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LOEWY SERIES OF PARABOLICALLY INDUCED G_1T-VERMA MODULES

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Abstract We show that the modules for the Frobenius kernel of a reductive algebraic group over an algebraically closed field of positive characteristic p induced from the p-regular blocks of its parabolic subgroups can be $\mathbb{Z}$-graded. In particular, we obtain that the modules induced from the simple modules of p-regular highest weights are rigid and determine their Loewy series, assuming the Lusztig conjecture on the irreducible characters for the reductive algebraic groups, which is now a theorem for large p. We say that a module is rigid if and only if it admits a unique filtration of minimal length with each subquotient semisimple, in which case the filtration is called the Loewy series.

Keywords: Loewy series; parabolic induction; graded induction; rigidity; Frobenius kernel

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Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ of positive characteristic p, P a parabolic subgroup of G, T a maximal torus of P, and G_1 (respectively, P_1) the Frobenius kernel of G (respectively, P). In this paper, we study the structure of G_1T-modules induced from the simple P_1T-modules of p-regular highest weights. Thus our study goes parallel to parabolically induced Verma modules in characteristic zero. When P is a Borel subgroup of G, assuming Lusztig's conjecture for the irreducible characters for G_1T, Andersen and the second author of the present paper showed that the induced modules are rigid, and determined their Loewy series [2]. If M is a finite-dimensional G_1T-module, we call the sum of its simple submodules the socle of M, and denote it by $\mathrm{soc}M = \mathrm{soc}^1M$. If $\pi : M \to M/\mathrm{soc}M$ is the quotient, we let $\mathrm{soc}^2M = \pi^{-1}\mathrm{soc}(M/\mathrm{soc}M)$, and repeat to construct a filtration $0 < \mathrm{soc}M < \mathrm{soc}^2M < \cdots < M$, called the socle series of M. Dually, we call the intersection of all its maximal submodules the radical of M, and denote it by $\mathrm{rad}M = \mathrm{rad}^1M$. Letting $\mathrm{rad}^iM = \mathrm{rad}(\mathrm{rad}^{i-1}M)$ for $i > 1$, one obtains a filtration $M > \mathrm{rad}M > \mathrm{rad}^2M > \cdots > 0$, called the radical series of M. It is known that the minimal i such that $\mathrm{soc}^iM = M$ and the minimal j such that

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$\mathrm{rad}^j M = 0$ coincide, called the Loewy length of M, and denoted $\ell\ell(M)$. By definition, each $\mathrm{soc}_i M = \mathrm{soc}^i M/\mathrm{soc}^{i-1} M$, called the ith socle layer of M, and $\mathrm{rad}_i M = \mathrm{rad}^i M/\mathrm{rad}^{i+1} M$, called the ith radical layer of M, is semisimple. Any filtration $0 < M^1 < M^2 < \cdots < M$ with each subquotient semisimple has length at least $\ell\ell(M)$. If the length of such a filtration $M^\bullet$ is $\ell\ell(M)$, then $\mathrm{soc}^i M \geqslant M^i \geqslant \mathrm{rad}^{\ell\ell(M)-i+1} M$ for each i, and the filtration is called a Loewy filtration. We say that M is rigid if and only if the socle series and the radical series of M coincide, in which case we call the filtration the Loewy series. We now show that our parabolically induced modules are also rigid, and describe their Loewy series. For that, we show that the parabolic induction is $\mathbb{Z}$-graded. The Lusztig conjecture is now a theorem for large p thanks to [1, 11, 13, 15], or more recently to [5].

To go into more detail, let B be a Borel subgroup of P containing T, Λ the character group of B, $R \subset \Lambda$ the root system of G relative to T, and R^+ the positive system of R such that the roots of B are $-R^+$. We let R^s denote the set of simple roots, and I a subset of R^s such that the root subgroups U_α of G associated with $\alpha \in I$ generate P together with B. Denote by $\hat{\nabla}_P$ the induction functor from the category of P_1T-modules to the category of G_1T-modules, and let $\hat{L}^P(\lambda)$ denote the simple P_1T-module of highest weight $\lambda \in \Lambda$. Our object of study is $\hat{\nabla}_P(\hat{L}^P(\lambda))$. After stating some generalities in §§1 and 2, we specialize into the case where λ is p-regular, i.e., when $p \nmid \langle \lambda + \rho, \alpha^\vee \rangle$ for the coroot $\alpha^\vee$ of each root α with $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. In §3, we employ graded representation theory from [1] to show that the induction functor $\hat{\nabla}_P$ is $\mathbb{Z}$-graded. Each block of finite-dimensional G_1T-modules is equivalent to the category of $p\mathbb{Z}R$-graded finite-dimensional modules over the $\Bbbk$-algebra E of the endomorphisms of a projective $p\mathbb{Z}R$-generator of the block. Andersen, Jantzen, and Soergel [1] showed that the algebra E for a p-regular block is $(p\mathbb{Z}R \times \mathbb{Z})$-graded, and that, assuming Lusztig's conjecture for the irreducible characters for G_1T, E is Koszul with respect to its $\mathbb{Z}$-gradation. We show in §4 that the rigidity of $\hat{\nabla}_P(\hat{L}^P(\lambda))$ for p-regular λ follows from a result in [4]. Unlike the case $P = B$, the number of G_1T-composition factors of $\hat{\nabla}_P(\hat{L}^P(\lambda))$ varies depending on the highest weight λ. Nonetheless, we show also in §4 that the Loewy length of $\hat{\nabla}_P(\hat{L}^P(\lambda))$ is uniformly $\ell(w_0 w_I) + 1$ with w_0 (respectively, w_I) the longest element of the Weyl group W (respectively, W_I) of G (respectively, P). In §5, we determine the composition factor multiplicities of the Loewy series of $\hat{\nabla}_P(\hat{L}^P(\lambda))$.

Given a category C and its objects X and Y, $C(X, Y)$ will denote the set of morphisms in C from X to Y. By $\otimes$ we will always mean $\otimes_\Bbbk$, unless otherwise specified.

1. Some generalities

Let G be a reductive algebraic group over an algebraically field $\Bbbk$ of positive characteristic p, B a Borel subgroup of G, T a maximal torus of B, Λ the character group of B, $R \subset \Lambda$ the root system of G relative to T, and R^+ the positive system of R such that the roots of B are $-R^+$. We let $R^s \subset R^+$ denote the set of simple roots, and Λ^+ the set of dominant weights of Λ. For each $\alpha \in R$ we let $\alpha^\vee$ denote the coroot of α. Let W be the Weyl group of G generated by the reflections s_α, $\alpha \in R$, and ℓ the length function on W with respect to the simple reflections. Let w_0 be the longest element of W. For simplicity,

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we will assume throughout the paper that G is semisimple and simply connected; cf. [7, Remarks II.3.15.2 and II.9.7].

For each $\alpha \in R$, let U_α denote the root subgroup of G associated with α. Let $I \subseteq R^s$ and $P = P_I = \langle B, U_\alpha \mid \alpha \in I \rangle$, the standard parabolic subgroup of G associated with I, and let L_I denote its standard Levi subgroup. Let $R_I \subseteq R$ denote the root system of L_I with its induced positive system R_I^+. Put $\Lambda_P = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle = 0 \ \forall \alpha \in I\}$ and $\Lambda_I^+ = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geqslant 0 \ \forall \alpha \in I\}$. Let W_I be the Weyl group of P and w_I its longest element. Put $w^I = w_0 w_I$. Let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ and $\rho_P = \frac{1}{2}\sum_{\alpha \in R^+ \setminus R_I^+} \alpha \in \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $W_p = W \ltimes p\mathbb{Z}R$, $W_{I,p} = W_I \ltimes p\mathbb{Z}R_I$, and $\rho_I = \frac{1}{2}\sum_{\alpha \in R_I^+} \alpha = \rho - \rho_P$. For $x \in W_p$, we will write $x \bullet \lambda$ for $x(\lambda + \rho) - \rho$. When $x \in W_{I,p}$, $x \bullet \lambda = x(\lambda + \rho_I) - \rho_I$. We will also let $(-x) \bullet \lambda$ stand for $-(x \bullet \lambda) - 2\rho = -x(\lambda + \rho) - \rho$.

(1.1) Let α_0 be the highest short root of R, and let $h = \langle \rho, \alpha_0^\vee \rangle + 1$, the Coxeter number of G.

Lemma. $2\rho_P = w_I\rho + \rho = w_0(w^I \bullet 0) \in \Lambda_P \cap \Lambda^+$ *with* $\langle 2\rho_P, \alpha^\vee \rangle \in [2, h] \ \forall \alpha \in R^s \setminus I$.

Proof. One has

$$w_0(w^I \bullet 0) = w_0(w_0 w_I \rho - \rho) = w_I\rho + \rho = \rho + w_I \frac{1}{2}\left(\sum_{\beta \in R^+ \setminus R_I^+} \beta + \sum_{\beta \in R_I^+} \beta \right)$$

$$= \frac{1}{2}\left(\sum_{\beta \in R^+ \setminus R_I^+} \beta + \sum_{\beta \in R_I^+} \beta \right) + \frac{1}{2}\left(\sum_{\beta \in R^+ \setminus R_I^+} \beta - \sum_{\beta \in R_I^+} \beta \right) = \sum_{\beta \in R^+ \setminus R_I^+} \beta = 2\rho_P.$$

If $\alpha \in I$, $\langle 2\rho_P, \alpha^\vee \rangle = \langle w_I\rho + \rho, \alpha^\vee \rangle = \langle \rho, w_I\alpha^\vee \rangle + 1 = 0$, and hence $2\rho_P \in \Lambda_P$. If $\alpha \in R^s \setminus I$, $\langle 2\rho_P, \alpha^\vee \rangle = \langle w_I\rho + \rho, \alpha^\vee \rangle = \langle \rho, w_I\alpha^\vee \rangle + 1 \leqslant \langle \rho, \alpha_0^\vee \rangle + 1 = h$. □

(1.2) If $H \leqslant K$ are closed subgroups of G, we let ind_H^K denote the induction functor from the category $H\mathbf{Mod}$ of rational H-modules to the category $K\mathbf{Mod}$ of rational K-modules: if $M \in H\mathbf{Mod}$, $\mathrm{ind}_H^K M = \{f \in \mathbf{Sch}_{\Bbbk}(K, M) \mid f(kh) = h^{-1}f(k) \ \forall k \in K \forall h \in H\}$. Here and elsewhere throughout the paper we will write the H-action on M simply as hm, $h \in H$, $m \in M$, though, precisely, one has to consider the action of $H(A)$ on $M \otimes A$, functorially on all $\Bbbk$-algebras A. We let $\mathrm{Dist}(H)$ (respectively, $\mathrm{Dist}(K)$) denote the algebra of distributions on H (respectively, K), and let $\mathrm{coind}_H^K = \mathrm{Dist}(K) \otimes_{\mathrm{Dist}(H)} ?$ denote the coinduction functor from the category $\mathrm{Dist}(H)\mathbf{Mod}$ of $\mathrm{Dist}(H)$-modules to the category $\mathrm{Dist}(K)\mathbf{Mod}$ of $\mathrm{Dist}(K)$-modules. For a finite-dimensional H-module M, we will mean by M^* the $\Bbbk$-linear dual of M. Let K_1 denote the Frobenius kernel of K. Then K_1H is a subgroup scheme of G isomorphic to $(K_1 \rtimes H)/(K_1 \cap H)$ [7, I.5.6.8 and 6.2.1]. Recall also for the finite group scheme K_1 that $\mathrm{Dist}(K_1) = \Bbbk[K_1]^*$ with multiplication given by $(\mu\nu)(f) = (\mu \otimes \nu) \circ \Delta(f)$, $f \in \Bbbk[K_1]$, μ and $\nu \in \mathrm{Dist}(K_1)$, Δ denoting the comultiplication on the coordinate Hopf algebra $\Bbbk[K_1]$ of K_1. Thus $\mathrm{Dist}(K_1)$ coincides with the restricted universal enveloping algebra of the Lie algebra of K [7, I.8, 9].

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If M is a P-module, $\mathrm{coind}_{P_1}^{G_1} M$ extends to a G_1P-module with P acting on $\mathrm{Dist}(G_1)$ and $\mathrm{Dist}(P_1)$ by the adjoint action and as given on M, in which case we will write $\mathrm{coind}_P^{G_1P} M$ for $\mathrm{coind}_{P_1}^{G_1} M$ [7, I.8.20]. Let $\mathrm{Ru}(P)$ denote the unipotent radical of P.

Proposition (Cf. [7, II.3.5]). *Let $M \in P\mathbf{Mod}$.*

(i) *There is an isomorphism of G_1P-modules $\mathrm{ind}_P^{G_1P} M \simeq \mathrm{coind}_P^{G_1P}(M \otimes 2(1-p)\rho_P)$.*

(ii) *If M is finite dimensional, there is an isomorphism of G_1P-modules,*

$$(\mathrm{ind}_P^{G_1P} M)^* \simeq \mathrm{ind}_P^{G_1P}(M^* \otimes 2(p-1)\rho_P).$$

Proof. Recall from [7, I.8.20] an isomorphism of G_1P-modules,

$$\mathrm{coind}_P^{G_1P} M \simeq \mathrm{ind}_P^{G_1P}(M \otimes \chi|_P(\chi')^{-1}), \tag{1}$$
$$(\mathrm{ind}_P^{G_1P} M)^* \simeq \mathrm{ind}_P^{G_1P}(M^* \otimes \chi|_P(\chi')^{-1}) \quad \text{if } \dim M < \infty, \tag{2}$$

where χ (respectively, χ') is a one-dimensional representation of G (respectively, P) through which G (respectively, P) acts on $\mathrm{Dist}(G_1)_\ell^{G_1} = \{\mu \in \mathrm{Dist}(G_1) | \nu\mu = \nu(1)\mu \ \forall \nu \in \mathrm{Dist}(G_1)\}$ (respectively, $\mathrm{Dist}(P_1)_\ell^{P_1} = \{\mu \in \mathrm{Dist}(P_1) | \nu\mu = \nu(1)\mu \ \forall \nu \in \mathrm{Dist}(P_1)\}$). As χ is trivial by [7, II.3.4/I.9.7], (1) and (2) read, respectively,

$$\mathrm{coind}_P^{G_1P} M \simeq \mathrm{ind}_P^{G_1P}(M \otimes (\chi')^{-1}), \tag{3}$$
$$(\mathrm{ind}_P^{G_1P} M)^* \simeq \mathrm{ind}_P^{G_1P}(M^* \otimes (\chi')^{-1}). \tag{4}$$

Recall from [7, I.9.7] that χ' is given by $g \mapsto \det(\mathrm{Ad}(g))^{p-1}$, $g \in P$. In particular, χ' factors through $P/\mathrm{Ru}(P)$, and is trivial on the derived subgroup of L_I. To compute χ', therefore, we have only to consider the adjoint representation of T on $\mathrm{Lie}(P) = \mathrm{Lie}(T) \oplus \bigoplus_{\beta \in R^+} \mathrm{Lie}(U_{-\beta}) \oplus \bigoplus_{\alpha \in R_I^+} \mathrm{Lie}(U_\alpha)$. Thus, for each $t \in T$,

$$\begin{aligned}\det(\mathrm{Ad}(t)) &= \left(\sum_{\beta \in R^+} -\beta + \sum_{\alpha \in R_I^+} \alpha\right)(t) = \left\{-\left(\sum_{\beta \in R^+} \beta - \sum_{\alpha \in R_I^+} \alpha\right)\right\}(t) \\ &= \left(-\sum_{\beta \in R^+ \setminus R_I^+} \beta\right)(t) = (-2\rho_P)(t).\end{aligned}$$

It follows that $\chi' = (p-1)(-2\rho_P)$, and hence also the assertions. □

(1.3) Likewise, write $\mathrm{coind}_{P_1T}^{G_1T} M$ for the G_1T-module $\mathrm{coind}_{P_1}^{G_1} M$ in the case when M is a P_1T-module.

Proposition. *Let $M \in P_1T\mathbf{Mod}$.*

(i) *There is an isomorphism of G_1T-modules, $\mathrm{ind}_{P_1T}^{G_1T} M \simeq \mathrm{coind}_{P_1T}^{G_1T}(M \otimes 2(1-p)\rho_P)$.*

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(ii) *If M is finite dimensional, there is an isomorphism of G_1T-modules,*

$$(\mathrm{ind}_{P_1T}^{G_1T} M)^* \simeq \mathrm{ind}_{P_1T}^{G_1T}(M^* \otimes 2(p-1)\rho_P).$$

(1.4) If L is a simple P-module, the P-action on L factors through $P/\mathrm{Ru}(P)$, affording a simple L_I-module of highest weight belonging to Λ_I^+. For each $\lambda \in \Lambda^+$ (respectively, $\lambda \in \Lambda_I^+$), we let $L(\lambda)$ (respectively, $L^P(\lambda)$) denote the simple G-module (respectively, P-module) of highest weight λ. Likewise for simple P_1T-modules. For each $\lambda \in \Lambda$, we let $\hat{L}(\lambda)$ (respectively, $\hat{L}^P(\lambda)$) denote the simple G_1T- module (respectively, P_1T- module) of highest weight λ. Let $\Lambda_p = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \in [0, p[\ \forall \alpha \in R^s\}$. Each $\lambda \in \Lambda$ admits a decomposition $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in \Lambda_p$ and $\lambda^1 \in \Lambda$. Thus $\hat{L}^P(\lambda) \simeq L^P(\lambda^0) \otimes p\lambda^1$; if $\lambda^0 = \lambda_I^0 + \lambda_I^c$ with $\lambda_I^c \in \Lambda_P$, then $\hat{L}^P(\lambda) \simeq L^P(\lambda_I^0) \otimes (\lambda_I^c + p\lambda^1)$. In particular,

$$\{\mathrm{ind}_{P_1T}^{G_1T}(\hat{L}^P(\lambda))\}^* \simeq \mathrm{ind}_{P_1T}^{G_1T}(\hat{L}^P((-w_I)\bullet\lambda)) \otimes p(2\rho_P + w_I\lambda^1 - \lambda^1) \tag{1}$$

with $2\rho_P + w_I\lambda^1 - \lambda^1 \in \mathbb{Z}R$.

If H is a closed subgroup of G and if M is an H-module, we let $\mathrm{soc}_H M$ (respectively, $\mathrm{rad}_H M$) denote the socle (respectively, the radical) of M, and put $\mathrm{hd}_H M = M/(\mathrm{rad}_H M)$.

Proposition. *For each $\lambda \in \Lambda$,*

$$\begin{aligned}\mathrm{soc}_{G_1T}\mathrm{ind}_{P_1T}^{G_1T}(\hat{L}^P(\lambda)) &= \hat{L}(\lambda),\\ \mathrm{hd}_{G_1T}\mathrm{ind}_{P_1T}^{G_1T}(\hat{L}^P(\lambda)) &= \hat{L}(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P)^*\\ &= \hat{L}(w^I\bullet\lambda) \otimes p\{\lambda^1 - 2\rho_P - w_I\lambda^1 + w_0((-w_I)\bullet\lambda)^1 - ((-w_I)\bullet\lambda)^1\}.\end{aligned}$$

Proof. For each $\lambda \in \Lambda$, we have $\mathrm{soc}_{P_1T}(\mathrm{ind}_{B_1T}^{P_1T}\lambda) \simeq \mathrm{soc}_{P_1T}(\mathrm{ind}_{(B/\mathrm{Ru}(P))_1T}^{(P/\mathrm{Ru}(P))_1T}\lambda) = \hat{L}^P(\lambda)$. Then

$$\mathrm{ind}_{P_1T}^{G_1T}\hat{L}^P(\lambda) \leqslant \mathrm{ind}_{P_1T}^{G_1T}\mathrm{ind}_{B_1T}^{P_1T}(\lambda) \simeq \mathrm{ind}_{B_1T}^{G_1T}\lambda.$$

It follows that $\mathrm{soc}_{G_1T}(\mathrm{ind}_{P_1T}^{G_1T}\hat{L}^P(\lambda)) = \{\mathrm{ind}_{P_1T}^{G_1T}\hat{L}^P(\lambda)\} \cap \mathrm{soc}_{G_1T}(\mathrm{ind}_{B_1T}^{G_1T}\lambda) = \hat{L}(\lambda)$. Then

$$\begin{aligned}\mathrm{hd}_{G_1T}(\mathrm{ind}_{P_1T}^{G_1T}\hat{L}^P(\lambda)) &\simeq \{\mathrm{soc}_{G_1T}((\mathrm{ind}_{P_1T}^{G_1T}\hat{L}^P(\lambda))^*)\}^*\\ &\simeq \{\mathrm{soc}_{G_1T}(\mathrm{ind}_{B_1T}^{G_1T}(\hat{L}^P(\lambda)^* \otimes 2(p-1)\rho_P))\}^* \quad \text{by (1.2)(ii)}.\end{aligned}$$

Now $\hat{L}^P(\lambda)^* = (L^P(\lambda^0) \otimes p\lambda^1)^* = L^P(\lambda^0)^* \otimes -p\lambda^1 = L^P(-w_I\lambda^0) \otimes -p\lambda^1 = \hat{L}^P(-w_I\lambda^0 - p\lambda^1)$. Also, $\forall \nu \in \Lambda$,

$$\begin{aligned}\hat{L}^P(\nu) \otimes 2(p-1)\rho_P &\leqslant (\mathrm{ind}_{B_1T}^{P_1T}\nu) \otimes 2(p-1)\rho_P\\ &\simeq \mathrm{ind}_{B_1T}^{P_1T}(\nu \otimes 2(p-1)\rho_P) \quad \text{by the tensor identity},\end{aligned}$$

and hence

$$\hat{L}^P(\nu) \otimes 2(p-1)\rho_P \leqslant \mathrm{soc}_{P_1T}\mathrm{ind}_{B_1T}^{P_1T}(\nu \otimes 2(p-1)\rho_P) = \hat{L}^P(\nu \otimes 2(p-1)\rho_P).$$

It follows that

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$$\mathrm{hd}_{G_1T}(\mathrm{ind}_{P_1T}^{G_1T}\hat{L}^P(\lambda)) \simeq \{\mathrm{soc}_{G_1T}(\mathrm{ind}_{B_1T}^{G_1T}(\hat{L}^P(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P)))\}^* \\ = \hat{L}(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P)^*.$$

Finally,

$$\begin{aligned} -w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P &= -w_I\lambda^0 - p\lambda^1 + (p-1)(w_I\rho + \rho) \quad \text{by (1.1)} \\ &= -w_I(\lambda^0 + \rho) - \rho + p(w_I\rho + \rho - \lambda^1) \\ &= (-w_I)\bullet\lambda + p(w_I\lambda^1 + w_I\rho + \rho - \lambda^1) \\ &= (-w_I)\bullet\lambda + p(w_I\lambda^1 + 2\rho_P - \lambda^1) \quad \text{by (1.1) again.} \end{aligned}$$

Thus

$$\hat{L}(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P)^* = \{\hat{L}((-w_I)\bullet\lambda + p(w_I\lambda^1 + 2\rho_P - \lambda^1))\}^* \\ = \hat{L}(-w_0((-w_I)\bullet\lambda)) \otimes p\{-w_I\lambda^1 - 2\rho_P + \lambda^1 + w_0((-w_I)\bullet\lambda)^1 - ((-w_I)\bullet\lambda)^1\}$$

with $-w_0((-w_I)\bullet\lambda) = -w_0(-w_I(\lambda+\rho)-\rho) = w_0w_I(\lambda+\rho)-\rho = w_0w_I\bullet\lambda = w^I\bullet\lambda$. □

(1.5) Corollary. *Let* $\lambda \in \Lambda$.

(i) $\mathrm{hd}_{P_1T}\hat{L}(\lambda) = \hat{L}^P(\lambda)$ *while* $\mathrm{soc}_{P_1T}\hat{L}(\lambda) = \hat{L}^P(w_Iw_0\lambda^0 + p\lambda^1)$.

(ii) *If* $\lambda \in \Lambda^+$, $\mathrm{hd}_P L(\lambda) = L^P(\lambda)$ *while* $\mathrm{soc}_P L(\lambda) = L^P(w_Iw_0\lambda)$.

