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Converge rates towards stationary solutions for the outflow problem of planar magnetohydrodynamics on a half line

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In this paper, convergence rates of solutions towards stationary solutions for the outflow problem of planar magnetohydrodynamics (MHD) are investigated. Inspired by the relationship between MHD and Navier-Stokes, we prove that the global solutions of the planar MHD converge to the corresponding stationary solutions of Navier-Stokes equations. We obtain the corresponding convergence rates based on the weighted energy method when the initial perturbation belongs to some weighted Sobolev space.

Keywords: magnetohydrodynamics; stationary solutions; outflow problem; convergence rates; weighted energy method

2010 *Mathematics subject classification:* Primary: 35B40; 34K21; 76W05

1. Introduction

1.1. The problem

Magnetohydrodynamics (MHD) concerns the motion of a conducting fluid in an electro-magnetic field and has very wide range applications in astrophysics, plasma, and so on. There is a complex interaction between the magnetic and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. The planar MHD on a half line $\mathbb{R}_+ =: (0, +\infty)$ is governed by the following equations in Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \rho(\partial_t u + u\partial_x u) + \partial_x \left(p + \dfrac{1}{2}|\mathbf{b}|^2\right) = \lambda \partial_x^2 u, \\ \rho(\partial_t \mathbf{w} + u\partial_x \mathbf{w}) - \partial_x \mathbf{b} = \mu \partial_x^2 \mathbf{w}, \\ \partial_t \mathbf{b} + \partial_x(u\mathbf{b} - \mathbf{w}) = \nu \partial_x^2 \mathbf{b}, \\ \dfrac{R}{\gamma - 1}\rho(\partial_t \theta + u\partial_x \theta) + p\partial_x u = \lambda(\partial_x u)^2 + \kappa \partial_x^2 \theta + \nu|\partial_x \mathbf{b}|^2 + \mu|\partial_x \mathbf{w}|^2, \end{cases} \tag{1.1}$$

where $\rho(x,t) \in \mathbb{R}$, $u(x,t) \in \mathbb{R}$, $\mathbf{w}(x,t) \in \mathbb{R}^2$, $\mathbf{b}(x,t) \in \mathbb{R}^2$ and $\theta(x,t) \in \mathbb{R}$ denote, respectively, the mass density, longitudinal velocity, transverse velocity, transverse magnetic field and temperature of the fluids. The longitudinal magnetic field is a constant which is taken to be one in (1.1). Here, the constants $\lambda > 0$ and $\mu > 0$ are the viscosity coefficients of the fluids; the constants $\nu > 0$ and $\kappa > 0$ are the resistivity coefficient acting as the magnetic diffusion coefficient of the magnetic field and the heat conductivity coefficient, respectively. In fact, the system (1.1) arises from a 3-D MHD with a special structure: the flow depends on only one space variable $x \in \mathbb{R}$ and does not change in the transverse directions; however, the velocity and magnetic field still have three components. For the detailed derivation of planar MHD (1.1), please refer to [**3**, **4**, **25**] and references therein. Liu, Yin and Zhu in [**15**] studied Euler–Maxwell equations which have a similarly special structure.

Assume that the conducting fluid is perfect. Hence, for pressure p, we have the state equation:

$$p = R\rho\theta, \tag{1.2}$$

where $R > 0$ is a gas constant.

The initial data for the system (1.1) is given by

$$(\rho, u, \mathbf{w}, \mathbf{b}, \theta)(x, 0) = (\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)(x), \quad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0, \quad \inf_{x\in\mathbb{R}_+} \theta_0(x) > 0. \tag{1.3}$$

We assume that the initial data in the far field $x = +\infty$ is constant, namely

$$\lim_{x\to+\infty} (\rho_0, u_0, \mathbf{w}_0, \mathbf{b}_0, \theta_0)(x) = (\rho_+, u_+, \mathbf{w}_+, \mathbf{b}_+, \theta_+), \tag{1.4}$$

and the boundary data for u, $\mathbf{w}$, $\mathbf{b}$ and θ at $x = 0$ is given by

$$(u, \mathbf{w}, \mathbf{b}, \theta)(0, t) = (u_-, \mathbf{w}_-, \mathbf{b}_-, \theta_-), \quad \forall\, t \geqslant 0, \tag{1.5}$$

where $u_- < 0$, $\theta_- > 0$ are constants, and $\mathbf{w}_-$, $\mathbf{b}_-$ are constant vectors, and the following compatibility conditions hold

$$u_0(0) = u_-, \quad \mathbf{w}_0(0) = \mathbf{w}_-, \quad \mathbf{b}_0(0) = \mathbf{b}_-, \quad \theta_0(0) = \theta_-. \tag{1.6}$$

The assumption $u_- < 0$ means that fluid blows out from the boundary $x = 0$ with the velocity u_-. Thus this problem is called the outflow problem (see [**17**]). The outflow boundary condition implies that the characteristic of the hyperbolic equation $(1.1)_1$ for the density ρ is negative around the boundary so that boundary conditions on u, $\mathbf{w}$, $\mathbf{b}$ and θ to parabolic equations $(1.1)_2$, $(1.1)_3$, $(1.1)_4$ and $(1.1)_5$ are necessary and sufficient for the wellposedness of the outflow problem. Motivated by the relationship between MHD and Navier-Stokes, we temporarily assume that $\mathbf{w}_\pm = \mathbf{b}_\pm = \mathbf{0}$, and can consider the large time behaviour of solutions to the outflow problem (1.1), (1.3), (1.4), (1.5), (1.6) in the setting of $\mathbf{w}(x, t) = \mathbf{b}(x, t) = \mathbf{0}$.Then the outflow problem is reduced to consider the following Navier-Stokes system in the form of

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \rho(\partial_t u + u\partial_x u) + \partial_x p = \lambda \partial_x^2 u, \\ \dfrac{R}{\gamma - 1}\rho(\partial_t \theta + u\partial_x \theta) + p\partial_x u = \lambda\left(\partial_x u\right)^2 + \kappa \partial_x^2 \theta, \end{cases} \tag{1.7}$$

with the initial data

$$(\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x) \to (\rho_+, u_+, \theta_+), \quad as \quad x \to +\infty, \tag{1.8}$$

and the boundary data

$$(u, \theta)(0, t) = (u_- < 0, \theta_- > 0), \quad \forall\, t \geqslant 0. \tag{1.9}$$

Hence, when time tends to infinity, it is reasonable for us to expect that the solutions to the outflow problem (1.1), (1.3), (1.4), (1.5), (1.6) asymptotically converge to the stationary solutions defined in § 1.2. Moreover, the cases for $\mathbf{w}_+ \neq \mathbf{w}_-$ and $\mathbf{b}_+ \neq \mathbf{b}_-$ which lead to more complex structures are left for study in future.

1.2. The existence of stationary solutions

We define stationary solutions $(\tilde\rho, \tilde u, \tilde\theta, \tilde{\mathbf{w}}, \tilde{\mathbf{b}})(x)$ by

$$\begin{cases} \partial_x(\tilde\rho\tilde u) = 0, \quad x \in \mathbb{R}_+, \\ \tilde\rho\tilde u\partial_x\tilde u + \partial_x\tilde p = \lambda\partial_x^2\tilde u, \quad x \in \mathbb{R}_+, \\ \dfrac{R}{\gamma - 1}\tilde\rho\tilde u\partial_x\tilde\theta + \tilde p\partial_x\tilde u = \lambda\left(\partial_x\tilde u\right)^2 + \kappa\partial_x^2\tilde\theta, \quad x \in \mathbb{R}_+, \\ \tilde u(0) = u_- < 0, \quad \tilde\theta(0) = \theta_-, \quad (\tilde\rho, \tilde u, \tilde\theta)(+\infty) = (\rho_+, u_+, \theta_+), \\ \inf_{x\in\mathbb{R}_+}\tilde\rho(x) > 0, \quad \inf_{x\in\mathbb{R}_+}\tilde\theta(x) > 0, \end{cases} \tag{1.10}$$

with $\tilde{\mathbf{w}} = \tilde{\mathbf{b}} = \mathbf{0}$, where $\tilde p = R\tilde\rho\tilde\theta$.

From the fact $\tilde\rho(x) > 0$ and $u_- < 0$, we have

$$\rho_- := \tilde\rho(0) = \frac{\rho_+ u_+}{u_-}, \quad \tilde\rho(x) = \frac{\rho_+ u_+}{\tilde u(x)}, \quad \tilde u(x) < 0, \quad u_+ < 0. \tag{1.11}$$

Thus, (1.10) is equivalent to the coupling of (1.11) and the following ODE system:

$$\begin{cases} \partial_x\tilde u = \dfrac{\rho_+ u_+}{\lambda}\left[(\tilde u - u_+) + R\left(\dfrac{\tilde\theta}{\tilde u} - \dfrac{\theta_+}{u_+}\right)\right], \quad x \in \mathbb{R}_+, \\ \partial_x\tilde\theta = \dfrac{\rho_+ u_+}{\kappa}\left[\dfrac{R}{\gamma - 1}(\tilde\theta - \theta_+) - \dfrac{1}{2}(\tilde u - u_+)^2 + \dfrac{R\theta_+}{u_+}(\tilde u - u_+)\right], \quad x \in \mathbb{R}_+, \\ \tilde u(0) = u_-, \quad \tilde\theta(0) = \theta_-, \quad (\tilde u, \tilde\theta)(+\infty) = (u_+, \theta_+). \end{cases} \tag{1.12}$$

The strength of the stationary solutions is measured by $\tilde\delta = |u_+ - u_-| + |\theta_+ - \theta_-|$.

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We define the pressure in the far field: $p_+ := R\rho_+\theta_+$. We also introduce the Mach number M_+ in the far field $x = +\infty$: $M_+ = ((|u_+|)/(c_+))$, where $c_+ = \sqrt{R\gamma\theta_+}$ is the sound speed. Then one has the following lemmas.

LEMMA 1.1. *Suppose that the boundary data* (u_-, θ_-) *satisfies*

$$(u_-, \theta_-) \in \mathcal{M}^+ := \left\{(u, \theta) \in \mathbb{R}^2; |(u - u_+, \theta - \theta_+)| < \delta_0\right\} \tag{1.13}$$

for a certain positive constant δ_0. *Notice that* (1.13) *is equivalent to the inequality* $\tilde{\delta} < \delta_0$.

(i) *For the supersonic case* $M_+ > 1$, *there exist unique smooth solutions* $(\tilde{u}, \tilde{\theta})(x)$ *to the problem* (1.12) *satisfying*

$$|\partial_x^k(\tilde{u}(x) - u_+, \tilde{\theta}(x) - \theta_+)| \leqslant C\tilde{\delta}e^{-cx}, \quad k = 0,\ 1,\ 2, \ldots, \tag{1.14}$$

where c *and* C *are positive constants.*

(ii) *For the transonic case* $M_+ = 1$, *if the boundary data* $(u_-, \theta_-) \in \mathcal{M}^0$ *which is defined in* (A.19), *then there exist unique smooth solutions* $(\tilde{u}, \tilde{\theta})(x)$ *to the problem* (1.12) *satisfying*

$$|\partial_x^k(\tilde{u}(x) - u_+, \tilde{\theta}(x) - \theta_+)| \leqslant C\frac{\tilde{\delta}^{k+1}}{(1 + \tilde{\delta}x)^{k+1}} + C\tilde{\delta}e^{-cx}, \quad k = 0,\ 1,\ 2, \ldots \tag{1.15}$$

(iii) *For the subsonic case* $M_+ < 1$, *if the boundary data* $(u_-, \theta_-) \in \mathcal{M}^-$ *which is defined in* (A.20), *then there exist unique smooth solutions* $(\tilde{u}, \tilde{\theta})(x)$ *to the problem* (1.12) *satisfying* (1.14).

LEMMA 1.2. *Suppose that* $M_+ = 1$. *Namely, the same conditions as in lemma 1.1 (ii) hold. Then the degenerate stationary solutions* $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$ *satisfy*

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) = (\rho_+, u_+, \theta_+) + \left(-\frac{\rho_+}{\theta_+(\gamma - 1)}, \frac{u_+}{\theta_+(\gamma - 1)}, -1\right)\tilde{z}(x) + O(\tilde{z}^2 + \tilde{\delta}e^{-cx})(1, 1, 1), \tag{1.16}$$

$$(\tilde{u}_x, \tilde{\theta}_x) = \frac{\gamma^2(\gamma + 1)R^2\rho_+}{2(\gamma - 1)[\lambda\gamma R + \kappa(\gamma - 1)^2]u_+}\left(\frac{u_+}{\theta_+(\gamma - 1)}, -1\right)\tilde{z}^2(x) + O(\tilde{z}^3 + \tilde{\delta}e^{-cx})(1, 1), \tag{1.17}$$

$$|\partial_x^k(\tilde{u}, \tilde{\theta})| \leqslant C\tilde{z}^{k+1}(x) + C\tilde{\delta}e^{-cx}, \quad k = 1,\ 2, \ldots \tag{1.18}$$

REMARK. $\tilde{z}(x)$ is defined in (A.11). For the sake of completeness, the detailed proofs of lemmas 1.1 and 1.2 are given in the appendix.

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1.3. The main result

Before stating our main result, we first introduce the following notation. A norm with algebraic weight is defined as follows:

$$\|f\|_{\alpha,\xi,i} := \left(\int W_{\alpha,\xi} \sum_{j\leqslant i} (\partial^j f)^2 \mathrm{d}x\right)^{1/2}, \quad i,j\in\mathbb{Z}, \quad i,j\geqslant 0,$$

$$W_{\alpha,\xi} := (1+\xi x)^{\alpha}, \quad \alpha > 0.$$

Note that this norm is equivalent to the norm defined by $\|(1+\xi x)^{\alpha/2} f\|_i$. The last subscript i is often dropped for the case of $i=0$, that is, $\|f\|_{\alpha,\xi} := \|f\|_{\alpha,\xi,0}$.

The main result of our paper is stated as follows.

