Ergod. Th. & Dynam. Sys., (2021), **41**, 3288–3306
doi:10.1017/etds.2020.106

L^q-spectra of measures on planar non-conformal attractors

KENNETH J. FALCONER, JONATHAN M. FRASER and LAWRENCE D. LEE

School of Mathematics & Statistics, University of St Andrews, St Andrews, KY16 9SS, UK
(e-mail: kjf@st-andrews.ac.uk, jmf32@st-andrews.ac.uk, ldl@st-andrews.ac.uk)

(*Received* 19 *May* 2020 *and accepted in revised form* 7 *August* 2020)

Abstract. We study the L^q-spectrum of measures in the plane generated by certain nonlinear maps. In particular, we consider attractors of iterated function systems consisting of maps whose components are $C^{1+\alpha}$ and for which the Jacobian is a lower triangular matrix at every point subject to a natural domination condition on the entries. We calculate the L^q-spectrum of Bernoulli measures supported on such sets by using an appropriately defined analogue of the singular value function and an appropriate pressure function.

Key words: L^q-spectrum, generalized q-dimensions, non-conformal attractor, modified singular value function, self-affine measure
2020 Mathematics subject classification: 28A80, 37C45 (Primary); 15A18 (Secondary)

1. *Introduction*

The study of fractals generated by iterated function systems (IFSs) consisting of nonlinear maps, which can often be identified with repellers of corresponding dynamical systems, has a rich history. In 1994, Falconer [4] calculated the dimension of mixing repellers for non-conformal mappings. To do this he applied techniques from thermodynamic formalism, in particular developing a subadditive version of the theory and also a 'bounded distortion' principle. Further work on nonlinear IFSs was done by Hu who in 1996 calculated the box and Hausdorff dimensions of invariant sets of expanding C^2 maps [13]. More recent work includes that of Cao *et al*. [3] as well as that of Feng and Simon [10]. Cao *et al*. studied the Hausdorff dimension of non-conformal repellers corresponding to $C^{1+\alpha}$ maps. By studying certain subadditive and superadditive pressures they were able to obtain bounds for the Hausdorff dimension of repellers. Feng and Simon proved that the upper box dimension of the attractor of any C^1 IFS in $\mathbb{R}^n$ is bounded above by its singularity dimension.

Other notable work in this area was done in 2007 by Manning and Simon [16] who investigated the subadditive pressure of nonlinear maps developed by Falconer and considered cases where bounded distortion does not hold. The work of Falconer as well as that of Manning and Simon and also Miao [9] was generalized by Barany [1] who used the subadditive pressure to calculate the Hausdorff dimension of fractals generated by IFSs whose maps have triangular Jacobians. Other authors to have considered IFSs generated by triangular mappings include Kolossváry and Simon [15]. In particular, they looked at a family of planar self-affine carpets with overlaps generated by lower triangular matrices and considered whether dimension drop occurs.

In terms of multifractal analysis Falconer studied the L^q-spectrum of self-affine measures [6] and almost self-affine measures [7]. In the case of self-affine measures he was able to establish a generic formula in the region $1 < q \leq 2$ in terms of a subadditive pressure expression. Barral and Feng [2] then generalized this in certain cases to calculate the L^q-spectrum for a wider range of q and were also able to verify the multifractal formalism in some cases. For results on the L^q-spectrum of measures on self-affine carpets, see Feng and Wang [11] and Fraser [12].

In this paper we calculate the L^q-spectra of Bernoulli measures in the plane supported on sets generated by IFSs consisting of $C^{1+\alpha}$ maps whose Jacobian matrices are lower triangular. Our approach is based on setting up certain 'almost additive' pressure functionals. As a corollary we calculate the box dimension of the supports of these measures. Our results on L^q-dimensions are new, even in the (non-diagonal) self-affine case.

Standard background on IFSs may be found, for example, in [8, 14]. We introduce further definitions, in particular *nonlinear attractors* and *nonlinear measures* which have a particular meaning in this paper as shorthand for the types of non-conformal attractors and measures we consider.