Proof. (i) For each $\nu \in \Lambda$,

$$P_1T\mathbf{Mod}(\hat{L}(\lambda), \hat{L}^P(\nu)) \simeq G_1T\mathbf{Mod}(\hat{L}(\lambda), \mathrm{ind}_{P_1T}^{G_1T}\hat{L}^P(\nu)) \\ = \delta_{\lambda\nu}\Bbbk \quad \text{by (1.4)}.$$

It follows that $\mathrm{hd}_{P_1T}\hat{L}(\lambda) = \hat{L}^P(\lambda)$. Then

$$\mathrm{soc}_{P_1T}\hat{L}(\lambda) \simeq \{\mathrm{hd}_{P_1T}(\hat{L}(\lambda)^*)\}^* = \{\mathrm{hd}_{P_1T}\hat{L}(-w_0\lambda^0 - p\lambda^1)\}^* = \hat{L}^P(-w_0\lambda^0 - p\lambda^1)^* \\ = \{L^P(-w_0\lambda^0) \otimes -p\lambda^1\}^* \simeq L^P(w_Iw_0\lambda^0) \otimes p\lambda^1 = \hat{L}^P(w_Iw_0\lambda^0 + p\lambda^1).$$

(ii) For each $\mu \in \Lambda_I^+$,

$$P\mathbf{Mod}(L(\lambda), L^P(\mu)) \simeq G\mathbf{Mod}(L(\lambda), \mathrm{ind}_P^G L^P(\mu)) \leqslant G\mathbf{Mod}(L(\lambda), \mathrm{ind}_P^G\mathrm{ind}_B^P(\mu)) \\ \simeq G\mathbf{Mod}(L(\lambda), \mathrm{ind}_B^G(\mu)) = \delta_{\lambda\mu}\Bbbk.$$

It follows that $\mathrm{hd}_P L(\lambda) = L^P(\lambda)$. Then

$$\mathrm{soc}_P L(\lambda) \simeq \{\mathrm{hd}_P(L(\lambda)^*)\}^* = \{\mathrm{hd}_P L(-w_0\lambda)\}^* = L^P(-w_0\lambda)^* = L^P(w_Iw_0\lambda). \quad \square$$

(1.6) Let H be a closed subgroup of G and ϕ an automorphism of G. If M is an H-module, by ${}^{\phi}M$ we will mean a $\phi(H)$-module of ambient $\Bbbk$-linear space M with the $\phi(H)$-action twisted by ϕ^{-1} [7, I.2.15/3.5]: $\forall h \in \phi(H)$ $\forall m \in M$, the action of h on m in ${}^{\phi}M$ is given by $\phi^{-1}(h)m$. In particular, under the conjugate action of W on T, $\forall w \in W$ and $\forall\lambda \in \Lambda$,

$${}^{w}\lambda = w\lambda. \tag{1}$$

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If K is a closed subgroup of H and V is a K-module, there is an isomorphism of $\phi(H)$-modules [7, I.3.5.4],

$$^{\phi}\mathrm{ind}_K^H(V) \simeq \mathrm{ind}_{\phi(K)}^{\phi(H)}(\phi V). \tag{2}$$

Throughout the rest of the paper we will abbreviate $\mathrm{ind}_{P_1T}^{G_1T}$ (respectively, $\mathrm{ind}_{B_1T}^{P_1T}$) as $\hat{\nabla}_P$ (respectively, $\hat{\nabla}^P$). More generally, for $w \in W$, let $^wP = wPw^{-1}$, and put $\hat{\nabla}_{^wP} = \mathrm{ind}_{(^wP)_1T}^{G_1T}$, $\hat{\nabla}^{^wP} = \mathrm{ind}_{(^wB)_1T}^{(^wP)_1T}$. Let also $\hat{\nabla}_w = \mathrm{ind}_{(^wB)_1T}^{G_1T}$; we will abbreviate $\hat{\nabla}_e$ as $\hat{\nabla}$. For each $\lambda \in \Lambda$ and $w \in W$, we will write $\lambda\langle w\rangle$ for $\lambda + (p-1)(w \bullet 0)$, after [1]. Then

$$\begin{aligned}
{}^w\hat{\nabla}_P(\hat{L}^P(\lambda)) &\simeq \hat{\nabla}_{^wP}(^w\hat{L}^P(\lambda)) \quad \text{by (2)}\\
&\leqslant \hat{\nabla}_{^wP}(^w\hat{\nabla}^P(\lambda))\\
&\simeq \hat{\nabla}_{^wP}(\hat{\nabla}^{^wP}_{^wB}(^w\lambda)) \quad \text{by (2) again}\\
&\simeq \hat{\nabla}_w(w\lambda) \quad \text{by (1)}\\
&= \hat{\nabla}_w(w\bullet\lambda - w\bullet 0) \simeq \hat{\nabla}_w(w\bullet\lambda + (p-1)(w\bullet 0)) \otimes -p(w\bullet 0)\\
&= \hat{\nabla}_w((w\bullet\lambda)\langle w\rangle) \otimes -p(w\bullet 0).
\end{aligned} \tag{3}$$

(1.7) Let τ be the Chevalley antiinvolution of G such that $\tau|_T = \mathrm{id}_T$ [7, II.1.16], and hence $\tau(U_\alpha) = U_{-\alpha}$ for each $\alpha \in R$. If H is a subgroup of G, and if M is a finite-dimensional H-module, let M^τ be the $\tau(H)$-module with the ambient space M^* and the $\tau(H)$-action twisted by τ: $\forall x \in \tau(H)$, $\forall f \in M^*$, $\forall m \in M$, $(xf)(m) = f(\tau(x)m)$. Put $B^+ = \tau B$ and $\hat{\Delta} = \mathrm{coind}_{B_1^+T}^{G_1T}$. Recall from [7, II.9.3.5] that there is an isomorphism of functors $(?^\tau) \circ \hat{\nabla} \simeq \hat{\Delta} \circ (?^\tau)$ from the category of finite-dimensional B_1T-modules to the category of G_1T-modules. More generally, put $P^+ = \tau P = \langle B^+, U_{-\alpha} | \alpha \in I\rangle$, and let $\hat{\Delta}_P = \mathrm{coind}_{P_1^+T}^{G_1T}$. If M is a finite-dimensional P_1T-module, there is an isomorphism of G_1T-modules,

$$(\hat{\nabla}_P(M))^\tau \simeq \hat{\Delta}_P(M^\tau). \tag{1}$$

Let $U_1^+(w_I) = \prod_{\beta \in R^+ \setminus R_I} U_{\beta,1}$ be the Frobenius kernel of the unipotent radical of P^+. If V is a G_1T-module, let $V^{U_1^+(w_I)} = \{v \in V | xv = v \ \forall x \in U_1^+(w_I)\}$. If M is a B_1T-module, as $G_1 = U_1^+(w_I)P_1$, $\hat{\nabla}(M)^{U_1^+(w_I)} = \{\mathbf{Sch}_{\Bbbk}(G_1T, M)^{B_1T}\}^{U_1^+(w_I)} = \mathbf{Sch}_{\Bbbk}(P_1T, M)^{B_1T}$ maintains the structure of a P_1T-module such that

$$\hat{\nabla}(M)^{U_1^+(w_I)} = \hat{\nabla}^P(M). \tag{2}$$

Recall also that each $\hat{\nabla}(\lambda)$, $\lambda \in \Lambda$, is projective/injective as a B_1^+T-module [7, II.9.5]. As $U_1^+(w_I)$ is a normal subgroup of B_1^+, $U_1^+(w_I)$ is exact in B_1^+ [7, I.6.5.2], and hence $\hat{\nabla}(\lambda)$ remains injective/projective as a $U_1^+(w_I)$-module.

2. Translation functors

For $\lambda, \mu \in \Lambda$, let T_λ^μ denote the translation functor on the G_1T-modules. If M is a $L_{I,1}T$-module, we say that M belongs to λ if and only if all the $L_{I,1}T$-composition factors

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of M have highest weights belonging to $W_{I,p}\bullet\lambda$. We let $\mathrm{T}^{\mu}_{I,\lambda}$ denote the translation functor on the $L_{I,1}T$-modules.

For each $\alpha\in R$ and $n\in\mathbb{Z}$, let $H_{\alpha,n}=\{v\in\Lambda\otimes_{\mathbb{Z}}\mathbb{R}\mid\langle v+\rho,\alpha^\vee\rangle=pn\}$. We call a connected component of $\Lambda\otimes_{\mathbb{Z}}\mathbb{R}\setminus\cup_{\alpha\in R,n\in\mathbb{Z}}H_{\alpha,n}$ an alcove. If $F\subseteq\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$, $\overline{F}$ will denote the closure of F in $\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$. We say that $\lambda\in\Lambda$ is p-regular if and only if λ lies in an alcove. If $x\in W_p$ and A is an alcove, we will write xA to mean $x\bullet A$.

(2.1) Lemma. *Let $\eta\in\Lambda$, and let E be a simple G-module of extremal weight η. If $w\eta\in\Lambda_I^+$, $w\in W_I$, and if $\alpha\in I$, then $w\eta+\alpha$ is not a weight of E.*

Proof. Let $x\in W$ with $x\eta\in\Lambda^+$, and put $\nu=x\eta$, $\nu'=w\eta$. Let $J=\{\beta\in I\mid\langle\nu',\beta^\vee\rangle=0\}$, $W_J=\langle s_\beta\mid\beta\in J\rangle$, $W^J=\{y\in W\mid y\beta>0\ \forall\beta\in J\}$, and write $xw^{-1}=y_1y_2$ with $y_1\in W^J$, $y_2\in W_J$. Just suppose that $w\eta+\alpha$ is a weight of E. Then $\nu+y_1\alpha=y_1(\nu'+\alpha)$ would also be a weight of E. As ν is the highest weight of E, $y_1\alpha<0$, and hence $\alpha\notin J$. Then $0<\langle\nu',\alpha^\vee\rangle=\langle y_1\nu',y_1\alpha^\vee\rangle=\langle\nu,y_1\alpha^\vee\rangle$, and hence $y_1\alpha>0$, which is absurd. □

(2.2) Proposition. *Let $\lambda,\mu\in\Lambda$ with μ lying in the closure of the facette that λ belongs to with respect to W_p. Regarding an $L_{I,1}T$-module as a P_1T-module through the quotient $P\to P/\mathrm{Ru}(P)$, there is an isomorphism of functors from the category of $L_{I,1}T$-modules to the category of G_1T-modules,*

$$\mathrm{T}^{\mu}_{\lambda}\hat{\nabla}_P(?)\simeq\hat{\nabla}_P(\mathrm{T}^{\mu}_{I,\lambda}?).$$

Proof. Let M be an $L_{I,1}T$-module belonging to the λ-block, and E a simple G-module of extremal weight $\mu-\lambda$. Let pr_μ (respectively, $\mathrm{pr}_{I,\mu}$) be the projection to the μ-block of G_1T- (respectively, $L_{I,1}T$-) modules. Thus $\mathrm{T}^{\mu}_{\lambda}\hat{\nabla}_P(M)=\mathrm{pr}_\mu(E\otimes\hat{\nabla}_P(M))$. If $w(\mu-\lambda)\in\Lambda_I^+$ with $w\in W_I$ and $v\in E\setminus 0$ is of weight $w(\mu-\lambda)$, then $\mathrm{Dist}(L_I)v$ is by (2.1) an L_I-module of highest weight $w(\mu-\lambda)$. If we put $E'=\mathrm{Dist}(L_I)v$, $\mathrm{T}^{\mu}_{I,\lambda}M=\mathrm{pr}_{I,\mu}(E'\otimes M)$ [7, Remark II.7.6.1]. Thus

$$\begin{aligned}\mathrm{T}^{\mu}_{\lambda}\hat{\nabla}_P(M)&=\mathrm{pr}_\mu(E\otimes\hat{\nabla}_P(M))\simeq\mathrm{pr}_\mu(\hat{\nabla}_P(E\otimes M))\\&\geqslant\mathrm{pr}_\mu(\hat{\nabla}_P(E'\otimes M))\geqslant\hat{\nabla}_P(\mathrm{pr}_{I,\mu}(E'\otimes M))=\hat{\nabla}_P(\mathrm{T}^{\mu}_{I,\lambda}(M)).\end{aligned}$$

As it becomes an isomorphism for $M=\hat{\nabla}^P(x\bullet\lambda)$ and $x\in W_{I,p}$, the isomorphism for general M follows using the five lemma. □

(2.3) Corollary. *Let $\lambda,\mu\in\Lambda$. Assume that μ lies in the closure of the facette that λ belongs to with respect to W_p. Let F_I be the facette that λ belongs to with respect to $W_{I,p}$, and let $\hat{F}_I$ be its upper closure with respect to $W_{I,p}$. Then*

$$\mathrm{T}^{\mu}_{\lambda}\hat{\nabla}_P(\hat{L}^P(\lambda))\simeq\begin{cases}\hat{\nabla}_P(\hat{L}^P(\mu)) & \text{if } \mu\in\hat{F}_I,\\ 0 & \text{else,}\end{cases}$$

in the first case of which one has a commutative diagram of G_1T-modules

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$$\begin{array}{ccc} \mathrm{T}_\lambda^\mu \hat{\nabla}(\lambda) & \xrightarrow{\sim} & \hat{\nabla}(\mu) \\ \uparrow & & \uparrow \\ \mathrm{T}_\lambda^\mu \hat{\nabla}_P(\hat{L}^P(\lambda)) & \xrightarrow[\sim]{} & \hat{\nabla}_P(\hat{L}^P(\mu)). \end{array}$$

(2.4) For $\alpha \in R$ and $n \in \mathbb{Z}$, let $s_{\alpha,n}$ denote the reflection in the wall $H_{\alpha,n}$.

Proposition. *Let $\lambda, \mu \in \Lambda$ with λ lying in an alcove A and $\mu \in \overline{A}$. Assume that $\{x \in W_p | x \bullet \mu = \mu\} = \{e,\ s_{\alpha,n}\}$ for some $\alpha \in R_I^+$ and $n \in \mathbb{Z}$. If M is an $L_{I,1}T$-module belonging to μ, there is an isomorphism of G_1T-modules*

$$\mathrm{T}_\mu^\lambda \hat{\nabla}_P(M) \simeq \hat{\nabla}_P(\mathrm{T}_{I,\mu}^\lambda M),$$

regarding M and $\mathrm{T}_{I,\mu}^\lambda M$ as P_1T-modules via the quotient $P \to P/\mathrm{Ru}(P)$.

Proof. Arguing as in (2.2) yields $\mathrm{T}_\mu^\lambda \hat{\nabla}_P(M) \geqslant \hat{\nabla}_P(\mathrm{T}_{I,\mu}^\lambda M)$. On the other hand, if $M = \hat{\nabla}^P(x \bullet \mu)$ for some $x \in W_{I,p}$,

$$\mathrm{ch}\,\mathrm{T}_\mu^\lambda \hat{\nabla}_P(\hat{\nabla}^P(x \bullet \mu)) = \mathrm{ch}\,\mathrm{T}_\mu^\lambda \hat{\nabla}(x \bullet \mu) = \mathrm{ch}\,\hat{\nabla}(x \bullet \lambda) + \mathrm{ch}\,\hat{\nabla}(x s_{\alpha,n} \bullet \lambda),$$

while

$$\begin{aligned} \mathrm{ch}\,\hat{\nabla}_P(\mathrm{T}_{I,\mu}^\lambda \hat{\nabla}^P(x \bullet \mu)) &= \mathrm{ch}\,\hat{\nabla}_P(\hat{\nabla}^P(x \bullet \mu)) + \mathrm{ch}\,\hat{\nabla}_P(\hat{\nabla}^P(x s_{\alpha,n} \bullet \lambda)) \quad \text{as } s_{\alpha,n} \in W_{I,p} \\ &= \mathrm{ch}\,\hat{\nabla}(x \bullet \lambda) + \mathrm{ch}\,\hat{\nabla}(x s_{\alpha,n} \bullet \lambda). \end{aligned}$$

By additivity, the character equality holds for general M, and hence the assertion holds. □

(2.5) Corollary. *Let $\lambda, \mu \in \Lambda$, and keep the assumptions on λ and μ from (2.4).*

(i) *$\mathrm{T}_\mu^\lambda \hat{\nabla}_P(\hat{L}^P(\mu))$ admits a G_1T-filtration whose subquotients are $\hat{\nabla}_P(\hat{L}^P(x \bullet \lambda))$, $x \in W_{I,p}$, with multiplicity $m_x \in \mathbb{N}$ such that $\mathrm{ch}\,\mathrm{T}_{I,\mu}^\lambda \hat{L}^P(\mu) = \sum_{x \in W_{I,p}} m_x \mathrm{ch}\,\hat{L}^P(x \bullet \lambda)$.*

(ii) *If $\lambda < s_{\alpha,n} \bullet \lambda$, then $\mathrm{soc}_{G_1T} \mathrm{T}_\mu^\lambda \hat{\nabla}_P(\hat{L}^P(\mu)) = \hat{L}(\lambda)$.*

Proof. For (i), argue as in (2.2). As $\hat{\nabla}_P(\hat{L}^P(\mu)) \leqslant \hat{\nabla}(\mu)$, (ii) follows from the fact that $\mathrm{soc}_{G_1T} \mathrm{T}_\mu^\lambda \hat{\nabla}(\mu) = \hat{L}(\lambda)$. □

3. Grading the induction functor

In this section, we employ graded representation theory from [1] to show that our induction functor $\hat{\nabla}_P$ can be graded on p-regular blocks. To facilitate reference to [1], we will adapt to their notation except for $\Bbbk = k$, $\Lambda = X$, and $\hat{L} = L_\Bbbk$. In particular, we will write $Z_\Bbbk$ for $\hat{\Delta}$, and more generally we let $Z_\Bbbk^w = \mathrm{coind}_{{}^wB_1^+T}^{G_1T}$ for $w \in W$. For each $\lambda \in \Lambda$ and $\beta \in R^+$, let $m \in \mathbb{Z}$ with $\langle \lambda + \rho, \beta^\vee \rangle \in](m-1)p, mp]$, and set $\beta \uparrow \lambda = s_{\beta,m} \bullet \lambda =$

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$\lambda + (mp - \langle \lambda + \rho, \beta^\vee \rangle)\beta$. Throughout the section we will assume that $p > h$, the Coxeter number of G.

Let Ω be a p-regular block of G_1T-modules, $\mathcal{C}_\Bbbk(\Omega)$ be the category of finite-dimensional G_1T-modules, and put $Y = p\mathbb{Z}R$. We recall in (3.1) that there is a $(Y \times \mathbb{Z})$-graded finite-dimensional $\Bbbk$-algebra $E_{\Omega,\Bbbk}$ such that $\mathcal{C}_\Bbbk(\Omega)$ is equivalent to the category $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ of finite-dimensional Y-graded $E_{\Omega,\Bbbk}$-modules. Denote by $E_{\Omega,\Bbbk}\mathbf{modgr}_{Y\times\mathbb{Z}}$ the category of finite-dimensional $(Y \times \mathbb{Z})$-graded $E_{\Omega,\Bbbk}$-modules, and let $\bar{v}$ be the composite of the forgetful functor from $E_{\Omega,\Bbbk}\mathbf{modgr}_{Y\times\mathbb{Z}}$ to $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ and the equivalence from $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ to $\mathcal{C}_\Bbbk(\Omega)$. If Ω_I, $\Omega_I \subseteq \Omega$, is a p-regular block for $L_{I,1}T$-modules, we define in (3.2) with $Y_I = p\mathbb{Z}R_I$ the corresponding $(Y_I \times \mathbb{Z})$-graded finite-dimensional $\Bbbk$-algebra $E_{\Omega_I,\Bbbk}$ such that $\mathcal{C}_\Bbbk(\Omega_I)$ is equivalent to the category $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}$ of finite-dimensional Y_I-graded $E_{\Omega_I,\Bbbk}$-modules. Denote by $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I\times\mathbb{Z}}$ the category of finite-dimensional $(Y_I \times \mathbb{Z})$-graded $E_{\Omega_I,\Bbbk}$-modules, and let $\bar{v}_I$ be the composite of the forgetful functor from $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I\times\mathbb{Z}}$ to $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}$ and the equivalence from $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}$ to $\mathcal{C}_\Bbbk(\Omega_I)$. We will show that there is a $(Y \times \mathbb{Z})$-graded left $E_{\Omega,\Bbbk}$- and $(Y_I \times \mathbb{Z})$-graded right $E_{\Omega_I,\Bbbk}$-bimodule $J_\Bbbk$ to yield an isomorphism of functors $\bar{v} \circ (J_\Bbbk \otimes_{E_{\Omega_I,\Bbbk}} ?) \simeq \hat{\nabla}_P \circ \bar{v}_I$:

$$\begin{array}{ccc}
E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I\times\mathbb{Z}} & \xrightarrow{J_\Bbbk \otimes_{E_{\Omega_I,\Bbbk}} ?} & E_{\Omega,\Bbbk}\mathbf{modgr}_{Y\times\mathbb{Z}} \\
\downarrow \bar{v}_I & \circlearrowleft & \downarrow \bar{v} \\
\mathcal{C}_\Bbbk(\Omega_I) & \searrow & \\
\downarrow & & \mathcal{C}_\Bbbk(\Omega) \\
L_{I,1}T\mathbf{Mod} & \circlearrowleft & \downarrow \\
\downarrow \text{inflation} & & \\
P_1T\mathbf{Mod} & \xrightarrow[\hat{\nabla}_P]{} & G_1T\mathbf{Mod},
\end{array}$$

where the inflation functor $L_{I,1}T\mathbf{Mod} \to P_1T\mathbf{Mod}$ is defined by inflation along the quotient $P \to L_I$. Thus, if M is a $P_{I,1}T$-module belonging to the block Ω_I which lifts to a $(Y_I \times \mathbb{Z})$-graded object, then $\hat{\nabla}_P(M)$ admits a $(Y \times \mathbb{Z})$-graded lift. The category $E_{\Omega,\Bbbk}\mathbf{modgr}_{Y\times\mathbb{Z}}$ (respectively, $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I\times\mathbb{Z}}$) will be denoted $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ (respectively, $\tilde{\mathcal{C}}_\Bbbk(\Omega_I)$) for short, after [1, 18.18], in what follows.

(3.1) Let us first recall [1, § 18] to suit our objectives. Let $S_\Bbbk$ be the symmetric algebra on $\mathbb{Z}R \otimes_\mathbb{Z} \Bbbk$ over $\Bbbk$, and $\hat{S}_\Bbbk$ its completion along the maximal ideal $\mathfrak{m}$ generated by R. We will denote each $\alpha \in R$ in $S_\Bbbk$ by h_α, after [1, 14.3]. Thus $\hat{S}_\Bbbk$ is the formal poser series $\Bbbk$-algebra in the indeterminates h_α, $\alpha \in R^s$. We will regard $S_\Bbbk$ as a $\mathbb{Z}$-graded algebra with each h_α, $\alpha \in R$, having degree 2. Fix a p-regular weight λ^+ belonging to the bottom dominant alcove, and put $\Omega = W_p \bullet \lambda^+$, $Y = p\mathbb{Z}R$. The category of finite-dimensional G_1T-modules belonging to the block Ω may be identified with $\mathcal{C}_\Bbbk(\Omega)$ from [1]. For each $\lambda \in \Omega$, let $Q_\Bbbk(\lambda)$ be the projective cover and the injective hull of $\hat{L}(\lambda)$ in $\mathcal{C}_\Bbbk(\Omega)$. If $Q = \bigoplus_{w\in W} Q_\Bbbk(w \bullet \lambda^+)$, Q is a projective Y-generator of $\mathcal{C}_\Bbbk(\Omega)$, i.e., every object of

$\mathcal{C}_{\Bbbk}(\Omega)$ is a quotient of a finite direct sum of some $Q[\gamma] = \bigoplus_w Q_{\Bbbk}(w \bullet \lambda^+ + \gamma)$, $\gamma \in Y$ [1, 18.1]. Thus, if $E_{\Omega,\Bbbk} = \mathcal{C}_{\Bbbk}(\Omega)^{\sharp}(Q, Q)^{\mathrm{op}} = \{\bigoplus_{\gamma \in Y} \mathcal{C}_{\Bbbk}(\Omega)(Q[\gamma], Q)\}^{\mathrm{op}}$, it is equipped with the structure of a finite-dimensional Y-graded $\Bbbk$-algebra [1, E.3]. Denoting the category of Y-graded $E_{\Omega,\Bbbk}$-modules of finite type by $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$, the functor $H_{\Omega,\Bbbk} = \mathcal{C}_{\Bbbk}(\Omega)^{\sharp}(Q, ?) = \bigoplus_{\gamma \in Y} \mathcal{C}_{\Bbbk}(\Omega)(Q[\gamma], ?)$ gives an equivalence of categories from $\mathcal{C}_{\Bbbk}(\Omega)$ to $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ with quasi-inverse $v = Q \otimes_{E_{\Omega,\Bbbk}} ?$ [1, E.4].