THEOREM 1.3. *Suppose that the stationary solutions* $(\tilde\rho, \tilde u, \tilde{\mathbf{w}} = \tilde{\mathbf{b}} = \mathbf{0}, \tilde\theta)$ *exist.*

(i) *Assume that* $M_+ > 1$ *and* $p_+ > 1/\gamma$ *hold. For an arbitrary positive constant* $\hat\lambda$*, there exist positive constants* β *and* ε_0 *such that if* $(1+\beta x)^{\hat\lambda/2}(\rho_0-\tilde\rho)$, $(1+\beta x)^{\hat\lambda/2}(u_0-\tilde u), (1+\beta x)^{\hat\lambda/2}\mathbf{w}_0, (1+\beta x)^{\hat\lambda/2}\mathbf{b}_0, (1+\beta x)^{\hat\lambda/2}(\theta_0-\tilde\theta)$ *respectively belongs to the Lebesgue space* $L^2(\mathbb{R}_+)$ *and* $\|[\rho_0-\tilde\rho, u_0-\tilde u, \mathbf{w}_0, \mathbf{b}_0, \theta_0-\tilde\theta]\|_{H^1} + \beta + \tilde\delta \leqslant \varepsilon_0$*, then the outflow problem* (1.1)*,* (1.3)*,* (1.4)*,* (1.5)*,* (1.6) *has unique solutions* $[\rho, u, \mathbf{w}, \mathbf{b}, \theta]$ *verifying the decay estimate*

$$\|[\rho-\tilde\rho, u-\tilde u, \mathbf{w}, \mathbf{b}, \theta-\tilde\theta](t)\|_\infty \leqslant C(1+t)^{-\hat\lambda/2}. \tag{1.19}$$

(ii) *Assume that* $M_+ < 1$ *holds. There exists a positive constant* ε_0 *such that if* $\|[\rho_0-\tilde\rho, u_0-\tilde u, \mathbf{w}_0, \mathbf{b}_0, \theta_0-\tilde\theta]\|_{H^1} + \tilde\delta \leqslant \varepsilon_0$*, the outflow problem* (1.1)*,* (1.3)*,* (1.4)*,* (1.5)*,* (1.6) *has unique solutions* $[\rho, u, \mathbf{w}, \mathbf{b}, \theta]$ *verifying the decay estimate*

$$\|[\rho-\tilde\rho, u-\tilde u, \mathbf{w}, \mathbf{b}, \theta-\tilde\theta](t)\|_{H^1} \leqslant C\varepsilon_0. \tag{1.20}$$

Moreover, the solutions $[\rho, u, \mathbf{w}, \mathbf{b}, \theta]$ *converge to the stationary solutions* $[\tilde\rho, \tilde u, \mathbf{0}, \mathbf{0}, \tilde\theta]$ *uniformly as time tends to infinity:*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}_+} \left|[\rho, u, \mathbf{w}, \mathbf{b}, \theta](x,t) - [\tilde\rho, \tilde u, \mathbf{0}, \mathbf{0}, \tilde\theta](x)\right| = 0. \tag{1.21}$$

(iii) *Assume that* $M_+ = 1$ *and* $p_+ > 1/\gamma$ *hold. For some positive constant* $1 \leqslant \hat\lambda < 2(1+\sqrt{2})$*, there exists a positive constant* ε_0 *such that if* $(1+\tilde\delta x)^{\hat\lambda/2} (\rho_0-\tilde\rho), (1+\tilde\delta x)^{\hat\lambda/2}(u_0-\tilde u), (1+\tilde\delta x)^{\hat\lambda/2}\mathbf{w}_0, (1+\tilde\delta x)^{\hat\lambda/2}\mathbf{b}_0, (1+\tilde\delta x)^{\hat\lambda/2}(\theta_0-\tilde\theta)$ *respectively belongs to the Sobolev space* $H^1(\mathbb{R}_+)$ *and* $\tilde\delta^{-1/2}\|[\rho_0-\tilde\rho, u_0-\tilde u, \mathbf{w}_0, \mathbf{b}_0, \theta_0-\tilde\theta]\|_{\hat\lambda,\tilde\delta,1} + \tilde\delta \leqslant \varepsilon_0$*, then the outflow problem* (1.1)*,* (1.3)*,* (1.4)*,* (1.5)*,* (1.6) *has unique solutions* $[\rho, u, \mathbf{w}, \mathbf{b}, \theta]$ *verifying the decay estimate*

$$\|[\rho-\tilde\rho, u-\tilde u, \mathbf{w}, \mathbf{b}, \theta-\tilde\theta](t)\|_\infty \leqslant C(1+t)^{-\hat\lambda/4}. \tag{1.22}$$

Remark. For the supersonic case $M_+ > 1$ and transonic case $M_+ = 1$, we prove that an exponential convergence rate

$$\|[\rho - \tilde{\rho}, u - \tilde{u}, \mathbf{w}, \mathbf{b}, \theta - \tilde{\theta}](t)\|_\infty \leqslant Ce^{-\hat{\lambda} t} \tag{1.23}$$

holds provided that the initial data satisfies the conditions as in theorem 1.3 (i) and (iii) with the exponential weight function $e^{-\hat{\lambda} x}$ instead of the algebraic weight function $(1 + \xi x)^\alpha$. Since the estimates for the exponential weight function are easier than that for the algebraic weight function, we only prove theorem 1.3 (i) and (iii) for the algebraic weight function in the sequel.

There have been a lot of studies on MHD equations by physicists and mathematicians because of their physical importance, complexity, rich phenomenology, and mathematical challenges; see [**2**–**8**, **14**, **16**, **24**–**26**] and the references cited therein. Here we only listed some related paper. For the initial boundary value problem, we refer to [**3**, **4**, **25**] for the global existence of large solutions of non-isentropic planar MHD equations with a special structure. In [**8**], Hu and Wang studied the global weak solutions to the three-dimensional MHD equations with large initial data, and investigated the fundamental problems such as global existence and large-time behaviour. Lv and Huang in [**16**] studied strong solutions to the Cauchy problem of the two-dimensional compressible MHD equations with vacuum. See [**6**] and [**26**] for some interesting results on the vanishing shear viscosity limit for the isentropic or non-isentropic planar MHD equations with special structures. Li, Xu and Zhang in [**14**] proved the global well-posedness of a classical solution of the Cauchy problem of three-dimensional isentropic compressible MHD equations, where the flow density is allowed to contain vacuum states. The authors in [**2**] and [**24**] proved the global existence of smooth solutions near the constant states for Cauchy problem to the three-dimensional isentropic or non-isentropic compressible MHD equations by energy method and meanwhile obtained convergence rates of the L^p-norm of these solutions to the constant states.

In fact, equations (1.1) reduce to the classical Navier-Stokes equations if we ignore the effect of the magnetic field. As far as we know, there have been a great number of mathematical studies about the outflow problem, impermeable wall problem and inflow problem on Navier-Stokes equations, please see [**9**, **12**, **18**–**21**] and the references therein. Three problems mentioned above are still important topics in the theory of fluid dynamics and plasma physics, for example, see [**10**, **27**]. Hence, it is important and meaningful for us to study the corresponding problem for MHD equations. Here, in this paper, we only discuss the outflow problem for MHD equations. The other two problems remain to be discussed in future. In this paper, we obtain convergence rates of solutions towards nontrivial stationary solutions by employing the weighted energy method, provided that the initial perturbation belongs to the weighted Sobolev space. According to our knowledge, this paper is the first result in this direction. It should be pointed out that the outflow problem is divided into three cases for discussion according to the value of Mach number M in the far field, that is, M_+. Compared with [**13**] for compressible Navier-Stokes equations, the outflow problem for compressible MHD equations is more complicated. Due to the strong interaction between the fluid motion and the magnetic field, the main trouble arising in this paper is that we must deal with the coupled term $\mathbf{w}\cdot\mathbf{b}$ under an extra assumption $p_+ > 1/\gamma$ for both supersonic case $M_+ > 1$ and transonic case $M_+ = 1$. One can see (2.12) and (2.51) for details. So far it is unclear how to remove such restriction for the stability of stationary solutions on MHD equations. Moreover, it is also interesting to obtain the convergence rates of solutions towards stationary solutions for the outflow problem of the planar MHD in the current setting.

The rest of the paper is arranged as follows. In the main part § 2, we give the *a priori* estimates on the solutions of the perturbative equations for the supersonic case $M_+ > 1$, subsonic case $M_+ < 1$ and transonic case $M_+ = 1$, respectively. The proof of theorem 1.3 is concluded in § 3. In the Appendix, we present the detailed proofs of lemmas 1.1 and 1.2 for completeness of the paper.

NOTATION. *Throughout the paper, we denote positive constants (generally large) and (generally small) independent of* t *by* C *and* c, *respectively. And the character '*C*' and '*c*' may take different values in different places.* $L^p = L^p(\mathbb{R}_+)$ $(1 \leqslant p \leqslant \infty)$ *denotes the usual Lebesgue space on* $[0,\infty)$ *with its norm* $\|\cdot\|_{L^p}$, *and when* $p = 2$, *we write* $\|\cdot\|_{L^2(\mathbb{R}_+)} = \|\cdot\|$. $H^s(\mathbb{R}_+)$ *denotes the usual* s*-th order Sobolev space with its norm* $\|f\|_{H^s(\mathbb{R}_+)} = \|f\|_s = (\sum_{i=0}^{s}\|\partial^i f\|^2)^{1/2}$.

2. Energy estimates

To prove theorem 1.3, we use the energy method. Define the perturbation as

$$[\varphi,\psi,\zeta](x,t) = [\rho-\tilde{\rho}, u-\tilde{u}, \theta-\tilde{\theta}],$$

then $[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](x,t)$ satisfies

$$\begin{cases} \partial_t\varphi + u\partial_x\varphi + \rho\partial_x\psi = -\partial_x\tilde{u}\varphi - \partial_x\tilde{\rho}\psi, \\ \rho(\partial_t\psi + u\partial_x\psi) + R\partial_x(\rho\theta - \tilde{\rho}\tilde{\theta}) + \partial_x(\frac{1}{2}|\mathbf{b}|^2) = \lambda\partial_x^2\psi - \partial_x\tilde{u}(\tilde{u}\varphi + \rho\psi), \\ \rho(\partial_t\mathbf{w} + u\partial_x\mathbf{w}) - \partial_x\mathbf{b} = \mu\partial_x^2\mathbf{w}, \\ \partial_t\mathbf{b} + \partial_x(u\mathbf{b} - \mathbf{w}) = \nu\partial_x^2\mathbf{b}, \\ \dfrac{R}{\gamma-1}\rho(\partial_t\zeta + u\partial_x\zeta) + R\rho\theta\partial_x\psi = \kappa\partial_x^2\zeta + \lambda(\partial_x\psi)^2 \\ -\dfrac{R}{\gamma-1}\partial_x\tilde{\theta}(\tilde{u}\varphi + \rho\psi) - \partial_x\tilde{u}R(\rho\theta - \tilde{\rho}\tilde{\theta}) \\ +2\lambda\partial_x\tilde{u}\partial_x\psi + \nu|\partial_x\mathbf{b}|^2 + \mu|\partial_x\mathbf{w}|^2 \end{cases} \tag{2.1}$$

with the initial data

$$[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](x,0) = [\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0](x) \tag{2.2}$$

and the boundary condition

$$\psi(0,t) = \zeta(0,t) = 0, \quad \mathbf{w}(0,t) = \mathbf{b}(0,t) = \mathbf{0}. \tag{2.3}$$

In the paper, to prove theorem 1.3, for brevity, we only devote ourselves to obtaining the global-in-time *a priori* estimates in the following.

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Lemma 2.1 plays an important role in the proof of the *a priori* estimates for the supersonic case $M_+ > 1$, subsonic case $M_+ < 1$ and transonic case $M_+ = 1$, respectively.

LEMMA 2.1.

(i) *For any function* $h(\cdot,t) \in H^1(\mathbb{R}_+)$, *there is a positive constant* C *such that*

$$\int_{\mathbb{R}_+} e^{-cx}|h|^2\mathrm{d}x \leqslant C(h^2(0,t) + \|\partial_x h(t)\|^2). \tag{2.4}$$

(ii) *Let* $\hat{\nu} \geqslant 1$. *For a function* $h(x,t)$ *satisfying* $(1+\tilde{\delta}x)^{\hat{\nu}/2}h$ *and* $(1+\tilde{\delta}x)^{\hat{\nu}/2}\partial_x h \in L^2(\mathbb{R}_+)$, *we have*

$$\int_{\mathbb{R}_+} (1+\tilde{\delta}x)^{\hat{\nu}-1}|h|^3\mathrm{d}x \leqslant C\tilde{\delta}^{-\hat{\nu}}\tilde{\delta}^{-1/2}\|h\|_{\hat{\nu},\tilde{\delta}}[h^2(0,t) + \tilde{\delta}^2\|h\|^2_{\hat{\nu}-2,\tilde{\delta}} + \|\partial_x h\|^2_{\hat{\nu},\tilde{\delta}}]. \tag{2.5}$$

Proof.

(i) (2.4) can be derived from the following Poincaré type inequality:

$$|h(x,t)| \leqslant |h(0,t)| + x^{1/2}\|\partial_x h\|. \tag{2.6}$$

(ii) Letting $\hat{\nu} \geqslant 1$ and using Hölder inequality, we compute as

$$\begin{aligned}(1+\tilde{\delta}x)^{\hat{\nu}}h - h(0,t) &= \int_0^x \partial_y[(1+\tilde{\delta}y)^{\hat{\nu}}h]\mathrm{d}y = \int_0^x (1+\tilde{\delta}y)^{\hat{\nu}}\partial_y h\mathrm{d}y \\ &\quad + \hat{\nu}\tilde{\delta}\int_0^x (1+\tilde{\delta}y)^{\hat{\nu}-1}h\mathrm{d}y \\ &\leqslant \left(\int_0^x (1+\tilde{\delta}y)\right)^{\hat{\nu}} |\partial_y h|^2\mathrm{d}y)^{1/2}\left(\int_0^x (1+\tilde{\delta}y)^{\hat{\nu}}\mathrm{d}y\right)^{1/2} \\ &\quad + \hat{\nu}\tilde{\delta}\left(\int_0^x (1+\tilde{\delta}y)\right)^{\hat{\nu}-2} h^2\mathrm{d}y)^{1/2}\left(\int_0^x (1+\tilde{\delta}y)^{\hat{\nu}}\mathrm{d}y\right)^{1/2} \\ &\leqslant C\tilde{\delta}^{-1/2}(1+\tilde{\delta}x)^{\hat{\nu}/2+1/2}(\|\partial_x h\|_{\hat{\nu},\tilde{\delta}} + \tilde{\delta}\|h\|_{\hat{\nu}-2,\tilde{\delta}}).\end{aligned}$$

Thus we have

$$\begin{aligned}(1+\tilde{\delta}x)^{((\hat{\nu}-1)/(2))}h &\leqslant (1+\tilde{\delta}x)^{-\hat{\nu}/2-1/2}h(0,t) \\ &\quad + C\tilde{\delta}^{-1/2}(\|\partial_x h\|_{\hat{\nu},\tilde{\delta}} + \tilde{\delta}\|h\|_{\hat{\nu}-2,\tilde{\delta}}).\end{aligned} \tag{2.7}$$

Using (2.7) and the Hölder inequality, we can see

$$\begin{aligned}&\int_{\mathbb{R}_+} (1+\tilde{\delta}x)^{\hat{\nu}-1}|h|^3\mathrm{d}x \\ &\leqslant h(0,t)\int_{\mathbb{R}_+} (1+\tilde{\delta}x)^{-1}h^2\mathrm{d}x + C\tilde{\delta}^{-1/2}(\|\partial_x h\|_{\hat{\nu},\tilde{\delta}} \\ &\quad + \tilde{\delta}\|h\|_{\hat{\nu}-2,\tilde{\delta}})\|h\|_{1,\tilde{\delta}}\|h\|_{\hat{\nu}-2,\tilde{\delta}}\end{aligned}$$

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$$
\begin{aligned}
&\leqslant h(0,t)\|h\|_{1,\tilde{\delta}}\left(\int_{\mathbb{R}_+}(1+\tilde{\delta}x)\right)^{-3}h^2\mathrm{d}x)^{1/2}\\
&\quad+C\tilde{\delta}^{-3/2}\|h\|_{1,\tilde{\delta}}[\tilde{\delta}^2\|h\|^2_{\hat{\nu}-2,\tilde{\delta}}+\|\partial_x h\|^2_{\hat{\nu},\tilde{\delta}}]\\
&\leqslant C\tilde{\delta}^{3/2}\|h\|_{1,\tilde{\delta}}\int_{\mathbb{R}_+}(1+\tilde{\delta}x)^{-3}h^2\mathrm{d}x\\
&\quad+C\tilde{\delta}^{-3/2}\|h\|_{1,\tilde{\delta}}[h^2(0,t)+\tilde{\delta}^2\|h\|^2_{\hat{\nu}-2,\tilde{\delta}}+\|\partial_x h\|^2_{\hat{\nu},\tilde{\delta}}]\\
&\leqslant C\tilde{\delta}^{-3/2}\|h\|_{1,\tilde{\delta}}[h^2(0,t)+\tilde{\delta}^2\|h\|^2_{\hat{\nu}-2,\tilde{\delta}}+\|\partial_x h\|^2_{\hat{\nu},\tilde{\delta}}]\\
&\leqslant C\tilde{\delta}^{-\nu}\tilde{\delta}^{-1/2}\|h\|_{\hat{\nu},\tilde{\delta}}[h^2(0,t)+\tilde{\delta}^2\|h\|^2_{\hat{\nu}-2,\tilde{\delta}}+\|\partial_x h\|^2_{\hat{\nu},\tilde{\delta}}],
\end{aligned}
$$

where we have used $-3\leqslant\hat{\nu}-2$ and $\tilde{\delta}^{3/2}\leqslant\tilde{\delta}^{1/2}$ in the fourth inequality and $\hat{\nu}\geqslant 1$ in the last inequality. Thus we complete the proof of lemma 2.1 (ii).