Definition 1.1. (Nonlinear attractor) Let $\mathcal{I}$ be a finite index set with $|\mathcal{I}| \geq 2$ and let $\{S_i\}_{i\in\mathcal{I}}$ be an IFS consisting of contractions on $[0, 1]^2$. Suppose also that each $S_i : [0, 1]^2 \to [0, 1]^2$ is of the form $S_i(a_1, a_2) = (f_i(a_1), g_i(a_1, a_2))$, where the f_i and g_i are $C^{1+\alpha}$ contractions $(0 < \alpha \leq 1)$ on $[0, 1]$ and $[0, 1]^2$ respectively, that is, their derivatives satisfy Hölder conditions of exponent α. (We use one-sided derivatives on the boundary of $[0, 1]^2$.) By Hutchinson's theorem [14] there is a unique non-empty, compact set F satisfying

$$F = \bigcup_{i\in\mathcal{I}} S_i(F)$$

which for the purposes of this paper we call the *nonlinear attractor* associated to $\{S_i\}_{i\in\mathcal{I}}$.

We are interested in the natural Bernoulli measures supported on nonlinear attractors F; see [5, 14].

Definition 1.2. (Nonlinear measure) Let F be a nonlinear attractor given by $\{S_i\}_{i\in\mathcal{I}}$ on $[0, 1]^2$, and let $\{p_i\}_{i\in\mathcal{I}}$ be a probability vector with each $p_i \in (0, 1)$. Then there is a unique Borel probability measure μ supported on F which satisfies

$$\mu = \sum_{i\in\mathcal{I}} p_i \, \mu \circ S_i^{-1}$$

which we call the *nonlinear measure* associated to $\{S_i\}_{i\in\mathcal{I}}$ and $\{p_i\}_{i\in\mathcal{I}}$.

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Our aim is to calculate the L^q*-spectra* of these measures. Let $\delta > 0$ and write $\mathcal{D}_\delta$ to denote the set of closed cubes in the δ-mesh on $\mathbb{R}^n$ that have positive μ-measure. Write

$$\mathcal{D}_\delta^q(\mu) = \sum_{Q \in \mathcal{D}_\delta} \mu(Q)^q. \tag{1.1}$$

Definition 1.3. If μ is a compactly supported Borel probability measure on $\mathbb{R}^n$ then for $q \geq 0$ the *upper* and *lower* L^q*-spectrum* of μ are defined to be

$$\overline{\tau}_\mu(q) = \overline{\lim}_{\delta \to 0} \frac{\log \mathcal{D}_\delta^q(\mu)}{-\log \delta} \tag{1.2}$$

and

$$\underline{\tau}_\mu(q) = \underline{\lim}_{\delta \to 0} \frac{\log \mathcal{D}_\delta^q(\mu)}{-\log \delta}, \tag{1.3}$$

respectively. If these values coincide then define the L^q*-spectrum* of μ, denoted by $\tau_\mu(q)$, to be their common value.

The L^q-spectrum can be thought of as an analogue of box-counting dimension for measures; indeed, the upper and lower box dimensions of the support of μ are easily seen to be given by $\overline{\tau}_\mu(0)$ and $\underline{\tau}_\mu(0)$, respectively. Note that $\tau_\mu(1) = 0$ (as μ is a probability measure) and that τ_μ is decreasing in q. Furthermore, the L^q-spectrum is central in multifractal analysis: in certain key cases the fine multifractal spectrum of μ can be obtained by taking the Legendre transform of τ_μ, in which case we say that the multifractal formalism holds (see, for instance, [8, 18]). Another useful property of the L^q-spectrum is that if it is differentiable at 1 then the Hausdorff dimension of the measure μ is given by $\dim_H \mu = -\tau_\mu'(1)$ [17].

For our calculations of L^q-spectra for nonlinear measures we require the following separation condition for the IFS.

Definition 1.4. (Rectangular open set condition) An IFS on $\mathbb{R}^2$ satisfies the *rectangular open set condition* (ROSC) if $\{S_i((0,1)^2)\}_{i \in \mathcal{I}}$ are pairwise disjoint subsets of the open unit square $(0,1)^2$.

Fraser [12] calculated the L^q-spectrum $\tau_\mu(q)$ of a class of *self-affine* measures in the plane. We broadly follow his approach, although there are several technical challenges which arise due to the nonlinearity, as well as the maps giving rise to non-diagonal Jacobians.

Our main result, Theorem 3.8, requires some more assumptions and technical details, in particular that the $\{S_i\}_{i \in \mathcal{I}}$ contract more in the vertical direction than in the horizontal direction. The theorem is stated fully in §3, but the essence of it is captured in the following version.

