On the Use of the term *Power* in Geometry, and on the treatment of the "doubtful sign."

By R. F. MUIRHEAD.

Amongst the "technical terms" that have come into use in connection with Coordinate Geometry, not the least convenient is the word *Power*. The only definition of a general kind for this term that I have met with is the following :

"DEF.—*The result of substituting the coordinates of any point in the equation of any line or curve is called the* POWER *of that point with respect to the line or curve.*

"[This definition, first given by Steiner, is now employed by all the French and German writers.]"

This quotation is from Casey's *Treatise on Conic Sections*, page 26.

I have consulted Steiner's published works, and have found therein two definitions of the "Potenz" or Power of a point with respect to a circle, but no general definition such as that given by Casey. The earlier of the two occurs in a paper of Steiner's in the first volume of "Crelle," where "Potenz des Punktes in Bezug auf den Kreis, oder Potenz des Kreises in Bezug auf den Punkt" is defined as the difference between the square on the distance of the point from the centre, and the square on the radius, and is distinguished as the *interior* or *exterior* Power, according as the point is within or without the circle, being positive in both cases. The later improved form of the definition is to be found in Steiner's "Synthetische Geometrie," and differs from the former in replacing the interior power and exterior power by a single power, thus : *If P be the point, and A, B the points in which the circle is intersected by any line through P, then the rectangle PA . PB is defined as the power of the point P with reference to the circle.* This definition makes the power positive for external, negative for internal points.

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Going back to Casey's definition, we find that it leaves a good deal to be desired. By inadvertence, as I suppose, the word *equation* is used for "that expression which, being equated to zero, gives the equation." Even if this correction were made, the definition would be incomplete until we had fixed the form of the equation still further. We should have to agree that in the equation $f(x, y) = 0$, the function $f(x, y)$ should be rational and integral as to x and y. And even then, for a given curve the "power" would be indeterminate to the extent of an arbitrary constant factor.

In the next two pages we find Casey interpreting his definition in two inconsistent senses, first taking $ax + by + c$ as the *power* of (x, y) as to the straight line $ax + by + c = 0$, and afterwards making a statement as to the power of a point, which is only true for the special form $x\cos\alpha + y\sin\alpha - p = 0$. Again, on p. 72 there is a statement as to the power of the point (x, y) with reference to a circle which is inconsistent with the definition to which reference is given.

Seeing that an authority like Casey has left the definition so indefinite, I feel at liberty to make suggestions as to what would be the most expedient usage of the term in question.

In the first place, a definition of Casey's type refers not simply to a line or curve, but to the particular form in which its equation is written. Would it not, therefore, be better to speak of the power of a point with respect to a curve as represented by a certain equation, or more briefly, the power of a point with reference to an equation? We could, for instance, speak of the power of (x, y) as to the equation $Ax + By + C = 0$, and the definition required would be:

The power of a point (x_1, y_1) *with respect to an equation* $f(x, y) = 0$ *is the value of* $f(x_1, y_1)$.

This might be generalised as follows: The power of $(x_1, y_1, z_1, \dots)$ as to the equation $f(x, y, z, \dots) = 0$ is the value of $f(x_1, y_1, z_1, \dots)$

But, in addition to the term Power of a point as to an *equation*, we may also use the term Power of a point as to a line, curve, or locus.

The expression "Power of a point as to a locus" (where the locus might be a line, a curve, or a surface) would appropriately get a purely geometrical definition. The later form of Steiner's definition of the power of a point with reference to a circle, which is purely geometrical, will no doubt continue to hold the field.

In seeking a complete geometrical definition of *the Power of a point with respect to a straight line* we meet a further complication which does not arise in connection with the Power of a point as to the *equation*. If we define it to be the perpendicular distance, reckoned as positive when the point and the origin are on opposite sides of the line, the definition will fail in the case of a line passing through the origin.

It would be best, I think, to give the definition with reference to a *directed* straight line thus: The power of a point with reference to a directed straight line is its distance from that line to the right of a traveller walking along it in the positive direction.

To make this agree with the usual convention, we have only to add that the *direction* of the line is to be such that the origin is to the left of the line. But I think this ought to be treated as a *special* convention, and not put into the definition, for in many cases such a restricted convention is disadvantageous.

Returning now to the analytical definition of the Power of a point with reference to an *equation*, we may note the following geometrical interpretations.

Let O be the origin and P the point (x, y) and S the point in which OP cuts the line whose equation is $x/a + y/b - 1 = 0$. The power of P with respect to the equation is equal to the ratio SP : OS.

Again, with reference to the equation $Ax + By + C = 0$

the power of P is $\qquad -C \times SP : OS.$

The power of (x, y) for the equation $(x-a)^2 + (y-b)^2 - r^2 = 0$ is the rectangle PQ . PR, when PQR is any straight line passing through P and cutting in Q, R the circle whose centre is (a, b) and radius r.

The power of (x, y) for the equation $Ax^2 + Ay^2 + Bx + Cy + D = 0$ is obviously equal to A × PQ . PR.

The power of (x, y) for the equation $x^2/a^2 + y^2/b^2 - 1 = 0$ is the ratio − PQ . PR : (OD . OD′), or PQ . PR : OD^2, where PQR is any secant through P, and DOD′ is a diameter parallel to it.