□

REMARK. In order to estimate the last term $\int_{\mathbb{R}_+}\tilde{z}^{-\hat{\nu}+1}(|\varphi|^3+|\psi|^3+|\zeta|^3)\mathrm{d}x$ in (2.54), we need the smallness of $\tilde{\delta}^{-1/2}\|h\|_{\hat{\nu},\tilde{\delta}}$. Moreover, we need the condition $\hat{\lambda}\geqslant 1$ in theorem 1.3 (iii).

2.1. The a priori estimates for $M_+>1$

The key to the proof of our main theorem 1.3 (i) is to derive the uniform *a priori* estimates of solutions to the initial boundary value problem (2.1), (2.2) and (2.3).

PROPOSITION 2.2. *Assume the same conditions as in theorem 1.3(i) hold. Let $\hat{\lambda}$ and $\hat{\kappa}$ be the positive constants. Suppose $[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta]$ is a solution to (2.1), (2.2) and (2.3) which satisfies $(1+\beta x)^{\hat{\lambda}/2}\varphi,(1+\beta x)^{\hat{\lambda}/2}\psi,(1+\beta x)^{\hat{\lambda}/2}\mathbf{w},(1+\beta x)^{\hat{\lambda}/2}\mathbf{b}$, $(1+\beta x)^{\hat{\lambda}/2}\zeta\in C([0,T];L^2(\mathbb{R}_+))$ for a certain positive constant T. For arbitrary $\hat{\nu}\in[0,\hat{\lambda}]$, there exist positive constants C and ε_1 independent of T such that if*

$$\sup_{0\leqslant t\leqslant T}\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|_1+\tilde{\delta}+\beta\leqslant\varepsilon_1 \tag{2.8}$$

is satisfied, it holds for an arbitrary $t\in[0,T]$ that

$$
\begin{aligned}
&(1+t)^{\hat{\lambda}-\hat{\nu}+\hat{\kappa}}\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|_1^2+\int_0^t(1+\tau)^{\hat{\lambda}-\hat{\nu}+\hat{\kappa}}\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|_1^2\mathrm{d}\tau\\
&\leqslant C(1+t)^{\hat{\kappa}}(\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|_1^2+\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat{\lambda},\beta}).
\end{aligned}\tag{2.9}
$$

Now, we prove proposition 2.2 by the following three steps.

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Step 1. The zero-order energy estimates.

Set $\Phi(s) = s - 1 - \ln s$, and define $\eta = R\rho\tilde{\theta}\Phi(\tilde{\rho}/\rho) + \rho/2\psi^2 + \rho/2|\mathbf{w}|^2 + 1/2$ $|\mathbf{b}|^2 + ((R)/(\gamma-1))\rho\tilde{\theta}\Phi(\theta/\tilde{\theta})$. Direct calculations give rise to

$$\begin{aligned}\partial_t\eta + \lambda\frac{\tilde{\theta}}{\theta}(\partial_x\psi)^2 + \kappa\frac{\tilde{\theta}}{\theta^2}(\partial_x\zeta)^2 + \nu|\partial_x\mathbf{b}|^2 + \mu|\partial_x\mathbf{w}|^2 + \partial_x H_1 \\ = \partial_x H_2 + \partial_x\tilde{u}Q_1 + \partial_x\tilde{\theta}Q_2 + Q_3,\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}H_1 &= u\eta + R(\rho\theta - \tilde{\rho}\tilde{\theta})\psi + \frac{1}{2}|\mathbf{b}|^2\psi - \mathbf{w}\cdot\mathbf{b},\\ H_2 &= \lambda\psi\partial_x\psi + \kappa\frac{\zeta\partial_x\zeta}{\theta} + \mu\mathbf{w}\cdot\partial_x\mathbf{w} + \nu\mathbf{b}\cdot\partial_x\mathbf{b},\end{aligned}$$

$$Q_1 = \left(\frac{R\tilde{\theta}}{\tilde{u}} - \tilde{u}\right)\varphi\psi - \rho\psi^2 - \frac{1}{2}|\mathbf{b}|^2 - R(\rho\theta - \tilde{\rho}\tilde{\theta})\frac{\zeta}{\theta},$$

$$Q_2 = R\rho u\Phi\left(\frac{\tilde{\rho}}{\rho}\right) - \frac{R}{\gamma-1}\rho u\Phi\left(\frac{\tilde{\theta}}{\theta}\right) - \frac{R}{\gamma-1}(\tilde{u}\varphi + \rho\psi)\frac{\zeta}{\theta},$$

and

$$Q_3 = \frac{2\lambda}{\theta}\partial_x\tilde{u}\partial_x\psi\zeta + \kappa\partial_x\tilde{\theta}\frac{\zeta\partial_x\zeta}{\theta^2} + \nu\frac{\zeta}{\theta}|\partial_x\mathbf{b}|^2 + \mu\frac{\zeta}{\theta}|\partial_x\mathbf{w}|^2.$$

Multiplying (2.10) by $W_{\hat{\nu},\beta}$, then we integrate the resulting equality over $\mathbb{R}_+$ to get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}_+} W_{\hat{\nu},\beta}\eta \mathrm{d}x + \lambda\int_{\mathbb{R}_+} W_{\hat{\nu},\beta}\frac{\tilde{\theta}}{\theta}(\partial_x\psi)^2\mathrm{d}x + \kappa\int_{\mathbb{R}_+} W_{\hat{\nu},\beta}\frac{\tilde{\theta}}{\theta^2}(\partial_x\zeta)^2\mathrm{d}x \\ &\quad + \int_{\mathbb{R}_+} W_{\hat{\nu},\beta}[\nu|\partial_x\mathbf{b}|^2 + \mu|\partial_x\mathbf{w}|^2]\mathrm{d}x + R\rho(0,t)\theta_b|u_b|\phi \\ &\quad \times\left(\frac{\tilde{\rho}}{\rho}\right)(0,t) - \hat{\nu}\beta\int_{\mathbb{R}_+} W_{\hat{\nu}-1,\beta}H_1\mathrm{d}x \\ &= \underbrace{-\hat{\nu}\beta\int_{\mathbb{R}_+} W_{\hat{\nu}-1,\beta}H_2\mathrm{d}x}_{J_1} + \underbrace{\int_{\mathbb{R}_+} W_{\hat{\nu},\beta}Q_3\mathrm{d}x}_{J_2} + \underbrace{\int_{\mathbb{R}_+} W_{\hat{\nu},\beta}(\partial_x\tilde{u}Q_1 + \partial_x\tilde{\theta}Q_2)\mathrm{d}x}_{J_3}.\end{aligned} \tag{2.11}$$

Now we estimate each term in (2.11). We decompose ρ as $\rho = \varphi + (\tilde{\rho} - \rho_+) + \rho_+$, u as $u = \psi + (\tilde{u} - u_+) + u_+$ and θ as $\theta = \zeta + (\tilde{\theta} - \theta_+) + \theta_+$. Then we see, under the condition $M_+ > 1$ and $u_+ < 0$, that

$$
\begin{aligned}
&-u\eta - R(\rho\theta - \tilde{\rho}\tilde{\theta})\psi - \frac{1}{2}|\mathbf{b}|^2\psi + \mathbf{w}\cdot\mathbf{b} \\
&\geqslant [-\frac{R\theta_+u_+}{2\rho_+}\varphi^2 - \frac{\rho_+u_+}{2}\psi^2 - \frac{R\rho_+u_+}{2(\gamma-1)\theta_+}\zeta^2 - R\theta_+\varphi\psi - R\rho_+\zeta\psi] - \frac{\rho_+u_+}{2}|\mathbf{w}|^2 \\
&\quad - \frac{u_+}{2}|\mathbf{b}|^2 + \mathbf{w}\cdot\mathbf{b} - C(\varepsilon_1+\tilde{\delta})(\varphi^2+\psi^2+|\mathbf{b}|^2+|\mathbf{w}|^2+\zeta^2) \\
&= [\varphi,\psi,\zeta]M_1[\varphi,\psi,\zeta]^T + [\mathbf{w},\mathbf{b}]M_2[\mathbf{w},\mathbf{b}]^T - C(\varepsilon_1+\tilde{\delta})(\varphi^2+\psi^2+|\mathbf{b}|^2+|\mathbf{w}|^2+\zeta^2),
\end{aligned}
\tag{2.12}
$$

where $[\,]^T$ denotes the transpose of a row vector. The 3×3 real symmetric matrix M_1 and 2×2 real symmetric matrix M_2 are respectively given by

$$
\begin{pmatrix}
-\frac{R\theta_+u_+}{2\rho_+} & -\frac{R\theta_+}{2} & 0 \\
-\frac{R\theta_+}{2} & -\frac{\rho_+u_+}{2} & -\frac{R\rho_+}{2} \\
0 & -\frac{R\rho_+}{2} & -\frac{R\rho_+u_+}{2(\gamma-1)\theta_+}
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
-\frac{\rho_+u_+}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{u_+}{2}
\end{pmatrix}.
$$

One can compute all the leading principal minors Δ_{ll} $(1 \leqslant l \leqslant 3)$ of M_1 as follows:

$$
\Delta_{11} = -\frac{R\theta_+u_+}{2\rho_+} > 0, \Delta_{22} = \frac{R\theta_+}{4}(u_+^2 - R\theta_+) > 0,
$$

and

$$
\Delta_{33} = -\frac{R^2\rho_+u_+}{8(\gamma-1)}[u_+^2 - \gamma R\theta_+] > 0,
$$

where we have used the condition $M_+ > 1$ and $u_+ < 0$.

Similarly, we can get all the leading principal minors Δ_{ll} $(1 \leqslant l \leqslant 2)$ of M_2 as follows:

$$
\Delta_{11} = -\frac{\rho_+u_+}{2} > 0, \quad \Delta_{22} = \frac{\rho_+u_+^2 - 1}{4} > \frac{\gamma p_+ - 1}{4} > 0,
$$

where we have used the condition $M_+ > 1$, $u_+ < 0$ and $p_+ > 1/\gamma$.

Thus we have

$$
-\hat{\nu}\beta\int_{\mathbb{R}_+} W_{\hat{\nu}-1,\beta}H_1\mathrm{d}x \geqslant c\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta]\|_{\hat{\nu}-1,\beta}^2,
$$

where we take ε_1 and $\tilde{\delta}$ small enough.

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It is easy to obtain that

$$R\rho(0,t)\theta_b|u_b|\phi\left(\frac{\tilde{\rho}}{\rho}\right)(0,t)\geqslant c\varphi(0,t)^2.$$

Using lemma 2.1(i), (2.8), (2.3) and the Sobolev inequality, we have

$$|J_1|\leqslant c\beta\|[\psi,\mathbf{w},\mathbf{b},\zeta]\|^2_{\hat{\nu}-1,\beta}+c\beta\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta]\|^2_{\hat{\nu}-1,\beta},$$

$$|J_2|\leqslant C\tilde{\delta}\|\zeta\|^2_{\hat{\nu}-1,\beta}+C\tilde{\delta}\|\partial_x[\psi,\zeta]\|^2_{\hat{\nu}-1,\beta}+C\varepsilon_1\|\partial_x[\mathbf{w},\mathbf{b}]\|^2_{\hat{\nu},\beta},$$

$$|J_3|\leqslant C\tilde{\delta}\varphi(0,t)^2+C\tilde{\delta}\|\partial_x[\varphi,\psi,\mathbf{b},\zeta]\|^2.$$

Inserting the above estimations into (2.11) and then choosing $\varepsilon_1,\tilde{\delta}$ and β suitably small, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}W_{\hat{\nu},\beta}\eta\mathrm{d}x+c\varphi(0,t)^2+c\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta]\|^2_{\hat{\nu}-1,\beta}&\\ +c\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta]\|^2_{\hat{\nu},\beta}\leqslant C\tilde{\delta}\|\partial_x\varphi\|^2.&\qquad(2.13)\end{aligned}$$

Multiplying (2.13) by $(1+t)^\xi$ and integrating in τ over $[0,t]$ for any $0\leqslant t\leqslant T$, we have

$$\begin{aligned}&(1+t)^\xi\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|^2_{\hat{\nu},\beta}+\int_0^t(1+\tau)^\xi\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu}-1,\beta}\mathrm{d}\tau\\ &\quad+\int_0^t(1+\tau)^\xi\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu},\beta}\mathrm{d}\tau+\int_0^t(1+\tau)^\xi\varphi^2(0,\tau)\mathrm{d}\tau\\ &\leqslant C\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat{\lambda},\beta}+\xi\int_0^t(1+\tau)^{\xi-1}\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu},\beta}\mathrm{d}\tau\\ &\quad+C\tilde{\delta}\int_0^t(1+\tau)^\xi\|\partial_x\varphi\|^2(\tau)\mathrm{d}\tau.\qquad(2.14)\end{aligned}$$

Step 2. *Dissipation of* $\|\partial_x\varphi\|^2$.

We first differentiate $(2.1)_1$ with respect to x, multiplying the resulting equations and $(2.1)_2$ by $\Lambda((\partial_x\varphi)/(\rho^2))$ and $((\partial_x\varphi)/(\rho))$ respectively to obtain

$$\begin{aligned}&\lambda\frac{\partial_x\varphi}{\rho^2}\partial_t\partial_x\varphi+\lambda\partial_xu\frac{(\partial_x\varphi)^2}{\rho^2}+\lambda u\frac{\partial_x\varphi\partial_x^2\varphi}{\rho^2}+\lambda\frac{\partial_x\varphi}{\rho^2}\partial_x\rho\partial_x\psi+\lambda\partial_x\tilde{u}\frac{(\partial_x\varphi)^2}{\rho^2}\\ &\quad=-\lambda\partial_x^2\psi\frac{\partial_x\varphi}{\rho}-\lambda\partial_x^2\tilde{u}\varphi\frac{\partial_x\varphi}{\rho^2}-\lambda\partial_x\tilde{\rho}\partial_x\psi\frac{\partial_x\varphi}{\rho^2}-\lambda\partial_x^2\tilde{\rho}\psi\frac{\partial_x\varphi}{\rho^2},\qquad(2.15)\end{aligned}$$

$$\begin{aligned}&\partial_t\psi\partial_x\varphi+u\partial_x\psi\partial_x\varphi+R\frac{\partial_x(\rho\theta-\tilde{\rho}\tilde{\theta})}{\rho}\partial_x\varphi+\frac{\partial_x|\mathbf{b}|^2}{2\rho}\partial_x\varphi\\ &\quad=\lambda\partial_x^2\psi\frac{\partial_x\varphi}{\rho}-\frac{\tilde{u}\partial_x\tilde{u}}{\rho}\varphi\partial_x\varphi-\partial_x\tilde{u}\psi\partial_x\varphi.\qquad(2.16)\end{aligned}$$

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The summation of (2.15) and (2.16), and then taking integration over $\mathbb{R}_+$ further imply

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left[\psi\partial_x\varphi+\lambda\frac{(\partial_x\varphi)^2}{2\rho^2}\right]\mathrm{d}x+\int_{\mathbb{R}_+}\left[\lambda\partial_x\tilde{u}\frac{(\partial_x\varphi)^2}{\rho^2}+\frac{R\theta}{\rho}(\partial_x\varphi)^2\right]\mathrm{d}x\\
&=\int_{\mathbb{R}_+}\psi\partial_t\partial_x\varphi\mathrm{d}x-\lambda\int_{\mathbb{R}_+}(\partial_x\varphi)^2\rho^{-3}\partial_t\rho\mathrm{d}x-\int_{\mathbb{R}_+}\frac{R\tilde{\rho}}{\rho}\partial_x\zeta\partial_x\varphi\mathrm{d}x-\int_{\mathbb{R}_+}R\varphi\frac{\partial_x\theta}{\rho}\partial_x\varphi\mathrm{d}x\\
&\quad-\int_{\mathbb{R}_+}R\zeta\frac{\partial_x\tilde{\rho}}{\rho}\partial_x\varphi\mathrm{d}x-\int_{\mathbb{R}_+}u\partial_x\psi\partial_x\varphi\mathrm{d}x-\int_{\mathbb{R}_+}\frac{\tilde{u}\partial_x\tilde{u}}{\rho}\varphi\partial_x\varphi\mathrm{d}x-\int_{\mathbb{R}_+}\partial_x\tilde{u}\psi\partial_x\varphi\mathrm{d}x\\
&\quad-\lambda\int_{\mathbb{R}_+}\partial_x u\frac{(\partial_x\varphi)^2}{\rho^2}\mathrm{d}x-\lambda\int_{\mathbb{R}_+}u\frac{\partial_x\varphi\partial_x^2\varphi}{\rho^2}\mathrm{d}x-\lambda\int_{\mathbb{R}_+}\frac{\partial_x\varphi}{\rho^2}\partial_x\rho\partial_x\psi\mathrm{d}x-\lambda\int_{\mathbb{R}_+}\partial_x^2\tilde{u}\varphi\frac{\partial_x\varphi}{\rho^2}\mathrm{d}x\\
&\quad-\lambda\int_{\mathbb{R}_+}\partial_x\tilde{\rho}\partial_x\psi\frac{\partial_x\varphi}{\rho^2}\mathrm{d}x-\lambda\int_{\mathbb{R}_+}\partial_x^2\tilde{\rho}\psi\frac{\partial_x\varphi}{\rho^2}\mathrm{d}x-\int_{\mathbb{R}_+}\frac{\partial_x|\mathbf{b}|^2}{2\rho}\partial_x\varphi\mathrm{d}x=\sum_{l=4}^{18}J_l,
\end{aligned}\tag{2.17}$$

where we have used $R(\rho\theta-\tilde{\rho}\tilde{\theta})=R\theta\varphi+R\tilde{\rho}\zeta$ and J_l $(4\leqslant l\leqslant 18)$ denote the corresponding terms on the left of (2.17).