THEOREM 1.5. *Let μ be a nonlinear measure which satisfies a natural domination condition and the ROSC and let $q \geq 0$. Then there exists a function $\gamma : [0, \infty) \to \mathbb{R}$, defined in terms of the probability vector $\{p_i\}_{i \in \mathcal{I}}$, the singular values of Jacobian matrices*

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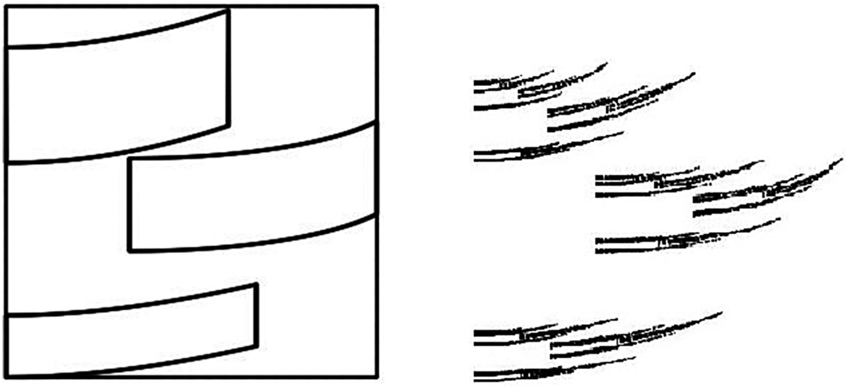

FIGURE 1. The image of the unit square $[0, 1]^2$ under the maps S_1, S_2 and S_3 is shown on the left; the IFS satisfies the ROSC (1.4). On the right is the corresponding nonlinear attractor.

of iterates of the $\{S_i\}_{i\in\mathcal{I}}$ *and the* L^q*-spectrum of the projection of* μ *onto the x-axis, such that*

$$\tau_\mu(q) = \gamma(q).$$

We set up the pressure formalism that enables us to define γ in §3 and prove the theorem in §4. A simple corollary is that if μ satisfies the ROSC then the box dimension of the support of μ is given by $\gamma(0)$.

2. *An example*

Here we provide an example of a nonlinear IFS and corresponding nonlinear attractor generated by three maps. The maps are

$$S_1(x, y) = \left(\frac{3x}{5} + \frac{3x^2}{40}, \frac{x^2}{12} + \frac{y}{6}\right),$$
$$S_2(x, y) = \left(\frac{4x}{5} - \frac{4x^3}{30} + \frac{1}{3}, \frac{x^2}{10} + \frac{y}{4} + \frac{17}{50}\right),$$
$$S_3(x, y) = \left(\frac{3x}{5}, \frac{x^2}{10} + \frac{y}{5} + \frac{y^3}{9} + \frac{26}{45}\right).$$

These maps satisfy the conditions for Theorems 1.5 and 3.8. The ROSC and domination condition (3.1) (stated formally in §3) are easy to check. Indeed, using Maple software gives, with $f_{i,x}$ and $g_{i,y}$ as at the start of §3,

$$\inf_{\mathbf{a}\in[0,1]^2} |f_{1,x}(\mathbf{a})| = 3/5 > \sup_{\mathbf{a}\in[0,1]^2}|g_{1,y}(\mathbf{a})| = 1/6 \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{1,y}(\mathbf{a})| = 1/6,$$
$$\inf_{\mathbf{a}\in[0,1]^2} |f_{2,x}(\mathbf{a})| = 2/5 > \sup_{\mathbf{a}\in[0,1]^2}|g_{2,y}(\mathbf{a})| = 1/4 \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{2,y}(\mathbf{a})| = 1/4,$$
$$\inf_{\mathbf{a}\in[0,1]^2} |f_{3,x}(\mathbf{a})| = 3/5 > \sup_{\mathbf{a}\in[0,1]^2}|g_{3,y}(\mathbf{a})| = 8/15 \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{3,y}(\mathbf{a})| = 1/5,$$

with $d = 1/6$, say. Thus any nonlinear measures supported on the attractor of this IFS would fall under the class considered. This example is displayed visually in Figure 1.

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3. *A singular value function and pressure*

In [12] Fraser introduced a *q-modified singular value function*. As he was dealing with self-affine measures he needed to consider the singular values of the linear part of each affine map in the IFS. In our nonlinear setting we shall instead consider singular values of Jacobian matrices.