Let now $S_{\Bbbk}^{\emptyset} = S_{\Bbbk}[h_{\alpha}^{-1} \mid \alpha \in R^+]$, and $S_{\Bbbk}^{\beta} = S_{\Bbbk}[h_{\alpha}^{-1} \mid \alpha \in R^+ \setminus \{\beta\}]$, $\beta \in R^+$, in the field of fractions of $S_{\Bbbk}$, and put $\hat{S}_{\Bbbk}^{\emptyset} = \hat{S}_{\Bbbk} \otimes_{S_{\Bbbk}} S_{\Bbbk}^{\emptyset}$, $\hat{S}_{\Bbbk}^{\beta} = \hat{S}_{\Bbbk} \otimes_{S_{\Bbbk}} S_{\Bbbk}^{\beta}$. Let also $d_{\alpha} \in \{1, 2, 3\}$ minimal such that the matrix $[\![d_{\alpha}\langle \gamma, \alpha^{\vee}\rangle]\!]_{\alpha,\gamma \in R^s}$ be symmetric [1, 2.4 and 14.4]. For $A \in \{S_{\Bbbk}, \hat{S}_{\Bbbk}, S_{\Bbbk}^{\emptyset}, \hat{S}_{\Bbbk}^{\emptyset}, S_{\Bbbk}^{\beta}, \hat{S}_{\Bbbk}^{\beta} \mid \beta \in R^+\}$, we define the deformation category $\mathcal{C}_A(\Omega)$ over A of $\mathcal{C}_{\Bbbk}(\Omega)$ as follows. Let $H_{\alpha} \in \mathrm{Lie}(T)$, $\alpha \in R^s$, $X_{\beta} \in \mathrm{Lie}(U_{\beta})$, $\beta \in R$, be a Chevalley basis of $\mathrm{Lie}(G)$. We introduce first category $\mathcal{C}_A$ of Λ-graded $\mathrm{Dist}(G_1) \otimes A$-modules. An element of $\mathcal{C}_A$ is a Λ-graded $\mathrm{Dist}(G_1) \otimes A$-module $M = \bigoplus_{\nu \in \Lambda} M_{\nu}$ graded as an A-module with each M_{ν} an A-submodule of finite type such that $X_{\beta}M_{\nu} \subseteq M_{\nu+\beta}$ for each $\beta \in R$ while H_{α}, $\alpha \in R^s$, acting on M_{ν} by the scalar $\langle \nu, \alpha^{\vee}\rangle + d_{\alpha}^{-1}h_{\alpha} \in A$. A morphism of $\mathcal{C}_A$ is a homomorphism of $\mathrm{Dist}(G_1) \otimes A$-modules preserving the Λ-gradings. For each $\lambda \in \Lambda$ and $w \in W$, we equip $Z_A^w(\lambda) = \mathrm{coind}_{{}^wB_1^+T}^{G_1T}(\lambda) \otimes A$ with the structure of $\mathcal{C}_A$ as follows. Using an A-linear isomorphism $\mathrm{coind}_{{}^wB_1^+T}^{G_1T}(\lambda) \otimes A \simeq \mathrm{Dist}({}^wU_1) \otimes A$, we define a Λ-grading on $Z_A^w(\lambda)$ such that $Z_A^w(\lambda)_{\nu} = \mathrm{Dist}({}^wU_1)_{\nu-\lambda} \otimes A$. The action of $\mathrm{Dist}(G_1)$ on $Z_A^w(\lambda)$ is given by regarding $Z_A^w(\lambda)$ as $\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}({}^wB_1^+)} A$ with the structure of a $\mathrm{Dist}({}^wB_1^+)$-module on A defined by $H_{\alpha}1 = \langle \lambda, \alpha^{\vee}\rangle + d_{\alpha}^{-1}h_{\alpha}$, $\alpha \in R^s$, while $X_{\beta}1 = 0$ $\forall \beta \in wR^+$; cf. [1, 2.10]. When $w = e$, we will simply write $Z_A(\lambda)$ for $Z_A^e(\lambda)$. Now $\mathcal{C}_A(\Omega)$ is a full subcategory of $\mathcal{C}_A$ consisting of the homomorphic images of all $M \in \mathcal{C}_A$ which admits a filtration whose subquotients are all of the form $Z_A(\lambda)$, $\lambda \in \Omega$ [1, 6.10]. For each $\lambda \in \Omega$ and $w \in W$, one has $Z_A^w(\lambda\langle w\rangle) \in \mathcal{C}_A(\Omega)$ [1, 6.11]. We will always regard $\Bbbk$ as an A-algebra via the quotient $A \to A/(h_{\alpha}|\alpha \in R^+)$. Then $Z_A(\lambda) \otimes_A \Bbbk \simeq \hat{\Delta}(\lambda)$ and $Z_A^{w_0}(\lambda\langle w_0\rangle) \otimes_A \Bbbk \simeq \hat{\nabla}(\lambda)$. We will sometimes write $\mathcal{C}(\Omega, A)$ for $\mathcal{C}_A(\Omega)$ for notational convenience.

Next, let $A \in \{S_{\Bbbk}, \hat{S}_{\Bbbk}\}$, and define a combinatorial category $\mathcal{K}(\Omega, A)$ as follows. For each $M \in \mathcal{C}_A$, put $M^{\emptyset} = M \otimes_A A^{\emptyset}$ and $M^{\beta} = M \otimes_A A^{\beta}$. We will write $Z_A^{\emptyset}(\lambda)$ (respectively, $Z_A^{\beta}(\lambda)$) for $Z_A(\lambda)^{\emptyset}$ (respectively, $Z_A(\lambda)^{\beta}$). An object of $\mathcal{K}(\Omega, A)$ is a family $\mathcal{M} = (\mathcal{M}(\lambda), \mathcal{M}(\lambda, \beta))_{\lambda \in \Omega, \beta \in R^+}$ of $A^{\emptyset}$-modules $\mathcal{M}(\lambda)$ of finite type, almost all equal to zero, and finitely generated A^{β}-submodules $\mathcal{M}(\lambda, \beta)$ of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ (respectively, $\mathcal{M}(\lambda)$) if $\beta \uparrow \lambda \neq \lambda$ (respectively, if $\beta \uparrow \lambda = \lambda$). A morphism ψ of $\mathcal{K}(\Omega, \hat{S}_{\Bbbk})$ from $\mathcal{M}$ to $\mathcal{M}'$ is a family $(\psi_{\lambda})_{\lambda \in \Omega}$ of $A^{\emptyset}$-linear maps $\psi_{\lambda} : \mathcal{M}(\lambda) \to \mathcal{M}'(\lambda)$ such that, $\forall \beta \in R^+$, $(\psi_{\lambda} \oplus \psi_{\beta\uparrow\lambda})\mathcal{M}(\lambda, \beta) \subseteq \mathcal{M}'(\lambda, \beta)$ (respectively, $\psi_{\lambda}\mathcal{M}(\lambda, \beta) \subseteq \mathcal{M}'(\lambda, \beta)$) if $\beta \uparrow \lambda \neq \lambda$ (respectively, if $\beta \uparrow \lambda = \lambda$).

If $\mathcal{F}(\Omega, \hat{S}_{\Bbbk})$ is the full subcategory of $\mathcal{C}(\Omega, \hat{S}_{\Bbbk})$ consisting of the objects that are free over $\hat{S}_{\Bbbk}$, there is a fully faithful functor $\mathcal{V}_{\Omega}$ from $\mathcal{F}(\Omega, \hat{S}_{\Bbbk})$ to the combinatorial category $\mathcal{K}(\Omega, \hat{S}_{\Bbbk})$ [1, 9.4]. To describe it, put $A = \hat{S}_{\Bbbk}$ for the moment. When $\beta \uparrow \lambda \neq \lambda$, fix a generator $e^{\beta}(\lambda)$ of A^{β}-module $\mathrm{Ext}^1_{\mathcal{C}_{A^{\beta}}}(Z_A^{\beta}(\lambda), Z_A^{\beta}(\beta \uparrow \lambda))$ chosen according to the Theorem of Good Choices [1, 13.4]. Then the functor $\mathcal{V}_{\Omega} : \mathcal{F}(\Omega, \hat{S}_{\Bbbk}) \to \mathcal{K}(\Omega, \hat{S}_{\Bbbk})$ is defined by

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$(\mathcal{V}_\Omega M)(\lambda) = \mathcal{C}_A(Z_A^\emptyset(\lambda), M^\emptyset)$ for each $\lambda \in \Omega$ and by $(\mathcal{V}_\Omega M)(\lambda, \beta) = \mathcal{C}_A(Z_A^\beta(\lambda), M^\beta)$ if $\beta \uparrow \lambda = \lambda$, $\beta \in R^+$. When $\beta \uparrow \lambda \neq \lambda$, the definition of $(\mathcal{V}_\Omega M)(\lambda, \beta)$ is more elaborate: let $0 \to Z_A^\beta(\beta \uparrow \lambda) \xrightarrow{f} Q \xrightarrow{g} Z_A^\beta(\lambda) \to 0$ be an exact sequence representing $e^\beta(\lambda)$. As the sequence splits uniquely over $A^\emptyset$, let $g' \in \mathcal{C}_A(Z_A^\emptyset(\lambda), Q^\emptyset)$ with $g' \circ g^\emptyset = \mathrm{id}_{Z_A^\emptyset(\lambda)}$, $g^\emptyset = g \otimes_{A^\beta} A^\emptyset$. We define $(\mathcal{V}_\Omega M)(\lambda, \beta)$ to be the image of $\mathcal{C}_A(Q, M^\beta)$ in $(\mathcal{V}_\Omega M)(\lambda) \oplus (\mathcal{V}_\Omega M)(\beta \uparrow \lambda)$ under the map $\phi \mapsto (\phi \circ g', \phi \circ f^\emptyset)$.

The category $\mathcal{K}(\Omega, \hat{S}_\Bbbk)$ has a graded version $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ [1, 15.2]: an object of $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ is an object $\mathcal{M}$ of $\mathcal{K}(\Omega, S_\Bbbk)$ with a grading $\mathcal{M}(\lambda) = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}(\lambda)_i$, $\lambda \in \Omega$, as an $S_\Bbbk^\emptyset$-module such that each $\mathcal{M}(\lambda, \beta)$ is a homogeneous $S_\Bbbk^\beta$-submodule of $\mathcal{M}(\lambda) \oplus \mathcal{M}(\beta \uparrow \lambda)$ (respectively, $\mathcal{M}(\lambda)$) if $\beta \uparrow \lambda \neq \lambda$ (respectively, $\beta \uparrow \lambda = \lambda$). The morphisms of $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ are those of $\mathcal{K}(\Omega, S_\Bbbk)$ that preserve the gradings. We will use the same notation for an object of $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ and for its image in $\mathcal{K}(\Omega, S_\Bbbk)$ under the forgetful functor. For $\mathcal{M} \in \tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ and $r \in \mathbb{Z}$, let $\mathcal{M}\langle r \rangle$ be $\mathcal{M}$ with the grading shifted by r, i.e., $\forall \lambda \in \Omega$ $\forall i \in \mathbb{Z}$, $\{\mathcal{M}\langle r \rangle(\lambda)\}_i = \{\mathcal{M}(\lambda)\langle r \rangle\}_i = \mathcal{M}(\lambda)_{i-r}$. Also, for each $\gamma \in Y$ define $\mathcal{M}[\gamma] \in \tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ by the formulae $\mathcal{M}[\gamma](\lambda) = \mathcal{M}(\lambda - \gamma)\langle -2\mathrm{ht}(\gamma/p) \rangle$ and $\mathcal{M}[\gamma](\lambda, \beta) = \mathcal{M}(\lambda - \gamma, \beta)\langle -2\mathrm{ht}(\gamma/p) \rangle$ $\lambda \in \Omega$, $\beta \in R^+$, where $\mathrm{ht}(\sum_{\alpha \in R^s} m_\alpha \alpha) = \sum_{\alpha \in R^s} m_\alpha$. If $\mathcal{N}$ is another object of $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)$, for each $r \in \mathbb{Z}$ let $\mathcal{K}(\Omega, S_\Bbbk)(\mathcal{M}, \mathcal{N})_r = \tilde{\mathcal{K}}(\Omega, S_\Bbbk)(\mathcal{M}\langle r \rangle, \mathcal{N})$, and put $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{M}, \mathcal{N}) = \bigoplus_{\gamma \in Y, i \in \mathbb{Z}} \tilde{\mathcal{K}}(\Omega, S_\Bbbk)(\mathcal{M}, \mathcal{N})_{\gamma, i}$ with $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)(\mathcal{M}, \mathcal{N})_{\gamma, i} = \tilde{\mathcal{K}}(\Omega, S_\Bbbk)(\mathcal{M}[\gamma]\langle i \rangle, \mathcal{N}) = \mathcal{K}(\Omega, S_\Bbbk)(\mathcal{M}[\gamma], \mathcal{N})_i$.

If $\mathcal{M} \in \mathcal{K}(\Omega, \hat{S}_\Bbbk)$, a graded $S_\Bbbk$-form of $\mathcal{M}$ is an object $\tilde{\mathcal{M}}$ of $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ such that $\tilde{\mathcal{M}} \otimes_{S_\Bbbk} \hat{S}_\Bbbk \simeq \mathcal{M}$. Each $Q_\Bbbk(\lambda)$ lifts to a projective object $Q_{\hat{S}_\Bbbk}(\lambda)$ of $\mathcal{C}(\Omega, \hat{S}_\Bbbk)$, and $\mathcal{V}_\Omega Q_{\hat{S}_\Bbbk}(\lambda)$ admits a graded $S_\Bbbk$-form $\mathcal{Q}(\lambda)$ in the graded combinatorial category $\tilde{\mathcal{K}}(\Omega, S_\Bbbk)$ [1, 17.6]. If $\mathcal{P} = \bigoplus_{w \in W} \mathcal{Q}(w \bullet \lambda^+)$, and if $E_\Omega = \tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{P}, \mathcal{P})^{\mathrm{op}}$, then E_Ω is a $(Y \times \mathbb{Z})$-graded $S_\Bbbk$-algebra of finite type, and there is an isomorphism of Y-graded $\Bbbk$-algebras $E_\Omega \otimes_{S_\Bbbk} \Bbbk \simeq E_{\Omega,\Bbbk}$ [1, 18.17.1]. Thus $E_{\Omega,\Bbbk}$ comes equipped with the structure of a finite-dimensional $(Y \times \mathbb{Z})$-graded $\Bbbk$-algebra. We denote by $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ the category of finite-dimensional $(Y \times \mathbb{Z})$-graded $E_{\Omega,\Bbbk}$-modules, and let $\bar{v}$ denote the functor from $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ to $\mathcal{C}_\Bbbk(\Omega)$ composite of the forgetful functor from $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ to the category $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ of Y-graded $E_{\Omega,\Bbbk}$-modules of finite type and $v = Q \otimes_{E_{\Omega,\Bbbk}} ?$. Each $Q_\Bbbk(\lambda)$, $Z_\Bbbk^w(\lambda\langle w \rangle)$, and $\hat{L}(\lambda)$, $\lambda \in \Omega$, $w \in W$, admits a graded object $\tilde{Q}_\Bbbk(\lambda)$, $\tilde{Z}_\Bbbk^w(\lambda)$, and $\tilde{L}_\Bbbk(\lambda)$ in $\tilde{\mathcal{C}}_\Bbbk(\Omega)$, respectively, such that $\bar{v}\tilde{Q}_\Bbbk(\lambda) \simeq Q_\Bbbk(\lambda)$, $\bar{v}\tilde{Z}_\Bbbk^w(\lambda) \simeq Z_\Bbbk^w(\lambda\langle w \rangle)$, and $\bar{v}\tilde{L}_\Bbbk(\lambda) \simeq \hat{L}(\lambda)$ in $\mathcal{C}_\Bbbk(\Omega)$ [1, 18.8 and 18.10].

(3.2) Fix $\lambda_I^+ \in \Lambda_I^+ \cap W_p \bullet \lambda^+$ with $\langle \lambda_I^+ + \rho, \alpha^\vee \rangle < p$ $\forall \alpha \in R_I^+$. Let $\Omega_I = W_{I,p} \bullet \lambda_I^+$, and let $\mathcal{C}_\Bbbk(\Omega_I)$ denote the category of finite-dimensional $L_{I,1}T$-modules belonging to the block Ω_I. Put $Y_I = p\mathbb{Z}I$. For each $\nu \in \Lambda$, let $Q_{I,\Bbbk}(\nu)$ be the projective cover of $\hat{L}^P(\nu)$ as $L_{I,1}T$-module. If $Q_I = \bigoplus_{w \in W_I} Q_{I,\Bbbk}(w \bullet \lambda_I^+)$, it is a projective Y_I-generator of $\mathcal{C}_\Bbbk(\Omega_I)$. Let $E_{\Omega_I,\Bbbk} = \mathcal{C}_\Bbbk(\Omega_I)^\sharp(Q_I, Q_I)^{\mathrm{op}}$, and denote the category of finite-dimensional Y_I-graded $E_{\Omega_I,\Bbbk}$-modules by $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}$. If $Q_I[\gamma] = \bigoplus_{w \in W_I} Q_\Bbbk(w \bullet \lambda_I^+ + \gamma)$, $\gamma \in Y_I$, the functor $H_{\Omega_I,\Bbbk} = \mathcal{C}_\Bbbk(\Omega_I)^\sharp(Q_I, ?) = \bigoplus_{\gamma \in Y_I} \mathcal{C}_\Bbbk(\Omega_I)(Q_I[\gamma], ?)$ gives an equivalence of categories from $\mathcal{C}_\Bbbk(\Omega_I)$ to $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}$ with quasi-inverse $v_I = Q_I \otimes_{E_{\Omega_I,\Bbbk}} ?$.

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Now the category $\mathcal{C}_{\Bbbk}(\Omega_I)$ can be deformed much in the same way as $\mathcal{C}_{\Bbbk}(\Omega)$. Let $S_{I,\Bbbk} = S_{\Bbbk}(\mathbb{Z}R_I \otimes_{\mathbb{Z}} \Bbbk)$ be the symmetric algebra over $\Bbbk$ on $\mathbb{Z}R_I \otimes_{\mathbb{Z}} \Bbbk$. Denoting $\alpha \in R_I$ by h_α as an element of $S_{I,\Bbbk}$, define $S^{\emptyset}_{I,\Bbbk} = S_{I,\Bbbk}[h_\alpha^{-1} \mid \alpha \in R_I^+]$ and $S^{\beta}_{I,\Bbbk} = S_{I,\Bbbk}[h_\alpha^{-1} \mid \alpha \in R_I^+ \setminus \{\beta\}]$, $\beta \in R_I^+$, in the field of fractions of $S_{I,\Bbbk}$. Let $\hat{S}_{I,\Bbbk}$ be the completion of $S_{I,\Bbbk}$ along the maximal ideal generated by $\mathbb{Z}R_I \otimes_{\mathbb{Z}} \Bbbk$, and put $\hat{S}^{\emptyset}_{I,\Bbbk} = \hat{S}_{I,\Bbbk} \otimes_{S_{I,\Bbbk}} S^{\emptyset}_{I,\Bbbk}$, $\hat{S}^{\beta}_{I,\Bbbk} = \hat{S}_{I,\Bbbk} \otimes_{S_{I,\Bbbk}} S^{\beta}_{I,\Bbbk}$. For $A \in \{S_{I,\Bbbk}, \hat{S}_{I,\Bbbk}, S^{\emptyset}_{I,\Bbbk}, \hat{S}^{\emptyset}_{I,\Bbbk}, S^{\beta}_{I,\Bbbk}, \hat{S}^{\beta}_{I,\Bbbk} \mid \beta \in R_I^+\}$, define a category $\mathcal{C}_{I,A}$ of graded Λ-graded $\mathrm{Dist}(L_{I,1}) \otimes A$-modules just like $\mathcal{C}_A$ in (3.1). An object of $\mathcal{C}_{I,A}$ is a Λ-graded $\mathrm{Dist}(L_{I,1}) \otimes A$-module $M = \bigoplus_{\nu \in \Lambda} M_\nu$ of finite type over A with each M_ν an A-submodule of M such that $X_\beta M_\nu \subseteq M_{\nu+\beta}$ for each $\beta \in R$ while H_α, $\alpha \in I$, acting on M_ν by the scalar $\langle \nu, \alpha^\vee \rangle + d_\alpha^{-1} h_\alpha \in A$. We then go on to define the deformation category $\mathcal{C}_A(\Omega_I)$ exactly the same way as for $\mathcal{C}_A(\Omega)$. When R^s has two lengths, if a component I' of I with $I' \subseteq R^s$ consists only of long roots, the action of H_α, $\alpha \in I'$, on M_ν defined above deviates from the convention in [1, 14.4/p. 11], according to which H_α should have acted via $\langle \nu, \alpha^\vee \rangle + h_\alpha \in A$. For our application, however, the deviation causes no problem. For each $\lambda \in \Omega_I$ and $w \in W_I$, define a lift $Z^w_{I,A}(\lambda\langle w \rangle) \in \mathcal{C}_A(\Omega_I)$ of $Z^w_{I,\Bbbk}(\lambda\langle w \rangle) = \mathrm{coind}^{L_1T}_{({}^wB^+ \cap L)_1 T}(\lambda\langle w \rangle)$ as for G_1T in (3.1). Again we will simply write $Z_{I,\Bbbk}(\lambda)$ for $Z^e_{I,\Bbbk}(\lambda\langle e \rangle)$. Let $\mathcal{F}(\Omega_I, \hat{S}_{I,\Bbbk})$ be the full subcategory of $\mathcal{C}(\Omega_I, \hat{S}_{I,\Bbbk})$ consisting of the objects that are free over $\hat{S}_{I,\Bbbk}$, define a combinatorial category $\mathcal{K}(\Omega_I, \hat{S}_{I,\Bbbk})$ just like $\mathcal{K}(\Omega, \hat{S}_{\Bbbk})$, and define a fully faithful functor $\mathcal{V}_{\Omega_I} : \mathcal{F}(\Omega_I, \hat{S}_{I,\Bbbk}) \to \mathcal{K}(\Omega_I, \hat{S}_{I,\Bbbk})$ using the generators $e_I^\beta(\lambda)$, $\lambda \in \Omega_I$, $\beta \in R_I^+$, according to the Theorem of Good Choices [1, 13.4] for Ω_I, just like $\mathcal{V}_\Omega$. The category $\mathcal{K}(\Omega_I, \hat{S}_{I,\Bbbk})$ admits a graded version $\mathcal{K}(\Omega_I, S_{I,\Bbbk})$, Q_I lifts to $Q_{I,\hat{S}_{I,\Bbbk}} \in \mathcal{F}(\Omega_I, \hat{S}_{I,\Bbbk})$, and $\mathcal{V}_{\Omega_I} Q_{I,\hat{S}_{I,\Bbbk}}$ admits a graded $S_{I,\Bbbk}$-form $\mathcal{P}_I \in \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})$. If $E_{\Omega_I} = \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})^\sharp(\mathcal{P}_I, \mathcal{P}_I)^{\mathrm{op}} = \bigoplus_{\gamma \in Y_I, i \in \mathbb{Z}} \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})(\mathcal{P}_I[\gamma]\langle i \rangle, \mathcal{P}_I)$, it is a $(Y_I \times \mathbb{Z})$-graded $S_{I,\Bbbk}$-algebra of finite type, responsible for the structure of the $(Y_I \times \mathbb{Z})$-graded $\Bbbk$-algebra on $E_{\Omega_I,\Bbbk} \simeq E_{\Omega_I} \otimes_{S_{I,\Bbbk}} \Bbbk$. Let $\tilde{\mathcal{C}}_{\Bbbk}(\Omega_I)$ denote the category of $(Y_I \times \mathbb{Z})$-graded $E_{\Omega_I,\Bbbk}$-modules, and let $\bar{\upsilon}_I : \tilde{\mathcal{C}}_{\Bbbk}(\Omega_I) \to \mathcal{C}_{\Bbbk}(\Omega_I)$ be the functor composite of the forgetful functor $\tilde{\mathcal{C}}_{\Bbbk}(\Omega_I) \to E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}$ with υ_I. Each $Z^w_{I,\Bbbk}(\lambda\langle w \rangle)$, $\hat{L}^P(\lambda)$, $Q_{I,\Bbbk}(\lambda)$, $\lambda \in \Omega_I$, $w \in W_I$, admits an object $\tilde{Z}^w_{I,\Bbbk}(\lambda\langle w \rangle)$, $\tilde{L}_{I,\Bbbk}(\lambda)$, $\tilde{Q}_{I,\Bbbk}(\lambda)$ in $\tilde{\mathcal{C}}_{\Bbbk}(\Omega_I)$ such that $\bar{\upsilon}_I \tilde{Z}^w_{I,\Bbbk}(\lambda\langle w \rangle) \simeq Z^w_{I,\Bbbk}(\lambda\langle w \rangle)$, $\bar{\upsilon}_I \tilde{L}_{I,\Bbbk}(\lambda) \simeq \hat{L}^P(\lambda)$, $\bar{\upsilon}_I \tilde{Q}_{I,\Bbbk}(\lambda) \simeq Q_{I,\Bbbk}(\lambda)$.

(3.3) Unless otherwise specified, we will regard an $L_{I,1}T$-module as a P_1T-module via inflation along the quotient $P \to P/\mathrm{Ru}(P) \simeq L_I$.

Lemma. *There is an isomorphism of functors from the category $E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}$ of finite-dimensional Y_I-graded $E_{\Omega_I,\Bbbk}$-modules to the category $\mathcal{C}_{\Bbbk}(\Omega)$ of finite-dimensional G_1T-modules belonging to the block Ω,*

$$Q \otimes_{E_{\Omega,\Bbbk}} \mathcal{C}_{\Bbbk}(\Omega)^\sharp(Q, \hat{\nabla}_P(Q_I)) \otimes_{E_{\Omega_I,\Bbbk}} ? \simeq \hat{\nabla}_P(Q_I \otimes_{E_{\Omega_I,\Bbbk}} ?).$$

Proof. Let $\tilde{M}$ be a Y_I-graded $E_{\Omega_I,\Bbbk}$-module of finite type. As $Q \otimes_{E_{\Omega,\Bbbk}} ?$ and $\mathcal{C}_{\Bbbk}(\Omega)^\sharp(Q, ?)$ are quasi-inverse to each other, $Q \otimes_{E_{\Omega,\Bbbk}} \mathcal{C}_{\Bbbk}(\Omega)^\sharp(Q, \hat{\nabla}_P(Q_I)) \otimes_{E_{\Omega_I,\Bbbk}} \tilde{M} \simeq \hat{\nabla}_P(Q_I) \otimes_{E_{\Omega_I,\Bbbk}} \tilde{M}$, which is isomorphic to $\hat{\nabla}_P(Q_I \otimes_{E_{\Omega_I,\Bbbk}} \tilde{M})$ if $\tilde{M}$ is isomorphic to $E_{\Omega_I,\Bbbk}$. In general,

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apply the five lemma to a natural homomorphism of G_1T-modules, $\hat{\nabla}_P(Q_I)\otimes_{E_{\Omega_I,\Bbbk}}\tilde{M}\to\hat{\nabla}_P(Q_I\otimes_{E_{\Omega_I,\Bbbk}}\tilde{M})$.