Applying the Sobolev's inequality, the Young's inequality and the Cauchy-Schwarz's inequality with $0<\eta<1$ and using lemma 2.1, (2.8), one has

$$\begin{aligned}
J_4&=\int_{\mathbb{R}_+}\partial_x\psi\partial_x(\rho u-\tilde{\rho}\tilde{u})\mathrm{d}x\\
&=\int_{\mathbb{R}_+}\rho(\partial_x\psi)^2\mathrm{d}x+\int_{\mathbb{R}_+}\partial_x\tilde{\rho}\psi\partial_x\psi\mathrm{d}x+\int_{\mathbb{R}_+}\varphi\partial_x\tilde{u}\partial_x\psi\mathrm{d}x+\int_{\mathbb{R}_+}u\partial_x\psi\partial_x\varphi\mathrm{d}x\\
&\leqslant(\eta+C\tilde{\delta})\|\partial_x\varphi\|^2+(C_\eta+C\tilde{\delta})\|\partial_x\psi\|^2+C\tilde{\delta}\varphi(0,t)^2,
\end{aligned}$$

$$\begin{aligned}
&J_5+J_{12}+J_{13}\\
&=-\frac{\lambda|u_-|}{2\rho(0,t)^2}(\partial_x\varphi)^2(0,t)+\frac{\lambda}{2}\int_{\mathbb{R}_+}\partial_x\tilde{u}(\partial_x\varphi)^2\rho^{-2}\mathrm{d}x+\frac{\lambda}{2}\int_{\mathbb{R}_+}\partial_x\psi(\partial_x\varphi)^2\rho^{-2}\mathrm{d}x\\
&\leqslant-\frac{\lambda|u_-|}{2\rho(0,t)^2}(\partial_x\varphi)^2(0,t)+C(\tilde{\delta}+\varepsilon_1)\|\partial_x\varphi\|^2+C\varepsilon_1\|\partial_x^2\psi\|^2,
\end{aligned}$$

$$\begin{aligned}
|J_6|+|J_9|+|J_{14}|+|J_{16}|\leqslant{}&(\eta+C\tilde{\delta}+C\varepsilon_1)\|\partial_x\varphi\|^2\\
&+(C_\eta+C\tilde{\delta})\|\partial_x[\psi,\zeta]\|^2+C\varepsilon_1\|\partial_x^2\psi\|^2,
\end{aligned}$$

$$|J_7|+|J_8|+|J_{10}|+|J_{11}|+|J_{15}|+|J_{17}|\leqslant C(\varepsilon_1+\tilde{\delta})\|\partial_x[\varphi,\psi,\zeta]\|^2+C\tilde{\delta}\varphi^2(0,t),$$

$$|J_{18}|\leqslant C\varepsilon_1\|\partial_x[\varphi,\mathbf{b}]\|^2.$$

Inserting the above estimates for J_l $(4\leqslant l\leqslant 18)$ into (2.17) and then choosing $\varepsilon_1,\tilde{\delta}$ and η suitably small, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\left(\psi\partial_x\varphi+\lambda\frac{(\partial_x\varphi)^2}{2\rho^2}\right)\mathrm{d}x+\|\partial_x\varphi\|^2+(\partial_x\varphi)^2(0,t)\\
&\qquad\leqslant C\|\partial_x[\psi,\mathbf{b},\zeta]\|^2+C\varepsilon_1\|\partial_x^2\psi\|^2+\varphi^2(0,t).
\end{aligned}\tag{2.18}$$

Multiplying (2.18) by $(1+t)^{\xi}$ and integrating in τ over $[0,t]$ for any $0 \leqslant t \leqslant T$, using (2.14) and the Cauchy-Schwarz's inequality, one has

$$\begin{aligned}
&(1+t)^{\xi}\|\partial_x\varphi\|^2 + \int_0^t (1+\tau)^{\xi}\|\partial_x\varphi\|^2 \mathrm{d}\tau + \int_0^t (1+\tau)^{\xi}(\partial_x\varphi)^2(0,\tau)\mathrm{d}\tau \\
&\quad \leqslant C(\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat{\lambda},\beta} + \|\partial_x\varphi_0\|^2) \\
&\qquad + \xi\int_0^t (1+\tau)^{\xi-1}(\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu},\beta} + \|\partial_x\varphi\|^2)\mathrm{d}\tau \\
&\qquad + C\varepsilon_1\int_0^t (1+\tau)^{\xi}\|\partial_x^2\psi\|^2\mathrm{d}\tau.
\end{aligned} \tag{2.19}$$

Step 3. *Higher order energy estimates.*

Multiplying $(2.1)_2$ by $-((\partial_x^2\psi)/(\rho))$, and then integrating the resulting equations over $\mathbb{R}_+$, one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\frac{(\partial_x\psi)^2}{2}\mathrm{d}x + \lambda\int_{\mathbb{R}_+}\frac{(\partial_x^2\psi)^2}{\rho}\mathrm{d}x \\
&\quad = \underbrace{R\int_{\mathbb{R}_+}\frac{\partial_x(\rho\theta-\tilde{\rho}\tilde{\theta})}{\rho}\partial_x^2\psi\mathrm{d}x}_{J_{19}} + \underbrace{\int_{\mathbb{R}_+}u\partial_x\psi\partial_x^2\psi\mathrm{d}x}_{J_{20}} + \underbrace{\int_{\mathbb{R}_+}\frac{\tilde{u}\partial_x\tilde{u}}{\rho}\varphi\partial_x^2\psi\mathrm{d}x}_{J_{21}} \\
&\qquad \underbrace{+\int_{\mathbb{R}_+}\partial_x\tilde{u}\psi\partial_x^2\psi\mathrm{d}x}_{J_{22}} + \underbrace{\int_{\mathbb{R}_+}\frac{\partial_x|\mathbf{b}|^2}{2\rho}\partial_x^2\psi\mathrm{d}x}_{J_{23}}.
\end{aligned} \tag{2.20}$$

We utilize integration by parts, the Cauchy-Schwarz's inequality and lemma 2.1 to address the following estimates:

$$|J_{19}| + |J_{20}| \leqslant \eta\|\partial_x^2\psi\|^2 + C_\eta\|\partial_x[\varphi,\psi,\zeta]\|^2,$$

$$|J_{21}| + |J_{22}| \leqslant \eta\|\partial_x^2\psi\|^2 + C_\eta\tilde{\delta}\|\partial_x[\varphi,\psi]\|^2 + C\tilde{\delta}\varphi^2(0,t),$$

and

$$|J_{23}| \leqslant C\varepsilon_1\|\partial_x[\mathbf{b},\partial_x\psi]\|^2.$$

Substituting the above estimates for J_l $(19 \leqslant l \leqslant 23)$ into (2.20) and taking η small enough, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}\frac{(\partial_x\psi)^2}{2}\mathrm{d}x + \|\partial_x^2\psi\|^2 \leqslant C\|\partial_x[\varphi,\psi,\mathbf{b},\zeta]\|^2 + C\varphi^2(0,t). \tag{2.21}$$

Multiplying $(2.1)_3$ and $(2.1)_4$ by $-((\partial_x^2\mathbf{w})/(\rho))$ and $-\partial_x^2\mathbf{b}$ respectively, and integrating the resulting equality over $\mathbb{R}_+$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}(|\partial_x\mathbf{w}|^2+|\partial_x\mathbf{b}|^2)\mathrm{d}x+\mu\int_{\mathbb{R}_+}\frac{|\partial_x^2\mathbf{w}|^2}{\rho}\mathrm{d}x+\nu\int_{\mathbb{R}_+}|\partial_x^2\mathbf{b}|^2\mathrm{d}x\\ =\underbrace{\int_{\mathbb{R}_+}u\partial_x\mathbf{w}\cdot\partial_x^2\mathbf{w}\mathrm{d}x}_{J_{24}}-\underbrace{\int_{\mathbb{R}_+}\frac{1}{\rho}\partial_x\mathbf{b}\cdot\partial_x^2\mathbf{w}\mathrm{d}x}_{J_{25}}+\underbrace{\int_{\mathbb{R}_+}\partial_x(u\mathbf{b}-\mathbf{w})\cdot\partial_x^2\mathbf{b}\mathrm{d}x}_{J_{26}}.\end{aligned}\tag{2.22}$$

To obtain the estimates for J_{24}-J_{26}, we use the Cauchy-Schwarz's inequality with $0<\eta<1$ to get

$$|J_{24}|+|J_{25}|+|J_{26}|\leqslant\eta\|\partial_x^2[\mathbf{w},\mathbf{b}]\|^2+C_\eta\|\partial_x[\mathbf{w},\mathbf{b},\psi]\|^2.$$

Then we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}(|\partial_x\mathbf{w}|^2+|\partial_x\mathbf{b}|^2)\mathrm{d}x+\|\partial_x^2[\mathbf{w},\mathbf{b}]\|^2\leqslant C\|\partial_x[\mathbf{w},\mathbf{b},\psi]\|^2\tag{2.23}$$

if η is small enough.

Similarly, multiplying $(2.1)_5$ by $-\frac{\partial_x^2\zeta}{\rho}$, and integrating the resulting equations over $\mathbb{R}_+$, one has

$$\begin{aligned}&\frac{R}{2(\gamma-1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}(\partial_x\zeta)^2\mathrm{d}x+\kappa\int_{\mathbb{R}_+}\frac{(\partial_x^2\zeta)^2}{\rho}\mathrm{d}x\\ &=\underbrace{\frac{R}{\gamma-1}\int_{\mathbb{R}_+}u\partial_x\zeta\partial_x^2\zeta\mathrm{d}x}_{J_{27}}+\underbrace{\int_{\mathbb{R}_+}R\theta\partial_x\psi\partial_x^2\zeta\mathrm{d}x}_{J_{28}}\underbrace{-\lambda\int_{\mathbb{R}_+}(\partial_x\psi)^2\rho^{-1}\partial_x^2\zeta\mathrm{d}x}_{J_{29}}\\ &\quad\underbrace{-\int_{\mathbb{R}_+}2\lambda\partial_x\tilde{u}\rho^{-1}\partial_x\psi\partial_x^2\zeta\mathrm{d}x}_{J_{30}}+\underbrace{\frac{R}{\gamma-1}\int_{\mathbb{R}_+}\tilde{u}\partial_x\tilde{\theta}\rho^{-1}\varphi\partial_x^2\zeta\mathrm{d}x}_{J_{31}}+\underbrace{\frac{R}{\gamma-1}\int_{\mathbb{R}_+}\partial_x\tilde{\theta}\psi\partial_x^2\zeta\mathrm{d}x}_{J_{32}}\\ &\quad\underbrace{+R\int_{\mathbb{R}_+}\partial_x\tilde{u}(\rho\theta-\tilde{\rho}\tilde{\theta})\rho^{-1}\partial_x^2\zeta\mathrm{d}x}_{J_{33}}\underbrace{-\int_{\mathbb{R}_+}\frac{\nu|\partial_x\mathbf{b}|^2+\mu|\partial_x\mathbf{w}|^2}{\rho}\partial_x^2\zeta\mathrm{d}x}_{J_{34}}.\end{aligned}\tag{2.24}$$

Performing the similar calculations as J_l $(19\leqslant l\leqslant 23)$, we have

$$|J_{27}|+|J_{28}|\leqslant\eta\|\partial_x^2\zeta\|^2+C_\eta\|\partial_x[\zeta,\psi]\|^2,$$

$$|J_{29}|\leqslant C\varepsilon_1\|\partial_x[\psi,\partial_x\psi,\partial_x\zeta]\|^2,\quad |J_{30}|\leqslant C\tilde{\delta}\|\partial_x[\psi,\partial_x\zeta]\|^2,$$

$$|J_{31}|+|J_{32}|+|J_{33}|\leqslant\eta\|\partial_x^2\zeta\|^2+C_\eta\tilde{\delta}\|\partial_x[\varphi,\psi,\zeta]\|^2+C\tilde{\delta}\varphi^2(0,t),$$

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$$
\begin{aligned}
|J_{34}| &\leqslant C\|\partial_x[\mathbf{w},\mathbf{b}]\|_\infty\|\partial_x[\mathbf{w},\mathbf{b}]\|\|\partial_x^2\zeta\|\\
&\leqslant C\|\partial_x[\mathbf{w},\mathbf{b}]\|^{\frac12}\|\partial_x^2[\mathbf{w},\mathbf{b}]\|^{\frac12}\|\partial_x[\mathbf{w},\mathbf{b}]\|\|\partial_x^2\zeta\|\\
&\leqslant C(\|\partial_x[\mathbf{w},\mathbf{b}]\|+\|\partial_x^2[\mathbf{w},\mathbf{b}]\|)\|\partial_x[\mathbf{w},\mathbf{b}]\|\|\partial_x^2\zeta\|\\
&\leqslant C\varepsilon_1(\|\partial_x[\mathbf{w},\mathbf{b}]\|^2+\|\partial_x^2[\mathbf{w},\mathbf{b}]\|^2+\|\partial_x^2\zeta\|^2).
\end{aligned}
$$

Plug the above estimates for J_l $(27 \leqslant l \leqslant 34)$ into (2.24), to derive

$$
\begin{aligned}
\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}(\partial_x\zeta)^2\mathrm{d}x+\|\partial_x^2\zeta\|^2 &\leqslant C\|\partial_x[\varphi,\psi,\zeta]\|^2\\
&\quad+C\varepsilon_1(\|\partial_x[\mathbf{w},\mathbf{b}]\|^2+\|\partial_x^2[\mathbf{w},\mathbf{b}]\|^2)+C\varphi^2(0,t).
\end{aligned}\tag{2.25}
$$

The summation of (2.21), (2.23) and (2.25) and multiplying the resulting inequality by $(1+t)^\xi$, then integrating the resulting inequality in τ over $[0,t]$ for any $0 \leqslant t \leqslant T$, using (2.14), (2.19) and the Cauchy-Schwarz's inequality, one has

$$
\begin{aligned}
&(1+t)^\xi\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta](t)\|^2+\int_0^t(1+\tau)^\xi\|\partial_x^2[\psi,\mathbf{w},\mathbf{b},\zeta]\|^2\mathrm{d}\tau\\
&\quad\leqslant C(\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat\lambda,\beta}+\|\partial_x[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2)\\
&\qquad+\xi\int_0^t(1+\tau)^{\xi-1}(\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat\nu,\beta}+\|\partial_x[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta]\|^2)\mathrm{d}\tau.
\end{aligned}\tag{2.26}
$$

Proof of proposition 2.2. Now, following the three steps above, we are ready to prove proposition 2.2. Summing up the estimates (2.14), (2.19) and (2.26), and taking $\tilde\delta$ and ε_1 suitably small, we have

$$
\begin{aligned}
&(1+t)^\xi(\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|^2_{\hat\nu,\beta}+\|\partial_x[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|^2)\\
&\quad+\int_0^t(1+\tau)^\xi(\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta,](\tau)\|^2_{\hat\nu-1,\beta}+\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat\nu,\beta})\mathrm{d}\tau\\
&\quad+\int_0^t(1+\tau)^\xi\|\partial_x[\varphi,\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta]](\tau)\|^2\mathrm{d}\tau\\
&\leqslant C(\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat\lambda,\beta}+\|\partial_x[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2)\\
&\quad+\xi\int_0^t(1+\tau)^{\xi-1}(\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat\nu,\beta}+\|\partial_x[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta]\|^2)\mathrm{d}\tau,
\end{aligned}\tag{2.27}
$$

where C is a positive constant independent of T, $\hat\nu$, β, ε_1 and $\tilde\delta$. Hence, similarly as in [**11**, **23**], applying an induction to (2.27) gives the desired estimate (2.9). □

2.2. The a priori estimates for $M_+ < 1$

The key to the proof of our main theorem 1.3 (ii) for the subsonic case $M_+ < 1$ is to derive the uniform *a priori* estimates of solutions to the initial boundary value problem (2.1), (2.2) and (2.3).