Let $\{S_i\}_{i\in\mathcal{I}}$ be an IFS of the form in Definition 1.1. For $\mathbf{a} = (a_1, a_2) \in [0, 1]^2$ and $i \in \mathcal{I}$ we denote the derivative of S_i by $D_{\mathbf{a}}S_i$. Note that as each S_i is of the form $S_i(a_1, a_2) = (f_i(a_1), g_i(a_1, a_2))$, the Jacobian matrix of $D_{\mathbf{a}}S_i$ is a lower triangular matrix. To simplify notation we will write $S_i(\mathbf{a}) = (f_i(\mathbf{a}), g_i(\mathbf{a}))$, where $\mathbf{a} = (a_1, a_2)$, even though f does not depend on a_2. If we now write f_x for the derivative of f, and g_x and g_y for the partial derivatives of g, then

$$D_{\mathbf{a}}S_i = \begin{pmatrix} f_{i,x}(\mathbf{a}) & 0 \\ g_{i,x}(\mathbf{a}) & g_{i,y}(\mathbf{a}) \end{pmatrix}.$$

From now on we assume that the IFS satisfies the following *domination condition*, which is our key technical assumption.

Definition 3.1. (Domination condition) We say that the IFS $\{S_i\}_{i\in\mathcal{I}}$ satisfies the *domination condition* if for each map S_i the following inequalities on the derivatives hold:

$$\inf_{\mathbf{a}\in[0,1]^2} |f_{i,x}(\mathbf{a})| > \sup_{\mathbf{a}\in[0,1]^2} |g_{i,y}(\mathbf{a})| \geq \inf_{\mathbf{a}\in[0,1]^2} |g_{i,y}(\mathbf{a})| \geq d, \tag{3.1}$$

where $d > 0$.

Let

$$\eta := \sup_{i\in\mathcal{I},\mathbf{a},\mathbf{b}\in[0,1]^2} \left\{ \frac{|g_{i,y}(\mathbf{a})|}{|f_{i,x}(\mathbf{b})|} \right\} < 1, \tag{3.2}$$

using (3.1). In the obvious way we will say that μ and F satisfy the domination condition if their defining IFS does.

There is no requirement on $g_{i,x}$ to be non-zero; in particular, since this allows $g_{i,x}(\mathbf{a}) = 0$ for all $\mathbf{a}$ the class of measures we consider includes self-affine measures supported on Bedford-McMullen carpets, as well as measures supported on attractors of nonlinear 'diagonal' IFSs.

Let $\mathcal{I}^* = \bigcup_{k\geq 1} \mathcal{I}^k$ denote the set of all finite sequences with entries in $\mathcal{I}$. For $\mathfrak{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$, let $S_{\mathfrak{i}} = S_{i_1\ldots i_k} := S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}$ and let $p(\mathfrak{i}) = p_{i_1} p_{i_2} \ldots p_{i_k}$.

We write $0 < c < 1$ for the maximum contraction ratio of the S_i so, in particular,

$$|S_{i_1\ldots i_k}(\mathbf{a}) - S_{i_1\ldots i_k}(\mathbf{b})| \leq c^k |\mathbf{a} - \mathbf{b}| \quad ((i_1, \ldots, i_k) \in \mathcal{I}^k,\ \mathbf{a}, \mathbf{b} \in [0, 1]^2). \tag{3.3}$$

By the chain rule the Jacobian of the composed maps $S_{\mathfrak{i}}$ must be lower triangular, so let $f_{\mathfrak{i},x}(\mathbf{a})$, $g_{\mathfrak{i},x}(\mathbf{a})$, $g_{\mathfrak{i},y}(\mathbf{a})$ denote the entries of $D_{\mathbf{a}}S_{\mathfrak{i}}$, that is,

$$D_{\mathbf{a}}S_{\mathfrak{i}} = \begin{pmatrix} f_{\mathfrak{i},x}(\mathbf{a}) & 0 \\ g_{\mathfrak{i},x}(\mathbf{a}) & g_{\mathfrak{i},y}(\mathbf{a}) \end{pmatrix}. \tag{3.4}$$

We will show that the domination condition implies that these matrices satisfy a *bounded distortion* property which will be key in calculating the L^q-spectra.

