\boldsymbol{\varpi}_0=(s_0,\mathbf{z}_0)\in V,\quad s_0<0.\tag{109}$$

Let $\{t_n\}_{n\in\mathbb{N}}\subseteq\mathbb{Q}$ be a sequence such that $t_n\uparrow\infty$. For large enough n,

$$\left[(-t_n+s_0)_-^D-(s_0)_-^D\right]M_-\mathbf{z}_0$$
$$=\left(\left[\left(t_n+\Phi_1(\boldsymbol{\varpi}_0)\right)_+^D-\left(\Phi_1\left(\boldsymbol{\varpi}_0\right)\right)_+^D\right]M_+-\left(\Phi_1\left(\boldsymbol{\varpi}_0\right)\right)_-^D M_-\right)\Phi_2\left(\boldsymbol{\varpi}_0\right);$$

i.e.,

$$(-t_n+s_0)_-^D M_-\mathbf{z}_0-\left(t_n+\Phi_1(\boldsymbol{\varpi}_0)\right)_+^D M_+\Phi_2(\boldsymbol{\varpi}_0)$$
$$=(s_0)_-^D M_-\mathbf{z}_0-\left(\Phi_1(\boldsymbol{\varpi}_0)\right)_+^D M_+\Phi_2(\boldsymbol{\varpi}_0)-\left(\Phi_1(\boldsymbol{\varpi}_0)\right)_-^D M_-\Phi_2(\boldsymbol{\varpi}_0).\tag{110}$$

Consider the Jordan decomposition $D=PJ_DP^{-1}$. The right-hand side of (110) is a constant with respect to n. Therefore, after pre-multiplying both sides by P^{-1}, we can recast (110) in the form

$$(-t_n+s_0)_-^{J_D}P^{-1}M_-\mathbf{z}_0-(t_n+\Phi_1(\boldsymbol{\varpi}_0))_+^{J_D}P^{-1}M_+\Phi_2(\boldsymbol{\varpi}_0)=C\in M(p,\mathbb{R}).\tag{111}$$

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We want to show that

$$M_{-}\mathbf{z}_0 = M_{+}\Phi_2(\boldsymbol{\varpi}_0) \equiv M_{+}\Phi_2(s_0, \mathbf{z}_0). \tag{112}$$

Without loss of generality, we can assume J_D is a single Jordan block. In view of the condition (27), it suffices to consider two cases, namely, when J_D is a Jordan block associated with an eigenvalue d with positive real part or with negative real part. So, first assume $\Re(d) > 0$ and rewrite (111) as

$$\left(\frac{t_n + \Phi_1(\boldsymbol{\varpi}_0)}{t_n - s_0}\right)^{-J_D} P^{-1}M_{-}\mathbf{z}_0 - P^{-1}M_{+}\Phi_2(\boldsymbol{\varpi}_0) = (t_n + \Phi_1(\boldsymbol{\varpi}_0))^{-J_D}C.$$

If $C \neq \mathbf{0}$, by taking $n \to \infty$, we arrive at a contradiction, since

$$\lim_{n\to\infty} \frac{t_n + \Phi_1(\boldsymbol{\varpi}_0)}{t_n - s_0} = 1$$

and $\lim_{n\to\infty} \|(t_n - s_0)^{-J_D}C\| = \infty$. Therefore,

$$(-t_n + s_0)_{-}^{J_D} P^{-1}M_{-}\mathbf{z}_0 = (t_n + \Phi_1(\boldsymbol{\varpi}_0))_{+}^{J_D} P^{-1}M_{+}\Phi_2(\boldsymbol{\varpi}_0). \tag{113}$$

Alternatively, assume $\Re(d) < 0$. Rewrite (111) as

$$P^{-1}M_{-}\mathbf{z}_0 - \left(\frac{t_n + \Phi_1(\boldsymbol{\varpi}_0)}{t_n - s_0}\right)^{J_D} P^{-1}M_{+}\Phi_2(\boldsymbol{\varpi}_0) = (t_n - s_0)^{-J_D}C.$$

Again by taking $n \to \infty$, we arrive at a contradiction unless $C = \mathbf{0}$. So (113) also holds. Thus, in any case, by taking $n \to \infty$ we conclude that (112) holds, as we wanted to show.

Still for $s_0 < 0$, now let $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ be a sequence such that $t_n \downarrow -\infty$. Then, for large enough n,

$$\begin{aligned}&\Big\{\Big[\big(-t_n + s_0\big)_{+}^{D}\Big]M_{+} + \Big[-(s_0)_{-}^{D}\Big]M_{-}\Big\}\mathbf{z}_0\\ &= \Big\{\Big[-(\Phi_1(\boldsymbol{\varpi}_0))_{+}^{D}\Big]M_{+} + \Big[\big(t_n + \Phi_1(\boldsymbol{\varpi}_0)\big)_{-}^{D} - (\Phi_1(\boldsymbol{\varpi}_0))_{-}^{D}\Big]M_{-}\Big\}\Phi_2(\boldsymbol{\varpi}_0).\end{aligned}$$

By an argument analogous to the one leading to (112), we conclude that

$$M_{+}\mathbf{z}_0 = M_{-}\Phi_2(\boldsymbol{\varpi}_0) \equiv M_{-}\Phi_2(s_0, \mathbf{z}_0). \tag{114}$$

As a consequence of (112) and (114), for arbitrary $(s, \mathbf{z}) \in V$ with $s < 0$,

$$\Phi_2(s, \mathbf{z}) = M_{-}^{-1}M_{+}\mathbf{z} = M_{+}^{-1}M_{-}\mathbf{z}. \tag{115}$$

This establishes (57) for all $(s, \mathbf{z}) \in V$ such that $s < 0$. Now let

$$\boldsymbol{\varpi}_0 = (s_0, \mathbf{z}_0) \in V, \quad s_0 > 0. \tag{116}$$

Let $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ be a sequence such that $t_n \uparrow \infty$ as $n \to \infty$. By adapting the arguments for showing (113), we conclude that

$$(t_n - s_0)_{+}^{D}M_{-}\mathbf{z}_0 = (t_n + \Phi_1(\boldsymbol{\varpi}_0))_{+}^{D}M_{+}\Phi_2(\boldsymbol{\varpi}_0). \tag{117}$$

Thus, by adapting the argument we conclude that the relation (112) holds also for $s > 0$. Similarly, the relation (114) holds for $s > 0$.