If $\eta:(Q\otimes_{E_{\Omega,\Bbbk}}?)\circ\mathcal{C}_\Bbbk(\Omega)^\sharp(Q,?)\to\mathrm{id}_{\mathcal{C}_\Bbbk(\Omega)}$ is the natural equivalence, the composite $\eta_{\hat{\nabla}_P(Q_I)}\otimes_{E_{\Omega_I,\Bbbk}}\tilde{M}$ with the homomorphism $\hat{\nabla}_P(Q_I)\otimes_{E_{\Omega_I,\Bbbk}}\tilde{M}\to\hat{\nabla}_P(Q_I\otimes_{E_{\Omega_I,\Bbbk}}\tilde{M})$ via $\phi\otimes m\mapsto\phi(?)\otimes m$, $\phi\in\hat{\nabla}_P(Q_I)=\mathbf{Sch}_\Bbbk(G_1T,Q_I)^{P_1T}$, $m\in\tilde{M}$, is certainly functorial in $\tilde{M}$, and hence the assertion follows. □

(3.4) We wish to lift the isomorphism of the lemma along the functors

$$\bar{v}:\tilde{\mathcal{C}}_\Bbbk(\Omega)=E_{\Omega,\Bbbk}\mathbf{modgr}_{Y\times\mathbb{Z}}\xrightarrow{\text{forgetful}}E_{\Omega,\Bbbk}\mathbf{modgr}_Y\xrightarrow{Q\otimes_{E_{\Omega,\Bbbk}}?}\mathcal{C}_\Bbbk(\Omega)$$

and

$$\bar{v}_I:\tilde{\mathcal{C}}_\Bbbk(\Omega_I)=E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I\times\mathbb{Z}}\xrightarrow{\text{forgetful}}E_{\Omega_I,\Bbbk}\mathbf{modgr}_{Y_I}\xrightarrow{Q_I\otimes_{E_{\Omega_I,\Bbbk}}?}\mathcal{C}_\Bbbk(\Omega_I).$$

We will show that $\mathcal{C}_\Bbbk(\Omega)^\sharp(Q,\hat{\nabla}_P(Q_I))$ carries the structure of a $(Y\times\mathbb{Z})$-graded left $E_{\Omega,\Bbbk}$- and $(Y_I\times\mathbb{Z})$-graded right $E_{\Omega_I,\Bbbk}$-bimodule to yield a functor $\mathcal{C}_\Bbbk(\Omega)^\sharp(Q,\hat{\nabla}_P(Q_I))\otimes_{E_{\Omega_I,\Bbbk}}?:\tilde{\mathcal{C}}_\Bbbk(\Omega_I)\to\tilde{\mathcal{C}}_\Bbbk(\Omega)$. Then an isomorphism of functors

$$\tilde{\mathcal{C}}_\Bbbk(\Omega_I)\xrightarrow{\mathcal{C}_\Bbbk(\Omega)^\sharp(Q,\hat{\nabla}_P(Q_I))\otimes_{E_{\Omega_I,\Bbbk}}?}\tilde{\mathcal{C}}_\Bbbk(\Omega)\xrightarrow{\bar{v}}\mathcal{C}_\Bbbk(\Omega)\hookrightarrow G_1T\mathbf{Mod}$$

and

$$\tilde{\mathcal{C}}_\Bbbk(\Omega)\xrightarrow{\bar{v}_I}\mathcal{C}_\Bbbk(\Omega_I)\hookrightarrow L_1T\mathbf{Mod}\xrightarrow{\text{inflation}}P_1T\mathbf{Mod}\xrightarrow{\hat{\nabla}_P}G_1T\mathbf{Mod}$$

will follow, i.e., schematically,

$$\begin{array}{ccc}
\tilde{\mathcal{C}}_\Bbbk(\Omega_I) & \xrightarrow{\mathcal{C}_\Bbbk(\Omega)^\sharp(Q,\hat{\nabla}_P(Q_I))\otimes_{E_{\Omega_I,\Bbbk}}?} & \tilde{\mathcal{C}}_\Bbbk(\Omega)\\
\downarrow{\scriptstyle \bar{v}_I} & & \\
L_{I,1}T\mathbf{Mod} & \circlearrowleft & \downarrow{\scriptstyle \bar{v}}\\
\downarrow & & \\
P_1T\mathbf{Mod} & \xrightarrow[\hat{\nabla}_P]{} & G_1T\mathbf{Mod}.
\end{array}$$

Let us sketch our strategy before proceeding. Put $A_G=\hat{S}_\Bbbk$ and $A_L=\hat{S}_{I,\Bbbk}$. We know that Q lifts to an object Q_{A_G} of $\mathcal{C}(\Omega,A_G)$, and $\mathcal{V}_\Omega Q_{A_G}$ admits a graded $S_\Bbbk$-form $\mathcal{P}\in\tilde{\mathcal{K}}(\Omega,S_\Bbbk)$. Likewise, let $Q_{I,A_L}=\bigoplus_{w\in W_I}Q_{I,A_L}(w\bullet\lambda_I^+)\in\mathcal{C}(\Omega_I,A_L)$, with $Q_{I,A_L}(w\bullet\lambda^+)\in\mathcal{C}(\Omega_I,A_L)$ a lift of the projective cover of $\hat{L}^P(w\bullet\lambda^+)$ for $L_{I,1}T$, and let $\mathcal{P}_I$ be a graded $S_{I,\Bbbk}$-form of $\mathcal{V}_{\Omega_I}Q_{I,A_L}$. We say that an object M of $\mathcal{C}(\Omega,A_G)$ (respectively, $\mathcal{C}(\Omega_I,A_L)$) admits a Z_{A_G}-filtration (respectively, Z_{I,A_L}-filtration) if and only if there is a

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filtration of M such that each suquotient is of the form $Z_{A_G}(\lambda)$, $\lambda \in \Omega$ (respectively, $Z_{I,A_L}(\lambda)$, $\lambda \in \Omega_I$). We will lift the induction functor $\hat{\nabla}_P : P_1T\mathbf{Mod} \to G_1T\mathbf{Mod}$ to a functor $\hat{\nabla}_{P,A_L} : \mathcal{F}(\Omega_I, A_L) \to \mathcal{F}(\Omega, A_G)$. We then construct a functor $\mathcal{I} : \mathcal{K}(\Omega_I, A_L) \to \mathcal{K}(\Omega, A_G)$ to lift $\hat{\nabla}_{P,A_L}$, i.e., such that $\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L} \simeq \mathcal{I} \circ \mathcal{V}_{\Omega_I}$, and finally its graded version $\tilde{\mathcal{I}} : \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk}) \to \tilde{\mathcal{K}}(\Omega, S_\Bbbk)$. We will find that the $(Y \times \mathbb{Z})$-graded left E_Ω- and $(Y_I \times \mathbb{Z})$-graded right E_{Ω_I}-bimodule $J = \tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{P}, \mathcal{I}(\mathcal{P}_I))$ does the job of equipping $\mathcal{C}_\Bbbk(\Omega)^\sharp(Q, \hat{\nabla}_P(Q_I))$ with the structure of a $(Y \times \mathbb{Z})$-graded left $E_{\Omega,\Bbbk}$- and $(Y_I \times \mathbb{Z})$-graded right $E_{\Omega_I,\Bbbk}$-bimodule through the isomorphism $J \otimes_{S_\Bbbk} \Bbbk \simeq \mathcal{C}_\Bbbk(\Omega)^\sharp(Q, \hat{\nabla}_P(Q_I))$.

We start with deforming the functor $\hat{\nabla}_P$. Let $A_G^\emptyset = A_G[(1/h_\alpha) \mid \alpha \in R^+]$, $A_L^\emptyset = A_L[(1/h_\alpha) \mid \alpha \in R_I^+]$; for each $\beta \in R^+$ put $A_G^\beta = A_G[(1/h_\alpha) \mid \alpha \in R^+ \setminus \{\beta\}]$, and for each $\beta \in R_I^+$ put $A_L^\beta = A_L[(1/h_\alpha) \mid \alpha \in R_I^+ \setminus \{\beta\}]$. We will regard A_G as an A_L-algebra via the inclusion $R_I \hookrightarrow R$. Thus $A_G^\emptyset$ is an $A_L^\emptyset$-algebra, and, for each $\beta \in R^+$,

$$A_G^\beta \simeq \begin{cases} A_L^\beta \otimes_{A_L} A_G^\beta & \text{if } \beta \in R_I^+ \\ A_L^\emptyset \otimes_{A_L} A_G^\beta & \text{else.} \end{cases}$$

For $\nu \in \Lambda$, define $Z_{I,A_L}(\nu)$, $Z_{I,A_L}^\beta(\nu) = Z_{I,A_L^\beta}(\nu)$ for $\beta \in R_I^+$, $Z_{I,A_L}^\emptyset(\nu) = Z_{I,A_L^\emptyset}(\nu)$, $Z_{I,A_G}(\nu)$, $Z_{I,A_G^\beta}(\nu)$ for $\beta \in R^+$, and $Z_{I,A_G^\emptyset}(\nu)$, as for $\nu \in \Omega_I$ in (3.2). For a $W_{I,p}$-orbit Γ_I in Λ, define $\mathcal{C}(\Gamma_I, A_L)$, $\mathcal{C}(\Gamma_I, A_L^\beta)$ for $\beta \in R_I^+$, $\mathcal{C}(\Gamma_I, A_G)$, $\mathcal{C}(\Gamma_I, A_G^\beta)$ for $\beta \in R^+$, $\mathcal{F}(\Gamma_I, A_L)$, $\mathcal{K}(\Gamma_I, A_L)$, $\tilde{\mathcal{K}}(\Gamma_I, S_{I,\Bbbk})$ and $\tilde{\mathcal{C}}_\Bbbk(\Gamma_I)$ for $L_{I,1}T$, just as for Ω.

Recall from (1.7) the parabolic subgroup $P^+ = \langle B^+, U_{-\alpha} | \alpha \in I \rangle$. Let Γ be the W_p-orbit in Λ containing Γ_I. Regarding an object of $\mathcal{F}(\Gamma_I, A_L)$ as a $\mathrm{Dist}(P_1^+)$-module by the quotient $P^+ \to P^+/\mathrm{Ru}(P^+)$, define a functor $\hat{\nabla}_{P,A_L} : \mathcal{F}(\Gamma_I, A_L) \to \mathcal{F}(\Gamma, A_G)$ via

$$M \mapsto (\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} M^\tau)^\tau \otimes_{A_L} A_G \simeq \{\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} (M \otimes_{A_L} A_G)^\tau\}^\tau,$$

which reduces to $\hat{\nabla}_P$ by reduction to $\Bbbk$. For each $\nu \in \Gamma_I$, one has

$$\hat{\nabla}_{P,A_L}(Z_{I,A_L}^{w_I}(\nu\langle w_I\rangle)) \simeq \hat{\nabla}_{P,A_L}(Z_{I,A_L}(\nu)^\tau) \simeq Z_{A_G}(\nu)^\tau \simeq Z_{A_G}^{w_0}(\nu\langle w_0\rangle). \quad (1)$$

(3.5) Let $U_1(w_I) = \prod_{\beta \in R^+ \setminus R_I} U_{-\beta,1}$ be the Frobenius kernel of the unipotent radical of P and $\mathrm{Dist}^+(U_1(w_I))$ the augmentation ideal of $\mathrm{Dist}(U_1(w_I))$. Let Γ be an arbitrary W_p-orbit. For each $M \in \mathcal{C}(\Gamma, A_G)$, put

$$M_\mathfrak{n} = M/\mathrm{Dist}^+(U_1(w_I))M \simeq \{\mathrm{Dist}(U_1(w_I))/\mathrm{Dist}^+(U_1(w_I))\} \otimes_{\mathrm{Dist}(U_1(w_I))} M,$$

the module of $\mathrm{Dist}^+(U_1(w_I))$-coinvariants of M. If $M = Z_{A_G}(\nu)$, $\nu \in \Lambda$, taking the τ-dual of (1.7) yields an isomorphism in $\mathcal{C}(W_{I,p} \bullet \nu, A_G)$:

$$Z_{A_G}(\nu)_\mathfrak{n} \simeq Z_{I,A_L}(\nu) \otimes_{A_L} A_G \simeq Z_{I,A_G}(\nu). \quad (1)$$

Let $\beta \in R_I^+$, $\nu \in \Gamma$ with $\beta \uparrow \nu > \nu$, and put $\Gamma_I = W_{I,p} \bullet \nu$. One has from [1, 8.6], as $d_\beta \in \Bbbk^\times$ by the standing hypothesis on p,

$$\mathrm{Ext}^1_{\mathcal{C}(\Gamma, A_G^\beta)}(Z_{A_G}^\beta(\nu), Z_{A_G}^\beta(\beta \uparrow \nu)) \simeq A_G^\beta h_\beta^{-1}/A_G^\beta \simeq (A_L^\beta h_\beta^{-1}/A_L^\beta) \otimes_{A_L} A_G^\beta$$

$$\simeq \mathrm{Ext}^1_{\mathcal{C}(\Gamma_I, A_L^\beta)}(Z^\beta_{I,A_L}(\nu), Z^\beta_{I,A_L}(\beta \uparrow \nu)) \otimes_{A_L} A_G^\beta$$
$$\simeq \mathrm{Ext}^1_{\mathcal{C}(\Gamma_I, A_G^\beta)}(Z_{I,A_G^\beta}(\nu), Z_{I,A_G^\beta}(\beta \uparrow \nu)) \quad \text{as } A_G^\beta \text{ is flat over } A_L. \tag{2}$$

Lemma. *Assume that $\beta \uparrow \nu > \nu$. If $0 \to Z^\beta_{A_G}(\beta \uparrow \nu) \to M \to Z^\beta_{A_G}(\nu) \to 0$ is exact in $\mathcal{C}(\Gamma, A_G^\beta)$, applying $?_{\mathfrak{n}}$ to the sequence yields an exact sequence $0 \to Z_{I,A_G^\beta}(\beta \uparrow \nu) \to M_{\mathfrak{n}} \to Z_{I,A_G^\beta}(\nu) \to 0$ with M projective in $\mathcal{C}(\Gamma, A_G^\beta)$ if and only if $M_{\mathfrak{n}}$ projective in $\mathcal{C}(\Gamma_I, A_G^\beta)$. Conversely, applying $\mathrm{Dist}(G_1)\otimes_{\mathrm{Dist}(P_1^+)}?$ to the latter sequence recovers the former. Likewise, if $0 \to Z^\beta_{I,A_L}(\beta \uparrow \nu) \to M' \to Z^\beta_{I,A_L}(\nu) \to 0$ is an exact sequence in $\mathcal{C}(\Gamma_I, A_L^\beta)$ with M' projective, then applying $\mathrm{Dist}(G_1)\otimes_{\mathrm{Dist}(P_1^+)}?\otimes_{A_L} A_G^\beta$ yields an exact sequence $0 \to Z^\beta_{A_G}(\beta \uparrow \nu) \to \mathrm{Dist}(G_1)\otimes_{\mathrm{Dist}(P_1^+)} M' \otimes_{A_L} A_G^\beta \to Z^\beta_{A_G}(\nu) \to 0$ with $\mathrm{Dist}(G_1)\otimes_{\mathrm{Dist}(P_1^+)} M' \otimes_{A_L} A_G^\beta$ projective in $\mathcal{C}(\Gamma, A_G^\beta)$.*

Proof. Assume that the sequence $0 \to Z^\beta_{A_G}(\beta \uparrow \nu) \to M \to Z^\beta_{A_G}(\nu) \to 0$ is exact. As $?_{\mathfrak{n}} \simeq \{\mathrm{Dist}(U_1(w_I))/\mathrm{Dist}^+(U_1(w_I))\}\otimes_{\mathrm{Dist}(U_1(w_I))}?$ and as $Z^\beta_{A_G}(\nu) \simeq \mathrm{Dist}(U_1) \simeq \mathrm{Dist}(U_1(w_I)) \otimes_{\Bbbk} \mathrm{Dist}((B \cap L_I)_1)$ is free over $\mathrm{Dist}(U_1(w_I))$, $0 \to Z^\beta_{A_G}(\beta \uparrow \nu)_{\mathfrak{n}} \to M_{\mathfrak{n}} \to Z^\beta_{A_G}(\nu)_{\mathfrak{n}} \to 0$ remains exact with $Z^\beta_{A_G}(\beta \uparrow \nu)_{\mathfrak{n}} \simeq Z_{I,A_G^\beta}(\beta \uparrow \nu)$ and $Z^\beta_{A_G}(\nu)_{\mathfrak{n}} \simeq Z_{I,A_G^\beta}(\nu)$.

Recall from [1, 12.4] how each M is constructed. Let $w_\beta \in W_I$ with $w_\beta^{-1}\beta \in I$. Let $v_\nu^{w_\beta} \in Z^\beta_{A_G}(\nu)$ of weight $\nu\langle w_\beta\rangle$ corresponding to the standard generator $1 \otimes 1$ of $Z^{w_\beta}_{A_G^\beta}(\nu\langle w_\beta\rangle)$ under the isomorphism $Z^\beta_{A_G}(\nu) = Z_{A_G^\beta}(\nu) \simeq Z^{w_\beta}_{A_G^\beta}(\nu\langle w_\beta\rangle)$, and define $v^{w_\beta}_{\beta\uparrow\nu} \in Z^\beta_{A_G}(\beta \uparrow \nu)$ likewise. Write $\langle \nu + \rho, \beta^\vee\rangle \equiv p - n \mod p$ with $n \in [0, p[$, and put $z_\nu = E^{(n)}_{-\beta} v^{w_\beta}_{\beta\uparrow\nu} b + v^{w_\beta}_\nu \in Z_K(\beta \uparrow \nu) \oplus Z_K(\nu)$ for each $b \in A_G^\beta h_\beta^{-1}$ with $K = \mathrm{Frac}(A_G)$, so z_ν is of weight $\nu\langle w_\beta\rangle$. Then M is of the form $M_\nu^{w_\beta}(b) = \mathrm{Dist}(G_1) v^{w_\beta}_{\beta\uparrow\nu} A_G^\beta + \mathrm{Dist}(G_1) z_\nu A_G^\beta$ living in $Z_K(\beta \uparrow \nu) \oplus Z_K(\nu)$, and the sequence reads $v^{w_\beta}_{\beta\uparrow\nu}$, mapping to itself while $z_\nu \mapsto v_\nu^{w_\beta}$. Now

$$\begin{aligned}\mathrm{Dist}(G_1) &\simeq \mathrm{Dist}({}^{w_\beta}U_1) \otimes \mathrm{Dist}({}^{w_\beta}B_1^+)\\ &\simeq \mathrm{Dist}({}^{w_\beta}U_1(w_I)) \otimes \mathrm{Dist}({}^{w_\beta}(B \cap L_I)_1) \otimes \mathrm{Dist}({}^{w_\beta}B_1^+)\\ &\simeq \mathrm{Dist}(U_1(w_I)) \otimes \mathrm{Dist}({}^{w_\beta}(B \cap L_I)_1) \otimes \mathrm{Dist}({}^{w_\beta}B_1^+)\end{aligned}$$

as ${}^{w_\beta}U(w_I) = \prod_{\alpha\in R^+\setminus R_I} {}^{w_\beta}U_{-\alpha} = \prod_{\alpha\in R^+\setminus R_I} U_{-\alpha} = U(w_I)$. Thus $(\mathrm{Dist}(G_1)v^{w_\beta}_{\beta\uparrow\nu})_{\mathfrak{n}} \simeq \mathrm{Dist}({}^{w_\beta}(B \cap L_I)_1) v^{w_\beta}_{\beta\uparrow\nu}$, $(\mathrm{Dist}(G_1)z_\nu)_{\mathfrak{n}} \simeq \mathrm{Dist}({}^{w_\beta}(B \cap L_I)_1) z_\nu$, and hence $M_\nu^{w_\beta}(b)_{\mathfrak{n}} = \mathrm{Dist}(L_{I,1}) v^{w_\beta}_{\beta\uparrow\nu} A_G^\beta + \mathrm{Dist}(L_{I,1}) z_\nu A_G^\beta$. It follows from [1, 8.7] that $M_\nu^{w_\beta}(b)$ is projective in $\mathcal{C}(\Omega, A_G^\beta)$ if and only if $A_G^\beta b = A_G^\beta h_\beta^{-1}/A_G^\beta$ if and only if $M_\nu^{w_\beta}(b)_{\mathfrak{n}}$ is projective in $\mathcal{C}(\Omega_I, A_G^\beta)$. Likewise the last assertion follows from (1). □

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(3.6) We now transfer from $\mathcal{F}(\Omega, A_G)$ (respectively, $\mathcal{F}(\Omega_I, A_L)$) to the combinatorial category $\mathcal{K}(\Omega, A_G)$ (respectively, $\mathcal{K}(\Omega_I, A_L)$) via the fully faithful functor $\mathcal{V}_\Omega$ (respectively, $\mathcal{V}_{\Omega_I}$). Define a functor $\mathcal{I}: \mathcal{K}(\Omega_I, A_L) \to \mathcal{K}(\Omega, A_G)$ as follows: for each $\mathcal{M} \in \mathcal{K}(\Omega_I, A_L)$ and $\lambda \in \Omega$, set

$$(\mathcal{I}\mathcal{M})(\lambda) = \begin{cases} \mathcal{M}(\lambda) \otimes_{A_L} A_G^{\emptyset} & \text{if } \lambda \in \Omega_I \\ 0 & \text{else,} \end{cases}$$

and for each $\beta \in R^+$ set

$$(\mathcal{I}\mathcal{M})(\lambda, \beta) = \begin{cases} \mathcal{M}(\lambda, \beta) \otimes_{A_L} A_G^{\beta} & \text{if } \lambda \in \Omega_I \text{ and } \beta \in R_I^+ \\ \mathcal{M}(\lambda) \otimes_{A_L} A_G^{\beta} & \text{if } \lambda \in \Omega_I \text{ and } \beta \notin R_I^+ \\ \mathcal{M}(\beta \uparrow \lambda) \otimes_{A_L} A_G^{\beta} & \text{if } \beta \uparrow \lambda \in \Omega_I \text{ and } \beta \notin R_I^+ \\ 0 & \text{else.} \end{cases}$$

Note that the second and the third cases above are exclusive to each other. We want to show an isomorphism of functors $\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L} \simeq \mathcal{I} \circ \mathcal{V}_{\Omega_I}$ from $\mathcal{F}(\Omega_I, A_L)$ to $\mathcal{F}(\Omega, A_G)$.

Recall that $\mathcal{V}_\Omega$ and $\mathcal{V}_{\Omega_I}$ are defined with specific choice of extensions according to Theorem of Good Choices [1, 13.4]. In the following crucial lemma the extensions $Y_{A_G}^{\beta}(\lambda)$ and $Y_{I,A_L}^{\beta}(\lambda)$ are constructed as specified in the proof of (3.5). Also, the extensions $Y_{I,A_L}^{\beta}(\lambda) \otimes_{A_L} A_G^{\beta}$ and $Y_{A_G}^{\beta}(\lambda)_{\mathfrak{n}}$ (respectively, $\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} Y_{I,A_L}^{\beta}(\lambda) \otimes_{A_L} A_G^{\beta}$ and $Y_{A_G}^{\beta}(\lambda)$) are isomorphic not just as modules but as extensions of modules, to emphasize which we denote the isomorphisms by equalities.