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For $x, y \in \mathbb{R}^+$, we write $x \lesssim y$ to mean that $x \leq Cy$ for some absolute constant $C > 0$. If we wish to emphasize that this constant depends on some other parameter, θ say, we write $x \lesssim_\theta y$. If both $x \lesssim y$ and $y \lesssim x$ we write $x \asymp y$. In this case we say that x and y are *comparable*.

Using the chain rule, the diagonal entries of (3.4) can be written in terms of derivatives of the individual f_i and g_i as follows:

$$f_{\mathtt{i},x}(\mathbf{a}) = \prod_{j=1}^{k} f_{i_j,x}(S_{i_{j+1}\ldots i_k}\mathbf{a}), \quad g_{\mathtt{i},y}(\mathbf{a}) = \prod_{j=1}^{k} g_{i_j,y}(S_{i_{j+1}\ldots i_k}\mathbf{a}). \tag{3.5}$$

(Here and elsewhere, we make the natural convention that $S_{i_{k+1}} S_{i_k}$ is the identity.) Note that from (3.2), using these expansions,

$$\frac{|g_{\mathtt{i},y}(\mathbf{a})|}{|f_{\mathtt{i},x}(\mathbf{b})|} \leq \eta^k \tag{3.6}$$

for all $\mathtt{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ and all $\mathbf{a}, \mathbf{b} \in [0, 1]^2$. For the bottom left term direct expansion or induction gives

$$g_{\mathtt{i},x}(\mathbf{a}) = \sum_{j=1}^{k} G_j(\mathbf{a}) \tag{3.7}$$

where, using the chain rule,

$$\begin{aligned} G_j(\mathbf{a}) &= g_{i_1,y}(S_{i_2\cdots i_k}\mathbf{a}) \cdots g_{i_{j-1},y}(S_{i_j\cdots i_k}\mathbf{a}) g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a}) \\ &\quad \times f_{i_{j+1},x}(S_{i_{j+2}\cdots i_k}\mathbf{a}) \cdots f_{i_{k-1},x}(S_{i_k}\mathbf{a}) f_{i_k,x}(\mathbf{a}) \end{aligned} \tag{3.8}$$

$$= \left(\prod_{l=1}^{j-1} g_{i_l,y}(S_{i_{l+1}\cdots i_k}\mathbf{a})\right) g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a}) \left(\prod_{l=j+1}^{k} f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{a})\right) \tag{3.9}$$

$$= g_{i_1\cdots i_{j-1},y}(S_{i_j\cdots i_k}\mathbf{a}) g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a}) f_{i_{j+1}\cdots i_k,x}(S_{i_{j+2}\cdots i_k}\mathbf{a}). \tag{3.10}$$

The next two lemmas obtain estimates on the entries of (3.4) that are uniform in $\mathtt{i}$ and $\mathbf{a}$.

LEMMA 3.2. *There exists a constant $R > 0$ such that, for all $\mathtt{i} \in \mathcal{I}^*$ and all $\mathbf{a}, \mathbf{b} \in [0, 1]^2$,*

$$R^{-1} \leq \frac{|f_{\mathtt{i},x}(\mathbf{a})|}{|f_{\mathtt{i},x}(\mathbf{b})|}, \frac{|g_{\mathtt{i},y}(\mathbf{a})|}{|g_{\mathtt{i},y}(\mathbf{b})|} \leq R. \tag{3.11}$$

Proof. Note that since each f_{i_j} is a $C^{1+\alpha}$ map there is a number B such that

$$|f_{i,x}(\mathbf{a}') - f_{i,x}(\mathbf{b}')| \leq B|\mathbf{a}' - \mathbf{b}'|^\alpha$$

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for all $i \in \mathcal{I}$ and all $\mathbf{a}', \mathbf{b}' \in [0,1]^2$. For $\mathtt{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ and $\mathbf{a}, \mathbf{b} \in [0,1]^2$, identity (3.5) gives