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PROPOSITION 2.3. *Assume the same conditions as in theorem 1.3 (ii) hold. Suppose* $[\varphi, \psi, \mathbf{w}, \mathbf{b}, \zeta]$ *is a solution to* (2.1), (2.2) *and* (2.3) *on* $0 \leqslant t \leqslant T$ *for* $T > 0$. *There exist positive constants* C *and* ε_2 *independent of* T *such that if*

$$\sup_{0\leqslant t\leqslant T} \|(\varphi, \psi, \mathbf{w}, \mathbf{b}, \zeta)(t)\|_1 + \tilde{\delta} \leqslant \varepsilon_2 \tag{2.28}$$

is satisfied, it holds that

$$\sup_{0\leqslant t\leqslant T} \|[\varphi, \psi, \mathbf{w}, \mathbf{b}, \zeta](t)\|_1^2 + \int_0^T \|\partial_x[\psi, \mathbf{w}, \mathbf{b}, \zeta](t)\|_1^2 \mathrm{d}t \\ + \int_0^T |[\varphi, \partial_x\varphi](0,t)|^2 \,\mathrm{d}t \leqslant C\|[\varphi_0, \psi_0, \mathbf{w}_0, \mathbf{b}_0, \zeta_0]\|_1^2. \tag{2.29}$$

For the subsonic case $M_+ < 1$, one characteristic is positive. Due to this, it is difficult to obtain a convergence rate for the subsonic case by using the weighted energy method. Hence we only need to re-estimate step 1 in the proof of supersonic case $M_+ > 1$. We integrate (2.10) over $\mathbb{R}_+$ to get

$$\frac{d}{dt}\int_{\mathbb{R}_+} \eta \mathrm{d}x + \lambda \int_{\mathbb{R}_+} \frac{\tilde{\theta}}{\theta}(\partial_x\psi)^2 \mathrm{d}x + \kappa \int_{\mathbb{R}_+} \frac{\tilde{\theta}}{\theta^2}(\partial_x\zeta)^2 \mathrm{d}x + \int_{\mathbb{R}_+} (\nu|\partial_x \mathbf{b}|^2 + \mu|\partial_x \mathbf{w}|^2)\mathrm{d}x \\ + R\rho(0,t)\theta_-|u_-|\phi\left(\frac{\tilde{\rho}}{\rho}\right)(0,t) = \int_{\mathbb{R}_+} (\partial_x\tilde{u}Q_1 + \partial_x\tilde{\theta}Q_2)\mathrm{d}x + \int_{\mathbb{R}_+} Q_3 \mathrm{d}x. \tag{2.30}$$

Using lemma 2.1 and the Sobolev inequality, we have

$$\left|\int_{\mathbb{R}_+} Q_3 \mathrm{d}x\right| \leqslant C\tilde{\delta}\|\partial_x[\psi, \zeta]\|^2 + C\varepsilon_2\|\partial_x[\mathbf{w}, \mathbf{b}]\|^2,$$

$$\left|\int_{\mathbb{R}_+} (\partial_x\tilde{u}Q_1 + \partial_x\tilde{\theta}Q_2)\mathrm{d}x\right| \leqslant C\tilde{\delta}\varphi(0,t)^2 + C\tilde{\delta}\|\partial_x[\varphi, \psi, \mathbf{b}, \zeta]\|^2.$$

Inserting the above estimates into (2.30) and then choosing $\tilde{\delta}$ and ε_2 suitably small, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \eta \mathrm{d}x + \varphi^2(0,t) + \|\partial_x[\psi, \mathbf{w}, \mathbf{b}, \zeta]\|^2 \leqslant C\tilde{\delta}\|\partial_x\varphi\|^2. \tag{2.31}$$

Proof of proposition 2.3. Now we are ready to prove proposition 2.3. Summing up (2.18), (2.21), (2.23), (2.25) and (2.31) together and integrating the resulting inequality over $[0, T]$, we get the desired estimate (2.29) when we take $\tilde{\delta}$ and ε_2 small enough. Thus the proof of proposition 2.3 is complete. □

2.3. The a priori estimates for $M_+ = 1$

PROPOSITION 2.4. *Assume the same conditions as in theorem 1.3 (iii) hold. Let* $1 \leqslant \hat{\lambda} < 2(1+\sqrt{2})$ *and* $\hat{\kappa}$ *be positive constants. Suppose* $[\varphi, \psi, \mathbf{w}, \mathbf{b}, \zeta]$ *is a solution*

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to (2.1), (2.2) *and* (2.3) *which satisfies* $(1+\tilde{\delta}x)^{\hat{\lambda}/2}\varphi, (1+\tilde{\delta}x)^{\hat{\lambda}/2}\psi, (1+\tilde{\delta}x)^{\hat{\lambda}/2}\mathbf{w}$, $(1+\tilde{\delta}x)^{\hat{\lambda}/2}\mathbf{b}, (1+\tilde{\delta}x)^{\hat{\lambda}/2}\zeta \in C([0,T]; H^1(\mathbb{R}_+))$ *for a certain positive constant* T. *For arbitrary* $\hat{\nu} \in [0, \hat{\lambda}]$, *there exist positive constants* C *and* ε_3 *independent of* T *such that if*

$$\sup_{0\leqslant t\leqslant T} \tilde{\delta}^{-1/2}\|(\varphi,\psi,\mathbf{w},\mathbf{b},\zeta)(t)\|_{\hat{\lambda},\tilde{\delta},1} + \tilde{\delta} \leqslant \varepsilon_3 \tag{2.32}$$

is satisfied, it holds for an arbitrary $t \in [0, T]$ *that*

$$\begin{aligned}(1+t)^{\hat{\lambda}-\hat{\nu}+\hat{\kappa}}\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|^2_{\hat{\nu},\tilde{\delta},1} + \int_0^t (1+\tau)^{\hat{\lambda}-\hat{\nu}+\hat{\kappa}}\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu},\tilde{\delta},1}\mathrm{d}\tau \\ \leqslant C(1+t)^{\hat{\kappa}}\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat{\lambda},\tilde{\delta},1}.\end{aligned} \tag{2.33}$$

Step 1. The zero-order energy estimates.

From (2.10), we have

$$\begin{aligned}\partial_t\eta + \partial_x(H_1 - H_2) + \lambda\frac{\tilde{\theta}}{\theta}(\partial_x\psi)^2 + \kappa\frac{\tilde{\theta}}{\theta^2}(\partial_x\zeta)^2 \\ + \nu|\partial_x\mathbf{b}|^2 + \mu|\partial_x\mathbf{w}|^2 - \partial_x\tilde{u}Q_1 - \partial_x\tilde{\theta}Q_2 = Q_3.\end{aligned} \tag{2.34}$$

In the sequel, we employ a space weight function $w(x) := \tilde{z}(x)^{-\hat{\nu}}$. Notice that $w(x) \sim \tilde{\delta}^{-\hat{\nu}}(1+\tilde{\delta}x)^{\hat{\nu}}$ holds due to (A.15). Furthermore, we have $w(0) = \tilde{z}(0)^{-\hat{\nu}} \sim \tilde{\delta}^{-\hat{\nu}}$ and

$$\partial_x w = \frac{\gamma^2(\gamma+1)R^2\rho_+}{2(\gamma-1)[\lambda\gamma R + \kappa(\gamma-1)^2]|u_+|}\hat{\nu}\tilde{z}^{-\hat{\nu}+1} + O(\hat{\nu}\tilde{z}^{-\hat{\nu}+2}) \tag{2.35}$$

due to (A.13). Then, multiplying (2.34) by the weight function $w(x)$ to get

$$\begin{aligned}\partial_t[w(x)\eta] + \partial_x[w(x)(H_1-H_2)] \underbrace{-\partial_x w(H_1-H_2)}_{I_1} \underbrace{+w(x)[\partial_x\tilde{u}(-Q_1) + \partial_x\tilde{\theta}(-Q_2)]}_{I_2} \\ \underbrace{+w(x)\left[\lambda\frac{\tilde{\theta}}{\theta}(\partial_x\psi)^2 + \kappa\frac{\tilde{\theta}}{\theta^2}(\partial_x\zeta)^2\right]}_{I_3} + w(x)[\nu|\partial_x\mathbf{b}|^2 + \mu|\partial_x\mathbf{w}|^2] = w(x)Q_3,\end{aligned} \tag{2.36}$$

where

$$\begin{aligned}I_1 = \underbrace{\partial_x w[-u\eta - R(\rho\theta - \tilde{\rho}\tilde{\theta})\psi]}_{I_1^1} \underbrace{+\partial_x w\left[\lambda\psi\partial_x\psi + \kappa\frac{\zeta\partial_x\zeta}{\theta}\right]}_{I_1^2} \\ + \partial_x w\left[-\frac{1}{2}|\mathbf{b}|^2\psi + \mathbf{w}\cdot\mathbf{b} + \mu\mathbf{w}\cdot\partial_x\mathbf{w} + \nu\mathbf{b}\cdot\partial_x\mathbf{b}\right].\end{aligned} \tag{2.37}$$

By using (1.16), we have

$$\begin{aligned}(\rho,u,\theta) = (\rho_+,u_+,\theta_+) + \left(-\frac{\rho_+}{\theta_+(\gamma-1)}, \frac{u_+}{\theta_+(\gamma-1)}, -1\right)\tilde{z}(x) \\ + O(\tilde{z}^2 + \tilde{\delta}e^{-cx})(1,1,1) + (\varphi,\psi,\zeta).\end{aligned} \tag{2.38}$$

Then we see, under the condition $M_+ = 1$ and $u_+ < 0$, and using (1.16) and (2.38) that

$$
\begin{aligned}
&-u\eta - R(\rho\theta - \tilde{\rho}\tilde{\theta})\psi \\
&\quad = \left[-\frac{R\theta_+ u_+}{2\rho_+}\varphi^2 - \frac{\rho_+ u_+}{2}\psi^2 - \frac{R\rho_+ u_+}{2(\gamma-1)\theta_+}\zeta^2 - R\theta_+\varphi\psi - R\rho_+\zeta\psi\right] - \frac{\rho u}{2}|\mathbf{w}|^2 - \frac{u}{2}|\mathbf{b}|^2 \\
&\qquad + \left[\frac{Ru_+(\gamma-3)}{2\rho_+(\gamma-1)}\varphi^2 - \frac{R\rho_+ u_+}{2(\gamma-1)\theta_+^2}\zeta^2 + \frac{R\rho_+}{(\gamma-1)\theta_+}\zeta\psi + R\varphi\psi\right]\tilde{z} \\
&\qquad + O(|\varphi| + |\psi| + |\zeta| + \tilde{z}^2 + \tilde{\delta}e^{-cx})(\varphi^2 + \psi^2 + \zeta^2).
\end{aligned}
$$

Combining this with (2.35), (1.16) and (2.38), we arrive at

$$
\begin{aligned}
I_1^1 = {} & \partial_x w \underbrace{\begin{matrix}\left[-\frac{R\theta_+ u_+}{2\rho_+}\varphi^2 - \frac{\rho_+ u_+}{2}\psi^2\right. \\ \left. - \frac{R\rho_+ u_+}{2(\gamma-1)\theta_+}\zeta^2 - R\theta_+\varphi\psi - R\rho_+\zeta\psi\right]\end{matrix}}_{F_1} + \partial_x w\left(-\frac{\rho u}{2}|\mathbf{w}|^2 - \frac{u}{2}|\mathbf{b}|^2\right) \\
& + \underbrace{\begin{matrix}\frac{\gamma^2(\gamma+1)R^3\rho_+}{2(\gamma-1)[\lambda\gamma R + \kappa(\gamma-1)^2]u_+^2}\left[\frac{-u_+^2(\gamma-3)}{2\rho_+(\gamma-1)}\varphi^2\right. \\ \left. + \frac{\rho_+ u_+^2}{2(\gamma-1)\theta_+^2}\zeta^2 - \frac{\rho_+ u_+}{(\gamma-1)\theta_+}\zeta\psi - u_+\varphi\psi\right]\end{matrix}}_{F_2} \hat{\nu}\tilde{z}^{-\hat{\nu}+2} \\
& + O(|\varphi| + |\psi| + |\zeta| + \tilde{z}^2 + \tilde{\delta}e^{-cx})\tilde{z}^{-\hat{\nu}+1}(\varphi^2 + \psi^2 + \zeta^2)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 + I_1^2 = {} & \left(\sqrt{\lambda}\partial_x\psi + \frac{\sqrt{\lambda}\gamma^2(\gamma+1)R^2\rho_+}{4(\gamma-1)[\lambda\gamma R + \kappa(\gamma-1)^2]|u_+|}\hat{\nu}\tilde{z}\psi\right)^2 \tilde{z}^{-\hat{\nu}} \\
& + \left(\frac{\sqrt{\kappa}}{\sqrt{\theta_+}}\partial_x\zeta + \frac{\gamma^2(\gamma+1)R^2\rho_+\sqrt{\kappa}}{4(\gamma-1)[\lambda\gamma R + \kappa(\gamma-1)^2]|u_+|\sqrt{\theta_+}}\hat{\nu}\tilde{z}\zeta\right)^2 \tilde{z}^{-\hat{\nu}} \\
& \underbrace{- \frac{\gamma^4(\gamma+1)^2R^4\rho_+^2}{16(\gamma-1)^2[\lambda\gamma R + \kappa(\gamma-1)^2]^2u_+^2}\left(\lambda\psi^2 + \frac{\kappa}{\theta_+}\zeta^2\right)}_{F_3}\hat{\nu}^2\tilde{z}^{-\hat{\nu}+2} \\
& + O(|\varphi| + |\psi| + |\zeta| + \tilde{z})[\tilde{z}^{-\hat{\nu}+2}(\varphi^2 + \psi^2 + \zeta^2) + \tilde{z}^{-\hat{\nu}}(\partial_x\psi)^2 + \tilde{z}^{-\hat{\nu}}(\partial_x\zeta)^2].
\end{aligned}
$$