Lemma. *Let $\lambda \in \Omega_I$ and $\beta \in R_I^+$ with $\beta \uparrow \lambda > \lambda$. Let $e^{\beta}(\lambda) \in \mathrm{Ext}^1_{\mathcal{C}(\Omega, A_G^{\beta})}(Z_{A_G}^{\beta}(\lambda), Z_{A_G}^{\beta}(\beta \uparrow \lambda))$ and $e_I^{\beta}(\lambda) \in \mathrm{Ext}^1_{\mathcal{C}(\Omega_I, A_L^{\beta})}(Z_{I,A_L}^{\beta}(\lambda), Z_{I,A_L}^{\beta}(\beta \uparrow \lambda))$, chosen according to the Theorem of Good Choices. Let $Y_{A_G}^{\beta}(\lambda) \in \mathcal{C}(\Omega, A_G^{\beta})$ (respectively, $Y_{I,A_L}^{\beta}(\lambda) \in \mathcal{C}(\Omega_I, A_L^{\beta})$) be the module representing $e^{\beta}(\lambda)$ (respectively, $e_I^{\beta}(\lambda)$). Then $Y_{I,A_L}^{\beta}(\lambda) \otimes_{A_L} A_G^{\beta} = Y_{A_G}^{\beta}(\lambda)_{\mathfrak{n}}$ and $\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} Y_{I,A_L}^{\beta}(\lambda) \otimes_{A_L} A_G^{\beta} = Y_{A_G}^{\beta}(\lambda)$ both as extensions of modules.*

Proof. Write $\lambda = w_1 \bullet \lambda^+ + p\gamma_1 = w_2 \bullet \lambda_I^+ + p\gamma_2$ with $w_1 \in W$, $w_2 \in W_I$, $\gamma_1 \in \mathbb{Z}R$, $\gamma_2 \in \mathbb{Z}R_I$. By [1, 13.25], we may assume that $\lambda_I^+ = w_2^{-1} w_1 \bullet \lambda^+$. Then, for each $\alpha \in R_I^+$,

$$(w_2^{-1} w_1)^{-1} \alpha = w_1^{-1} w_2 \alpha > 0, \tag{1}$$

and hence

$$w_1^{-1} \alpha > 0 \quad \text{if and only if} \quad w_2^{-1} \alpha > 0. \tag{2}$$

Recall from [1, 13.2.5] that $e^{\beta}(\lambda) = b^{\beta}(\lambda) e_0^{\beta}(\lambda)$, and thus $e_I^{\beta}(\lambda) = b_I^{\beta}(\lambda) e_{I,0}^{\beta}(\lambda)$ likewise,

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with the right-hand sides specified as follows. By [1, 13.2.3, 4],

$$b^{\beta}(\lambda)=\begin{cases}\varepsilon^{\beta}_{w_1\bullet\lambda^+,-\rho}d(w_1\bullet\lambda^+,-\rho,\ s_\beta)\kappa(\beta) & \text{if } w_1^{-1}\beta>0\\ \varepsilon^{\beta}_{w_1\bullet\lambda^+,-\rho}d(w_1\bullet\lambda^+,-\rho,\ s_\beta)h_\beta & \text{else}\end{cases}\tag{3}$$

with $\kappa(\beta)=\prod_{\substack{\alpha\in R^+\\ s_\beta\alpha<0,w_\beta^{-1}\alpha<0}} h_\alpha^{-\langle\beta,\alpha^\vee\rangle}$, where $w_\beta\in W_I$ such that $w_\beta^{-1}\beta\in I$, and thus

$$b_I^{\beta}(\lambda)=\begin{cases}\varepsilon^{\beta}_{I,w_2\bullet\lambda_I^+,-\rho_I}d_I(w_2\bullet\lambda_I^+,-\rho_I,\ s_\beta)\kappa_I(\beta) & \text{if } w_2^{-1}\beta>0\\ \varepsilon^{\beta}_{I,w_2\bullet\lambda_I^+,-\rho_I}d_I(w_2\bullet\lambda_I^+,-\rho_I,\ s_\beta)h_\beta & \text{else}\end{cases}\tag{4}$$

with $\kappa_I(\beta)=\prod_{\substack{\alpha\in R_I^+\\ s_\beta\alpha<0,w_\beta^{-1}\alpha<0}} h_\alpha^{-\langle\beta,\alpha^\vee\rangle}$. By (2) two cases in (3) and (4) agree, and $\kappa(\beta)=\kappa_I(\beta)$; note that any $\alpha\in R^+$ with $s_\beta\alpha<0$ must belong to R_I^+ as $\beta\in R_I^+$, which will be used repeatedly in the following computations. By [1, 12.12.5]

$$\varepsilon^{\beta}_{w_1\bullet\lambda^+,-\rho}=\prod_{\substack{\alpha\in R^+\\ s_\beta\alpha<0}}(-1)^{\langle-\rho-w_1\bullet\lambda^+,\alpha^\vee\rangle\bar{\alpha}(-\rho-w_1\bullet\lambda^+)}\prod_{\substack{\alpha\in R^+\setminus\{\beta\}\\ s_\beta\alpha<0,w_\beta^{-1}\alpha>0}}(-1)^{\langle-\rho-w_1\bullet\lambda^+,\alpha^\vee\rangle}$$

with $\bar{\alpha}(\nu)=\begin{cases}1 & \text{if } \langle\nu,\alpha^\vee\rangle>0\\ 0 & \text{else}\end{cases}$ for each $\nu\in\Lambda$ [1, A.1.1], and thus

$$\varepsilon^{\beta}_{I,w_2\bullet\lambda_I^+,-\rho_I}=\prod_{\substack{\alpha\in R_I^+\\ s_\beta\alpha<0}}(-1)^{\langle-\rho_I-w_2\bullet\lambda_I^+,\alpha^\vee\rangle\bar{\alpha}(-\rho_I-w_2\bullet\lambda_I^+)}\prod_{\substack{\alpha\in R_I^+\setminus\{\beta\}\\ s_\beta\alpha<0,w_\beta^{-1}\alpha>0}}(-1)^{\langle-\rho_I-w_2\bullet\lambda_I^+,\alpha^\vee\rangle}.$$

One has $-\rho-w_1\bullet\lambda^+=-\rho-w_1\bullet(w_2^{-1}w_1)^{-1}\bullet\lambda_I^+=-\rho_P-\rho_I-w_2\bullet\lambda_I^+$. If $\alpha\in R^+$ with $s_\beta\alpha<0$, $\alpha\in R_I^+$, and hence $\langle-\rho-w_1\bullet\lambda^+,\alpha^\vee\rangle=\langle-\rho_P-\rho_I-w_2\bullet\lambda_I^+,\alpha^\vee\rangle=\langle-\rho_I-w_2\bullet\lambda_I^+,\alpha^\vee\rangle$ and $\bar{\alpha}(-\rho-w_1\bullet\lambda^+)=\bar{\alpha}(-\rho_I-w_2\bullet\lambda_I^+)$. Thus $\varepsilon^{\beta}_{w_1\bullet\lambda^+,-\rho}=\varepsilon^{\beta}_{I,w_2\bullet\lambda_I^+,-\rho_I}$. By [1, 13.2.2]

$$d(w_1\bullet\lambda^+,-\rho,\ s_\beta)=\prod_{\substack{\alpha\in R^+\\ s_\beta\alpha<0,w_1^{-1}\alpha<0}}\frac{[k_\alpha;w_1\bullet\lambda^++\rho]}{h_\alpha},$$

$$d_I(w_2\bullet\lambda_I^+,-\rho_I,\ s_\beta)=\prod_{\substack{\alpha\in R_I^+\\ s_\beta\alpha<0,w_2^{-1}\alpha<0}}\frac{[k_\alpha;w_2\bullet\lambda_I^++\rho_I]_I}{h_\alpha}.$$

By (2) again, the products run over the same subset of R_I^+. By [1, 13.1.4],

$$[k_\alpha;w_1\bullet\lambda^++\rho]=(H_\alpha+\langle w_1\bullet\lambda^++\rho,\alpha^\vee\rangle)H_\alpha^{-1}h_\alpha$$

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with $\langle w_1 \bullet \lambda^+ + \rho, \alpha^\vee\rangle = \langle w_1 \bullet (w_2^{-1}w_1)^{-1} \bullet \lambda_I^+ + \rho, \alpha^\vee\rangle = \langle w_2 \bullet \lambda_I^+ + \rho, \alpha^\vee\rangle = \langle w_2 \bullet \lambda_I^+ + \rho_I, \alpha^\vee\rangle$, and hence $d(w_1 \bullet \lambda^+, -\rho,\ s_\beta) = d_I(w_2 \bullet \lambda_I^+, -\rho_I,\ s_\beta)$ and $b^\beta(\lambda) = b_I^\beta(\lambda)$.

We compare next $e_0^\beta(\lambda)$ and $e_{I,0}^\beta(\lambda)$. Take $\omega \in \Lambda$ in the upper closure of the facette that λ belongs to with respect to $\langle s_{\beta,r} \mid r \in \mathbb{Z}\rangle$. By [1, 12.13.1],

$$t_0^\beta[\omega,\lambda]e_0^\beta(\lambda) = \varepsilon_{\lambda\omega}^\beta d(\omega,\lambda,\ s_\beta)h_\beta^{-1} + A_G^\beta, \qquad (5)$$
$$t_{I,0}^\beta[\omega,\lambda]e_{I,0}^\beta(\lambda) = \varepsilon_{I,\lambda,\omega}^\beta d_I(\omega,\lambda,\ s_\beta)h_\beta^{-1} + A_L^\beta.$$

By definition [1, 12.12.5] again,

$$\varepsilon_{\lambda\omega}^\beta = \prod_{\substack{\alpha\in R^+\\ s_\beta\alpha<0}} (-1)^{\langle\omega-\lambda,\alpha^\vee\rangle\bar{\alpha}(\omega-\lambda)} \prod_{\substack{\alpha\in R^+\setminus\{\beta\}\\ s_\beta\alpha<0,\, w_\beta^{-1}\alpha>0}} (-1)^{\langle\omega-\lambda,\alpha^\vee\rangle} = \varepsilon_{I,\lambda,\omega}^\beta. \qquad (6)$$

By definition [1, A.7.1 and A.2.1],

$$d(\omega,\lambda,\ s_\beta) = \prod_{\substack{\alpha\in R^+\\ s_\beta\alpha<0}} d(\omega,\lambda,\alpha) = \prod_{\substack{\alpha\in R^+\\ s_\beta\alpha<0}} \left(\frac{H_\alpha + \langle\omega+\rho,\alpha^\vee\rangle}{H_\alpha + \langle\lambda+\rho,\alpha^\vee\rangle}\right)^{\bar{\alpha}(\omega-\rho)} = d_I(\omega,\lambda,\ s_\beta).$$

It follows in (5) that $\varepsilon_{\lambda\omega}^\beta d(\omega,\lambda,\ s_\beta)h_\beta^{-1} = \varepsilon_{I,\lambda,\omega}^\beta d_I(\omega,\lambda,\ s_\beta)h_\beta^{-1}$. By [1, 12.12.1],

$$t_0^\beta[\omega,\lambda] = t[\omega,\lambda,a_{\lambda\omega}], \quad t_{I,0}^\beta[\omega,\lambda] = t_I[\omega,\lambda,a_{I,\lambda,\omega}]$$

with

$$\begin{aligned} a_{\lambda\omega} &= a'_{\lambda\omega}\varepsilon_{\lambda\omega}^\beta \\ &= \varepsilon_{\lambda\omega}^\beta \quad \text{by [1, A.12]} \\ &= \varepsilon_{I,\lambda,\omega}^\beta \quad \text{by (6)} \\ &= a'_{I,\lambda,\omega}\varepsilon_{I,\lambda,\omega}^\beta = a_{I,\lambda,\omega}. \end{aligned} \qquad (7)$$

We have $t[\omega,\lambda,a_{\lambda\omega}] = t[\omega,\lambda,e,\bar{e}]$ by [1, 12.8.2] with $e \in E_{\omega-\lambda}\setminus 0$, E a simple G-module of extremal weight $\omega-\lambda$ [1, 11.1], and with $\bar{e} = a_{\lambda\omega}(-1)^n E_{-\beta}^{(n)}e \in E_{s_\beta(\omega-\lambda)}\setminus 0$, $\langle\lambda+\rho,\beta^\vee\rangle \equiv p-n \mod p$, $n \in]0,p[$, by [1, 12.3.1]. Recall from [1, 12.6] the definition of $t[\omega,\lambda,e,\bar{e}] : \mathrm{Ext}^1_{\mathcal{C}(\Omega,A_G^\beta)}(Z_{A_G}^\beta(\lambda), Z_{A_G}^\beta(\beta\uparrow\lambda)) \to H_\beta^{-1}A_G^\beta/A_G^\beta = h_\beta^{-1}A_G^\beta/A_G^\beta$. Let $\xi \in \mathrm{Ext}^1_{\mathcal{C}(\Omega,A_G^\beta)}(Z_{A_G}^\beta(\lambda), Z_{A_G}^\beta(\beta\uparrow\lambda))$ be represented by a short exact sequence

$$0 \to Z_{A_G}^\beta(\beta\uparrow\lambda) \xrightarrow{i} M \xrightarrow{j} Z_{A_G}^\beta(\lambda) \to 0. \qquad (8)$$

As $H_\beta\xi = 0$, there is $j' \in \mathcal{C}(\Omega,A_G^\beta)(Z_{A_G}^\beta(\lambda), M)$ with $j\circ j' = H_\beta \mathrm{id}_{Z_{A_G}^\beta(\lambda)}$. Apply the translation functor T_λ^ω to (8) to obtain a split exact sequence

$$0 \to T_\lambda^\omega Z_{A_G}^\beta(\beta\uparrow\lambda) \xrightarrow{T_\lambda^\omega i} T_\lambda^\omega M \xrightarrow{T_\lambda^\omega j} T_\lambda^\omega Z_{A_G}^\beta(\lambda) \to 0.$$

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Let $i' \in \mathcal{C}(A_G^\beta)(T_\lambda^\omega Z_{A_G}^\beta(\lambda), T_\lambda^\omega M)$ with $i' \circ T_\lambda^\omega j = \mathrm{id}_{T_\lambda^\omega Z_{A_G}^\beta(\lambda)}$. Recall from [1, 11.2.1] isomorphisms $f_e : Z_{A_G}^\beta(\omega) \to T_\lambda^\omega Z_{A_G}^\beta(\lambda) = \mathrm{pr}(E \otimes Z_{A_G}^\beta(\lambda))$ via $1 \otimes 1 \mapsto \mathrm{pr}(e \otimes 1 \otimes 1)$ and $f_{\bar{e}} : Z_{A_G}^\beta(\omega) \to T_\lambda^\omega Z_{A_G}^\beta(\beta \uparrow \lambda) = \mathrm{pr}(E \otimes Z_{A_G}^\beta(\beta \uparrow \lambda))$ via $1 \otimes 1 \mapsto \mathrm{pr}(\bar{e} \otimes 1 \otimes 1)$. If $a \in A_G^\beta$ with $f_{\bar{e}}^{-1} \circ i' \circ T_\lambda^\omega j' \circ f_e = a\mathrm{id}_{Z_{A_G}^\beta(\omega)}$, then $t[\omega, \lambda, e, \bar{e}]\xi = aH_\beta^{-1} + A_G^\beta$. Now recall the L_I-submodule E' of E from (2.2), and choose $e_I = e \in E'$ and $\bar{e}_I = \bar{e} \in E'$ to define $t_I[\omega, \lambda, e_I, \bar{e}_I] : \mathrm{Ext}^1_{\mathcal{C}(\Omega_I, A_L^\beta)}(Z_{I,A_L}^\beta(\lambda), Z_{I,A_L}^\beta(\beta \uparrow \lambda)) \to H_\beta^{-1} A_L^\beta / A_L^\beta = h_\beta^{-1} A_L^\beta / A_L^\beta$ likewise. As we have natural isomorphisms from (2.2), or rather from its τ-dual

$$
\begin{aligned}
\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} Z_{I,A_L}^\beta(\lambda) \otimes_{A_L} A_G^\beta &\simeq Z_{A_G}^\beta(\lambda),\\
\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} Z_{I,A_L}^\beta(\beta \uparrow \lambda) \otimes_{A_L} A_G^\beta &\simeq Z_{A_G}^\beta(\beta \uparrow \lambda),\\
\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} T_{I,\lambda}^\omega Z_{I,A_L}^\beta(\lambda) \otimes_{A_L} A_G^\beta &\simeq T_\lambda^\omega Z_{A_G}^\beta(\lambda),\\
\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} T_{I,\lambda}^\omega Z_{I,A_L}^\beta(\beta \uparrow \lambda) \otimes_{A_L} A_G^\beta &\simeq T_\lambda^\omega Z_{A_G}^\beta(\beta \uparrow \lambda),
\end{aligned}
$$

the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Ext}^1_{\mathcal{C}(\Omega_I, A_L^\beta)}(Z_{I,A_L}^\beta(\lambda), Z_{I,A_L}^\beta(\beta \uparrow \lambda)) & \xrightarrow{t_{I,0}^\beta[\omega,\lambda]} & H_\beta^{-1} A_L^\beta / A_L^\beta \\
{\scriptstyle \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} ? \otimes_{A_L} A_G^\beta} \downarrow & & \downarrow \\
\mathrm{Ext}^1_{\mathcal{C}(\Omega, A_G^\beta)}(Z_{A_G}^\beta(\lambda), Z_{A_G}^\beta(\beta \uparrow \lambda)) & \xrightarrow{t_0^\beta[\omega,\lambda]} & H_\beta^{-1} A_G^\beta / A_G^\beta
\end{array}
$$

follows. More precisely, if $Y_{A_G}^\beta(\lambda)$ (respectively, $Y_{I,A_L}^\beta(\lambda)$) is the module representing $e^\beta(\lambda)$ (respectively, $e_I^\beta(\lambda)$), then $\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} Y_{I,A_L}^\beta(\lambda) \otimes_{A_L} A_G^\beta = Y_{A_G}^\beta(\lambda)$ with $Y_{I,A_L}^\beta(\lambda) \otimes_{A_L} A_G^\beta = Y_{A_G}^\beta(\lambda)_{\mathfrak{n}}$ by (3.5). □

(3.7) We are now ready to show the following.

Theorem. *There is an isomorphism of functors from $\mathcal{F}(\Omega_I, A_L)$ to $\mathcal{K}(\Omega, A_G)$:*

$$\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L} \simeq \mathcal{I} \circ \mathcal{V}_{\Omega_I}.$$

Proof. For each $X \in \mathcal{D}(\Omega, A_G)$ and $M \in \mathcal{D}(\Omega_I, A_L)$, one has, as X (respectively, M) is free of finite rank over A_G (respectively, A_L),

$$
\begin{aligned}
\mathcal{C}(\Omega, A_G)(X, \hat{\nabla}_{P,A_G}(M)) &\simeq \mathcal{C}(\Omega, A_G)(X, (\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} M^\tau)^\tau \otimes_{A_L} A_G)\\
&\simeq \mathcal{C}(\Omega, A_G)(X, \{\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} (M \otimes_{A_L} A_G)^\tau\}^\tau)\\
&\simeq \mathcal{C}(\Omega, A_G)(\mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1^+)} (M \otimes_{A_L} A_G)^\tau, X^\tau) \quad \text{by [1, 4.5.5]}\\
&\simeq \mathcal{C}_{L_I}(A_G)((M \otimes_{A_L} A_G)^\tau, \mathrm{Ann}_{X^\tau}(\mathrm{Dist}^+(U_1^+(w_I)))),
\end{aligned}
$$

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with

$$\begin{aligned}\mathrm{Ann}_{X^\tau}(\mathrm{Dist}^+(U_1^+(w_I))) &= \{f \in \mathbf{Mod}_{A_G}(X, A_G) \mid 0 = xf = f(\tau(x)?)\ \forall x \in \mathrm{Dist}^+(U_1^+(w_I))\}\\ &\simeq \{X/\tau(\mathrm{Dist}^+(U_1^+(w_I)))X\}^\tau\\ &= \{X/(\mathrm{Dist}^+(U_1(w_I))X)\}^\tau = (X_{\mathfrak{n}})^\tau.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{C}(\Omega, A_G)(X, \hat{\nabla}_{P,A_L}(M)) &\simeq \mathcal{C}(\Omega_I, A_G)((M \otimes_{A_L} A_G)^\tau, (X_{\mathfrak{n}})^\tau)\\ &\simeq \mathcal{C}(\Omega_I, A_G)(X_{\mathfrak{n}}, M \otimes_{A_L} A_G).\end{aligned} \tag{1}$$

The isomorphism is functorial in both X and M: $\forall \phi \in \mathcal{C}(\Omega, A_G)(X, X')$ and $\forall \psi \in \mathcal{C}(\Omega_I, A_L)(M, M')$, ϕ induces $\bar{\phi} \in \mathcal{C}(\Omega, A_G)(X_{\mathfrak{n}}, X'_{\mathfrak{n}})$ to yield the commutative diagrams

$$\begin{array}{ccc}
\mathcal{C}(\Omega, A_G)(X, \hat{\nabla}_{P,A_L}(M')) & \xleftarrow{\sim} & \mathcal{C}(\Omega_I, A_G)(X_{\mathfrak{n}}, M' \otimes_{A_L} A_G)\\
{\scriptstyle \mathcal{C}(\Omega,A_G)(X,\hat{\nabla}_{P,A_L}(\psi))}\uparrow & & \uparrow{\scriptstyle \mathcal{C}(\Omega,A_G)(X_{\mathfrak{n}},\psi\otimes_{A_L}A_G)}\\
\mathcal{C}(\Omega, A_G)(X, \hat{\nabla}_{P,A_L}(M)) & \xleftarrow{\sim} & \mathcal{C}(\Omega_I, A_G)(X_{\mathfrak{n}}, M \otimes_{A_L} A_G)\\
{\scriptstyle \mathcal{C}(\Omega,A_G)(\phi,\hat{\nabla}_{P,A_L}(M))}\uparrow & & \uparrow{\scriptstyle \mathcal{C}(\Omega,A_G)(\bar{\phi},M\otimes_{A_L}A_G)}\\
\mathcal{C}(\Omega, A_G)(X', \hat{\nabla}_{P,A_L}(M)) & \xleftarrow{\sim} & \mathcal{C}(\Omega_I, A_G)(X'_{\mathfrak{n}}, M \otimes_{A_L} A_G).
\end{array}$$

It follows for each $\lambda \in \Omega$ that

$$\begin{aligned}(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L})(M)(\lambda) &= \mathcal{C}(\Omega, A_G^\emptyset)(Z^\emptyset(\lambda), \hat{\nabla}_{P,A_L}(M)^\emptyset)\\ &\simeq \mathcal{C}(\Omega_I, A_G^\emptyset)(Z_{I,A_G^\emptyset}(\lambda), M^\emptyset \otimes_{A_L} A_G^\emptyset) \quad \text{by (3.5)(1)}\\ &\simeq \mathcal{C}(\Omega_I, A_L^\emptyset)(Z_{I,A_L}^\emptyset(\lambda), M^\emptyset) \otimes_{A_L} A_G^\emptyset\\ &= \begin{cases}\mathcal{V}_{\Omega_I}(M)(\lambda) \otimes_{A_L} A_G^\emptyset & \text{if } \lambda \in \Omega_I\\ 0 & \text{else}\end{cases}\\ &= (\mathcal{I} \circ \mathcal{V}_{\Omega_I})(M)(\lambda).\end{aligned}$$

Let now $Y_{A_G}^\beta(\lambda)$, $\lambda \in \Omega$, $\beta \in R^+$, be the extension of $Z_{A_G}^\beta(\beta \uparrow \lambda)$ by $Z_{A_G}^\beta(\lambda)$ constructed according to the Theorem of Good Choices. Assume first that $\lambda \in \Omega_I$. If $\beta \in R_I^+$,

$$\begin{aligned}(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L})(M)(\lambda, \beta) &= \mathcal{C}(\Omega, A_G^\beta)(Y_{A_G}^\beta(\lambda), \hat{\nabla}_{P,A_L}(M)^\beta)\\ &\simeq \mathcal{C}(\Omega_I, A_L^\beta)(Y_{I,A_L}^\beta(\lambda), M^\beta) \otimes_{A_L} A_G^\beta \quad \text{likewise by (3.6)}\\ &\simeq \mathcal{V}_{\Omega_I}(M)(\lambda, \beta) \otimes_{A_L} A_G^\beta = (\mathcal{I} \circ \mathcal{V}_{\Omega_I})(M)(\lambda, \beta).\end{aligned}$$

If $\beta \in R^+ \setminus R_I^+$, $\beta \uparrow \lambda \notin \Omega_I$. As $A_G^\beta \simeq A_L^\emptyset \otimes_{A_L} A_G^\beta$, $Y_{A_G}^\beta(\lambda)_{\mathfrak{n}} \simeq \{Z_{I,A_L}^\emptyset(\lambda) \oplus Z_{I,A_L}^\emptyset(\beta \uparrow$

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$\lambda)\} \otimes_{A_L} A_G^\beta$ by (3.5), and hence

$$\begin{aligned}(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L})(M)(\lambda,\beta) &= \mathcal{C}(\Omega, A_G^\beta)(Y_{A_G}^\beta(\lambda), \hat{\nabla}_{P,A_L}(M)^\beta)\\ &\simeq \mathcal{C}(\Omega_I, A_G^\beta)((M^\beta \otimes_{A_L} A_G^\beta)^\tau, (Y_{A_G}^\beta(\lambda)_{\mathfrak{n}})^\tau)\\ &\simeq \mathcal{C}(\Omega_I, A_G^\beta)((M^\emptyset \otimes_{A_L} A_G^\beta)^\tau, (Z_{I,A_L}^\emptyset(\lambda) \otimes_{A_L} A_G^\beta)^\tau)\\ &\quad \text{as } M \in \mathcal{D}(\Omega_I, A_L)\\ &\simeq \mathcal{C}(\Omega_I, A_L^\beta)(Z_{I,A_L}^\emptyset(\lambda), M^\emptyset) \otimes_{A_L} A_G^\beta\\ &= \mathcal{V}_{\Omega_I}(M)(\lambda) \otimes_{A_L} A_G^\beta = (\mathcal{I} \circ \mathcal{V}_{\Omega_I})(M)(\lambda,\beta).\end{aligned}$$

If $\lambda \in \Omega \setminus \Omega_I$, and if $\beta \in R^+ \setminus R_I^+$ with $\beta \uparrow \lambda \in \Omega_I$, we have likewise

$$(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L})(M)(\lambda,\beta) \simeq \mathcal{V}_{\Omega_I}(M)(\beta \uparrow \lambda) \otimes_{A_L} A_G^\beta = (\mathcal{I} \circ \mathcal{V}_{\Omega_I})(M)(\lambda,\beta).$$

If $\lambda \in \Omega \setminus \Omega_I$, and if $\beta \uparrow \lambda \notin \Omega_I$,

$$(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L})(M)(\lambda,\beta) = 0 = (\mathcal{I} \circ \mathcal{V}_{\Omega_I})(M)(\lambda,\beta).$$

There follows an isomorphism $(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L})(M) \simeq (\mathcal{I} \circ \mathcal{V}_{\Omega_I})(M)$ functorially in M. □

(3.8) Recall from (3.1) (respectively, (3.2)) the graded version $\tilde{\mathcal{K}}(\Omega, S_{\Bbbk})$ (respectively, $\tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})$) of the combinatorial category $\mathcal{K}(\Omega, \hat{S}_{\Bbbk})$ (respectively, $\mathcal{K}(\Omega_I, \hat{S}_{I,\Bbbk})$), and define finally a functor $\tilde{\mathcal{I}} : \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk}) \to \tilde{\mathcal{K}}(\Omega, S_{\Bbbk})$ just like $\mathcal{I}$ as follows: for each $\mathcal{M} \in \mathcal{K}(\Omega_I, A_L)$ and $\lambda \in \Omega$, set

$$(\tilde{\mathcal{I}}(\mathcal{M}))(\lambda) = \begin{cases} \mathcal{M}(\lambda) \otimes_{S_{I,\Bbbk}} S_{\Bbbk}^\emptyset & \text{if } \lambda \in \Omega_I\\ 0 & \text{else,}\end{cases}$$

and for each $\beta \in R^+$ set

$$(\tilde{\mathcal{I}}(\mathcal{M}))(\lambda,\beta) = \begin{cases} \mathcal{M}(\lambda,\beta) \otimes_{S_{I,\Bbbk}} S_{\Bbbk}^\beta & \text{if } \lambda \in \Omega_I \text{ and } \beta \in R_I^+\\ \mathcal{M}(\lambda) \otimes_{S_{I,\Bbbk}} S_{\Bbbk}^\beta & \text{if } \lambda \in \Omega_I \text{ and } \beta \notin R_I^+\\ \mathcal{M}(\beta \uparrow \lambda) \otimes_{S_{I,\Bbbk}} S_{\Bbbk}^\beta & \text{if } \beta \uparrow \lambda \in \Omega_I \text{ and } \beta \notin R_I^+\\ 0 & \text{else.}\end{cases}$$

Thus $\tilde{\mathcal{I}} \otimes_{S_{\Bbbk}} \hat{S}_{\Bbbk} \simeq \mathcal{I}$. For each $w \in W$, recall from [1, 14.10 and 15.3] an $S_{\Bbbk}$-form $\mathcal{Z}_\lambda^w \in \tilde{\mathcal{K}}(\Omega, S_{\Bbbk})$ of $\mathcal{V}_\Omega Z_{A_G}^w(\lambda\langle w\rangle)$ defined by setting, for each $\mu \in \Lambda$,

$$\mathcal{Z}_\lambda^w(\mu) = \begin{cases} A_G^\emptyset & \text{if } \mu = \lambda\\ 0 & \text{else,}\end{cases}$$

and, for each $\beta \in R^+$,

$$\mathcal{Z}^w_\lambda(\mu, \beta) = \begin{cases} A^\beta_G & \text{if } \mu = \lambda = \beta \uparrow \lambda \\ A^\beta_G(1,0) & \text{if } \mu = \lambda \neq \beta \uparrow \lambda \\ A^\beta(0, h_\beta) & \text{if } \mu = \beta \downarrow \lambda, \lambda \neq \beta \uparrow \lambda \text{ and } w^{-1}\beta > 0 \\ A^\beta(0,1) & \text{if } \mu = \beta \downarrow \lambda, \lambda \neq \beta \uparrow \lambda \text{ and } w^{-1}\beta < 0 \\ 0 & \text{else,} \end{cases}$$

where $\beta \downarrow \lambda \in \Lambda$ such that $\beta \uparrow (\beta \downarrow \lambda) = \lambda$. For $\lambda \in \Omega_I$ and $w \in W_I$, define an $S_{I,\Bbbk}$-form $\mathcal{Z}^w_{I,\lambda} \in \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})$ of $\mathcal{V}_{\Omega_I} Z^w_{A_L}(\lambda\langle w\rangle)$ likewise. One has, in particular, for each $\lambda \in \Omega_I$,

$$\tilde{\mathcal{I}}(\mathcal{Z}^{w_I}_{I,\lambda}) \simeq \mathcal{Z}^{w_0}_\lambda. \tag{1}$$

Let $Q_{I,A_L} = \bigoplus_{w \in W_I} Q_{I,A_L}(w \bullet \lambda^+_I) \in \mathcal{C}(\Omega_I, A_L)$ with $Q_{I,A_L}(w \bullet \lambda^+_I)$ the lift of the projective cover of $\hat{L}^P(w \bullet \lambda^+_I)$ for L_I over A_L. Let $\mathcal{P}_I$ be a graded $S_{I,\Bbbk}$-form of $\mathcal{V}_{\Omega_I}(Q_{I,A_L})$.