The power of (x, y) for the equation $y^2 - 4ax = 0$ is the rectangle PQ . PR where PQR is a line parallel to the directrix, intersecting the parabola in Q, R; it is also equal to $4a$. PV, where PV is a line parallel to the axis of the parabola and meeting the parabola in V.

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The power of (x, y) as to the general equation of the second degree,

$$u \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

is $= -\mathrm{PQ}\,.\,\mathrm{PR}\,.\,u_0/\mathrm{OD}^2$; where u_0 is the power of the centre, and may be written $gx_0 + fy_0 + c$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \div \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$$

The power of (x, y) as to any rational integral algebraic equation $f(x, y) = 0$ receives its interpretation through the equation

$$\frac{f(x, y)}{f(x_1, y_1)} = \frac{\mathrm{PA}\,.\,\mathrm{PB}\,.\,\mathrm{PC}\ldots\ldots}{\mathrm{P_1A}\,.\,\mathrm{P_1B}\,.\,\mathrm{P_1C}\ldots},$$

where A, B, C... are the points in which the line joining the points $\mathrm{P}(x, y)$ and $\mathrm{P}_1(x_1, y_1)$ intersects the locus of $f(x, y) = 0$.
Here (x_1, y_1) is an arbitrarily chosen point.

A similar interpretation holds in solid geometry for the power of (x, y, z) as to the equation $f(x, y, z) = 0$, if it be algebraic, rational, and integral.

The power of P, (ξ, η, ζ) as to the equation $\mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C}\zeta = 0$ which represents a straight line, ξ, η, ζ being any point-coordinates whose invariable relation is $\lambda\xi + \mu\eta + \nu\zeta = 1$, may be interpreted thus: $\dfrac{\mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C}\zeta}{\mathrm{A}\xi_1} = \dfrac{p}{p_1}$, where p is the perpendicular from (ξ, η, ζ) on the line, and p_1 the perpendicular from $(\xi_1, 0, 0)$ one of the angular points of the fundamental triangle. Since $\lambda\xi_1 = 1$, we get the power of (ξ, η, ζ) to be $= \dfrac{\mathrm{A}p}{\lambda p_1} = \dfrac{\mathrm{A}ap}{2\lambda\Delta}$, where a is one side and Δ the area of the fundamental triangle.

The following application of the 'power of an equation' is given because it involves a point of interest with reference to the *sense* of the perpendicular.

It is required to write down the equations of the lines bisecting the internal angles of a triangle, the equations of the sides being given.

Let these equations be
$$u \equiv a\,x + by + c = 0,$$
$$u' \equiv a'x + b'y + c' = 0,$$
$$u'' \equiv a''x + b''y + c'' = 0.$$

Take (x_1, y_1), (x_2, y_2), (x_3, y_3) to represent the vertices of the triangle and let $ax_1 + by_1 + c \equiv u_1$, etc.

Then we have
$$a\,x_1 + by_1 + c = u_1,$$
$$a'x_1 + b'y_1 + c' = 0,$$
$$a''x_1 + b''y_1 + c'' = 0.$$

Hence
$$\begin{vmatrix} a, & b, & c - u_1 \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = 0\,; \quad \therefore \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = u_1 \begin{vmatrix} a, & b, & 1 \\ a', & b', & 0 \\ a'', & b'', & 0 \end{vmatrix}.$$

Now for points on the same side of the line $u = 0$ as the vertex P_1, u will have the same sign as u_1.

Hence $\quad u_1(ax + by + c) \div \sqrt{u_1^2(a^2 + b^2)}$, or $u_1 u \div \sqrt{u_1^2(a^2 + b^2)}$

is the perpendicular distance of (x, y) from $u = 0$, reckoned as positive when on the same side as P_1.

Hence the equations to the in-centre of $P_1P_2P_3$ are
$$uu_1 \div \sqrt{u_1^2(a^2 + b^2)} = u'u_2' \div \sqrt{u_2'(a'^2 + b'^2)} = u''u_3'' \div \sqrt{u_3''(a''^2 + b''^2)},$$
or, explicitly in terms of the given coefficients,
$$(ax + by + c).\begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix}.\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} \div \sqrt{\left\{\begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix}.\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}.(a^2 + b^2)\right\}}$$
$$= \text{etc.} = \text{etc.}$$

Remark suggested by the above: If $ax + by + c = 0$ be the equation of a line, and we write it in the form
$$(ax_1 + by_1 + c)(ax + by + c) = 0,$$
then the power of (x, y) with respect to the latter equation will be *positive* for points on the same side of the line as the point (x_1, y_1).

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Thus for $-c(ax+by+c)=0$, (x, y) has *positive* power for points on the side remote from the origin, and

$$-c(ax+by+c) \div \sqrt{(a^2+b^2)c^2}=0 \text{ is the } \textit{standard}$$

form of the equation of the straight line.

Of course we might use here instead of the factor (ax_1+by_1+c), any power of that factor whose index is an odd integer, positive or negative, or in fact any odd function of that factor.

More general remark: To express "that value of $\pm a$ which has the same sign as b" we may use $b\sqrt{\dfrac{a^2}{b^2}}$, or $\dfrac{\sqrt{a^2b^2}}{b}$. For example, that value of ± 1 which has the same sign as b, is $b/\sqrt{b^2}$ or $\sqrt{b^2}/b$.

Note that so long as x is *real*, the symbols $\sqrt{x^2}$ and $|x|$ have the same meaning.

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