$$\begin{aligned}
\frac{|f_{\mathtt{i},x}(\mathbf{a})|}{|f_{\mathtt{i},x}(\mathbf{b})|} &= \prod_{j=1}^{k} \frac{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|} \\
&= \prod_{j=1}^{k} \bigg(1 + \frac{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})| - |f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}\bigg) \\
&\le \prod_{j=1}^{k} \bigg(1 + \frac{||f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})| - |f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})||}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}\bigg) \\
&\le \prod_{j=1}^{k} \bigg(1 + \frac{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a}) - f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|}\bigg) \\
&\le \prod_{j=1}^{k} \bigg(1 + \frac{B|S_{i_{j+1}\cdots i_k}\mathbf{a} - S_{i_{j+1}\cdots i_k}\mathbf{b}|^\alpha}{d}\bigg) \\
&\le \prod_{j=1}^{k} \bigg(1 + \frac{B\, c^{(k-j)\alpha}\, |\mathbf{a} - \mathbf{b}|^\alpha}{d}\bigg) \\
&\le \prod_{j=1}^{k} \exp\bigg(\frac{2^\alpha B}{d} c^{(k-j)\alpha}\bigg) \\
&\le \exp \sum_{j=1}^{k} \bigg(\frac{2^\alpha B}{d} c^{(k-j)\alpha}\bigg) \\
&\le \exp\bigg(\frac{2^\alpha B}{d(1-c^\alpha)}\bigg),
\end{aligned}$$

using that $|\mathbf{a} - \mathbf{b}| \le 2$. Setting $R = \exp(2^\alpha B/d(1-c^\alpha))$ gives (3.11) for $f_{\mathtt{i},x}$, with the left-hand estimate obtained by reversing the roles of $\mathbf{a}$ and $\mathbf{b}$. A similar argument using (3.5) applies for $g_{\mathtt{i},y}$. □

We turn to the bottom left entries $g_{\mathtt{i},x}$.

LEMMA 3.3. *There exists $C > 0$ such that, for all $\mathtt{i} \in \mathcal{I}^*$ and all $\mathbf{a}, \mathbf{b} \in [0,1]^2$,*

$$\bigg|\frac{g_{\mathtt{i},x}(\mathbf{a})}{f_{\mathtt{i},x}(\mathbf{b})}\bigg| \le C. \tag{3.12}$$

Proof. Let $\mathtt{i} = (i_1, \ldots, i_k) \in \mathcal{I}^k$ and $\mathbf{a}, \mathbf{b} \in [0,1]^2$. Then, for $1 \le j \le k$, identities (3.5) and (3.9) give

$$\bigg|\frac{G_j(\mathbf{a})}{f_{\mathtt{i},x}(\mathbf{b})}\bigg| = \bigg|\bigg(\prod_{l=1}^{j-1} \frac{g_{i_l,y}(S_{i_{l+1}\cdots i_k}\mathbf{a})}{f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{b})}\bigg) \frac{g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})}{f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})} \bigg(\prod_{l=j+1}^{k} \frac{f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{a})}{f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{b})}\bigg)\bigg|$$

$$= \left(\prod_{l=1}^{j-1} \frac{|g_{i_l,y}(S_{i_{l+1}\cdots i_k}\mathbf{a})|}{|f_{i_l,x}(S_{i_{l+1}\cdots i_k}\mathbf{b})|} \right) \frac{|g_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{a})|}{|f_{i_j,x}(S_{i_{j+1}\cdots i_k}\mathbf{b})|} \frac{|f_{i_{j+1}\cdots i_k,x}(\mathbf{a})|}{|f_{i_{j+1}\cdots i_k,x}(\mathbf{b})|}$$
$$\leq \eta^{j-1}\frac{1}{d}R,$$

using (3.2) and where R is as in (3.11). Hence, by (3.7),

$$\left|\frac{g_{\mathrm{i},x}(\mathbf{a})}{f_{\mathrm{i},x}(\mathbf{b})}\right| = \left|\frac{\sum_{j=1}^{k} G_j(\mathbf{a})}{f_{\mathrm{i},x}(\mathbf{b})}\right| \leq \sum_{j=1}^{k} \left|\frac{G_j(\mathbf{a})}{f_{\mathrm{i},x}(\mathbf{b})}\right| \leq \sum_{j=1}^{k} \frac{R}{d}\,\eta^{j-1} < \frac{R}{d(1-\eta)},$$

giving (3.12) with $C = R/d(1-\eta)$. □

Recall that the singular values of an $n \times n$ matrix A are defined to be the positive square roots of the eigenvalues of $A^T A$. For $\mathbf{a} = (a_1, a_2) \in [0, 1]^2$, write $\alpha_1(D_{\mathbf{a}}S_{\mathrm{i}}) \geq \alpha_2(D_{\mathbf{a}}S_{\mathrm{i}})$ for the singular values of $D_{\mathbf{a}}S_{\mathrm{i}}$.