By using (1.15), we have

$$
(\tilde{\rho}, \tilde{u}, \tilde{\theta}) = (\rho_+, u_+, \theta_+) + O(\tilde{\delta}), \tag{2.39}
$$

and

$$
(\rho, u, \theta) = (\rho_+, u_+, \theta_+) + O(\varepsilon_3 + \tilde{\delta}). \tag{2.40}
$$

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By using (2.39) and (2.40), we have

$$-Q_1 = \frac{(\gamma-1)R\theta_+}{u_+}\varphi\psi + \rho_+\psi^2 + \frac{R\rho_+}{\theta_+}\zeta^2 + R\varphi\zeta + \frac{1}{2}|\mathbf{b}|^2 + O(\varepsilon_3+\tilde{\delta})(\varphi^2+\psi^2+\zeta^2). \tag{2.41}$$

Similarly, we have

$$-Q_2 = -\frac{Ru_+}{2\rho_+}\varphi^2 + \frac{R\rho_+u_+}{2(\gamma-1)\theta_+^2}\zeta^2 + \frac{Ru_+}{(\gamma-1)\theta_+}\varphi\zeta + \frac{R\rho_+}{(\gamma-1)\theta_+}\psi\zeta + O(\varepsilon_3+\tilde{\delta})(\varphi^2+\psi^2+\zeta^2). \tag{2.42}$$

Combining this with (1.17) and noticing $w(x) = \tilde{z}(x)^{-\hat{\nu}}$, we have

$$\begin{aligned} I_2 = {} & \underbrace{\frac{\gamma^2(\gamma+1)R^3\rho_+}{2(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]u_+^2}\left[\frac{u_+^2}{2\rho_+}\varphi^2 + \frac{\gamma\rho_+}{\gamma-1}\psi^2 + \frac{R\gamma\rho_+\zeta^2}{2(\gamma-1)\theta_+} + u_+\varphi\psi - \frac{\rho_+u_+\psi\zeta}{(\gamma-1)\theta_+}\right]}_{F_4}\tilde{z}^{-\hat{\nu}+2} \\ & + \frac{\gamma^2(\gamma+1)R^2\rho_+|\mathbf{b}|^2\tilde{z}^{-\hat{\nu}+2}}{4(\gamma-1)^2[\lambda\gamma R+\kappa(\gamma-1)^2]\theta_+} + [O(\varepsilon_3+\tilde{\delta})\tilde{z}^{-\hat{\nu}+2} \\ & + O(\tilde{\delta})\tilde{z}^{-\hat{\nu}}e^{-cx}](\varphi^2+\psi^2+\zeta^2+|\mathbf{b}|^2), \end{aligned} \tag{2.43}$$

where we have used the transonic condition $M_+ = 1 \Longleftrightarrow u_+^2 = R\gamma\theta_+$.

Now we make a simple conclusion from the above analysis. We can rewrite (2.36) as follows:

$$\begin{aligned} & \partial_t[w(x)\eta] + \partial_x[w(x)(H_1-H_2)] + \partial_x wF_1 + \partial_x w\left(-\frac{\rho_+u_+}{2}|\mathbf{w}|^2 - \frac{u_+}{2}|\mathbf{b}|^2 + \mathbf{w}\cdot\mathbf{b}\right) \\ & + (F_2\hat{\nu} + F_3\hat{\nu}^2 + F_4)\tilde{z}^{-\hat{\nu}+2} + \frac{\gamma^2(\gamma+1)R^2\rho_+}{4(\gamma-1)^2[\lambda\gamma R+\kappa(\gamma-1)^2]\theta_+}|\mathbf{b}|^2\tilde{z}^{-\hat{\nu}+2} \\ & + [\nu|\partial_x\mathbf{b}|^2 + \mu|\partial_x\mathbf{w}|^2]\tilde{z}^{-\hat{\nu}} + \left(\sqrt{\lambda}\partial_x\psi + \frac{\sqrt{\lambda}\gamma^2(\gamma+1)R^2\rho_+}{4(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]|u_+|}\hat{\nu}\tilde{z}\psi\right)^2\tilde{z}^{-\hat{\nu}} \\ & + \left(\frac{\sqrt{\kappa}}{\sqrt{\theta_+}}\partial_x\zeta + \frac{\gamma^2(\gamma+1)R^2\rho_+\sqrt{\kappa}}{4(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]|u_+|\sqrt{\theta_+}}\hat{\nu}\tilde{z}\zeta\right)^2\tilde{z}^{-\hat{\nu}} \\ = {} & w(x)Q_3 + \partial_x w\left[\frac{1}{2}|\mathbf{b}|^2\psi - \mu\mathbf{w}\cdot\partial_x\mathbf{w} - \nu\mathbf{b}\cdot\partial_x\mathbf{b}\right] + O(\varepsilon_3+\tilde{\delta})\partial_x w(|\mathbf{b}|^2+|\mathbf{w}|^2) \\ & + O(\varepsilon_3+\tilde{\delta})[(\partial_x\psi)^2 + (\partial_x\zeta)^2]\tilde{z}^{-\hat{\nu}} + [O(|\varphi|+|\psi|+|\zeta|)\tilde{z}^{-\hat{\nu}+1} \\ & + O(\tilde{\delta})\tilde{z}^{-\hat{\nu}}e^{-cx}](\varphi^2+\psi^2+\zeta^2). \end{aligned} \tag{2.44}$$

We rewrite

$$F_1 = [\varphi,\psi,\zeta]M_3[\varphi,\psi,\zeta]^T,$$

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where $[\,]^T$ denotes the transpose of a row vector, and the 3×3 real symmetric matrix M_3 is given by

$$\begin{pmatrix} -\dfrac{R\theta_+ u_+}{2\rho_+} & -\dfrac{R\theta_+}{2} & 0 \\ -\dfrac{R\theta_+}{2} & -\dfrac{\rho_+ u_+}{2} & -\dfrac{R\rho_+}{2} \\ 0 & -\dfrac{R\rho_+}{2} & -\dfrac{R\rho_+ u_+}{2(\gamma-1)\theta_+} \end{pmatrix}.$$

One can compute all the leading principal minors Δ_{ll} $(1 \leqslant l \leqslant 3)$ of M_3 as follows:

$$\Delta_{11} = -\frac{R\theta_+ u_+}{2\rho_+} > 0, \quad \Delta_{22} = \frac{R^2\theta_+^2(\gamma-1)}{4} > 0, \quad \Delta_{33} = 0,$$

where we have used the transonic condition $M_+ = 1 \Longleftrightarrow u_+^2 = R\gamma\theta_+$. Then we see that the matrix M_3 admits three eigenvalues 0, λ_- and λ_+ satisfying

$$0 < \lambda_- < \lambda_+.$$

Let q_1, q_2 and q_3 be unit eigenvectors of M_3 corresponding to the eigenvalues 0, λ_- and λ_+, respectively. We define $Q := (q_1, q_2, q_3)$ which is an orthogonal matrix. Especially, we obtain

$$q_1 = (\rho_+, -u_+, (\gamma-1)\theta_+)^T\bar{q}, \quad \bar{q} = detQ > 0.$$

Furthermore, we employ a new function $(\hat{\varphi}, \hat{\psi}, \hat{\zeta})^T$ defined by

$$(\hat{\varphi}, \hat{\psi}, \hat{\zeta})^T := Q^{-1}(\varphi, \psi, \zeta)^T.$$

Using the fact that

$$Q^T M_3 Q = Q^{-1} M_3 Q = diag(0, \lambda_-, \lambda_+),$$

we see that the quadratic form F_1 satisfies the estimate from below as

$$F_1 = (Q(\hat{\varphi}, \hat{\psi}, \hat{\zeta})^T)^T M_3 Q(\hat{\varphi}, \hat{\psi}, \hat{\zeta})^T = \lambda_-\hat{\psi}^2 + \lambda_+\hat{\zeta}^2 \geqslant c(\hat{\psi}^2 + \hat{\zeta}^2).$$

Combining this estimate with the inequality $\partial_x w \geqslant c\hat{\nu}\tilde{z}^{-\hat{\nu}+1}$, which follows from (2.35) with $\tilde{\delta} \ll 1$, we have

$$\partial_x w F_1 \geqslant c\hat{\nu}\tilde{z}^{-\hat{\nu}+1}(\hat{\psi}^2 + \hat{\zeta}^2). \tag{2.45}$$

We rewrite

$$F_2\hat{\nu} + F_3\hat{\nu}^2 + F_4 = \frac{\gamma^2(\gamma+1)R^3\rho_+}{2(\gamma-1)[\lambda\gamma R + \kappa(\gamma-1)^2]u_+^2}[\varphi, \psi, \zeta]M_4[\varphi, \psi, \zeta]^T, \tag{2.46}$$

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where $[\,]^T$ denotes the transpose of a row vector, and the 3×3 real symmetric matrix M_4 is given by

$$\begin{pmatrix} \frac{u_+^2}{2\rho_+}[1-\frac{\gamma-3}{\gamma-1}\hat{\nu}] & \frac{u_+}{2}(1-\hat{\nu}) & 0 \\ \frac{u_+}{2}(1-\hat{\nu}) & \frac{\gamma\rho_+}{\gamma-1}-\frac{\lambda\gamma^2(\gamma+1)R\rho_+\hat{\nu}^2}{8(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]} & -\frac{\rho_+u_+(\hat{\nu}+1)}{2(\gamma-1)\theta_+} \\ 0 & -\frac{\rho_+u_+(\hat{\nu}+1)}{2(\gamma-1)\theta_+} & \frac{\rho_+u_+^2(\hat{\nu}+1)}{2(\gamma-1)\theta_+^2}-\frac{\kappa\gamma^2(\gamma+1)R\rho_+\hat{\nu}^2}{8(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]\theta_+} \end{pmatrix}.$$

Noticing $\tilde{z}\leqslant\tilde{\delta}$, we make use of the semi-positive definition matrix M_3 to control M_4. In fact, let $\hat{M}_4 := (\hat{a}_{ij})_{ij} := Q^TM_4Q$. Then we see that

$$\begin{aligned} F_2\hat{\nu}+F_3\hat{\nu}^2+F_4 &= \frac{\gamma^2(\gamma+1)R^3\rho_+}{2(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]u_+^2}(Q(\hat{\varphi},\hat{\psi},\hat{\zeta})^T)^TM_4Q(\hat{\varphi},\hat{\psi},\hat{\zeta})^T \\ &= \frac{\gamma^2(\gamma+1)R^3\rho_+}{2(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]u_+^2}(\hat{\varphi},\hat{\psi},\hat{\zeta})\hat{M}_4(\hat{\varphi},\hat{\psi},\hat{\zeta})^T \\ &= \hat{a}_{11}\hat{\varphi}^2+O(|(\hat{\psi},\hat{\zeta})|^2+|\hat{\varphi}(\hat{\psi}+\hat{\zeta})|). \end{aligned} \tag{2.47}$$

Since the sign of $\hat{a}_{11}$ will play an important role later, we obtain it explicitly:

$$\begin{aligned} \hat{a}_{11} &= \frac{\gamma^2(\gamma+1)R^3\rho_+}{2(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]u_+^2}q_1^TM_4q_1 \\ &= \frac{\gamma^3(\gamma+1)^2R^3\rho_+^2}{16(\gamma-1)^2[\lambda\gamma R+\kappa(\gamma-1)^2]}[4+4\hat{\nu}-\hat{\nu}^2]\bar{q}^2. \end{aligned} \tag{2.48}$$

When $0\leqslant\hat{\nu}<2(1+\sqrt{2})$, we see that $\hat{a}_{11}>0$. When $\hat{\nu}=0$, the real symmetric matrix M_4 becomes a real symmetric matrix M_5 defined by

$$\begin{pmatrix} \frac{u_+^2}{2\rho_+} & \frac{u_+}{2} & 0 \\ \frac{u_+}{2} & \frac{\gamma\rho_+}{\gamma-1} & -\frac{\rho_+u_+}{2(\gamma-1)\theta_+} \\ 0 & -\frac{\rho_+u_+}{2(\gamma-1)\theta_+} & \frac{\rho_+u_+^2}{2(\gamma-1)\theta_+^2} \end{pmatrix}.$$

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One can compute all the leading principal minors Δ_{ll} $(1 \leqslant l \leqslant 3)$ of M_5 as follows:

$$\Delta_{11} = \frac{u_+^2}{2\rho_+} > 0, \quad \Delta_{22} = \frac{(\gamma+1)u_+^2}{4(\gamma-1)} > 0, \quad \Delta_{33} = \frac{\gamma\rho_+ u_+^4}{8(\gamma-1)^2\theta_+^2} > 0.$$

Hence matrix M_5 is positive definite.

Now we claim that there exists a non-negative constant $\hat{\nu}$ satisfying $0 \leqslant \hat{\nu} < 2(1+\sqrt{2})$ such that

$$\partial_x w F_1 + (F_2\hat{\nu} + F_3\hat{\nu}^2 + F_4)\tilde{z}^{-\hat{\nu}+2} \geqslant c\tilde{z}^{-\hat{\nu}+2}(\varphi^2 + \psi^2 + \zeta^2). \tag{2.49}$$

When $\hat{\nu} = 0$, $\partial_x w F_1 + (F_2\hat{\nu} + F_3\hat{\nu}^2 + F_4)\tilde{z}^{-\hat{\nu}+2} = ((\gamma^2(\gamma+1)R^3\rho_+)/(2(\gamma-1)[\lambda\gamma R + \kappa(\gamma-1)^2]u_+^2))[\varphi,\psi,\zeta]M_5[\varphi,\psi,\zeta]^T\tilde{z}^2$. Since matrix M_5 is positive definite, (2.49) holds for $\hat{\nu} = 0$. Owing to the continuous dependency on $\hat{\nu}$, there exists a positive constant $\hat{\nu}_* > 0$ such that (2.49) holds for $0 \leqslant \hat{\nu} \leqslant \hat{\nu}_*$. In fact, the constant $\hat{a}_{11} > 0$ when $0 < \hat{\nu}_* \leqslant \hat{\nu} < 2(1+\sqrt{2})$. Thus using (2.45) and (2.47), we have

$$\begin{aligned}
&\partial_x w F_1 + (F_2\hat{\nu} + F_3\hat{\nu}^2 + F_4)\tilde{z}^{-\hat{\nu}+2} \geqslant c\hat{\nu}_*\tilde{z}^{-\hat{\nu}+1}(\hat{\psi}^2 + \hat{\zeta}^2) \\
&+ \hat{a}_{11}\tilde{z}^{-\hat{\nu}+2}\hat{\varphi}^2 - C\tilde{z}^{-\hat{\nu}+2}(\hat{\psi}^2 + \hat{\zeta}^2 + |\hat{\varphi}(\hat{\psi} + \hat{\zeta})|) \\
&\qquad \geqslant (c\hat{\nu}_* - C\sqrt{\tilde{\delta}})\tilde{z}^{-\hat{\nu}+1}(\hat{\psi}^2 + \hat{\zeta}^2) + (\hat{a}_{11} - C\sqrt{\tilde{\delta}})\tilde{z}^{-\hat{\nu}+2}\hat{\varphi}^2,
\end{aligned} \tag{2.50}$$

which yields (2.49) if $\tilde{\delta}$ is sufficiently small. Therefore, we have shown that (2.49) holds for $0 \leqslant \hat{\nu} < 2(1+\sqrt{2})$.

Similar to the real symmetric matrix M_2 and recalling (2.35), we have

$$\partial_x w \left(-\frac{\rho_+ u_+}{2}|\mathbf{w}|^2 - \frac{u_+}{2}|\mathbf{b}|^2 + \mathbf{w}\cdot\mathbf{b}\right) \geqslant c\tilde{z}^{-\hat{\nu}+1}(|\mathbf{w}|^2 + |\mathbf{b}|^2) \tag{2.51}$$

provided that conditions $M_+ = 1$ and $p_+ > 1/\gamma$ hold.