Lemma. *If $\mathcal{M}$ is a graded $S_{I,\Bbbk}$-form of $\mathcal{V}_{\Omega_I} M$ for $M \in \mathcal{C}(\Omega_I, A_L)$, there are isomorphisms in $\mathcal{K}(\Omega, A_G)$,*

$$\tilde{\mathcal{I}}(\mathcal{M}) \otimes_{S_\Bbbk} A_G \simeq \mathcal{I}(\mathcal{V}_{\Omega_I}(M)) \simeq \mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L}(M).$$

In particular, $\tilde{\mathcal{I}}(\mathcal{P}_I) \otimes_{S_\Bbbk} A_G \simeq \mathcal{I}(\mathcal{V}_{\Omega_I}(Q_{I,A_L})) \simeq \mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_L}(Q_{I,A_L})$.

Proof. The first isomorphism follows from the definition that $\mathcal{M} \otimes_{S_{I,\Bbbk}} A_L \simeq \mathcal{V}_{\Omega_I}(M)$, and the second from (3.7). □

(3.9) Recall from (3.1) the $(Y \times \mathbb{Z})$-graded $S_\Bbbk$-algebra of finite type $E_\Omega = \tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{P}, \mathcal{P})^{\text{op}}$ inducing the structure of a $(Y \times \mathbb{Z})$-graded $\Bbbk$-algebra on $E_{\Omega,\Bbbk} = \mathcal{C}(\Omega)^\sharp(Q, Q)^{\text{op}} \simeq E_\Omega \otimes_{S_\Bbbk} \Bbbk$, and from (3.2) the $(Y_I \times \mathbb{Z})$-graded $S_{I,\Bbbk}$-algebra of finite type $E_{\Omega_I} = \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})^\sharp(\mathcal{P}_I, \mathcal{P}_I)^{\text{op}}$, responsible for the structure of the $(Y_I \times \mathbb{Z})$-graded $\Bbbk$-algebra on $E_{\Omega_I,\Bbbk} = \mathcal{C}(\Omega_I)^\sharp(Q, Q)^{\text{op}} \simeq E_{\Omega_I} \otimes_{S_{I,\Bbbk}} \Bbbk$. Now set $J = \tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{P}, \tilde{\mathcal{I}}(\mathcal{P}_I))$, which comes equipped with the structure of a $(Y \times \mathbb{Z})$-graded left E_Ω and $(Y_I \times \mathbb{Z})$-graded right $E_{\Omega,I}$-bimodule. If $J_\Bbbk = J \otimes_{S_\Bbbk} \Bbbk$, it is thus a $(Y \times \mathbb{Z})$-graded left $E_{\Omega,\Bbbk}$ and $(Y_I \times \mathbb{Z})$-graded right $E_{\Omega_I,\Bbbk}$-bimodule.

For each $\lambda \in \Omega$ and $\alpha \in R^+$, let $n_\alpha \in \mathbb{Z}$ such that $\langle \lambda + \rho, \alpha^\vee \rangle \in]n_\alpha p, (n_\alpha + 1)p[$, and put $\delta(\lambda) = \sum_{\alpha \in R^+} n_\alpha$. If $\lambda \in \Omega_I$, define $\delta_I(\lambda) = \sum_{\alpha \in R^+_I} n_\alpha$ likewise. For each $w \in W$ (respectively, $w \in W_I$) and $\lambda \in \Omega$ (respectively, $\lambda \in \Omega_I$), set $\tilde{Z}^w_\Bbbk(\lambda) = \tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{P}, \mathcal{Z}^w_\lambda\langle -\delta(\lambda)\rangle) \otimes_{S_\Bbbk} \Bbbk \in \tilde{\mathcal{C}}_\Bbbk(\Omega)$ (respectively, $\tilde{Z}^w_{I,\Bbbk}(\lambda) = \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})^\sharp(\mathcal{P}_I, \mathcal{Z}^w_{I,\lambda}\langle -\delta_I(\lambda)\rangle) \otimes_{S_\Bbbk} \Bbbk \in \tilde{\mathcal{C}}_\Bbbk(\Omega_I)$), which is a graded form of $Z^w_\Bbbk(\lambda\langle w\rangle) \in \mathcal{C}_\Bbbk(\Omega)$ (respectively, $Z^w_{I,\Bbbk}(\lambda\langle w\rangle) \in \mathcal{C}_\Bbbk(\Omega_I)$). Put $\tilde{\nabla}_\Bbbk(\lambda) = \tilde{Z}^{w_0}_\Bbbk(\lambda)$ and $\tilde{\nabla}_{I,\Bbbk}(\lambda) = \tilde{Z}^{w_I}_\Bbbk(\lambda)$ for simplicity.

Corollary. *The parabolic induction functor $\hat{\nabla}_P$ is $\mathbb{Z}$-graded by the bimodule $J_\Bbbk$, i.e., one has an isomorphism of functors $\bar{v} \circ (J_\Bbbk \otimes_{E_{\Omega_I,\Bbbk}} ?) \simeq \hat{\nabla}_P \circ \bar{v}_I : \tilde{\mathcal{C}}_\Bbbk(\Omega_I) \to \mathcal{C}_\Bbbk(\Omega)$:*

$$\begin{array}{ccc}
\tilde{\mathcal{C}}_{\Bbbk}(\Omega_I) & \xrightarrow{J_{\Bbbk}\otimes_{E_{\Omega_I,\Bbbk}} ?} & \tilde{\mathcal{C}}_{\Bbbk}(\Omega) \\
{\scriptstyle \bar{v}_I}\downarrow & & \\
L_{I,1}T\mathbf{Mod} & \circlearrowleft & \downarrow{\scriptstyle \bar{v}} \\
\downarrow & & \\
P_1T\mathbf{Mod} & \xrightarrow[\hat{\nabla}_P]{} & G_1T\mathbf{Mod}.
\end{array}$$

For each $\lambda \in \Omega_I$, there is an isomorphism in $\tilde{\mathcal{C}}_{\Bbbk}(\Omega)$:

$$J_{\Bbbk} \otimes_{E_{\Omega_I,\Bbbk}} \tilde{\nabla}_{I,\Bbbk}(\lambda) \simeq \tilde{\nabla}_{\Bbbk}(\lambda)\langle \delta(\lambda) - \delta_I(\lambda)\rangle.$$

Proof. The commutativity of the diagram, i.e., the naturality of the isomorphism, follows from (3.3) by the isomorphism of left $E_{\Omega,\Bbbk}$- and right $E_{\Omega_I,\Bbbk}$-bimodules

$$\begin{aligned}
J_{\Bbbk} &\simeq J \otimes_{S_{\Bbbk}} A_G \otimes_{A_G} \Bbbk \\
&\simeq \mathcal{K}(\Omega, A_G)^{\sharp}(\mathcal{V}_{\Omega}(Q_{A_G}), \tilde{\mathcal{I}}(\mathcal{P}_I) \otimes_{S_{\Bbbk}} A_G)) \otimes_{A_G} \Bbbk \quad \text{by [1, 18.9.3]} \\
&\simeq \mathcal{K}(\Omega, A_G)^{\sharp}(\mathcal{V}_{\Omega}(Q_{A_G}), \mathcal{V}_{\Omega} \circ \hat{\nabla}_{P,A_L}(Q_{I,A_L})) \otimes_{A_G} \Bbbk \quad \text{by (3.8)} \\
&\simeq \mathcal{C}(\Omega, A_G)^{\sharp}(Q_{A_G}, \hat{\nabla}_{P,A_L}(Q_{I,A_L})) \otimes_{A_G} \Bbbk \quad \text{by [1, 18.9.5/6]} \\
&\simeq \mathcal{C}_{\Bbbk}(\Omega)^{\sharp}(Q_{A_G} \otimes_{A_G} \Bbbk, \hat{\nabla}_{P,A_L}(Q_{I,A_L}) \otimes_{A_G} \Bbbk) \quad \text{by [1, 3.3]} \\
&\simeq \mathcal{C}_{\Bbbk}(\Omega)^{\sharp}(Q, \hat{\nabla}_P(Q_I)) \quad \text{from (3.4)(3)}.
\end{aligned}$$

For $\lambda \in \Omega_I$, one has

$$J_{\Bbbk} \otimes_{E_{\Omega_I,\Bbbk}} \tilde{\nabla}_{I,\Bbbk}(\lambda) \simeq \{\tilde{\mathcal{K}}(\Omega, S_{\Bbbk})^{\sharp}(\mathcal{P}, \tilde{\mathcal{I}}(\mathcal{P}_I)) \otimes_{E_{\Omega_I}} \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})^{\sharp}(\mathcal{P}_I, \mathcal{Z}^{w_I}_{I,\lambda}\langle -\delta_I(\lambda)\rangle)\} \otimes_{S_{I,\Bbbk}} \Bbbk. \tag{1}$$

Consider a natural map of $\tilde{\mathcal{C}}(\Omega)$,

$$\begin{aligned}
&\tilde{\mathcal{K}}(\Omega, S_{\Bbbk})^{\sharp}(\mathcal{P}, \tilde{\mathcal{I}}(\mathcal{P}_I)) \otimes_{E_{\Omega_I}} \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})^{\sharp}(\mathcal{P}_I, \mathcal{Z}^{w_I}_{I,\lambda}\langle -\delta_I(\lambda)\rangle) \otimes_{S_{I,\Bbbk}} \Bbbk \\
&\quad \to \tilde{\mathcal{K}}(\Omega, S_{\Bbbk})^{\sharp}(\mathcal{P}, \tilde{\mathcal{I}}(\mathcal{Z}^{w_I}_{I,\lambda}\langle -\delta_I(\lambda)\rangle)) \otimes_{S_{\Bbbk}} \Bbbk.
\end{aligned}$$

It reads as a $\Bbbk$-linear map,

$$\begin{aligned}
&\tilde{\mathcal{K}}(\Omega, S_{\Bbbk})^{\sharp}(\mathcal{P}, \tilde{\mathcal{I}}(\mathcal{P}_I)) \otimes_{E_{\Omega_I}} \tilde{\mathcal{K}}(\Omega_I, S_{I,\Bbbk})^{\sharp}(\mathcal{P}_I, \mathcal{Z}^{w_I}_{I,\lambda}\langle -\delta_I(\lambda)\rangle) \otimes_{S_{I,\Bbbk}} A_G \otimes_{A_G} \Bbbk \\
&\quad \to \tilde{\mathcal{K}}(\Omega, S_{\Bbbk})^{\sharp}(\mathcal{P}, \tilde{\mathcal{I}}(\mathcal{Z}^{w_I}_{I,\lambda}\langle -\delta_I(\lambda)\rangle)) \otimes_{S_{\Bbbk}} A_G \otimes_{A_G} \Bbbk,
\end{aligned}$$

which in turn reads, by (3.7), as

$$\begin{aligned}
&\mathcal{K}(\Omega, A_G)^{\sharp}(\mathcal{V}_{\Omega}\mathcal{P}, \mathcal{V}_{\Omega}(\hat{\nabla}_{P,A_L}(Q_{I,A_L}))) \otimes_{E_{\Omega_I} \otimes_{S_{I,\Bbbk}} A_L} \\
&\qquad\qquad \mathcal{K}(\Omega_I, A_L)^{\sharp}(\mathcal{V}_{\Omega_I}\mathcal{P}_I, \mathcal{V}_{\Omega_I}\mathcal{Z}^{w_I}_{I,\lambda}\langle -\delta_I(\lambda)\rangle) \otimes_{A_G} \Bbbk \\
&\qquad\qquad\qquad \to \mathcal{K}(\Omega, A_G)^{\sharp}(\mathcal{V}_{\Omega}\mathcal{P}, \mathcal{V}_{\Omega}(\hat{\nabla}_{P,A_L}(Z^{w_I}_{I,A_L}(\lambda\langle w_I\rangle)))) \otimes_{A_G} \Bbbk.
\end{aligned}$$

As both $\mathcal{V}_{\Omega} : \mathcal{C}(\Omega, A_G) \to \mathcal{K}(\Omega, A_G)$ and $\mathcal{V}_{\Omega_I} : \mathcal{C}(\Omega_I, A_L) \to \mathcal{K}(\Omega_I, A_L)$ are fully faithful, the last map reads, with $Q_{A_G} = \mathcal{V}_\Omega \mathcal{P}$,

$$\mathcal{C}(\Omega, A_G)^\sharp(Q_{A_G}, \hat{\nabla}_{P,A_L}(Q_{I,A_L})) \otimes_{\mathcal{C}(\Omega_I,A_L)^\sharp(Q_{I,A_L},Q_{I,A_L})} \mathcal{C}(\Omega_I, A_L)^\sharp(Q_{I,A_L}, Z^{w_I}_{I,A_L}(\lambda\langle w_I\rangle)) \otimes_{A_G} \Bbbk \to \mathcal{C}(\Omega, A_G)^\sharp(Q_{A_G}, \hat{\nabla}_{P,A_L}(Z^{w_I}_{I,A_L}(\lambda\langle w_I\rangle))) \otimes_{A_G} \Bbbk,$$

which is bijective by the five lemma. Thus the isomorphism (1) continues as

$$\begin{aligned} J_\Bbbk \otimes_{E_{\Omega_I,\Bbbk}} \tilde{\nabla}_{I,\Bbbk}(\lambda) &\simeq \tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{P}, \tilde{\mathcal{I}}(\mathcal{Z}^{w_I}_{I,\lambda}\langle -\delta_I(\lambda)\rangle)) \otimes_{S_\Bbbk} \Bbbk \\ &\simeq \tilde{\mathcal{K}}(\Omega, S_\Bbbk)^\sharp(\mathcal{P}, \mathcal{Z}^{w_0}_\lambda\langle -\delta_I(\lambda)\rangle) \otimes_{S_\Bbbk} \Bbbk \quad \text{by (3.8)(1)} \\ &= \tilde{\nabla}_\Bbbk(\lambda)\langle \delta(\lambda) - \delta_I(\lambda)\rangle. \end{aligned}$$

□

4. Rigidity

Keep the notation of §3. We show first that all $\hat{\nabla}_P(\hat{L}^P(\lambda))$ for p-regular $\lambda \in \Lambda$ are $\mathbb{Z}$-graded. Recall Lusztig's conjecture on the irreducible character formulae for G_1T-modules [14]/[12]:

$$\operatorname{ch} \hat{L}(\lambda) = \sum_{\mu \in W_p \bullet \lambda} (-1)^{\mathrm{d}(\mu,\lambda)} \hat{P}_{\mu,\lambda}(1) \operatorname{ch} \hat{\nabla}(\mu),$$

where $\mathrm{d}(\mu, \lambda) = \delta(\lambda) - \delta(\mu)$, and $\hat{P}_{\mu,\lambda} = \hat{P}_{A,C}$ for alcove A containing μ and alcove C containing λ is Kato's periodic Kazhdan–Lusztig polynomial for W_p [12]. We will refer to the conjecture as (LG). It is now a theorem for large p thanks to [1, 11, 13, 15], and more recently to [5]. The conjecture for $L_{I,1}T$, to which we refer as (LP), reads likewise for p-regular λ with respect to $W_{I,p}$:

$$\operatorname{ch} \hat{L}^P(\lambda) = \sum_{\mu \in W_{I,p} \bullet \lambda} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu,\lambda}(1) \operatorname{ch} \hat{\nabla}^P(\mu),$$

where $\mathrm{d}_I(\mu, \lambda) = \delta_I(\lambda) - \delta_I(\mu)$, and $\hat{P}^I_{\mu,\lambda}$ is Kato's periodic Kazhdan–Lusztig polynomial for $W_{I,p}$. Conjecture (LP) follows in fact from (LG); one just checks an analogue of [7, II.5.21.2], namely, for each $\lambda, \mu \in \Lambda$ with $\lambda - \mu \in \mathbb{Z}I$, one has $[\hat{\nabla}(\lambda) : \hat{L}(\mu)] = [\hat{\nabla}^P(\lambda) : \hat{L}^P(\mu)]$. As the analogous equality holds for the corresponding quantum algebras, and as Lusztig's conjecture holds for the quantum algebras, $[\hat{\nabla}^P(\lambda) : \hat{L}^P(\mu)]$ should be equal to what is expected by (LP), which in turn implies (LP) by inversion.

Assuming (LG), [1] has shown that the endomorphism algebra of a projective Y-generator for the block of λ is Koszul. We show that the rigidity of $\hat{\nabla}_P(\hat{L}^P(\lambda))$ follows from a result of [4]. We will also find that the Loewy length of $\hat{\nabla}_P(\hat{L}^P(\lambda))$ for a p-regular $\lambda \in \Lambda$ is uniformly equal to $\ell(w^I) + 1$.

Thus fix a p-regular weight λ, and put $\Omega = W_p \bullet \lambda$. For $M \in \mathcal{C}_\Bbbk(\Omega)$, we let $[M : \hat{L}(\mu)]$, $\mu \in \Omega$, denote the multiplicity of simple $\hat{L}(\mu)$ among the $\mathcal{C}_\Bbbk(\Omega)$-composition factors of M.

(4.1) Let us first recall the construction of $\tilde{L}_\Bbbk(\lambda)$, slightly simplifying the one given in [1, 18.12]. As $\tilde{Z}_\Bbbk(\lambda)$ has a simple head in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ by the

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categorical equivalence v, the radical $\mathrm{rad}_{E_{\Omega,\Bbbk}\mathbf{modgr}_Y}\tilde{Z}_\Bbbk(\lambda)$ of $\tilde{Z}_\Bbbk(\lambda)$ in the category $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ is maximal. But $\mathrm{rad}_{E_{\Omega,\Bbbk}\mathbf{modgr}_Y}\tilde{Z}_\Bbbk(\lambda)$ belongs to $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ by [1, E.11], and hence coincides with the radical $\mathrm{rad}_{\tilde{\mathcal{C}}_\Bbbk(\Omega)}\tilde{Z}_\Bbbk(\lambda)$ of $\tilde{Z}_\Bbbk(\lambda)$ in the category of $\tilde{\mathcal{C}}_\Bbbk(\Omega)$. We set $\tilde{L}_\Bbbk(\lambda) = \tilde{Z}_\Bbbk(\lambda)/\mathrm{rad}_{\tilde{\mathcal{C}}_\Bbbk(\Omega)}\tilde{Z}_\Bbbk(\lambda)$. Then $\bar{v}\tilde{L}_\Bbbk(\lambda) = Q \otimes_{E_{\Omega,\Bbbk}} \tilde{L}_\Bbbk(\lambda) \simeq \{Q \otimes_{E_{\Omega,\Bbbk}} \tilde{Z}_\Bbbk(\lambda)\}/\{Q \otimes_{E_{\Omega,\Bbbk}} \mathrm{rad}_{E_{\Omega,\Bbbk}\mathbf{modgr}_Y}\tilde{Z}_\Bbbk(\lambda)\} \simeq Z_\Bbbk(\lambda)/\mathrm{rad}_{\mathcal{C}_\Bbbk(\Omega)}Z_\Bbbk(\lambda) \simeq \hat{L}(\lambda)$. Thus $\hat{\nabla}_P(\hat{L}^P(\lambda))$ is $(Y \times \mathbb{Z})$-graded by (3.9). In turn, $\tilde{L}_\Bbbk(\lambda) \simeq H_{\Omega,\Bbbk}\hat{L}(\lambda)$ in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ while $H_{\Omega,\Bbbk}\hat{L}(\lambda) = \mathcal{C}_\Bbbk(\Omega)^\sharp(Q, \hat{L}(\lambda)) \simeq \mathcal{C}_\Bbbk(\Omega)(Q_\Bbbk(\lambda), \hat{L}(\lambda))$ as $Q_\Bbbk(\lambda)$ is the projective cover of $\hat{L}(\lambda)$, and hence $\tilde{L}_\Bbbk(\lambda)$ is of dimension 1.

By the equivalence v, the $\tilde{L}_\Bbbk(\lambda)$, $\lambda \in \Omega$, exhaust the simple objects of $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$. If $\tilde{L}$ is a simple object of $\tilde{\mathcal{C}}_\Bbbk(\Omega)$, then

$$\begin{aligned} 0 &\neq E_{\Omega,\Bbbk}\mathbf{modgr}_Y(\tilde{L}, \tilde{L}_\Bbbk(\lambda)) \quad \text{for some } \lambda \in \Omega \\ &= \bigoplus_{i\in\mathbb{Z}} E_{\Omega,\Bbbk}\mathbf{modgr}_Y(\tilde{L}, \tilde{L}_\Bbbk(\lambda))_i = \bigoplus_{i\in\mathbb{Z}} \tilde{\mathcal{C}}_\Bbbk(\Omega)(\tilde{L}, \tilde{L}_\Bbbk(\lambda)\langle -i\rangle), \end{aligned}$$

and hence $\tilde{\mathcal{C}}_\Bbbk(\Omega)(\tilde{L}, \tilde{L}_\Bbbk(\lambda)\langle i\rangle) \neq 0$ for some i. Then $\tilde{L} \simeq \tilde{L}_\Bbbk(\lambda)\langle i\rangle$ in $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ by their simplicity. Such λ and i are unique, by [1, 18.8]. Thus we have obtained the first two parts of the following.