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In summary, we conclude that

$$\Phi_2(s, \mathbf{z}) = M_-^{-1} M_+ \mathbf{z} = M_+^{-1} M_- \mathbf{z}, \quad (s, \mathbf{z}) \in V, \quad s \neq 0. \tag{118}$$

In other words, (57) holds, as we wanted to show.

Next, we show that (58) holds. So, fix again $\boldsymbol{\varpi}_0 = (s_0, \mathbf{z}_0) \in V$, $s_0 < 0$, as in (109). Consider a sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ such that $t_n \uparrow \infty$. Then, for some fixed large n (depending on $(s_0, \mathbf{z}_0)$), the expressions (111) (with $C = \mathbf{0}$) and (115) imply that

$$\left(\frac{(-t_n + s_0)_-}{t_n + \Phi_1(s_0, \mathbf{z}_0)}\right)^D M_- \mathbf{z}_0 = \left(\frac{t_n - s_0}{t_n + \Phi_1(s_0, \mathbf{z}_0)}\right)^D M_- \mathbf{z}_0 = M_+ \Phi_2(s_0, \mathbf{z}_0) = M_- \mathbf{z}_0.$$

In particular, 1 is an eigenvalue of

$$\left(\frac{t_n - s_0}{t_n + \Phi_1(s, \mathbf{z}_0)}\right)^D$$

with corresponding eigenvector $M_- \mathbf{z}_0$. However, in view of the condition (27), $\mathrm{eig}(D) \cap \{0\} = \emptyset$. Hence,

$$\{1\} \in \mathrm{eig}\left(\left(\frac{t_n - s_0}{t_n + \Phi_1(s, \mathbf{z})}\right)^D\right) = \left\{w \in \mathbb{C} : w = \left(\frac{t_n - s_0}{t_n + \Phi_1(s, \mathbf{z})}\right)^d, \, d \in \mathrm{eig}(D)\right\}$$
$$\Leftrightarrow \frac{t_n - s_0}{t_n + \Phi_1(s_0, \mathbf{z}_0)} = 1.$$

Thus,

$$\Phi_1(s_0, \mathbf{z}_0) = -s_0.$$

Now fix again $\boldsymbol{\varpi}_0 = (s_0, \mathbf{z}_0) \in V$, $s_0 > 0$, as in (116), and let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \uparrow \infty$ as $n \to \infty$. Hence, (117) holds. Then, by the explicit expression (118) for Φ_2, we conclude that 1 is an eigenvalue of

$$\left(\frac{t_n - s_0}{t_n + \Phi_1(\boldsymbol{\varpi}_0)}\right)^D.$$

Thus, once again we arrive at the relation $t_n + s_0 = t_n - \Phi_1(\boldsymbol{\varpi}_0)$, i.e.,

$$\Phi_1(\boldsymbol{\varpi}_0) = -s_0.$$

Since $(s_0, \mathbf{z}_0)$ was arbitrary, we conclude that

$$\Phi_1(s, \mathbf{z}) = -s \quad \text{for all } (s, \mathbf{z}) \in V.$$

In particular,

$$\Phi(s, \mathbf{z}) = (-s, M_-^{-1} M_+ \mathbf{z}) \quad \kappa(ds, d\mathbf{z})\text{-a.e.}$$

However, by Proposition D.1(ii), the mapping Φ is a measure space isomorphism from the space $(\mathbb{R} \times \mathbb{R}^p, \mathcal{B}(\mathbb{R} \times \mathbb{R}^p), \kappa)$ to itself. In particular, the expression (134) holds with $\kappa_1 = \kappa = \kappa_2$. Since, in addition, $\eta[-1, 0] = 1 = \eta[0, 1]$, we have

$$\mu(B) = \kappa([0, 1] \times B) = \kappa \circ \Phi^{-1}\big([0, 1] \times B)\big) = \kappa\Big([-1, 0] \times \Phi_2^{-1}(B)\Big) = \mu\big(\Phi_2^{-1}(B)\big)$$

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for any Borel set $B \in \mathcal{B}(\mathbb{R}^p)$. Hence, the expression (58) holds. This shows that (i) $\Rightarrow$ (ii). Therefore, (i) $\Leftrightarrow$ (ii), as claimed.

We now show that (a) $\Leftrightarrow$ (a′). So, suppose (a) in (ii) holds. Then, for each $s, t \in \mathbb{R}$, $M_+\mathbf{z} = M_-M_+^{-1}M_-\mathbf{z}$ $\mu(d\mathbf{z})$-a.e. Thus,

$$\begin{aligned} g_{-t}(-s)\mathbf{z} &= \Big\{\Big[(-t+s)_+^D - s_+^D\Big]M_+ + \Big[(-t+s)_-^D - s_-^D\Big]M_-\Big\}\mathbf{z} \\ &= \Big\{\Big[(-t+s)_+^D - s_+^D\Big]M_- + \Big[(-t+s)_-^D - s_-^D\Big]M_+\Big\}M_+^{-1}M_-\mathbf{z} \\ &= \Big\{\Big[(t-s)_-^D - (-s)_-^D\Big]M_- + \Big[(t-s)_+^D - (-s)_+^D\Big]M_+\Big\}M_+^{-1}M_-\mathbf{z} = g_t(s)M_+^{-1}M_-\mathbf{z}. \end{aligned}$$

Analogously, $g_{-t}(-s)\mathbf{z} = g_t(s)M_-^{-1}M_+\mathbf{z}$ $\mu(d\mathbf{z})$-a.e. Hence, (a′) holds. In turn, assuming (a′), for fixed s, by taking large enough t, through reasoning similar to that of the argument leading to (113), we obtain the relation

$$\left(\frac{t+s}{t-s}\right)^D M_+M_-^{-1}M_+\mathbf{z} = M_-\mathbf{z} \quad \mu(d\mathbf{z})\text{-a.e.}$$

By taking the limit $t \to \infty$, we obtain (57). Thus, (a) holds. In other words, (a) $\Leftrightarrow$ (a′), as claimed.