LEMMA 3.4. *The singular values of the Jacobian matrices $D_{\mathbf{a}}S_{\mathrm{i}}$ satisfy*

$$\alpha_1(D_{\mathbf{a}}S_{\mathrm{i}}) \asymp |f_{\mathrm{i},x}(\mathbf{a})| \tag{3.13}$$

and

$$\alpha_2(D_{\mathbf{a}}S_{\mathrm{i}}) \asymp |g_{\mathrm{i},y}(\mathbf{a})| \tag{3.14}$$

for all $\mathbf{a} \in [0, 1]^2$ *and* $\mathrm{i} \in \mathcal{I}^*$.

Proof. Let

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

be a matrix with $|c| \leq |a|$ and $|b| \leq C|a|$ for some constant $C > 0$. Calculating the larger singular value $\alpha_1(A)$ of A gives

$$\alpha_1(A)^2 = \tfrac{1}{2}\big((a^2+b^2+c^2) + ((a^2+b^2+c^2)^2 - 4a^2c^2)^{1/2}\big).$$

Making obvious estimates,

$$\tfrac{1}{2}a^2 \;\leq\; \alpha_1(A)^2 \;\leq\; a^2+b^2+c^2 \;\leq\; (2+C^2)a^2.$$

Applying this to the matrix

$$D_{\mathbf{a}}S_{\mathrm{i}} = \begin{pmatrix} f_{\mathrm{i},x}(\mathbf{a}) & 0 \\ g_{\mathrm{i},x}(\mathbf{a}) & g_{\mathrm{i},y}(\mathbf{a}) \end{pmatrix},$$

where $|g_{\mathrm{i},y}(\mathbf{a})| \leq |f_{\mathrm{i},x}(\mathbf{a})|$ by (3.6) and $|g_{\mathrm{i},x}(\mathbf{a})| \leq C|f_{\mathrm{i},x}(\mathbf{a})|$ by (3.12), gives (3.13). Using that $\alpha_1(A)\alpha_2(A) = |\det A| = |a||c|$ for the matrix A, (3.14) follows immediately from (3.13). □

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An immediate consequence of Lemmas 3.2 and 3.4 is that the singular values of the Jacobian matrices satisfy

$$\alpha_1(D_{\mathbf{a}}S_{\mathrm{i}}) \asymp \alpha_1(D_{\mathbf{b}}S_{\mathrm{i}}) \quad \text{and} \quad \alpha_2(D_{\mathbf{a}}S_{\mathrm{i}}) \asymp \alpha_2(D_{\mathbf{b}}S_{\mathrm{i}}) \tag{3.15}$$

for all $\mathbf{a}, \mathbf{b} \in [0, 1]^2$ and $\mathrm{i} \in \mathcal{I}^*$.

We define the projection map onto the x-axis $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi(x, y) = x$. It is immediate that the projection of the nonlinear measure μ onto the x-axis, $\pi(\mu)$, is a self-conformal measure. It follows from a result of Peres and Solomyak [19] that the L^q-spectra of this projected measure, which we denote by

$$\beta(q) := \tau_{\pi(\mu)}(q), \tag{3.16}$$

exist for $q \geq 0$. Note that this holds even if there are complicated overlaps between the components of the projected measure, which is the typical situation for us.

For $s \in \mathbb{R}$, $q \geq 0$ and $\mathbf{a} \in [0, 1]^2$, we define the *q-modified singular value function*, $\psi_{\mathbf{a}}^{s,q} : \mathcal{I}^* \to (0, \infty)$, by

$$\psi_{\mathbf{a}}^{s,q}(\mathrm{i}) = p(\mathrm{i})^q \alpha_1(D_{\mathbf{a}}S_{\mathrm{i}})^{\beta(q)} \alpha_2(D_{\mathbf{a}}S_{\mathrm{i}})^{s-\beta(q)}. \tag{3.17}$$

It follows from (3.15) that, for all $\mathbf{a}, \mathbf{b} \in [0, 1]^2$ and $\mathrm{i} \in \mathcal{I}^*$, we have $\psi_{\mathbf{a}}^{s,q}(\mathrm{i}) \asymp_{s,q} \psi_{\mathbf{b}}^{s,q}(\mathrm{i})$. Moreover, by Lemma 3.2,

$$\psi_{\mathbf{a}}^{s,q}(\mathrm{i}) \asymp_{s,q} p(\mathrm{i})^q |f_{\mathrm{i},x}(\mathbf{a})|^{\beta(q)} |g_{\mathrm{i},y}(\mathbf{a})|^{s-\beta(q)}. \tag{3.18}$$