Proposition. (i) *Each $\tilde{L}_\Bbbk(\lambda)$, $\lambda \in \Omega$, is one dimensional.*

(ii) *Each simple object of $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ is isomorphic to some $\tilde{L}(\lambda)\langle i\rangle$ for unique $\lambda \in \Omega$ and $i \in \mathbb{Z}$. Any simple object of $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ is isomorphic to some $\tilde{L}(\lambda)$ for unique $\lambda \in \Omega$.*

(iii) *If $M \in \tilde{\mathcal{C}}_\Bbbk(\Omega)$, the radical (respectively, socle) series of M in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ and in $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ coincide.*

Proof. (iii) We show first that each radical layer $\mathrm{rad}^i_{E_{\Omega,\Bbbk}\mathbf{modgr}_Y} M/\mathrm{rad}^{i+1}_{E_{\Omega,\Bbbk}\mathbf{modgr}_Y} M$ remains semisimple in $\tilde{\mathcal{C}}_\Bbbk(\Omega)$. As it inherits the structure of $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ from M by [1, E.11], we may assume that M is semisimple in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$. If L is a simple component of M in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$, as L is one dimensional by (i), each $(E_{\Omega,\Bbbk})_{Y\times\{i\}}$, $i \neq 0$, annihilates L while each element of $(E_{\Omega,\Bbbk})_{Y\times\{0\}}$ is acting by a scalar, and hence M is semisimple also in $\tilde{\mathcal{C}}_\Bbbk(\Omega)$; each $\mathbb{Z}$-homogeneous component M_i of M must be $E_{\Omega,\Bbbk}$-stable. On the other hand, each $\mathrm{rad}^i_{\tilde{\mathcal{C}}_\Bbbk(\Omega)} M/\mathrm{rad}^{i+1}_{\tilde{\mathcal{C}}_\Bbbk(\Omega)} M$ is semisimple in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ as each simple component is one dimensional by (i) again. It now follows that the radical series of M in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$ and $\tilde{\mathcal{C}}_\Bbbk(\Omega)$ coincide.

The socle version of [1, E.11] holds, and hence also the assertion about the socle series of M. □

(4.2) Assume now Lusztig's conjecture (LG) on the irreducible characters of G_1T-modules. Then $E_{\Omega,\Bbbk}$ is Koszul with respect to its $\mathbb{Z}$-gradation, thanks to [1, 18.17]. In particular, $E_{\Omega,\Bbbk}$ is positively graded: $E_{\Omega,\Bbbk} = \bigoplus_{i\in\mathbb{N}}(E_{\Omega,\Bbbk})_i$ with $(E_{\Omega,\Bbbk})_0 = \prod_{w\in W} \Bbbk\pi_w$, and is generated by $(E_{\Omega,\Bbbk})_1$ over $\Bbbk$, by [4, Props. 2.1.3 and 2.3.1], where $\pi_w : \prod_{x\in W} Q_\Bbbk(x \bullet \lambda^+) \to Q_\Bbbk(w \bullet \lambda^+)$ is the projection. Let $E_{\Omega,\Bbbk}\mathbf{modgr}_\mathbb{Z}$ denote the category of finite-dimensional $\mathbb{Z}$-graded $E_{\Omega,\Bbbk}$-modules.

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Proposition. *Assume the Lusztig conjecture (LG).*

(i) *Each $\tilde{L}_{\Bbbk}(\lambda)$, $\lambda \in \Omega$, is homogeneous of degree 0 with respect to the $\mathbb{Z}$-grading. In particular, each $\tilde{L}_{\Bbbk}(w \bullet \lambda^+)$, $w \in W$, is isomorphic to $\Bbbk\pi_w$ in $E_{\Omega,\Bbbk}$**modgr**$_{\mathbb{Z}}$.*

(ii) *Each simple object of $E_{\Omega,\Bbbk}$**modgr**$_{\mathbb{Z}}$ is isomorphic to some $\tilde{L}_{\Bbbk}(w \bullet \lambda^+)\langle i\rangle$ for unique $w \in W$ and $i \in \mathbb{Z}$.*

(iii) *If $M \in \tilde{\mathcal{C}}_{\Bbbk}(\Omega)$, the radical (respectively, socle) series of M in $E_{\Omega,\Bbbk}$**modgr**$_{\mathbb{Z}}$ and in $\tilde{\mathcal{C}}_{\Bbbk}(\Omega)$ coincide.*

Proof. (i) Recall from (4.1) that the $\mathbb{Z}$-grading on $E_{\Omega,\Bbbk}$ arises from that of E_Ω. Thus $\Bbbk\pi_w = \Bbbk(\tilde{\pi}_w \otimes 1)$ if $\tilde{\pi}_w : \prod_{x\in W} \mathcal{Q}(x \bullet \lambda^+) \to \mathcal{Q}(w \bullet \lambda^+)$ is the projection. But

$$\begin{aligned}
&\tilde{\mathcal{K}}(\Omega, S_{\Bbbk})^\sharp\left(\bigoplus_{x\in W} \mathcal{Q}(x \bullet \lambda^+), \mathcal{Q}(w \bullet \lambda^+)\right)\\
&\quad = \bigoplus_{x\in W}\bigoplus_{\gamma\in Y}\bigoplus_{i\in\mathbb{Z}} \mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(x \bullet \lambda^+)[\gamma], \mathcal{Q}(w \bullet \lambda^+))_i \quad \text{by definition [1, E.1, E.3]}\\
&\quad = \bigoplus_{x\in W}\bigoplus_{\gamma\in Y} \mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(x \bullet \lambda^+)[\gamma], \mathcal{Q}(w \bullet \lambda^+)) \quad \text{by [1, E.1]}\\
&\quad \simeq \bigoplus_{x\in W}\bigoplus_{\gamma\in Y} \mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(x \bullet \lambda^+ + \gamma), \mathcal{Q}(w \bullet \lambda^+)) \quad \text{by [1, 17.6/18.5]}
\end{aligned}$$

with $\mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(x \bullet \lambda^+ + \gamma),\ \mathcal{Q}(w \bullet \lambda^+)) = \bigoplus_{i>0} \mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(x \bullet \lambda^+ + \gamma),\ \mathcal{Q}(w \bullet \lambda^+))_i$ unless $x \bullet \lambda^+ + \gamma = w \bullet \lambda^+$ while $\mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(w \bullet \lambda^+), \mathcal{Q}(w \bullet \lambda^+))_0 = S_{\Bbbk}\mathrm{id}_{\mathcal{Q}(w\bullet\lambda^+)}$ [1, 17.9]. On the other hand,

$$\begin{aligned}
\tilde{Z}_{\Bbbk}(w \bullet \lambda^+ + \gamma) &= \tilde{\mathcal{K}}(\Omega, S_{\Bbbk})^\sharp\left(\bigoplus_{x\in W} \mathcal{Q}(x \bullet \lambda^+), \mathcal{Z}_{w\bullet\lambda^+ +\gamma}\langle -\delta(w \bullet \lambda^+ + \gamma)\rangle\right) \otimes_{S_{\Bbbk}} \Bbbk\\
&\qquad \text{by definition [1, 18.10.1 and 18.12]}\\
&\simeq \bigoplus_{x\in W}\bigoplus_{\nu\in Y} \mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(x \bullet \lambda^+ + \nu), \mathcal{Z}_{w\bullet\lambda^+ +\gamma}\langle -\delta(w \bullet \lambda^+ + \gamma)\rangle) \otimes_{S_{\Bbbk}} \Bbbk \quad \text{as above.}
\end{aligned}$$

Each $\mathcal{K}(\Omega, S_{\Bbbk})$ $(\mathcal{Q}(x \bullet \lambda^+ + \nu),\ \mathcal{Z}_{w\bullet\lambda^+ +\gamma}\langle -\delta(w \bullet \lambda^+ + \gamma)\rangle)$ is a direct summand of $\tilde{\mathcal{K}}(\Omega, S_{\Bbbk})$ $(\mathcal{Q}(x \bullet \lambda^+ + \nu),\ \mathcal{Q}(w \bullet \lambda^+ + \gamma))$, by [1, 15.10 and 17.6.1/18.9.c], and hence $\mathcal{K}(\Omega, S_{\Bbbk})$ $(\mathcal{Q}(x \bullet \lambda^+ + \nu),\ \mathcal{Z}_{w\bullet\lambda^+ +\gamma}\langle -\delta(w \bullet \lambda^+ + \gamma)\rangle) = \mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(x \bullet \lambda^+ + \nu), \mathcal{Z}_{w\bullet\lambda^+ +\gamma}$ $\langle -\delta(w \bullet \lambda^+ + \gamma)\rangle)_{>0}$ unless $x \bullet \lambda^+ + \nu = w \bullet \lambda^+ + \gamma$, i.e., $x = w$ and $\nu = \gamma$, by [1, 17.9] again, while

$$\begin{aligned}
&\mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Q}(w \bullet \lambda^+ + \gamma), \mathcal{Z}_{w\bullet\lambda^+ +\gamma}\langle -\delta(w \bullet \lambda^+ + \gamma)\rangle)_0\\
&\quad \simeq \mathcal{K}(\Omega, S_{\Bbbk})(\mathcal{Z}'_{w\bullet\lambda^+ +\gamma}\langle 2|R^+| - \delta(w \bullet \lambda^+ + \gamma)\rangle, \mathcal{Z}_{w\bullet\lambda^+ +\gamma}\langle -\delta(w \bullet \lambda^+ + \gamma)\rangle)_0\\
&\qquad \text{by [1, 15.10 and 17.6.2]}\\
&\quad \simeq (S_{\Bbbk})_0 \quad \text{by [1, 15.10.2].}
\end{aligned}$$

Thus the epi $\tilde{Z}_{\Bbbk}(w \bullet \lambda^+ + \gamma)/\tilde{Z}_{\Bbbk}(w \bullet \lambda^+ + \gamma)_{>0} \to \tilde{L}_{\Bbbk}(w \bullet \lambda^+ + \gamma)$ is an isomorphism of $E_{\Omega,\Bbbk}$**modgr**$_{\mathbb{Z}}$ by dimension, and hence $\tilde{L}_{\Bbbk}(w \bullet \lambda^+ + \gamma)$ is of degree 0. In particular, $\tilde{L}_{\Bbbk}(w \bullet \lambda^+) \simeq \Bbbk(\tilde{\pi}_w \otimes 1)$.

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(ii) Let L be a simple object of $E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}$. As $(E_{\Omega,\Bbbk})_{>0}L = 0$, L is an $(E_{\Omega,\Bbbk})_0$-module. Then L is by its simplicity isomorphic to some $\Bbbk\pi_w\langle i\rangle$, $w \in W$, $i \in \mathbb{Z}$.

(iii) This now follows from (ii), just like (4.1)(iii), applying [1, E.11] to the pair $(Y \times \mathbb{Z}, \mathbb{Z})$ in place of $(Y \times \mathbb{Z}, Y)$. □

(4.3) We are now to obtain from [4, Proposition 2.4.1] the rigidity of $\hat{\nabla}_P(\hat{L}^P(\lambda))$, as well as $\hat{\nabla}(\lambda)$ and $\hat{Q}(\lambda) = Q_{\Bbbk}(\lambda)$ for each $\lambda \in \Omega$, demonstrated first in [2] by a different method using Vogan's version of the Lusztig conjecture. The result from [4] referred to above asserts that, for a finite-dimensional graded module M over a Koszul $\Bbbk$-algebra A, if M has a simple socle and a simple head, then both the socle series and the radical series of M coincides with the grading filtration on M up to degree shifts.

Lemma. *Assume the Lusztig conjecture (LG). Let $M \in \tilde{\mathcal{C}}_{\Bbbk}(\Omega)$. If M has a simple socle and a simple head as an object of $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$, then M is rigid in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$.*

Proof. By the hypothesis, M has a simple socle and a simple head in $(E_{\Omega,\Bbbk})\mathbf{modgr}_{\mathbb{Z}}$, by (4.1) and (4.2). If $\mathrm{hd}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M$ (respectively, $\mathrm{soc}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M$) is concentrated in degree j (respectively, k), then, from [4, Proposition 2.4.1],

$$\mathrm{rad}^i_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M = M_{\geqslant i+j} \quad \text{and} \quad \mathrm{soc}^i_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M = M_{\geqslant k-i+1} \quad \forall i.$$

Thus $M_{\geqslant k-i+1} = \mathrm{soc}^i_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M \geqslant \mathrm{rad}^{\ell\ell(M)-i}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M = M_{\geqslant \ell\ell(M)-i+j}$, and hence $k-i+1 \leqslant \ell\ell(M)-i+j$. As the equality holds for $i=0$, $k+1 = \ell\ell(M)+j$. Then $\forall i$, $k-i+1 \leqslant \ell\ell(M)-i+j = \ell\ell(M)-i+k+1-\ell\ell(M) = k-i+1$, and hence

$$\mathrm{soc}^i_{E_{\Omega,\Bbbk}\mathbf{modgr}_Y}M = \mathrm{soc}^i_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M = M_{\geqslant \ell\ell(M)+j-i} = \mathrm{rad}^{\ell\ell(M)-i}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}M = \mathrm{rad}^{\ell\ell(M)-i}_{E_{\Omega,\Bbbk}\mathbf{modgr}_Y}M.$$

□

(4.4) Recalling from (1.4) that each $\hat{\nabla}_P(\hat{L}^P(\lambda))$ has a simple socle and a simple head yields the following.

Theorem. *Assume the Lusztig conjecture (LG). Each $\hat{\nabla}_P(\hat{L}^P(\lambda))$ for p-regular λ is rigid.*

(4.5) To determine eventually the Loewy series of $\hat{\nabla}_P(\hat{L}^P(\lambda))$, we have to compute its Loewy length. As $\ell\ell(\hat{\nabla}_P(\hat{L}^P(\lambda))) = \ell\ell({}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda)))$, we will compute $\ell\ell({}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda)))$.

Lemma. $\mathrm{hd}_{G_1T}({}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))) = \hat{L}(w^I \bullet \lambda - p(w^I \bullet 0))$.

Proof. We may assume that $\lambda^1 = 0$. By (1.4),

$$\begin{aligned}
&\mathrm{hd}_{G_1T}({}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))) \\
&= {}^{w^I}\mathrm{hd}_{G_1T}(\hat{\nabla}_P(\hat{L}^P(\lambda))) \\
&= {}^{w^I}\{\hat{L}(w^I \bullet \lambda) \otimes p(-2\rho_P + w_0((-w_I)\bullet\lambda)^1 - ((-w_I)\bullet\lambda)^1)\}
\end{aligned}$$

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$$
\begin{aligned}
&= {}^{w^I}\{L((w^I\bullet\lambda)^0)\otimes p((w^I\bullet\lambda)^0-2\rho_P+w_0((-w_I)\bullet\lambda)^1-((-w_I)\bullet\lambda)^1)\}\\
&= L((w^I\bullet\lambda)^0)\otimes pw^I\{(w^I\bullet\lambda)^0-2\rho_P+w_0((-w_I)\bullet\lambda)^1-((-w_I)\bullet\lambda)^1\}
\end{aligned}
$$

while $\hat{L}(w^I\bullet\lambda)\otimes -p(w^I\bullet 0)=L((w^I\bullet\lambda)^0)\otimes p\{(w^I\bullet\lambda)^1-(w^I\bullet 0)\}$. Thus we have to show

$$(w^I\bullet\lambda)^1-(w^I\bullet 0)=w^I\{(w^I\bullet\lambda)^1-2\rho_P+w_0((-w_I)\bullet\lambda)^1-((-w_I)\bullet\lambda)^1\}. \qquad (1)$$

Write $w_I\bullet\lambda=\mu^0+p\mu^1$ with $\mu^0\in\Lambda_p$ and $\mu^1\in\Lambda$. Thus μ^0 is p-regular. As $w^I\bullet\lambda=w_0\bullet(\mu^0+p\mu^1)=w_0\bullet\mu^0+pw_0\mu^1$, $(w^I\bullet\lambda)^1=w_0\mu^1-\rho$. Likewise, as $(-w_I)\bullet\lambda=(-1)\bullet(\mu^0+p\mu^1)=(-1)\bullet\mu^0-p\mu^1$, $((-w_I)\bullet\lambda)^1=-\mu^1-\rho$. It follows that the left-hand side of (1) is equal by (1.1) to

$$w_0\mu^1-\rho-w_0w_I\bullet 0=w_0\mu^1-\rho-(w_0w_I\rho-\rho)=w_0\mu^1-\rho-w_02\rho_P=w_0(\mu^1+\rho-2\rho_P)$$

while the right-hand side of (1) is equal to

$$
\begin{aligned}
&w^I\{w_0\mu^1-\rho-2\rho_P+w_0(-\mu^1-\rho)-(-\mu^1-\rho)\}=w^I(-2\rho_P+\mu^1+\rho)\\
&\quad = w_0(w_I\mu^1+w_I\rho-2\rho_P) \quad \text{as } w_I2\rho_P=2\rho_P.
\end{aligned}
$$

Thus we are left to verify that $\mu^1+\rho=w_I(\mu^1+\rho)$, for which we have only to check $\langle\mu^1+\rho,\alpha^\vee\rangle=0\ \forall\alpha\in I$. But

$$
\begin{aligned}
&]0,p[+p\langle\mu^1+\rho,\alpha\rangle\ni\langle\mu^0+\rho,\alpha\rangle+p\langle\mu^1+\rho,\alpha\rangle \quad \text{as } \mu^0 \text{ is } p\text{-regular}\\
&\quad=\langle\mu^0+p\mu^1+p\rho+\rho,\alpha^\vee\rangle=\langle w_I\bullet\lambda+p\rho+\rho,\alpha^\vee\rangle=\langle\lambda+\rho,w_I\alpha^\vee\rangle+p\\
&\qquad\in-]0,p[+p \quad \text{as } w_I\alpha\in -I\\
&\quad=]0,p[,
\end{aligned}
$$

and hence $\langle\mu^1+\rho,\alpha^\vee\rangle=0$, as desired. □

(4.6) Recall from (1.6)(3) that ${}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))\leqslant\hat{\nabla}_{w^I}((w^I\bullet\lambda)\langle w^I\rangle)\otimes(-p(w^I\bullet 0))$. Recall also from [2] an intertwining homomorphism $\phi_w\in G_1T\mathbf{Mod}(\hat{\nabla}_w((w\bullet\lambda)\langle w\rangle),\hat{\nabla}(w\bullet\lambda))\setminus 0$ for each $w\in W$, which is unique up to $\Bbbk^\times$. As $1=[\hat{\nabla}(w^I\bullet\lambda):\hat{L}(w^I\bullet\lambda)]=[\hat{\nabla}_{w^I}((w^I\bullet\lambda)\langle w^I\rangle):\hat{L}(w^I\bullet\lambda)]$ by [2, 1.2.3], one obtains from (4.5) a commutative diagram of G_1T-modules,

$$
\begin{array}{ccc}
\hat{\nabla}_{w^I}((w^I\bullet\lambda)\langle w^I\rangle)\otimes(-p(w^I\bullet 0)) & \xrightarrow{\phi_{w^I}\otimes(-p(w^I\bullet 0))} & \hat{\nabla}(w^I\bullet\lambda)\otimes(-p(w^I\bullet 0))\\
\uparrow & & \uparrow\\
{}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda)) & \twoheadrightarrow & \hat{L}(w^I\bullet\lambda)\otimes(-p(w^I\bullet 0)).
\end{array}
\qquad (1)
$$

As $\phi_{w^I}(\mathrm{soc}^{\ell(w^I)}\hat{\nabla}_{w^I}((w^I\bullet\lambda)\langle w^I\rangle))=0$ [2], we must have

$$\ell\ell({}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda)))\geqslant\ell(w^I)+1. \qquad (2)$$

On the other hand, there is another intertwining homomorphism $\phi'_{w^I}\in G_1T\mathbf{Mod}(\hat{\nabla}_{w_0}((w^I\bullet\lambda)\langle w_0\rangle),\hat{\nabla}_{w^I}((w^I\bullet\lambda)\langle w^I\rangle))\setminus 0$. As

$$\mathrm{hd}_{G_1T}\hat{\nabla}_{w_0}((w^I\bullet\lambda)\langle w_0\rangle)\otimes -p(w^I\bullet 0)=\mathrm{hd}_{G_1T}\hat{\Delta}(w^I\bullet\lambda)\otimes -p(w^I\bullet 0) \quad \text{by [2, 1.2]}$$

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$$= \hat{L}(w^I \bullet \lambda) \otimes -p(w^I \bullet 0) = \mathrm{hd}_{G_1T}({}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))),$$

one obtains as in (1) another commutative diagram,

$$\begin{array}{ccc} \hat{\nabla}_{w_0}((w^I \bullet \lambda)\langle w_0\rangle) \otimes (-p(w^I \bullet 0)) & & \\ & \searrow\!\!\!\searrow & \\ \Big\downarrow {\scriptstyle \phi'_{w^I} \otimes (-p(w^I \bullet 0))} & & {}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda)) \\ & \swarrow\!\!\supset & \\ \hat{\nabla}_{w^I}((w^I \bullet \lambda)\langle w^I\rangle) \otimes (-p(w^I \bullet 0)) & & \end{array} \tag{3}$$

with $\phi'_{w^I}(\mathrm{soc}^{\ell(w_0)-\ell(w^I)}\hat{\nabla}_{w_0}((w^I \bullet \lambda)\langle w_0\rangle)) = 0$. Assuming the conjecture (LG), one has $\ell\ell(\hat{\nabla}_{w_0}((w^I \bullet \lambda)\langle w_0\rangle)) = \ell(w_0) + 1$. It follows that

$$\ell\ell({}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))) \leqslant \ell(w_0) + 1 - \{\ell(w_0) - \ell(w^I)\} = \ell(w^I) + 1.$$

Thus, together with (2), we have obtained the following.

Theorem. *Assume the Lusztig conjecture (LG). For any p-regular $\lambda \in \Lambda$,*

$$\ell\ell(\hat{\nabla}_P(\hat{L}^P(\lambda))) = \ell(w^I) + 1.$$

(4.7) Remark. This is a generalization of [8, 1.4] and [9], where we found for G of rank at most 2 or when $G = GL_{n+1}(\Bbbk)$, with P a maximal parabolic such that $G/P \simeq \mathbb{P}^n$ for any $n \in \mathbb{N}$, that $\ell\ell(\hat{\nabla}_P(\lambda)) = \ell(w^I) + 1$ for p-regular $\lambda \in \Lambda_P$. In fact, for $G/P \simeq \mathbb{P}^n$, we computed $\ell\ell(\hat{\nabla}_P(\lambda))$ for any $\lambda \in \Lambda_P$ in [9, 2.3], dispensing with the Lusztig conjecture.

(4.8) Recall that $\tilde{Z}_{\Bbbk}(\lambda)/\tilde{Z}_{\Bbbk}(\lambda)_{>0} \simeq \tilde{L}_{\Bbbk}(\lambda) \simeq \mathrm{hd}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}\tilde{Z}_{\Bbbk}(\lambda)$ for each $\lambda \in \Omega$. It follows that the $\mathbb{Z}$-gradation on $\tilde{Z}_{\Bbbk}(\lambda)$ is such that, for each $j \in \mathbb{N}$,

$$\begin{aligned} \tilde{Z}_{\Bbbk}(\lambda)_{\geqslant j} &= \mathrm{rad}^j_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}\tilde{Z}_{\Bbbk}(\lambda) = \mathrm{rad}^j_{E_{\Omega,\Bbbk}\mathbf{modgr}_{Y}}\tilde{Z}_{\Bbbk}(\lambda) \\ &= \mathrm{soc}^{|R^+|+1-j}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}}\tilde{Z}_{\Bbbk}(\lambda) = \mathrm{soc}^{|R^+|+1-j}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{Y}}\tilde{Z}_{\Bbbk}(\lambda), \end{aligned}$$

and hence

$$\mathrm{soc}^{|R^+|+1-j}_{\mathcal{C}_{\Bbbk}(\Omega)}Z_{\Bbbk}(\lambda) = \bar{v}(\tilde{Z}_{\Bbbk}(\lambda)_{\geqslant j}) = \mathrm{rad}^j_{\mathcal{C}_{\Bbbk}(\Omega)}Z_{\Bbbk}(\lambda).$$

More generally, we have the following.

Proposition. *Assume the Lusztig conjecture (LG). The $\mathbb{Z}$-gradation on each $\tilde{Z}^w_{\Bbbk}(\lambda)$, $\lambda \in \Omega$, $w \in W$, is such that, for each $i \in \mathbb{N}$,*

$$\begin{aligned} \mathrm{rad}^i_{\mathcal{C}_{\Bbbk}(\Omega)}Z^w_{\Bbbk}(\lambda\langle w\rangle) &= \mathrm{rad}^i_{G_1T}\hat{\nabla}_{ww_0}(\lambda\langle ww_0\rangle) = \bar{v}(\tilde{Z}^w_{\Bbbk}(\lambda)_{\geqslant -\ell(w)+i}) \\ &= \mathrm{soc}^{|R^+|+1-i}_{\mathcal{C}_{\Bbbk}(\Omega)}Z^w_{\Bbbk}(\lambda\langle w\rangle) = \mathrm{soc}^{|R^+|+1-i}_{G_1T}\hat{\nabla}_{ww_0}(\lambda\langle ww_0\rangle). \end{aligned}$$

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Thus, $\forall \mu \in \Omega$,

$$[\mathrm{rad}_{\mathcal{C}_{\Bbbk}(\Omega),i} Z^w_{\Bbbk}(\lambda\langle w\rangle) : \hat{L}(\mu)] = [\tilde{Z}^w_{\Bbbk}(\lambda) : \tilde{L}_{\Bbbk}(\mu)\langle -\ell(w)+i\rangle] = [\mathrm{soc}_{\mathcal{C}_{\Bbbk}(\Omega),|R^+|+1-i} Z^w_{\Bbbk}(\lambda\langle w\rangle) : \hat{L}(\mu)],$$

where the middle term is the multiplicity of simple $\tilde{L}_{\Bbbk}(\mu)\langle -\ell(w)+i\rangle$ *in* $\tilde{Z}^w_{\Bbbk}(\lambda)$ *considered as objects of* $E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}$.