Proof of Theorem 4.2: As in the proof of Theorem 4.1, let $\{\widetilde{Y}_H(t)\}_{t\in\mathbb{R}} = \{\widetilde{X}_H(-t)\}_{t\in\mathbb{R}}$ be the time-reversed process. First note that, if $\{f_t(\varpi), t \in \mathbb{R}\} = \{\widetilde{g}_t(x)\mathbf{z}, t \in \mathbb{R}\}$ is a minimal representation of $\widetilde{X}_H$ with respect to $\mathcal{B} \mod \kappa$, then

$$\{f_{-t}(\varpi), t \in \mathbb{R}\} \text{ is a minimal representation of } \widetilde{Y}_H \text{ with respect to } \mathcal{B} \mod \kappa. \tag{119}$$

First, we show that the condition (66) implies time-reversibility. Observe that

$$\begin{aligned} \Re\big(\widetilde{g}_t(x)\mathbf{z}\big) &= \Re\left(\frac{e^{\mathbf{i}tx}-1}{\mathbf{i}x}\left[x_+^{-D}A + x_-^{-D}\overline{A}\right]\mathbf{z}\right) \\ &= \Re\left(\left(\frac{e^{\mathbf{i}tx}-1}{-\mathbf{i}x}\right)\left[x_+^{-D}\overline{A}(-\overline{A}^{-1}A) + x_-^{-D}A(-A^{-1}\overline{A})\right]\mathbf{z}\right) \\ &= \Re\left(\left(\frac{e^{-\mathbf{i}tx'}-1}{\mathbf{i}x'}\right)\left[(x')_-^{-D}\overline{A} + (x')_+^{-D}A\right]\mathbf{z}'\right) = \Re\big(\widetilde{g}_{-t}(x')\mathbf{z}'\big), \end{aligned}$$

where $(x', \mathbf{z}') = \left(-x, -A^{-1}\overline{A}\mathbf{z}\mathbf{1}_{\{x<0\}} - \overline{A}^{-1}A\mathbf{z}\mathbf{1}_{\{x>0\}}\right) =: \Psi(x, \mathbf{z})$. Then,

$$\Re\big(\widetilde{g}_t(x)\mathbf{z}\big) = f_t(\varpi) = f_{-t}\big(\Psi^{-1}(\varpi)\big).$$

For $\widetilde{\kappa}(d\varpi) = dx \otimes \widetilde{\mu}(d\mathbf{z})$, recall that we define

$$\widetilde{\mu}(d\mathbf{z}) = \frac{\mu(d\mathbf{z}) + \mu\big(\overline{d\mathbf{z}}\big)}{2}.$$

Also observe that, by the condition (66), $\widetilde{\mu}(d\mathbf{z}) = \widetilde{\mu}\big(-A^{-1}\overline{A}d\mathbf{z}\big)$. We now show that

$$\widetilde{\kappa} = \widetilde{\kappa} \circ \Psi^{-1}. \tag{120}$$

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Indeed, if $I \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{B}(\mathbb{C}^p)$, $I\pm = I \cap \mathbb{R}_\pm$, then

$$\widetilde{\kappa}(I_+ \times B) = \eta(I_+)\widetilde{\mu}(B) = \eta(-I_+)\widetilde{\mu}\big(-A^{-1}\overline{A}B\big) = \widetilde{\kappa} \circ \Psi^{-1}(I_+ \times B)$$

and

$$\widetilde{\kappa}(I_- \times B) = \eta(I_-)\widetilde{\mu}(B) = \eta(-I_-)\widetilde{\mu}\big(-\overline{A}^{-1}AB\big) = \widetilde{\kappa} \circ \Psi^{-1}(I_- \times B).$$

This shows that $\widetilde{\kappa}(I \times B) = \widetilde{\kappa} \circ \Psi^{-1}(I \times B)$, which in turn implies that the measures coincide on $\mathcal{B}(\mathbb{R} \times \mathbb{C}^p)$. This establishes (120).

Therefore, starting from (78), by a change of variables,

$$\begin{aligned}
&\mathbb{E}\exp\left\{\mathbf{i}\sum_{k=1}^n \mathbf{u}_k^* \widetilde{X}_H(t_k)\right\}\\
&= \exp\left\{\int_{\mathbb{R}\times\mathbb{C}^p}\left[e^{\mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{t_k}(\boldsymbol{\varpi})\right)} - 1 - \mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{t_k}(\boldsymbol{\varpi})\right)\right]\widetilde{\kappa}(d\boldsymbol{\varpi})\right\}\\
&= \exp\left\{\int_{\mathbb{R}\times\mathbb{C}^p}\left[e^{\mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{-t_k}(\Psi^{-1}(\boldsymbol{\varpi}))\right)} - 1 - \mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{-t_k}\big(\Psi^{-1}(\boldsymbol{\varpi})\big)\right)\right]\widetilde{\kappa}(d\boldsymbol{\varpi})\right\}\\
&= \exp\left\{\int_{\mathbb{R}\times\mathbb{C}^p}\left[e^{\mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{-t_k}(\boldsymbol{\varpi})\right)} - 1 - \mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{-t_k}(\boldsymbol{\varpi})\right)\right]\big(\widetilde{\kappa}\circ\Psi^{-1}\big)(d\boldsymbol{\varpi})\right\}\\
&= \exp\left\{\int_{\mathbb{R}\times\mathbb{C}^p}\left[e^{\mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{-t_k}(\boldsymbol{\varpi})\right)} - 1 - \mathbf{i}2\Re\left(\sum_{k=1}^n \mathbf{u}_k^* f_{-t_k}(\boldsymbol{\varpi})\right)\right]\widetilde{\kappa}(d\boldsymbol{\varpi})\right\}\\
&= \mathbb{E}\exp\left\{\mathbf{i}\sum_{k=1}^n \mathbf{u}_k^* \widetilde{X}_H(-t_k)\right\}.
\end{aligned}$$

This shows that $\widetilde{X}_H$ is time-reversible.