Combining (2.49) and (2.51) with Cauchy-Schwarz's inequality, we deal with (2.44) as follows:

$$\begin{aligned}
&\partial_x w F_1 + (F_2\hat{\nu} + F_3\hat{\nu}^2 + F_4)\tilde{z}^{-\hat{\nu}+2} + \partial_x w\left(-\frac{\rho_+ u_+}{2}|\mathbf{w}|^2 - \frac{u_+}{2}|\mathbf{b}|^2 + \mathbf{w}\cdot\mathbf{b}\right) \\
&\quad + \frac{\gamma^2(\gamma+1)R^2\rho_+}{4(\gamma-1)^2[\lambda\gamma R + \kappa(\gamma-1)^2]\theta_+}|\mathbf{b}|^2\tilde{z}^{-\hat{\nu}+2} + [\nu|\partial_x\mathbf{b}|^2 + \mu|\partial_x\mathbf{w}|^2]\tilde{z}^{-\hat{\nu}} \\
&\quad + \left(\partial_x\psi + \frac{(K+1)^3\rho_+}{4[K(D+1)+1]|u_+|}\hat{\nu}\tilde{z}\psi\right)^2\tilde{z}^{-\hat{\nu}} \\
&\quad + \left(\frac{\sqrt{D}}{\sqrt{\theta_+}}\partial_x\zeta + \frac{(K+1)^3\rho_+\sqrt{D}}{4[K(D+1)+1]|u_+|\sqrt{\theta_+}}\hat{\nu}\tilde{z}\zeta\right)^2\tilde{z}^{-\hat{\nu}}
\end{aligned}$$

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$$\geqslant c\tilde{z}^{-\hat{\nu}+2}(\varphi^2+\psi^2+\zeta^2+|\mathbf{b}|^2)+c\tilde{z}^{-\hat{\nu}+1}(|\mathbf{w}|^2+|\mathbf{b}|^2)+c\tilde{z}^{-\hat{\nu}}(|\ \partial_x\mathbf{b}|^2 + |\partial_x\mathbf{w}|^2+(\partial_x\psi)^2+(\partial_x\zeta)^2) \tag{2.52}$$

and

The right hand side of (2.44)

$$\leqslant O(\varepsilon_3+\tilde{\delta}^{\frac{1}{2}})\tilde{z}^{-\hat{\nu}}[|\partial_x\mathbf{b}|^2+|\partial_x\mathbf{w}|^2+(\partial_x\psi)^2+(\partial_x\zeta)^2]+[O(\varepsilon_3 + \tilde{\delta})\tilde{z}^{-\hat{\nu}+2}+O(\tilde{\delta})\tilde{z}^{-\hat{\nu}}e^{-cx}](\varphi^2+\psi^2+\zeta^2) + O(\varepsilon_3+\tilde{\delta}^{\frac{1}{2}})\tilde{z}^{-\hat{\nu}+1}(|\mathbf{w}|^2+|\mathbf{b}|^2)+O(|\varphi|+|\psi|+|\zeta|)\tilde{z}^{-\hat{\nu}+1}(\varphi^2+\psi^2+\zeta^2). \tag{2.53}$$

Then using (2.52), (2.53) and boundary condition $u_b<0$ and (2.3), and integrate (2.44) over $\mathbb{R}_+$, and take $\tilde{\delta}$ and ε_3 small enough to derive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}w(x)\eta\mathrm{d}x+\tilde{z}(0)^{-\hat{\nu}}R\rho(0,t)\theta_b|u_b|\phi\left(\frac{\tilde{\rho}}{\rho}\right)(0,t)\\&\quad+c\int_{\mathbb{R}_+}\tilde{z}^{-\hat{\nu}+2}(\varphi^2+\psi^2+\zeta^2+|\mathbf{b}|^2)\mathrm{d}x\\&\quad+c\int_{\mathbb{R}_+}\tilde{z}^{-\hat{\nu}+1}(|\mathbf{w}|^2+|\mathbf{b}|^2)\mathrm{d}x+c\int_{\mathbb{R}_+}\tilde{z}^{-\hat{\nu}}(|\ \partial_x\mathbf{b}|^2\\&\quad+|\partial_x\mathbf{w}|^2+(\partial_x\psi)^2+(\partial_x\zeta)^2)\mathrm{d}x\\&\leqslant C\int_{\mathbb{R}_+}\tilde{\delta}\tilde{z}^{-\hat{\nu}}e^{-cx}(\varphi^2+\psi^2+\zeta^2)\mathrm{d}x+C\int_{\mathbb{R}_+}\tilde{z}^{-\hat{\nu}+1}(|\varphi|^3+|\psi|^3+|\zeta|^3)\mathrm{d}x.\end{aligned} \tag{2.54}$$

Combining the above estimates with (A.15), lemma 2.1 and (2.32), then multiplying the resulting inequality by $(1+t)^{\xi}$ and integrating in τ over $[0,t]$ for any $0\leqslant t\leqslant T$, we have

$$\begin{aligned}&(1+t)^{\xi}\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|^2_{\hat{\nu},\tilde{\delta}}+\tilde{\delta}^2\int_0^t(1+\tau)^{\xi}\|[\varphi,\psi,\zeta,\mathbf{b}](\tau)\|^2_{\hat{\nu}-2,\tilde{\delta}}\mathrm{d}\tau\\&\quad+\int_0^t(1+\tau)^{\xi}\varphi^2(0,\tau)\mathrm{d}\tau+\tilde{\delta}\int_0^t(1+\tau)^{\xi}\|[\mathbf{w},\mathbf{b}](\tau)\|^2_{\hat{\nu}-1,\tilde{\delta}}\mathrm{d}\tau\\&\quad+\int_0^t(1+\tau)^{\xi}\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu},\tilde{\delta}}\mathrm{d}\tau\\&\leqslant C\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat{\lambda},\tilde{\delta}}+\xi\int_0^t(1+\tau)^{\xi-1}\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu},\tilde{\delta}}\mathrm{d}\tau.\end{aligned} \tag{2.55}$$

Step 2. *The first order derivative estimates.*

We only show the estimate for $\partial_x\varphi$ as the other estimates for $\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta]$ can be established by similar computations. The summation of (2.15) and (2.16) further implies

$$\begin{aligned}
&\partial_t\left(\psi\partial_x\varphi+\lambda\frac{(\partial_x\varphi)^2}{2\rho^2}\right)+\partial_x\left[\frac{\lambda u}{2\rho^2}(\partial_x\varphi)^2-\psi\partial_t\varphi\right]+\left[\lambda\partial_x\tilde{u}\frac{(\partial_x\varphi)^2}{\rho^2}+\frac{R\theta}{\rho}(\partial_x\varphi)^2\right]\\
&=-\partial_x\psi\partial_t\varphi+\frac{\lambda\partial_x\tilde{u}}{2}(\partial_x\varphi)^2\rho^{-2}-\frac{R\tilde{\rho}}{\rho}\partial_x\zeta\partial_x\varphi-R\varphi\frac{\partial_x\theta}{\rho}\partial_x\varphi-R\partial_x\tilde{\rho}\frac{\zeta}{\rho}\partial_x\varphi\\
&\quad-u\partial_x\psi\partial_x\varphi-\tilde{u}\partial_x\tilde{u}\frac{\varphi}{\rho}\partial_x\varphi-\partial_x\tilde{u}\psi\partial_x\varphi+\frac{\lambda}{2}\partial_x\psi(\partial_x\varphi)^2\rho^{-2}-\frac{\lambda\partial_x\varphi}{\rho^2}\partial_x\rho\partial_x\psi\\
&\quad-\lambda\partial_x^2\tilde{u}\varphi\frac{\partial_x\varphi}{\rho^2}-\lambda\partial_x\tilde{\rho}\partial_x\psi\frac{\partial_x\varphi}{\rho^2}-\lambda\partial_x^2\tilde{\rho}\psi\frac{\partial_x\varphi}{\rho^2}-\frac{\partial_x|\mathbf{b}|^2}{2\rho}\partial_x\varphi=:\tilde{R},
\end{aligned}\tag{2.56}$$

where we have used $R(\rho\theta-\tilde{\rho}\tilde{\theta})=R\theta\varphi+R\tilde{\rho}\zeta$ and $\tilde{R}$ denotes all terms on the right of (2.56). Then, multiplying (2.56) by the weight function $W_{\hat{\nu},\tilde{\delta}}$ and integrating resulting equality over $\mathbb{R}_+$ to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+}[W_{\hat{\nu},\tilde{\delta}}(\psi\partial_x\varphi+\lambda\frac{(\partial_x\varphi)^2}{2\rho^2})]\mathrm{d}x+\frac{\lambda|u_b|}{2\rho(0,t)^2}(\partial_x\varphi)^2(0,t)+\int_{\mathbb{R}_+}W_{\hat{\nu},\tilde{\delta}}\frac{R\theta}{\rho}(\partial_x\varphi)^2\mathrm{d}x\\
&=\underbrace{\hat{\nu}\tilde{\delta}\int_{\mathbb{R}_+}W_{\hat{\nu}-1,\tilde{\delta}}\frac{\lambda u}{2\rho^2}(\partial_x\varphi)^2\mathrm{d}x}_{K_1}\underbrace{-\lambda\partial_x\tilde{u}\int_{\mathbb{R}_+}W_{\hat{\nu},\tilde{\delta}}\frac{(\partial_x\varphi)^2}{\rho^2}\mathrm{d}x}_{K_2}\\
&\quad\underbrace{-\hat{\nu}\tilde{\delta}\int_{\mathbb{R}_+}W_{\hat{\nu}-1,\tilde{\delta}}\psi\partial_t\varphi\mathrm{d}x}_{K_3}+\underbrace{\int_{\mathbb{R}_+}W_{\hat{\nu},\tilde{\delta}}\tilde{R}\mathrm{d}x}_{K_4}.
\end{aligned}\tag{2.57}$$

Applying the Sobolev's inequality, the Young's inequality and the Cauchy-Schwarz's inequality with $0<\eta<1$ and using (1.17), (A.15), and integration by parts, one has

$$|K_1|+|K_2|\leqslant C\tilde{\delta}\|\partial_x\varphi\|^2_{\hat{\nu},\tilde{\delta}},$$

$$\begin{aligned}
K_3&=-\hat{\nu}(\hat{\nu}-1)\tilde{\delta}^2\int_{\mathbb{R}_+}W_{\hat{\nu}-2,\tilde{\delta}}[\psi(\rho u-\tilde{\rho}\tilde{u})]\mathrm{d}x-\hat{\nu}\tilde{\delta}\int_{\mathbb{R}_+}W_{\hat{\nu}-1,\tilde{\delta}}[\partial_x\psi(\rho u-\tilde{\rho}\tilde{u})]\mathrm{d}x\\
&\leqslant C\tilde{\delta}^2\|[\varphi,\psi]\|^2_{\hat{\nu}-2,\tilde{\delta}}+C\|\partial_x\psi\|^2_{\hat{\nu},\tilde{\delta}},
\end{aligned}$$

$$\begin{aligned}
K_4&\leqslant C\tilde{\delta}^2\|[\varphi,\psi,\zeta]\|^2_{\hat{\nu}-2,\tilde{\delta}}+\eta\|\partial_x\varphi\|^2_{\hat{\nu},\tilde{\delta}}+C_\eta\|\partial_x[\psi,\zeta]\|^2_{\hat{\nu},\tilde{\delta}}\\
&\quad+C\|\partial_x\psi\|_\infty\|\partial_x\varphi\|^2_{\hat{\nu},\tilde{\delta}}+C\|\mathbf{b}\|_\infty\|\partial_x[\mathbf{b},\varphi]\|^2_{\hat{\nu},\tilde{\delta}}\\
&\leqslant C\tilde{\delta}^2\|[\varphi,\psi,\zeta]\|^2_{\hat{\nu}-2,\tilde{\delta}}+\eta\|\partial_x\varphi\|^2_{\hat{\nu},\tilde{\delta}}\\
&\quad+C_\eta\|\partial_x[\psi,\zeta]\|^2_{\hat{\nu},\tilde{\delta}}+(\|\partial_x\psi\|+\|\partial_x^2\psi\|)\|\partial_x\varphi\|^2_{\hat{\nu},\tilde{\delta}}+C\varepsilon_3\|\partial_x[\mathbf{b},\varphi]\|^2_{\hat{\nu},\tilde{\delta}}\\
&\leqslant C\tilde{\delta}^2\|[\varphi,\psi,\zeta]\|^2_{\hat{\nu}-2,\tilde{\delta}}+\eta\|\partial_x\varphi\|^2_{\hat{\nu},\tilde{\delta}}+C_\eta\|\partial_x[\psi,\zeta]\|^2_{\hat{\nu},\tilde{\delta}}+C\varepsilon_3\|\partial_x[\mathbf{b},\varphi,\partial_x\psi]\|^2_{\hat{\nu},\tilde{\delta}}.
\end{aligned}$$

Inserting the above estimates for K_l $(1\leqslant l\leqslant 4)$ into (2.57) and multiplying the resulting inequality by $(1+t)^\xi$ and integrating in τ over $[0,t]$ for any $0\leqslant t\leqslant T$, using (2.55), the Cauchy-Schwarz's inequality, and then choosing $\varepsilon_3, \tilde{\delta}$ and η suitably small, one has

$$
\begin{aligned}
&(1+t)^{\xi}\|\partial_x\varphi(t)\|^2_{\hat{\nu},\tilde{\delta}} + \int_0^t (1+\tau)^{\xi}\|\partial_x\varphi\|^2_{\hat{\nu},\tilde{\delta}}\mathrm{d}\tau + \int_0^t (1+\tau)^{\xi}(\partial_x\varphi)^2(0,\tau)\mathrm{d}\tau \\
&\leqslant C(\|[\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat{\lambda},\tilde{\delta}} + \|\varphi_0\|^2_{\hat{\lambda},\tilde{\delta},1}) + C\varepsilon_3\int_0^t (1+\tau)^{\xi}\|\partial_x^2\psi\|^2_{\hat{\nu},\tilde{\delta}}(\tau)\mathrm{d}\tau \\
&\quad + \xi\int_0^t (1+\tau)^{\xi-1}(\|\varphi(\tau)\|^2_{\hat{\nu},\tilde{\delta},1} + \|[\psi,\zeta,\mathbf{w},\mathbf{b}](\tau)\|^2_{\hat{\nu},\tilde{\delta}})\mathrm{d}\tau. \qquad (2.58)
\end{aligned}
$$

The estimates for $\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta]$ are obtained by similar computations to Step 3 for the supersonic case $M_+ > 1$. Thus we have

$$
\begin{aligned}
&(1+t)^{\xi}\|\partial_x[\psi,\mathbf{w},\mathbf{b},\zeta](t)\|^2_{\hat{\nu},\tilde{\delta}} + \int_0^t (1+\tau)^{\xi}\|\partial_x^2[\psi,\mathbf{w},\mathbf{b},\zeta]\|^2_{\hat{\nu},\tilde{\delta}}\mathrm{d}\tau \\
&\leqslant C\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|^2_{\hat{\lambda},\tilde{\delta},1} + \xi\int_0^t (1+\tau)^{\xi-1}\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](\tau)\|^2_{\hat{\nu},\tilde{\delta},1}\mathrm{d}\tau. \quad (2.59)
\end{aligned}
$$

Proof of proposition 2.4. Now we are ready to prove proposition 2.4. Summing up (2.55), (2.58) and (2.59) together, and similarly as in [**11**, **23**], applying an induction to the resulting inequality gives the desired estimate (2.33) when we take $\tilde{\delta}$ and ε_3 small enough. Thus the proof of proposition 2.4 is complete. □

3. Global existence and large time behaviour

We are now in a position to complete the proof of theorem 1.3.