Proof. One has, from [1, 15.3.2],

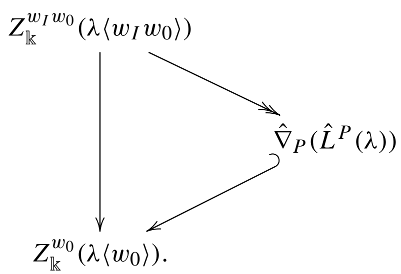
$$\Bbbk = \mathcal{C}_{\Bbbk}(\Omega)(Z^w_{\Bbbk}(\lambda\langle w\rangle), Z_{\Bbbk}(\lambda)) = \tilde{\mathcal{C}}_{\Bbbk}(\Omega)(\tilde{Z}^w_{\Bbbk}(\lambda), \tilde{Z}_{\Bbbk}(\lambda)\langle -2\ell(w)\rangle).$$

Let $j \in \mathbb{Z}$ minimal such that $\tilde{Z}^w_{\Bbbk}(\lambda)_j \neq 0$, so $\tilde{Z}^w_{\Bbbk}(\lambda)_{\geqslant j}/\tilde{Z}^w_{\Bbbk}(\lambda)_{>j} = \mathrm{hd}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}} \tilde{Z}^w_{\Bbbk}(\lambda) = \mathrm{hd}_{E_{\Omega,\Bbbk}\mathbf{modgr}_{Y}} \tilde{Z}^w_{\Bbbk}(\lambda) = H_{\Omega,\Bbbk}(\mathrm{rad}_{\mathcal{C}_{\Bbbk}(\Omega),0} Z^w_{\Bbbk}(\lambda\langle w\rangle))$, which is sent to

$$\begin{aligned} \tilde{Z}_{\Bbbk}(\lambda)_{\geqslant j+2\ell(w)}/\tilde{Z}_{\Bbbk}(\lambda)_{>j+2\ell(w)} &= (\tilde{Z}_{\Bbbk}(\lambda)\langle -2\ell(w)\rangle)_{\geqslant j}/(\tilde{Z}_{\Bbbk}(\lambda)\langle -2\ell(w)\rangle)_{>j} \\ &= H_{\Omega,\Bbbk}(\mathrm{rad}_{\mathcal{C}_{\Bbbk}(\Omega),\ell(w)} Z_{\Bbbk}(\lambda)) \\ &= \tilde{Z}_{\Bbbk}(\lambda)_{\geqslant \ell(w)}/\tilde{Z}_{\Bbbk}(\lambda)_{>\ell(w)} \quad \text{by the above.} \end{aligned}$$

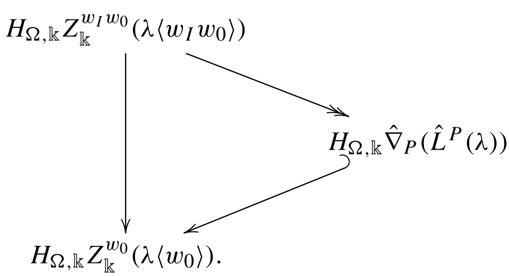
Thus $j = -\ell(w)$. As $\ell\ell(Z^w_{\Bbbk}(\lambda\langle w\rangle)) = |R^+|+1$, the assertion follows. □

(4.9) Untwisting w^I of (4.6)(3) reads



(1)

Thus one obtains a commutative diagram in $E_{\Omega,\Bbbk}\mathbf{modgr}_Y$



(2)

Recall that $\mathcal{C}_{\Bbbk}(\Omega)(Z^{w_I w_0}_{\Bbbk}(\lambda\langle w_I w_0\rangle), Z^{w_0}_{\Bbbk}(\lambda\langle w_0\rangle))$ is one dimensional. On the other hand, each $Z^w_{\Bbbk}(\lambda\langle w\rangle)$, $w \in W$, admits a graded object $\tilde{Z}^w_{\Bbbk}(\lambda) \in \tilde{\mathcal{C}}_{\Bbbk}(\Omega)$ such that $\bar{v}\tilde{Z}^w_{\Bbbk}(\lambda) \simeq Z^w_{\Bbbk}(\lambda\langle w\rangle)$. It follows that

$$E_{\Omega,\Bbbk}\mathbf{modgr}_Y(H_{\Omega,\Bbbk} Z^{w_I w_0}_{\Bbbk}(\lambda\langle w_I w_0\rangle), H_{\Omega,\Bbbk} Z^{w_0}_{\Bbbk}(\lambda\langle w_0\rangle))$$

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$$= E_{\Omega,\Bbbk}\mathbf{modgr}_Y(\tilde{Z}_{\Bbbk}^{w_I w_0}(\lambda), \tilde{Z}^{w_0}(\lambda))$$
$$= \bigoplus_{i\in\mathbb{Z}} E_{\Omega,\Bbbk}\mathbf{modgr}_Y(\tilde{Z}_{\Bbbk}^{w_I w_0}(\lambda), \tilde{Z}^{w_0}(\lambda))_i = \bigoplus_{i\in\mathbb{Z}} \tilde{\mathcal{C}}_{\Bbbk}(\Omega)(\tilde{Z}_{\Bbbk}^{w_I w_0}(\lambda), \tilde{Z}^{w_0}(\lambda)\langle -i\rangle)$$
$$= \tilde{\mathcal{C}}_{\Bbbk}(\Omega)(\tilde{Z}_{\Bbbk}^{w_I w_0}(\lambda), \tilde{Z}^{w_0}(\lambda)\langle -j\rangle)$$

for some single $j \in \mathbb{Z}$ by dimension; in fact, $j = 0$, by [1, 15.3.2]. Then, taking $\eta \in \tilde{\mathcal{C}}_{\Bbbk}(\Omega)(\tilde{Z}_{\Bbbk}^{w_I w_0}(\lambda), \tilde{Z}^{w_0}(\lambda)) \setminus 0$, $\mathrm{im}(\eta) \in \tilde{\mathcal{C}}_{\Bbbk}(\Omega)$ with $\bar{v}(\mathrm{im}(\eta)) = \hat{\nabla}_P(\hat{L}^P(\lambda))$. This gives another proof that $\hat{\nabla}_P(\hat{L}^P(\lambda))$ is $\mathbb{Z}$-graded, and hence is rigid.

Corollary. *Assume the Lusztig conjecture (LG). The* $\mathbb{Z}$*-gradation on* $\mathrm{im}(\eta)$ *is such that, for each* $i \in \mathbb{N}$,

$$\bar{v}((\mathrm{im}(\eta))_{\geqslant -i}) = \mathrm{rad}_{G_1T}^{\ell(w^I)-i}\hat{\nabla}_P(\hat{L}^P(\lambda)) = \mathrm{soc}_{G_1T}^{i+1}\hat{\nabla}_P(\hat{L}^P(\lambda)).$$

Proof. As

$$\begin{aligned}\mathrm{soc}_{G_1T}\hat{\nabla}_P(\hat{L}^P(\lambda)) &= \mathrm{soc}_{G_1T} Z_{\Bbbk}^{w_0}(\lambda\langle w_0\rangle) \quad \text{by (1)}\\ &= \bar{v}(\tilde{Z}^{w_0}(\lambda\langle w_0\rangle)_0) \quad \text{by (4.8)},\end{aligned}$$

$\bar{v}(\mathrm{im}(\eta)_0) = \mathrm{soc}_{G_1T}\hat{\nabla}_P(\hat{L}^P(\lambda))$, and hence the assertion follows. □

5. The Loewy series

Keep the notation of section 4. In particular, we continue to assume the Lusztig conjecture (LG). In this section, we will derive a formula to describe the socle series of $\hat{\nabla}_P(\hat{L}^P(\lambda))$.

(5.1) Let us first recall from [2] or from [1, 18.19] a formula for the socle series of $\hat{\nabla}(\lambda)$:

$$Q^{\mu,\lambda} = \sum_j q^{\frac{\mathrm{d}(\mu,\lambda)-j}{2}}[\mathrm{soc}_{j+1}\hat{\nabla}(\lambda) : \hat{L}(\mu)], \tag{1}$$

where $Q^{\mu,\lambda} = Q^{A,C}$ for alcove A containing μ and alcove C containing λ is a periodic inverse Kazhdan–Lusztig polynomial defined in [14]. We will prove a formula

$$\sum_j q^{\frac{\mathrm{d}(\mu,\lambda)-j}{2}}[\mathrm{soc}_{j+1}\hat{\nabla}_P(\hat{L}^P(\lambda)) : \hat{L}(\mu)] = \sum_{\nu\in W_{I,p}\bullet\lambda} Q^{\mu,\nu}(-1)^{\mathrm{d}_I(\nu,\lambda)}\hat{P}^I_{\nu,\lambda}. \tag{2}$$

The formula reduces to (1) when $I = \emptyset$, i.e., when P is a Borel subgroup. It also holds for $P = G$ by the inversion formula $\sum_\nu Q^{\mu,\nu}(-1)^{\mathrm{d}(\nu,\lambda)}\hat{P}_{\nu,\lambda} = \delta_{\mu,\lambda}$ [14, 11.10]/[12, p. 129].

(5.2) Recalling the abbreviation $\tilde{\nabla}_{\Bbbk}(\lambda) = \tilde{Z}_{\Bbbk}^{w_0}(\lambda)$ for each $\lambda \in \Omega$, we see by (4.8) that formula (5.1)(1) reads, with $q^{1/2} = t$,

$$Q^{\mu,\lambda}(t^2) = \sum_j t^{\mathrm{d}(\mu,\lambda)-j}[\tilde{\nabla}_{\Bbbk}(\lambda) : \tilde{L}_{\Bbbk}(\mu)\langle -j\rangle]. \tag{1}$$

If we write $Q^{\lambda,\nu}(t) = \sum_j Q_j^{\lambda\nu} t^{\frac{j}{2}}$ with $Q_j^{\lambda\nu} \in \mathbb{Z}$, formula (1) reads, in the Grothendieck group of $E_{\Omega,\Bbbk}\mathbf{modgr}_{\mathbb{Z}}$,

$$[\tilde{\nabla}_{\Bbbk}(\lambda)] = \sum_{j\in\mathbb{Z}}\sum_{\mu\in\Omega} Q^{\mu\lambda}_{\mathrm{d}(\mu,\lambda)-j}[\tilde{L}_{\Bbbk}(\mu)\langle -j\rangle], \tag{2}$$

inverting which reads, if we write $\hat{P}_{\mu,\lambda}(t) = \sum_{j\in\mathbb{Z}} \hat{P}_{\mu\lambda,j} t^{\frac{1}{2}}$,

$$[\tilde{L}_{\Bbbk}(\lambda)] = \sum_{j\in\mathbb{Z}} \sum_{\mu\in\Omega} (-1)^{\mathrm{d}(\mu,\lambda)} \hat{P}_{\mu\lambda,j+\mathrm{d}(\mu,\lambda)} [\tilde{\nabla}_{\Bbbk}(\mu)\langle j\rangle]. \tag{3}$$

We now verify formula (5.1)(2).

Theorem. *Assume the Lusztig conjecture (LG). If λ is a p-regular weight, the Loewy series of $\hat{\nabla}_P(\hat{L}^P(\lambda))$ is given by*

$$\sum_{j\in\mathbb{N}} q^{\frac{\mathrm{d}(\mu,\lambda)-j}{2}} [\mathrm{soc}_{j+1} \hat{\nabla}_P(\hat{L}^P(\lambda)) : \hat{L}(\mu)] = \sum_{\nu\in W_{I,p}\bullet\lambda} Q^{\mu,\nu} (-1)^{\mathrm{d}_I(\nu,\lambda)} \hat{P}^I_{\nu,\lambda} \quad \forall \mu \in \Lambda.$$

Proof. Put $\tilde{\nabla}_P = J_{\Bbbk} \otimes_{E_{\Omega_I,\Bbbk}} ?$ from (3.9) for simplicity. In the Grothendieck group of $\tilde{\mathcal{C}}_{\Bbbk}(\Omega_I)$, formula (3) reads $[\tilde{L}_{I,\Bbbk}(\lambda)] = \sum_{j\in\mathbb{Z},\mu\in\Omega} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda,j+\mathrm{d}_I(\mu,\lambda)} [\tilde{\nabla}_{I,\Bbbk}(\mu)\langle j\rangle]$. Put $n_\lambda = \delta(\lambda) - \delta_I(\lambda)$, so $\tilde{\nabla}_P(\tilde{\nabla}_{I,\Bbbk}(\lambda)) \simeq \tilde{\nabla}_{\Bbbk}(\lambda)\langle n_\lambda\rangle$ by (3.9). As $\hat{\nabla}_P$ is exact, so is $\tilde{\nabla}_P$ by (3.9) also. Then

$$\begin{aligned}
[\tilde{\nabla}_P(\tilde{L}_{I,\Bbbk}(\lambda))] &= \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda,j+\mathrm{d}_I(\mu,\lambda)} [\tilde{\nabla}_P(\tilde{\nabla}_{I,\Bbbk}(\mu))\langle j\rangle] \\
&= \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda,j+\mathrm{d}_I(\mu,\lambda)} [\tilde{\nabla}_{\Bbbk}(\mu)\langle n_\mu + j\rangle] \\
&= \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda,j+\mathrm{d}_I(\mu,\lambda)} \sum_{k,\nu} Q^{\nu\mu}_{\mathrm{d}(\nu,\mu)-k} [\tilde{L}_{\Bbbk}(\nu)\langle n_\mu + j - k\rangle] \\
&= \sum_{\mu,j,\nu,k} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda,j+\mathrm{d}_I(\mu,\lambda)} Q^{\nu\mu}_{\mathrm{d}(\nu,\mu)-k} [\tilde{L}_{\Bbbk}(\nu)\langle n_\mu + j - k\rangle].
\end{aligned}$$

Recall now $\mathrm{im}(\eta)$ from (4.9). As $\tilde{L}_{I,\Bbbk}(\lambda) \leqslant \tilde{\nabla}_{I,\Bbbk}(\lambda)$, $\tilde{\nabla}_P(\tilde{L}_{I,\Bbbk}(\lambda)) \leqslant \tilde{\nabla}_P(\tilde{\nabla}_{I,\Bbbk}(\lambda)) \simeq \tilde{\nabla}_{\Bbbk}(\lambda)\langle n_\lambda\rangle$. As $\mathrm{im}(\eta) \leqslant \tilde{\nabla}_{\Bbbk}(\lambda)$, it follows that $\tilde{\nabla}_P(\tilde{L}_{I,\Bbbk}(\lambda)) \simeq \mathrm{im}(\eta)\langle n_\lambda\rangle$. Thus

$$\begin{aligned}
[\mathrm{soc}_{i+1} \hat{\nabla}_P(\hat{L}^P(\lambda)) : \hat{L}(\nu)] &= [\mathrm{im}(\eta) : \tilde{L}_{\Bbbk}(\nu)\langle -i\rangle] \quad \text{by (4.9)} \\
&= [\tilde{\nabla}_P(\tilde{L}_{I,\Bbbk}(\lambda))\langle -n_\lambda\rangle : \tilde{L}_{\Bbbk}(\nu)\langle -i\rangle] \\
&= [\tilde{\nabla}_P(\tilde{L}_{I,\Bbbk}(\lambda)) : \tilde{L}_{\Bbbk}(\nu)\langle n_\lambda - i\rangle] \\
&= \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda,j+\mathrm{d}_I(\mu,\lambda)} Q^{\nu\mu}_{\mathrm{d}(\nu,\mu)+\delta(\lambda)-\delta_I(\lambda)-\delta(\mu)+\delta_I(\mu)-i-j} \\
&= \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda,j+\mathrm{d}_I(\mu,\lambda)} Q^{\nu\mu}_{\mathrm{d}(\nu,\lambda)-\mathrm{d}_I(\mu,\lambda)-i-j} \\
&= \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda j} Q^{\nu\mu}_{\mathrm{d}(\nu,\lambda)-i-j},
\end{aligned}$$

and hence

$$\sum_i t^{\mathrm{d}(\nu,\lambda)-i} [\mathrm{soc}_{i+1} \hat{\nabla}_P(\hat{L}^P(\lambda)) : \hat{L}(\nu)] = \sum_i t^{\mathrm{d}(\nu,\lambda)-i} \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda j} Q^{\nu\mu}_{\mathrm{d}(\nu,\lambda)-i-j}$$

$$
\begin{aligned}
&= \sum_i \sum_{\mu,j} (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda j} t^j Q^{\nu\mu}_{\mathrm{d}(\nu,\lambda)-i-j} t^{\mathrm{d}(\nu,\lambda)-i-j} \\
&= \sum_\mu \sum_j (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu\lambda j} t^j \sum_k Q^{\nu\mu}_k t^k \\
&= \sum_\mu (-1)^{\mathrm{d}_I(\mu,\lambda)} \hat{P}^I_{\mu,\lambda}(t^2) Q^{\nu,\mu}(t^2),
\end{aligned}
$$

as desired. □

(5.3) Given a simple G_1T-module, formula (5.1)(2) need not immediately locate a simple factor in the Loewy layers of $\hat{\nabla}_P(\hat{L}^P(\lambda))$. The following are particularly important factors in the study of the Frobenius direct image of the structure sheaf of G/P ([6], [8]). Let $W^I = \{w \in W \mid \ell(ww') = \ell(w) + \ell(w')\ \forall w' \in W_I\}$, which forms a complete set of representatives of W/W_I.

Proposition. *Assume the Lusztig conjecture (LG). Let $\lambda \in \Lambda$ be p-regular. If $w \in W^I$, $L((w \bullet \lambda)^0) \otimes p(w^{-1} \bullet (w \bullet \lambda)^1)$ appears in the $(\ell(w)+1)$-st socle layer of $\hat{\nabla}_P(\hat{L}^P(\lambda))$.*

Proof. In the commutative diagram (4.6)(1), put $\phi = \phi_{w^I}$. Write $w^I = s_1 s_2 \ldots s_m$ in a reduced expression with $m = \ell(w^I)$, and put $y_r = s_1 s_2 \ldots s_r$ for $r \leqslant m$. Then $y_r^{-1} w^I = s_{r+1} \ldots s_m \in W^I$. Recall from [2] that $\phi_{w^I} : \hat{\nabla}_{w^I}((w^I \bullet \lambda)\langle w^I \rangle) \to \hat{\nabla}(w^I \bullet \lambda)$ is the composite

$$
\begin{aligned}
&\hat{\nabla}_{w^I}((w^I \bullet \lambda)\langle w^I \rangle) = \hat{\nabla}_{s_1 \ldots s_m}((w^I \bullet \lambda)\langle s_1 \ldots s_m \rangle) \xrightarrow{\phi_m} \\
&\hat{\nabla}_{s_1 \ldots s_{m-1}}((w^I \bullet \lambda)\langle s_1 \ldots s_{m-1} \rangle) \xrightarrow{\phi_{m-1}} \hat{\nabla}_{s_1 \ldots s_{m-2}}((w^I \bullet \lambda)\langle s_1 \ldots s_{m-2} \rangle) \xrightarrow{\phi_{m-2}} \ldots \xrightarrow{\phi_2} \\
&\hat{\nabla}_{s_1}((w^I \bullet \lambda)\langle s_1 \rangle) \xrightarrow{\phi_1} \hat{\nabla}(w^I \bullet \lambda).
\end{aligned}
$$

Put $L = \mathrm{soc}\hat{\nabla}_{y_r}((w^I \bullet \lambda)\langle y_r \rangle) \otimes (-p(w^I \bullet 0))$ and $\phi'_r = \{(\phi_{r+1} \circ \cdots \circ \phi_m) \otimes (-p(w^I \bullet 0))\}|_{{}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))}$. As $\phi'_0 \neq 0$, and as each ϕ_i annihilates the socle of its domain, we must have $\ell\ell(\mathrm{im}\phi'_r) = \ell\ell(\hat{\nabla}_P(\hat{L}^P(\lambda))) - (m-r) = r+1$ by (4.6). Then $L = \mathrm{soc}(\mathrm{im}\phi'_r) = \mathrm{rad}_r(\mathrm{im}\phi'_r)$, which is a quotient of $\mathrm{rad}_r{}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))$. Thus L lies in $\mathrm{rad}_r{}^{w^I}\hat{\nabla}_P(\hat{L}^P(\lambda))$. It follows from the rigidity of $\hat{\nabla}_P(\hat{L}^P(\lambda))$ that ${}^{(w^I)^{-1}}L$ appears in its socle layer of level $\ell\ell(\hat{\nabla}_P(\hat{L}^P(\lambda))) - r = \ell(w^I) + 1 - r = m + 1 - r = \ell(y_r^{-1} w^I) + 1$. Recall now from [2, 1.2.4] that

$$
\begin{aligned}
L &= \hat{L}((y_r^{-1} \bullet (w^I \bullet \lambda))^0 + p(y_r \bullet (y_r^{-1} \bullet (w^I \bullet \lambda))^1)) \otimes (-p(w^I \bullet 0)) \\
&= \hat{L}((y_r^{-1} w^I \bullet \lambda)^0 + p(y_r \bullet (y_r^{-1} w^I \bullet \lambda)^1)) \otimes (-p(w^I \bullet 0)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
{}^{(w^I)^{-1}}L &= L((y_r^{-1} w^I \bullet \lambda)^0) \otimes p\{(w^I)^{-1}\{y_r \bullet (y_r^{-1} w^I \bullet \lambda)^1 - (w^I \bullet 0)\}\} \\
&= L((y_r^{-1} w^I \bullet \lambda)^0) \otimes p\{(w^I)^{-1} y_r \bullet (y_r^{-1} w^I \bullet \lambda)^1\} \\
&= L((y_r^{-1} w^I \bullet \lambda)^0) \otimes p\{(y_r^{-1} w^I)^{-1} \bullet (y_r^{-1} w^I \bullet \lambda)^1\}.
\end{aligned}
$$

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Finally, we check that any $w \in W^I$ may be realized as $y_r^{-1}w^I$ as above. Let $w \in W^I$. As $\ell(ww_I) = \ell(w) + \ell(w_I)$, one can write $w_0 = s_{j_1} \ldots s_{j_r} ww_I$ with $r = \ell(w_0) - \ell(w) - \ell(w_I)$. Then $w^I = w_0 w_I = s_{j_1} \ldots s_{j_r} w$ with $\ell(w^I) = r + \ell(w)$. Thus, putting $y_r = s_{j_1} \ldots s_{j_r}$ yields $w = y_r^{-1}w^I$, as desired. □

(5.4) Remark. This is a generalization of [8, 1.5], [9], and [10, 3.5]. For $\lambda = 0$, we constructed for G of rank at most 2 [8] or for $G = GL_{n+1}(\Bbbk)$ and P maximal parabolic such that $G/P \simeq \mathbb{P}^n$ for any $n \in \mathbb{N}$ [9], a Karoubian complete strongly exceptional sequence $\{\mathcal{E}_w \mid w \in W^I\}$ for the bounded derived category of coherent $\mathcal{O}_{G_\mathbb{C}/P_\mathbb{C}}$-modules out of $G_1\mathbf{Mod}(L((w \bullet 0)^0), \mathrm{soc}_{\ell(w)+1}\hat{\nabla}_P(0))$, where $G_\mathbb{C}$ and $P_\mathbb{C}$ are the groups over the complex number field corresponding to G and P, respectively. Our (5.3) ensures at least that $G_1\mathbf{Mod}(L((w \bullet 0)^0), \mathrm{soc}_{\ell(w)+1}\hat{\nabla}_P(0)) \neq 0$ in general for large p.

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References

1. H. H. Andersen, J. C. Jantzen and W. Soergel, Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p : independence of p, *Astérisque* **220** (1994), (SMF).
2. H. H. Andersen and M. Kaneda, Loewy series of modules for the first Frobenius kernel in a reductive algebraic group, *Proc. LMS* **59**(3) (1989), 74–98.
3. M. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra.* (Addison-Wesley, Reading, 1969).
4. A. Beilinson, D. Ginzburg and W. Soergel, Koszul duality patterns in representation theory, *J. Am. Math. Soc.* **9**(2) (1996), 473–527.
5. P. Fiebig, Sheaves on affine Schubert varieties, modular representations and Lusztig's conjecture, *J. Am. Math. Soc.* **24** (2011), 133–181.
6. Y. Hashimoto, M. Kaneda and D. Rumynin, On localization of $\bar{D}$-modules, in *Representations of Algebraic Groups, Quantum Groups, and Lie Algebras*, Contemp. Math., Volume volume 413, pp. 43–62. (2006).
7. J. C. Jantzen, *Representations of Algebraic Groups.* (American Math. Soc., 2003).
8. M. Kaneda and J. -C. Ye, Some observations on Karoubian complete strongly exceptional posets on the projective homogeneous varieties, arXiv:0911.2568v2 [math.RT].
9. M. Kaneda, The structure of Humphreys–Verma modules for projective spaces, *J. Algebra* **322** (2009), 237–244.
10. M. Kaneda, Homomorphisms between neighbouring G_1T-Verma modules, *Contemp. Math.* **565** (2012), 105–113.
11. M. Kashiwara and T. Tanisaki, Kazhdan–Lusztig conjecture for symmetrizable Kac–Moody Lie algebras with negative level II. Nonintegral case, *Duke Math. J.* **84** (1996), 771–813.

12. S. Kato, On the Kazhdan–Lusztig polynomials for affine Weyl groups, *Adv. Math.* **55** (1995), 103–130.
13. D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras I, II, *J. Am. Math. Soc.* **6** (1993), 905–1011 ; Tensor structures arising from affine Lie algebras III, IV. *J. Am. Math. Soc.* **7**, (1994) 335–453.
14. G. Lusztig, Hecke algebras and Jantzen's generic decomposition patterns, *Adv. Math.* **37** (1980), 121–164.
15. G. Lusztig, Monodromic systems on affine flag manifolds, *Proc. R. Soc. Lond.* A **445** (1994), 231–246 ; Errata, **450**, (1995) 731–732.
16. H. Matsumura, *Commutative ring theory.* (Cambridge Univ. Press, Cambridge, 1990).