Now suppose that $\widetilde{X}_H$ is time-reversible, i.e., that

$$\left\{\int_{\mathbb{R}^{p+1}} f_{-t}(\boldsymbol{\varpi})\widetilde{N}(d\boldsymbol{\varpi})\right\}_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{=} \left\{\int_{\mathbb{R}^{p+1}} f_t(\boldsymbol{\varpi})\widetilde{N}(d\boldsymbol{\varpi})\right\}_{t\in\mathbb{R}}.$$

Recall that $\widetilde{\kappa}(d\boldsymbol{\varpi}) \equiv dx \otimes \widetilde{\mu}(d\mathbf{z})$. Under the condition (64), Proposition D.1 implies that there exists a measurable bijection

$$\Phi : \mathbb{R}^{p+1} \to \mathbb{R}^{p+1}, \quad \boldsymbol{\varpi} \mapsto \big(\Phi_1(\boldsymbol{\varpi}), \Phi_2(\boldsymbol{\varpi})\big) \in \mathbb{R} \times \mathbb{R}^p,$$

such that, for all $t \in \mathbb{R}$,

$$\Re\big(\widetilde{g}_t(x)\mathbf{z}\big) = \Re\big(\widetilde{g}_{-t}\big(\Phi_1(x,\mathbf{z})\big)\Phi_2(x,\mathbf{z})\big) \quad dx \otimes \widetilde{\mu}(d\mathbf{z})\text{-a.e.} \tag{121}$$

Moreover, by Proposition D.1, the mapping Φ is a measure space isomorphism from the space $\big(\mathbb{R} \times \mathbb{C}^p, \mathcal{B}\big(\mathbb{R} \times \mathbb{C}^p\big), \widetilde{\kappa}\big)$ to itself. In particular, it also preserves the measure $\widetilde{\kappa}$, the same being true of Φ^{-1} (cf. (134)). Noting that the set $[0, 1]$ has Lebesgue measure 1, we have

$$\widetilde{\mu}(B) = \widetilde{\kappa}([0,1] \times B) = \widetilde{\kappa} \circ \Phi^{-1}([0,1] \times B), \quad \widetilde{\kappa}\Big([0,1] \times \big(-A^{-1}\overline{AB}\big)\Big) = \widetilde{\mu}\Big(-A^{-1}\overline{AB}\Big).$$

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Our goal is to show that

$$\widetilde{\kappa} \circ \Phi^{-1}([0,1] \times B) = \widetilde{\kappa}\Big([0,1] \times \big(-A^{-1}\overline{AB}\big)\Big), \tag{122}$$

whence (66) is established. For this purpose, we need to conveniently re-express Φ.

So, for each t, let

$$\widetilde{V}_t = \{\boldsymbol{\varpi} : (121) \text{ holds at } \boldsymbol{\varpi}\} \in \mathcal{B}(\mathbb{R} \times \mathbb{R}^p),$$

and set $\widetilde{V} = \bigcap_{t \in \mathbb{Q}} \widetilde{V}_t$. In particular, $\widetilde{\kappa}\big(\widetilde{V}_t^c\big) = 0$. Fix a vector $(x_0, \mathbf{z}_0) \in \widetilde{V}$ with

$$x_0 > 0, \tag{123}$$

and note that $A\mathbf{z}_0 \neq 0$ as a consequence of the condition (65). For notational simplicity, write

$$\Phi_1 = \Phi_1(x_0, \mathbf{z}_0), \qquad \Phi_2 = \Phi_2(x_0, \mathbf{z}_0).$$

By (121), for all $t \in \mathbb{Q}$,

$$\Re\left(\frac{e^{\mathbf{i}tx_0} - 1}{\mathbf{i}x_0}(x_0)_+^{-D} A\mathbf{z}_0\right) = \Re\left(\frac{e^{-\mathbf{i}t\Phi_1} - 1}{\mathbf{i}\Phi_1}\Big[(\Phi_1)_+^{-D} A + (\Phi_1)_-^{-D}\overline{A}\Big]\Phi_2\right). \tag{124}$$

By continuity in t, the relation (124) holds for all $t \in \mathbb{R}$. Taking derivatives in t, we find that for all integers $k \geq 0$,

$$\Re\Big((\mathbf{i}x)^k e^{\mathbf{i}tx_0}(x_0)_+^{-D} A\mathbf{z}_0\Big) = \Re\Big(-(-\mathbf{i}\Phi_1)^k e^{-\mathbf{i}t\Phi_1}\Big[(\Phi_1)_+^{-D} A + (\Phi_1)_-^{-D}\overline{A}\Big]\Phi_2\Big), \quad t \in \mathbb{R}. \tag{125}$$

By taking $k = 4$ and $k = 0$, respectively, we arrive at the equalities

$$\begin{aligned} x_0^4 \Re\Big(e^{\mathbf{i}tx_0}(x_0)_+^{-D} A\mathbf{z}_0\Big) &= \Phi_1^4 \Re\Big(-e^{-\mathbf{i}t\Phi_1}\Big[(\Phi_1)_+^{-D} A + (\Phi_1)_-^{-D}\overline{A}\Big]\Phi_2\Big) \\ &= \Phi_1^4 \Re\Big(e^{\mathbf{i}tx}(x_0)_+^{-D} A\mathbf{z}_0\Big). \end{aligned}$$

In particular,

$$\text{either } \Phi_1 = x_0 \text{ or } \Phi_1 = -x_0. \tag{126}$$

In view of (126), first suppose $\Phi_1 = x_0 > 0$. Then, from (125) with $k = 0$, and by the invertibility of the matrix $(x_0)_+^{-D}$,

$$\Re\big(e^{\mathbf{i}tx_0} A\mathbf{z}_0\big) = \Re\big(-e^{-\mathbf{i}tx} A\Phi_2\big), \qquad t \in \mathbb{R}. \tag{127}$$

Hence, by choosing $t = 0$ in (127), we see that $\Re(A\mathbf{z}_0) = \Re(-A\Phi_2)$. On the other hand, by choosing t such that $e^{\mathbf{i}tx_0} = \mathbf{i}$ in (127), we see that $\Im(A\mathbf{z}_0) = -\Im(-A\Phi_2)$. This implies that $A\mathbf{z}_0 = -\overline{A\Phi_2}$. Hence,

$$\Phi_2(x_0, \mathbf{z}_0) = -A^{-1}\overline{A\mathbf{z}_0}, \qquad \text{if } \Phi_1(x_0, \mathbf{z}_0) = x_0. \tag{128}$$

Alternatively, suppose $\Phi_1 = -x_0$ in (126). From (125) with $k = 0$, we obtain

$$\Re\big(e^{\mathbf{i}tx_0} A\mathbf{z}_0\big) = \Re\Big(-e^{\mathbf{i}tx_0}\overline{A}\Phi_2\Big), \qquad t \in \mathbb{R}.$$

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As with the relation (127), this implies that $\Re(A\mathbf{z}_0) = \Re(-\overline{A}\Phi_2)$, $\Im(A\mathbf{z}_0) = \Im(-\overline{A}\Phi_2)$ through an appropriate choice of t. In other words, $A\mathbf{z}_0 = -\overline{A}\Phi_2$. Hence,

$$\Phi_2(x_0, \mathbf{z}_0) = -\overline{A}^{-1} A\mathbf{z}_0, \quad \text{if } \Phi_1(x_0, \mathbf{z}_0) = -x_0. \tag{129}$$