Proof of theorem 1.3. Here we omit the proof of theorem 1.3 (i) and (ii). We only prove theorem 1.3 (iii) for the transonic case $M_+ = 1$. In view of the energy estimates obtained in proposition 2.4, one sees that

$$
\|[\varphi,\psi,\mathbf{w},\mathbf{b},\zeta](t)\|_1 \leqslant C(1+t)^{-\lambda/4}\|[\varphi_0,\psi_0,\mathbf{w}_0,\mathbf{b}_0,\zeta_0]\|_{\lambda,\tilde{\delta},1}. \qquad (3.1)
$$

The global existence of the solution to the initial boundary value problem (2.1), (2.2) and (2.3) follows from the standard continuation argument based on the local existence [**22**] and the *a priori* estimate (2.33). Moreover, (3.1) implies (1.22) with the aid of the Sobolev's inequality. Thus we complete the proof of theorem 1.3 (iii). □

Appendix A

In this appendix, we will give the detailed proofs of lemmas 1.1 and 1.2 for the completeness. One can see [**13**] for reference about proofs of lemmas 1.1 and 1.2 in the dimensionless form. Since the process of proof is borrowed from [**13**], so we use the same notations as in [**13**] for convenience.

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Proof of lemma 1.1. Define $(\bar{u}, \bar{\theta})(x) := (\tilde{u}, \tilde{\theta})(x) - (u_+, \theta_+)$, and then we rewrite system (1.12) into the following system

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\bar{u}\\ \bar{\theta}\end{pmatrix} = J\begin{pmatrix}\bar{u}\\ \bar{\theta}\end{pmatrix} + \begin{pmatrix}\bar{f}(\bar{u}, \bar{\theta})\\ \bar{g}(\bar{u}, \bar{\theta})\end{pmatrix}, (\bar{u}, \bar{\theta})(0) = (u_- - u_+, \theta_- - \theta_+),$$
$$(\bar{u}, \bar{\theta})(+\infty) = (0, 0), \tag{A.1}$$

where the matrix J and nonlinear terms $\bar{f}$ and $\bar{g}$ are defined by

$$J := \begin{pmatrix} \dfrac{(M_+^2\gamma - 1)p_+}{\lambda u_+} & \dfrac{R\rho_+}{\lambda} \\ \dfrac{p_+}{\kappa} & \dfrac{R\rho_+ u_+}{\kappa(\gamma-1)} \end{pmatrix}, \quad \bar{f}(\bar{u}, \bar{\theta}) := \frac{p_+ \bar{u}^2}{\lambda u_+(\bar{u} + u_+)} - \frac{R\rho_+ \bar{u}\bar{\theta}}{\lambda(\bar{u} + u_+)},$$
$$\bar{g}(\bar{u}, \bar{\theta}) := -\frac{\rho_+ u_+ \bar{u}^2}{2\kappa}.$$

To prove the existence of the stationary solutions $(\tilde{\rho}, \tilde{u}, \tilde{\theta})$, it suffices to show the existence of the solutions $(\bar{u}, \bar{\theta})$ to the boundary value problem (A.1). To this end, we first diagonalize the system (A.1). Let λ_1 and λ_2 be eigenvalues of the Jacobian matrix J. Since we see later that J has real eigenvalues, without loss of generality, we assume $\lambda_1 \geqslant \lambda_2$. Let r_1 and r_2 be eigenvectors of J corresponding to λ_1 and λ_2, respectively, and let $P := (r_1, r_2)$ be a matrix. Furthermore, using the matric P, we employ new unknown functions $U(x)$ and $\Theta(x)$ defined by

$$\begin{pmatrix}U(x)\\ \Theta(x)\end{pmatrix} := P^{-1}\begin{pmatrix}\bar{u}(x)\\ \bar{\theta}(x)\end{pmatrix}. \tag{A.2}$$

We also define a corresponding boundary data and nonlinear terms by

$$\begin{pmatrix}U_-\\ \Theta_-\end{pmatrix} := P^{-1}\begin{pmatrix}u_- - u_+\\ \theta_- - \theta_+\end{pmatrix}, \quad \begin{pmatrix}f(U,\Theta)\\ g(U,\Theta)\end{pmatrix} := P^{-1}\begin{pmatrix}\bar{f}(\bar{u}, \bar{\theta})\\ \bar{g}(\bar{u}, \bar{\theta})\end{pmatrix}.$$

Using these notations, we rewrite the system (A.1) to that for (U, Θ) in a diagonal form as

$$\frac{d}{\mathrm{d}x}\begin{pmatrix}U\\ \Theta\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}U\\ \Theta\end{pmatrix} + \begin{pmatrix}f(U,\Theta)\\ g(U,\Theta)\end{pmatrix}, \tag{A.3}$$

$$\lim_{x\to+\infty}(U, \Theta)(x) = (0, 0), \quad (U, \Theta)(0) = (U_-, \Theta_-). \tag{A.4}$$

Note that solving the problem (A.3) and (A.4) immediately yields the existence of a solution to (1.12). Hereafter, we consider the existence of a solution to the problem (A.3) and (A.4).

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(i) Firstly, we consider the case $M_+ > 1$. Since a discriminant of an eigen-equation of the matrix J satisfies

$$(TrJ)^2 - 4detJ = \left[\frac{R}{\kappa(\gamma-1)} - \frac{M_+^2\gamma - 1}{\lambda M_+^2\gamma}\right]^2 \rho_+^2 u_+^2 + \frac{4R}{\lambda\kappa M_+^2\gamma}\rho_+^2 u_+^2 > 0.$$

Moreover, by $u_+ < 0$ and $M_+ > 1$, we see

$$\lambda_1 + \lambda_2 = TrJ = \left[\frac{R}{\kappa(\gamma-1)} + \frac{M_+^2\gamma - 1}{\lambda M_+^2\gamma}\right]\rho_+ u_+ < 0,$$

$$\lambda_1\lambda_2 = detJ = \frac{(M_+^2 - 1)\gamma R\rho_+ p_+}{\lambda\kappa(\gamma-1)} > 0,$$

which show that $\lambda_1 < 0$ and $\lambda_2 < 0$. Thus, the equilibrium point $(0,0)$ of (A.3) is asymptotically stable. Consequently, if $|(U_-, \Theta_-)|$ is sufficiently small, the problem (A.3) and (A.4) has a unique smooth solution (U, Θ) satisfying

$$|\partial_x^k(U(x), \Theta(x))| \leqslant C\tilde{\delta}e^{-cx}, \quad k = 0,\ 1,\ 2, \ldots. \tag{A.5}$$

(ii) Secondly, we consider the case $M_+ = 1$. Since the matrix J satisfies

$$\lambda_1 + \lambda_2 = TrJ = \left[\frac{R}{\kappa(\gamma-1)} + \frac{\gamma-1}{\lambda\gamma}\right]\rho_+ u_+ < 0, \quad \lambda_1\lambda_2 = detJ = 0,$$

which show that $\lambda_1 = 0$ and $\lambda_2 = [((R/(\kappa(\gamma-1))) + ((\gamma-1)/(\lambda\gamma))]\rho_+ u_+ < 0$. The eigenvectors of λ_1 and λ_2 are explicitly given by

$$r_1 = \begin{pmatrix} \dfrac{u_+}{\theta_+(\gamma-1)} \\ -1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} -1 \\ -\dfrac{\lambda u_+}{\kappa(\gamma-1)} \end{pmatrix}.$$

Thus the matrix

$$P = \begin{pmatrix} \dfrac{u_+}{\theta_+(\gamma-1)} & -1 \\ -1 & -\dfrac{\lambda u_+}{\kappa(\gamma-1)} \end{pmatrix} \tag{A.6}$$

satisfies $detP = -((\lambda\gamma R + \kappa(\gamma-1)^2)/(\kappa(\gamma-1)^2)) < 0$. It is easy to compute that

$$P^{-1} = \begin{pmatrix} \dfrac{\lambda(\gamma-1)u_+}{\lambda\gamma R + \kappa(\gamma-1)^2} & -\dfrac{\kappa(\gamma-1)^2}{\lambda\gamma R + \kappa(\gamma-1)^2} \\ -\dfrac{\kappa(\gamma-1)^2}{\lambda\gamma R + \kappa(\gamma-1)^2} & -\dfrac{R\kappa\gamma(\gamma-1)}{[\lambda\gamma R + \kappa(\gamma-1)^2]u_+} \end{pmatrix}, \tag{A.7}$$

such that

$$P^{-1}JP = \begin{pmatrix} 0 & 0 \\ 0 & \left[\dfrac{R}{\kappa(\gamma-1)} + \dfrac{\gamma-1}{\lambda\gamma}\right]\rho_+ u_+ \end{pmatrix}. \tag{A.8}$$

We can see that the nonlinear terms f and g satisfy

$$f(U,\Theta)=\frac{\gamma^2(\gamma+1)R^2\rho_+}{2(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]u_+}U^2+O(|U|^3+|U\Theta|+|\Theta|^2), \quad (A.9)$$

and

$$g(U,\Theta)=\frac{\gamma R^2\rho_+[\lambda R\gamma^2-2\kappa(\gamma-1)^2-2\kappa(\gamma-1)]}{2\lambda(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]u_+^2}U^2+O(|U|^3+|U\Theta|+|\Theta|^2). \quad (A.10)$$

Recalling $\lambda_1=0$ and $\lambda_2=[((R/(\kappa(\gamma-1)))+((\gamma-1)/(\lambda\gamma))]\rho_+u_+<0$ in (A.3), thus the problem (A.3) and (A.4) has a local centre manifold $\Theta=h^c(U)$ and a local stable manifold $U=h^s(\Theta)$ corresponding to the eigenvalues $\lambda_1=0$ and $\lambda_2=[((R)/(\kappa(\gamma-1)))+((\gamma-1)/(\lambda\gamma))]\rho_+u_+<0$, respectively. In order to show the existence of the solution, we have to examine dynamics on the centre manifold. To this end, we employ a solution $\tilde z=\tilde z(x)$ to (A.3) restricted on the centre manifold satisfying the equation

$$\tilde z_x=f(\tilde z,h^c(\tilde z)). \quad (A.11)$$

By virtue of the centre manifold theory in [**1**], there exists a solution $\tilde z$ to (A.11) such that the solution (U,Θ) to (A.3) and (A.4) is given by

$$U(x)=\tilde z(x)+O(\tilde\delta e^{-cx}),\quad \Theta(x)=h^c(\tilde z(x))+O(\tilde\delta e^{-cx}). \quad (A.12)$$

Therefore, to obtain the solution (U,Θ) to (A.3) and (A.4), it suffices to show the existence of the solution to (A.11) satisfying $\tilde z_x\to 0$ as $x\to+\infty$. Substituting (A.9) into (A.11), we deduce (A.11) to

$$\tilde z_x=\frac{\gamma^2(\gamma+1)R^2\rho_+}{2(\gamma-1)[\lambda\gamma R+\kappa(\gamma-1)^2]u_+}\tilde z^2+O(|\tilde z|^3), \quad (A.13)$$

which yields that $\tilde z$ is monotonically decreasing for sufficiently small $\tilde z$. Thus, to satisfy $\tilde z_x\to 0$ as $x\to+\infty$, the boundary data $\tilde z(0)$ should be positive. Namely, for the existence of the solution (U,Θ), the boundary data (U_-,Θ_-) should be located in the right region from the local stable manifold, that is, (U_-,Θ_-) should satisfy a condition

$$U_-\geqslant h^s(\Theta_-). \quad (A.14)$$

From (A.13), we also see that the solution $\tilde z$ satisfies

$$0<c\frac{\tilde\delta}{1+\tilde\delta x}\leqslant\tilde z(x)\leqslant C\frac{\tilde\delta}{1+\tilde\delta x},\quad |\partial_x^k\tilde z(x)|\leqslant C\frac{\tilde\delta^{k+1}}{(1+\tilde\delta x)^{k+1}}. \quad (A.15)$$

Combining (A.12) and (A.15) with using $h^c(\tilde z)=O(\tilde z^2)$, we have the following decay property of (U,Θ):

$$|\partial_x^k(U,\Theta)|\leqslant C\frac{\tilde\delta^{k+1}}{(1+\tilde\delta x)^{k+1}}+C\tilde\delta e^{-cx},\quad k=0,\ 1,\ 2,\dots \quad (A.16)$$

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(iii) Thirdly, we consider the case $M_+ < 1$. For this case, the

eigenvalues of the matrix J are $\lambda_1 > 0$ and $\lambda_2 < 0$, so that there exist a local unstable manifold and a local stable manifold corresponding to the eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$, respectively. Therefore the problem (A.3) and (A.4) has a solution (U, Θ) satisfying (A.5) if the boundary data is located on the stable manifold, that is

$$U_- = h^s(\Theta_-). \tag{A.17}$$

Finally, we precisely define the regions $\mathcal{M}^0$ and $\mathcal{M}^-$ in lemma 1.1. Define $(\hat{U}(u,\theta))$ and $(\hat{\Theta}(u,\theta))$ by

$$\begin{pmatrix} \hat{U}(u,\theta) \\ \hat{\Theta}(u,\theta) \end{pmatrix} := P^{-1} \begin{pmatrix} u - u_+ \\ \theta - \theta_+ \end{pmatrix}. \tag{A.18}$$

Then we have $U(x) = \hat{U}(\tilde{u}, \tilde{\theta})$ and $\Theta(x) = \hat{\Theta}(\tilde{u}, \tilde{\theta})$ from (A.2) and (A.18). Then, define the regions $\mathcal{M}^0$ and $\mathcal{M}^-$ by

$$\mathcal{M}^0 := \left\{ (u,\theta) \in \mathcal{M}^+; \ \ \hat{\Theta}(u,\theta) = h^c(\hat{U}(u,\theta)), \ \ \hat{U}(u,\theta) \geqslant h^s(\hat{\Theta}(u,\theta)) \right\}, \tag{A.19}$$

and

$$\mathcal{M}^- := \left\{ (u,\theta) \in \mathcal{M}^+; \ \ \hat{U}(u,\theta) = h^s(\hat{\Theta}(u,\theta)) \right\}. \tag{A.20}$$

We see that conditions (A.14) and (A.17) are equivalent to $(u_-, \theta_-) \in \mathcal{M}^0$ and $(u_-, \theta_-) \in \mathcal{M}^-$ in lemma 1.1, respectively. Thus combining this with the above analysis, we complete the proof of lemma 1.1. □

Finally, we give the detailed proof of lemma 1.2.

Proof of lemma 1.2. The estimates for $(\tilde{u}, \tilde{\theta})$ in (1.16) are obtained by using (A.12) and

$$\begin{pmatrix} \tilde{u} \\ \tilde{\theta} \end{pmatrix} := \begin{pmatrix} u_+ \\ \theta_+ \end{pmatrix} + P \begin{pmatrix} U(x) \\ \Theta(x) \end{pmatrix}, \quad P = \begin{pmatrix} \dfrac{u_+}{\theta_+(\gamma-1)} & -1 \\ -1 & -\dfrac{\lambda u_+}{\kappa(\gamma-1)} \end{pmatrix}, \tag{A.21}$$

which follows from (A.2). Due to the fact that $\tilde{\rho}\tilde{u} = \rho_+ u_+$, we have the estimates for $\tilde{\rho}$ in (1.16). By using (A.12) and (A.13), we have

$$U_x = \frac{\gamma^2(\gamma+1)R^2\rho_+}{2(\gamma-1)[\lambda\gamma R + \kappa(\gamma-1)^2]u_+} \tilde{z}^2 + O(|\tilde{z}|^3 + \tilde{\delta} e^{-cx}), \quad \Theta_x = O(|\tilde{z}|^3 + \tilde{\delta} e^{-cx}). \tag{A.22}$$

Differentiating (A.21) in x and substituting (A.22) yield the desired estimate (1.17). We also have the estimates $|\partial_x^k (U, \Theta)| = O(\tilde{z}^{k+1} + \tilde{\delta} e^{-cx})$ inductively, which give the estimate (1.18) due to (A.21). Therefore, we complete the proof of lemma 1.2. □

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