Consequently, for each $(x, \mathbf{z}) \in \widetilde{V}$ with $x > 0$ (namely, under the condition (123)), either (128) or (129) holds. An analogous argument shows that $|\Phi_1(x, \mathbf{z})| = |x|$ for $x < 0$. It also shows that, for each fixed $(x_0, \mathbf{z}_0)$ with $x_0 < 0$,

$$\Phi_2(x_0, \mathbf{z}_0) = \begin{cases} -\overline{A}^{-1} A\overline{\mathbf{z}}, & \text{if } \Phi_1(x_0, \mathbf{z}_0) = x_0; \\ -A^{-1}\overline{A}\mathbf{z}, & \text{if } \Phi_1(x_0, \mathbf{z}_0) = -x_0, \end{cases} \quad \text{when } x_0 < 0. \tag{130}$$

Define the sets $\widetilde{V}_\pm = \{(x, \mathbf{z}) \in \widetilde{V}: \Phi_1(x, \mathbf{z}) = \pm x\}$. In view of (128), (129), and (130), $\widetilde{V}_-$ and $\widetilde{V}_+$ partition $\widetilde{V}$. Now define the functions

$$\Psi(s, \mathbf{z}) = \left(x, -\overline{A}^{-1} A\overline{\mathbf{z}}\mathbf{1}_{\{x<0\}} - A^{-1}\overline{A\mathbf{z}}\mathbf{1}_{\{x>0\}}\right), \qquad \gamma(x, \mathbf{z}) = (-x, \overline{\mathbf{z}}).$$

Then we can write

$$\Phi(x, \mathbf{z}) = \Psi(x, \mathbf{z})\mathbf{1}_{\widetilde{V}_+}(s, \mathbf{z}) + (\Psi \circ \gamma)(x, \mathbf{z})\mathbf{1}_{\widetilde{V}_-}(s, \mathbf{z}). \tag{131}$$

Observe that Ψ, γ are bijective, and also that

$$\widetilde{\kappa} \circ \gamma^{-1} = \widetilde{\kappa}. \tag{132}$$

Consider any set $B \in \mathcal{B}(\mathbb{R}^p)$. For notational simplicity, write $[0, 1] \times B = S_+ \cup S_-$, where $S_\pm = ([0, 1] \times B) \cap \widetilde{V}_\pm$. Since S_- and S_+ are disjoint, we have

$$\widetilde{\kappa} \circ \Phi^{-1}(S_+ \cup S_-) = \widetilde{\kappa} \circ \Phi^{-1}(S_+) + \widetilde{\kappa} \circ \Phi^{-1}(S_-). \tag{133}$$

Therefore, by the relations (131), (132), and (133),

$$\widetilde{\kappa}([0, 1] \times B) = \widetilde{\kappa} \circ \Phi^{-1}([0, 1] \times B) = \widetilde{\kappa} \circ \Phi^{-1}(S_+ \cup S_-) = \widetilde{\kappa} \circ \Psi^{-1}(S_+) + \widetilde{\kappa} \circ \gamma^{-1} \circ \Psi^{-1}(S_-)$$

$$= \widetilde{\kappa} \circ \Psi^{-1}(S_+) + \widetilde{\kappa} \circ \Psi^{-1}(S_-) = \widetilde{\kappa} \circ \Psi^{-1}(B) = \widetilde{\kappa}\big([0, 1] \times (-A^{-1}\overline{AB})\big).$$

This shows (122). Therefore, (66) holds.

Appendix D. On the uniqueness of stochastic integral representations

To characterize the uniqueness of stochastic integral representations, we first recap the concept of isomorphism between measurable spaces, as well as related notions.

Definition D.1. Consider the following definitions.

(i) An *isomorphism between two measurable spaces* $(\overline{\Omega}_i, \mathcal{B}_i)$, $i = 1, 2$, is a bijection $\Phi: \overline{\Omega}_1 \to \overline{\Omega}_2$ such that both Φ and Φ^{-1} are measurable.

(ii) A measurable space $(\overline{\Omega}, \mathcal{B})$ is called a *Borel space* if it is isomorphic (in the sense of (i)) to a complete separable metric space endowed with its Borel σ-algebra.

(iii) A Borel space endowed with a σ-finite measure is called a *σ-finite Borel space*.

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(iv) An *isomorphism modulo null sets between two measure spaces* $\big(\overline{\Omega}_i, \mathcal{B}_i, \kappa_i\big)$, $i = 1, 2$, is a bijection $\Phi{:}\overline{\Omega}_1 \setminus A_1 \to \overline{\Omega}_2 \setminus A_2$, where $A_1 \in \mathcal{B}_1$ and $A_2 \in \mathcal{B}_2$ are null sets, such that both Φ and Φ^{-1} are measurable and

$$\kappa_1(A) = \kappa_2(\Phi(A)) \tag{134}$$

for all measurable $A \subseteq \overline{\Omega}_1 \setminus A_1$. Two isomorphisms Φ, Ψ are considered *equal modulo null* sets if $\Phi(\boldsymbol{\varpi}) = \Psi(\boldsymbol{\varpi})$ for κ_1-almost all $\boldsymbol{\varpi} \in \overline{\Omega}_1$.

In the following proposition, we establish the uniqueness of finite-second-moment stochastic integral representations of processes based on compensated Poisson random measures. The proof is omitted since it is similar to that of Theorem 2.17 in Kabluchko and Stoev [32], originally involving univariate integrals.

Proposition D.1 *For $\emptyset \neq T \subseteq \mathbb{R}$, let $X = \{X(t)\}_{t\in T}$ be an ID stochastic process with stochastic representation of the form* (55). *For $i = 1, 2$, let*

$$\Big\{f_t^{(i)}\Big\}_{t\in T} \subseteq L^2\big(\overline{\Omega}_i, \mathcal{B}_i, \kappa_i\big) \tag{135}$$

be two minimal representations of X, where

$$\overline{\Omega}_i = \mathbb{R} \times \mathbb{R}^q \in \mathcal{B}\big(\mathbb{R} \times \mathbb{R}^q\big), \quad \mathcal{B}_i = \mathcal{B}\big(\overline{\Omega}_i\big). \tag{136}$$

(*i*) *If $\big(\overline{\Omega}_1, \mathcal{B}_1, \kappa_1\big)$ is a (σ-finite) Borel space, then there is a measurable map $\Phi : \overline{\Omega}_2 \to \overline{\Omega}_1$ such that $\kappa_1 = \kappa_2 \circ \Phi^{-1}$ and, for all $t \in T$,*

$$f_t^{(2)}(\boldsymbol{\varpi}) = f_t^{(1)}[\Phi(\boldsymbol{\varpi})] \quad \text{for } \kappa_2\text{-almost all } \boldsymbol{\varpi} \in \overline{\Omega}_2.$$

(*ii*). *If both $\big(\overline{\Omega}_i, \mathcal{B}_i, \kappa_i\big)$, $i = 1, 2$, are (σ-finite) Borel spaces, then the mapping Φ in (i) is a measure space isomorphism and it is unique modulo null sets.*

Appendix E. Auxiliary results

As pointed out in Remark 3.2, there are instances of ofLm—hence, stochastic processes with proper distributions—whose stochastic integral representations have integrands with deficient rank a.e. In fact, for $x \neq 0$, define the integrand

$$\begin{aligned} L^2(\mathbb{R}, M(p, \mathbb{R})) \ni \widetilde{h}_t(x) &= \Re\widetilde{g}_t(x) - \Im\widetilde{g}_t(x) \\ &= \left\{\frac{\sin(tx)}{x} - \frac{(1-\cos(tx))}{x}\right\}|x|^{-D}\Re(A) \\ &\quad - \left\{\frac{\sin(tx)}{x} + \frac{(1-\cos(tx))}{x}\right\}\left\{x_+^{-D} - x_-^{-D}\right\}\Im(A), \quad t \in \mathbb{R}. \end{aligned} \tag{137}$$

Also, in (36), suppose $\mu_{\mathbb{R}^{2p}}(d\mathbf{z}) = \delta_{\mathbf{1}}(d\mathbf{z})$, where $\mathbf{1} = (1, \ldots, 1)^* \in \mathbb{R}^{2p}$. Then, for $\mathbf{1} \in \mathbb{R}^p$ and by the relation (79), we can express the rhofLm $\widetilde{X}_H$ as

$$\mathbb{E}\widetilde{X}_H(t)\widetilde{X}_H(t)^* = \int_{\mathbb{R}} \widetilde{h}_t(x)\mathbf{1}\mathbf{1}^*\widetilde{h}_t(x)^* dx, \quad t \neq 0. \tag{138}$$

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Note that the integrand on the right-hand side of (138) has rank 1 a.e. However, the properness condition (30) may still be satisfied, as we show next.

Lemma E.1 *For* $p=2$, *let* $A=\Re(A)+\mathbf{i}\Im(A)\in M(2,\mathbb{C})$ *be such that* $\Re(A)\mathbf{1}=\mathbf{0}$ *and* $\Im(A)=I$. *Let* $\widetilde{X}_H=\{\widetilde{X}_H(t)\}_{t\in\mathbb{R}}$ *be as in (138). Then there exist* $-\frac{1}{2}<d_1, d_2<\frac{1}{2}$ *such that, for* $D=\operatorname{diag}(d_1, d_2)=H-(1/2)I$, *the matrix* $\mathbb{E}\widetilde{X}_H(t)\widetilde{X}_H(t)^*$ *has full rank matrix for any* $t\neq 0$.

Proof. Fix $t\neq 0$. Let

$$M:=\Im(A)\mathbf{1}\mathbf{1}^*\Im(A)^*=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}. \tag{139}$$

For $\widetilde{h}_t$ as in (137), the expression (138) implies that

$$\begin{aligned}\mathbb{E}\widetilde{X}_H(t)\widetilde{X}_H(t)^* &= \int_{\mathbb{R}}\left\{\left(\frac{\sin(tx)}{x}\right)^2+\left(\frac{1-\cos(tx)}{x}\right)^2\right\}|x|^{-D}M|x|^{-D^*}dx\\ &= \int_{\mathbb{R}} 2\frac{(1-\cos(tx))}{x^2}|x|^{-D}M|x|^{-D^*}dx,\end{aligned}$$

where we make use of the fact that

$$\frac{\sin(tx)(1-\cos(tx))}{x^2}$$

is an odd function. For $-1<\delta<1$, we can write

$$\mathbb{R}\ni\int_{\mathbb{R}}2\frac{(1-\cos(tx))}{x^2}|x|^{-\delta}dx=|t|^{1+\delta}\int_{\mathbb{R}}2\frac{(1-\cos(y))}{y^2}|y|^{-\delta}dy=:|t|^{1+\delta}\beta(\delta),$$

where we make the change of variable $y=tx$. Then,

$$\det\left(\mathbb{E}\widetilde{X}_H(t)\widetilde{X}_H(t)^*\right)=|t|^{2+2(d_1+d_2)}\left(\beta(2d_1)\beta(2d_2)-\beta^2(d_1+d_2)\right).$$

In particular, for all $t\neq 0$, $\det\left(\mathbb{E}\widetilde{X}_H(t)\widetilde{X}_H(t)^*\right)=0$ if and only if

$$\beta(2d_1)\beta(2d_2)-\beta^2(d_1+d_2)=0.$$

However, the latter condition cannot hold for every $-\frac{1}{2}<d_1, d_2<\frac{1}{2}$ (cf. Remark 4.1 in Didier and Pipiras [23]). This establishes the claim. □

The following proposition is used in the proof of Proposition 3.1. It establishes orthogonal-increment random measures that can be used to yield integral representations for maofLm and rhofLm (cf. Rozanov [59], Section 1.3).

Proposition E.1.

(*i*) *Let* $\mathcal{M}(ds)$ *be the random measure* (20) *under the assumption* (39). *Then the random measure*

$$\Phi_{\mathcal{M}}(a,b]:=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{\mathbf{i}sb}-e^{\mathbf{i}sa}}{\mathbf{i}s}\mathcal{M}(ds),\qquad -\infty<a\leq b<\infty,$$

defined on intervals, naturally extends to a $\mathbb{C}^p$*-valued orthogonal-increment random measure on* $\mathcal{B}(\mathbb{R})$. *In particular,*

$$\mathbb{E}\Phi_{\mathcal{M}}(B)=0,\qquad \text{for all } B\in\mathcal{B}(\mathbb{R}) \text{ with } \eta(B)<\infty, \tag{140}$$

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and

$$\mathbb{E}\Phi_{\mathcal{M}}(dy)\Phi_{\mathcal{M}}(dy)^* = dy \times I. \tag{141}$$

Furthermore, for $f \in L^2(\mathbb{R}; M(p, \mathbb{R}))$,

$$\int_{\mathbb{R}} f(s)\mathcal{M}(ds) = \int_{\mathbb{R}} \widehat{f}(x)\Phi_{\mathcal{M}}(dx) \quad \text{a.s.}, \tag{142}$$

where, in the Parseval-type relation (142)*, we define, entrywise,*

$$\widehat{f} = \mathcal{F}(f). \tag{143}$$

(*ii*) *Let* $\widetilde{\mathcal{M}}(dx)$ *be the random measure* (14) *under the assumption* (40)*. Then the expression*

$$\Phi_{\widetilde{\mathcal{M}}}(a, b] := \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-\mathbf{i}xa} - e^{-\mathbf{i}xb}}{\mathbf{i}x}\widetilde{\mathcal{M}}(dx), \tag{144}$$

defined on intervals, naturally extends to an $\mathbb{R}^p$*-valued orthogonal-increment random measure on* $\mathcal{B}(\mathbb{R})$*. In particular, the relations* (140) *and* (141) *also hold for* $\Phi_{\widetilde{\mathcal{M}}}$*. Furthermore, for* $\widehat{f} \in L^2_{\mathrm{Herm}}(\mathbb{R})$,

$$\int_{\mathbb{R}} \widehat{f}(x)\widetilde{\mathcal{M}}(dx) = \int_{\mathbb{R}} f(s)\Phi_{\widetilde{\mathcal{M}}}(ds) \quad \text{a.s.}, \tag{145}$$

where f satisfies (143).

Proof. Statement (i) can be shown by means of a direct adaptation of the statement of Theorem 3.5 in Marquardt and Stelzer [46], which in turn is based on a multivariate generalization of Rozanov [59], Theorem 2.1. Therefore we only show (140) and (141). It suffices to establish the statement over intervals $(a, b]$, $-\infty < a \le b < \infty$. Note that the characteristic function of $\Phi_{\widetilde{\mathcal{M}}}(a, b]$ at $\mathbf{u} \in \mathbb{R}^p$ is given by

$$\exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^p}\left(e^{\mathbf{i}\mathbf{u}^*\frac{1}{2\pi}\left(\frac{e^{\mathbf{i}sb}-e^{\mathbf{i}sa}}{\mathbf{i}s}\right)\mathbf{z}} - 1 - \mathbf{i}\mathbf{u}^*\frac{1}{2\pi}\left(\frac{e^{\mathbf{i}sb}-e^{\mathbf{i}sa}}{\mathbf{i}s}\right)\mathbf{z}\right)\mu(d\mathbf{z})ds\right\}.$$

By taking the first derivative with respect to $\mathbf{u}$ and setting $\mathbf{u} = \mathbf{0}$, we conclude that $\mathbb{E}\Phi_{\widetilde{\mathcal{M}}}(a, b] = \mathbf{0}$, which proves (140). Moreover, by taking the second derivative with respect to $\mathbf{u}$ and setting $\mathbf{u} = \mathbf{0}$, we obtain

$$\mathbb{E}\Phi_{\widetilde{\mathcal{M}}}(a, b]\Phi_{\widetilde{\mathcal{M}}}(a, b]^* = \frac{1}{4\pi^2}\left(\int_{\mathbb{R}}\left|\frac{e^{\mathbf{i}sb}-e^{\mathbf{i}sa}}{\mathbf{i}s}\right|^2 ds\right)\left(\int_{\mathbb{R}^p}\mathbf{z}\mathbf{z}^*\mu(d\mathbf{z})\right).$$

Thus, by setting $a = 0$ and $b = y$ under the condition (39), we further conclude that (141) holds, where we make use of the fact that

$$\frac{1}{4\pi^2}\int_{\mathbb{R}}\left|\frac{e^{\mathbf{i}xy}-1}{\mathbf{i}x}\right|^2 dx = y$$

(see the expression (9.7) in Taqqu [66]). Likewise, the orthogonality of the increments can be verified by Parseval's theorem.

Statement (ii) can be established by a similar argument. In particular, the random measure (144) is $\mathbb{R}^p$-valued because the integrand

$$\frac{e^{-\mathbf{i}xa} - e^{-\mathbf{i}xb}}{\mathbf{i}x}$$

is a Hermitian function. □

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Remark E.1. The integrals on the right-hand sides of (142) and (145) with respect to the orthogonal-increment random measures $\Phi_{\widetilde{\mathcal{M}}}$, $\Phi_{\mathcal{M}}$ are interpreted in the traditional L^2 sense. Note that although $\Phi_{\widetilde{\mathcal{M}}}$, $\Phi_{\mathcal{M}}$ have uncorrelated increments, $\widetilde{\mathcal{M}} \neq \Phi_{\mathcal{M}}$ and $\mathcal{M} \neq \Phi_{\widetilde{\mathcal{M}}}$.

The following lemma is used in Example 4.4.

Lemma E.2. *Under the assumptions laid out in Example 4.4, the expression (60) holds.*

Proof. Observe that $\Sigma = \int_{\mathbb{R}^p} \mathbf{z}\mathbf{z}^* \mu\big((M_-^{-1}M_+)d\mathbf{z}\big) = \big(M_+^{-1}M_-\big)\Sigma\big(M_+^{-1}M_-\big)^*$, by (58). If $\Sigma^{1/2}$ is the unique symmetric positive definite square root of Σ, it follows that $I = \big\{\Sigma^{-1/2}\big(M_+^{-1}M_-\big)\Sigma^{1/2}\big\}\big\{\Sigma^{1/2}\big(M_+^{-1}M_-\big)^*\Sigma^{-1/2}\big\}$. Therefore, $\Sigma^{-1/2}(M_-^{-1}M_+)\Sigma^{1/2} = O$ for some orthogonal matrix $O \in M(p, \mathbb{R})$, showing that $M_-^{-1}M_+ = \Sigma^{1/2}O\Sigma^{-1/2}$. Moreover, under the condition (57), $I = (M_-^{-1}M_+)^2 = \Sigma^{1/2}O^2\Sigma^{-1/2}$. This shows that $O^2 = I$, i.e., $O = O^*$. Hence, (60) holds, as claimed. □

The following lemma is used in Section 4.

Lemma E.3 *The rhofLm $\widetilde{X}_H$ as in (61) can also be represented based on (62) and (63).*

Proof. This is a consequence of the fact that, for any m and any $t_1, \ldots, t_m$,

$$\int_{\mathbb{R}}\int_{\mathbb{C}^p}\left(e^{\mathbf{i}2\sum_{j=1}^m \mathbf{u}_j^*\Re(\widetilde{g}_{t_j}(x)\mathbf{z})} - 1 - \mathbf{i}2\sum_{j=1}^m \mathbf{u}_j^*\Re(\widetilde{g}_{t_j}(x)\mathbf{z})\right)\mu(d\mathbf{z})dx$$

$$= \int_{\mathbb{R}}\int_{\mathbb{C}^p}\left(e^{\mathbf{i}2\sum_{j=1}^m \mathbf{u}_j^*\Re(\widetilde{g}_{t_j}(x')\mathbf{z}')} - 1 - \mathbf{i}2\sum_{j=1}^m \mathbf{u}_j^*\Re(\widetilde{g}_{t_j}(x')\mathbf{z}')\right)\mu(\overline{d\mathbf{z}'})dx,$$

where we make the change of variables $\big(x', \mathbf{z}'\big) = (-x, \overline{\mathbf{z}})$. □

Appendix F. Tempered operator-stable Lévy measures

More specifically, the framework of tempered operator-stable Lévy measures is used in Proposition 3.3. To recap it, we first recall the notion of operator-stable distributions (see, for instance, Meerschaert and Scheffler [50]). Let $B \in M(p, \mathbb{R})$, $\mathrm{eig}B \subseteq \{z \in \mathbb{C}: \Re z \in (1/2, \infty)\}$. Consider a norm $\|\cdot\|_B$ on $\mathbb{R}^q$ with unit sphere $S_0 := \{x: \|x\|_B = 1\}$ satisfying the following:

(i) for each $x \in \mathbb{R}^q \setminus \{0\}$, $r \mapsto \big\|r^B x\big\|$ is monotonically increasing for $r > 0$;

(ii) $(r, \boldsymbol{\theta}) \mapsto r^B\boldsymbol{\theta}$ from $\mathbb{R}_+ \times S_B$ to $\mathbb{R}^q \setminus \{0\}$ is a homeomorphism.

(See Lemma 6.1.5 in Meerschaert and Scheffler [50].) A *full operator-stable distribution* (recall that *full* means a distribution is not supported on any proper subspace of $\mathbb{R}^p$) has a Lévy measure μ that can be written as

$$\mu_B(A) = \int_{S_0}\int_{\mathbb{R}_+} 1_A\big(r^B\boldsymbol{\theta}\big)\frac{dr}{r^2}\lambda(d\boldsymbol{\theta}), \tag{146}$$

where λ is a finite Borel measure on S_0 (see Theorem 7.2.5 in Meerschaert and Scheffler [50]). Provided $\Re\lambda_q(B) < 1$, it can be shown that

$$\int_{\|\mathbf{z}\| \geq 1} \|\mathbf{z}\|\mu_B(d\mathbf{z}) < \infty$$

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(see Corollary 8.2.6 in Meerschaert and Scheffler [50] and Theorem 25.3 in Sato [62]). In particular, under the assumption $\Re\lambda_q(B) < 1$, the integral

$$\psi_B(\mathbf{u}) = \int_{\mathbb{R}^q} \left(e^{\mathbf{i}\langle \mathbf{u},\mathbf{z}\rangle} - 1 - \mathbf{i}\langle \mathbf{u}, \mathbf{z}\rangle\right)\mu_B(d\mathbf{z}), \quad \mathbf{u} \in \mathbb{R}^q \tag{147}$$

(cf. (82)), exists, and the Lévy symbol ψ_B satisfies $\psi(c^B\mathbf{u}) = c\psi(\mathbf{u})$; i.e., the function $e^{\psi(\mathbf{u})}$ is the characteristic function of a *strictly* operator-stable distribution ν_B (see Kremer and Scheffler [36], p. 4085). Based on ν_B, we define the random measures L_B used in Proposition 3.3 as $\mathbb{R}^p$-valued (in (47)) or $\mathbb{C}^p$-valued (in (50)) ID independently scattered random measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ generated by ν_B and the Lebesgue measure on $\mathbb{R}$ in the sense of Example 3.7(a) and Remark 3.8 in Kremer and Scheffler [35].

For the purposes of retaining second moments and constructing asymptotically o.s.s. instances of ofLm, we consider *tempered* counterparts of (146), where the associated Lévy measure is given by

$$\mu_{B,q}(A) = \int_{S_0}\int_{\mathbb{R}_+} 1_A\left(r^B\boldsymbol{\theta}\right)q(r,\boldsymbol{\theta})\frac{dr}{r^2}\lambda(d\boldsymbol{\theta}), \qquad r \in (0,\infty), \quad \boldsymbol{\theta} \in S_0. \tag{148}$$

In (148), $q : (0, \infty) \times S_0 \to [0, 1]$ is any Borel measurable function such that, for $\lambda(d\boldsymbol{\theta})$-a.e. $\boldsymbol{\theta} \in S_0$,

$$q(\cdot, \boldsymbol{\theta}) \text{ decays to } 0 \tag{149}$$

sufficiently fast as $r \to \infty$ to guarantee that $\mu_{B,q}$ has second moments and

$$q(0^+, \boldsymbol{\theta}) = 1 \tag{150}$$

for each $\boldsymbol{\theta} \in S_0$ (for instance, $q(r, \boldsymbol{\theta}) = \mathbf{1}_{\{|r|\leq 1\}}$, $\boldsymbol{\theta} \in S_0$). When q is a completely monotone function (that is,

$$(-1)^k \frac{d^k}{dt^k} q(t, \boldsymbol{\theta}) > 0$$

for all $t > 0$ and each $k \geq 0$), Lévy measures defined by (148) are called *tempered operator-stable Lévy measures*; they are studied in Ali [2] (see also the seminal work of Rosiński [57] for the tempered stable case). For conditions for the existence of moments in the tempered stable case, see Rosiński [57], Proposition 2.7. For the tempered operator-stable case, see Ali [2], *Korollar* 3.2.5.

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