For each $k \in \mathbb{N}$, define $\Psi_{\mathbf{a},k}^{s,q}$ by

$$\Psi_{\mathbf{a},k}^{s,q} = \sum_{\mathrm{i} \in \mathcal{I}^k} \psi_{\mathbf{a}}^{s,q}(\mathrm{i}). \tag{3.19}$$

The quantities $\psi_{\mathbf{a}}^{s,q}(\mathrm{i})$ and $\Psi_{\mathbf{a},k}^{s,q}$ satisfy some useful multiplicative properties, similar to those from [12, Lemma 2.2].

LEMMA 3.5. *Let* $s \in \mathbb{R}$, $q \geq 0$ *and* $\mathbf{a} \in [0, 1]^2$.
(a) *If* $\mathrm{i}, \mathrm{j} \in \mathcal{I}^*$ *then*

$$\psi_{\mathbf{a}}^{s,q}(\mathrm{ij}) \asymp_{s,q} \psi_{\mathbf{a}}^{s,q}(\mathrm{i})\psi_{\mathbf{a}}^{s,q}(\mathrm{j}). \tag{3.20}$$

(b) *If* $k, l \in \mathbb{N}$ *then*

$$\Psi_{\mathbf{a},k+l}^{s,q} \asymp_{s,q} \Psi_{\mathbf{a},k}^{s,q}\Psi_{\mathbf{a},l}^{s,q}. \tag{3.21}$$

Proof. By the chain rule applied to $f_{\mathrm{ij},x}$ and using (3.11),

$$|f_{\mathrm{ij},x}(\mathbf{a})| = |f_{\mathrm{i},x}(S_{\mathrm{j}}\mathbf{a})||f_{\mathrm{j},x}(\mathbf{a})| \asymp_{s,q} |f_{\mathrm{i},x}(\mathbf{a})||f_{\mathrm{j},x}(\mathbf{a})|,$$

and similarly

$$|g_{\mathrm{ij},y}(\mathbf{a})| \asymp_{s,q} |g_{\mathrm{i},y}(\mathbf{a})||g_{\mathrm{j},y}(\mathbf{a})|.$$

Using the form (3.18),

$$\begin{aligned}\psi_{\mathbf{a}}^{s,q}(\mathrm{ij}) &\asymp_{s,q} p(\mathrm{ij})^q|f_{\mathrm{ij},x}(\mathbf{a})|^{\beta(q)}|g_{\mathrm{ij},y}(\mathbf{a})|^{s-\beta(q)}\\ &\asymp_{s,q} p(\mathrm{i})^q p(\mathrm{j})^q|f_{\mathrm{i},x}(\mathbf{a})|^{\beta(q)}|f_{\mathrm{j},x}(\mathbf{a})|^{\beta(q)}|g_{\mathrm{i},y}(\mathbf{a})|^{s-\beta(q)}|g_{\mathrm{j},y}(\mathbf{a})|^{s-\beta(q)}\\ &\asymp_{s,q} \psi_{\mathbf{a}}^{s,q}(\mathrm{i})\psi_{\mathbf{a}}^{s,q}(\mathrm{j}),\end{aligned}$$

giving (3.20)

For part (b), if $k, l \in \mathbb{N}$ then

$$\Psi_{\mathbf{a},k+l}^{s,q} = \sum_{\mathrm{i}\in\mathcal{I}^{k+l}} \psi_{\mathbf{a}}^{s,q}(\mathrm{i}) = \sum_{\mathrm{i}\in\mathcal{I}^{k}}\sum_{\mathrm{j}\in\mathcal{I}^{l}} \psi_{\mathbf{a}}^{s,q}(\mathrm{ij})$$

and

$$\Psi_{\mathbf{a},k}^{s,q}\Psi_{\mathbf{a},l}^{s,q} = \bigg(\sum_{\mathrm{i}\in\mathcal{I}^{k}} \psi_{\mathbf{a}}^{s,q}(\mathrm{i})\bigg)\bigg(\sum_{\mathrm{j}\in\mathcal{I}^{l}} \psi_{\mathbf{a}}^{s,q}(\mathrm{j})\bigg) = \sum_{\mathrm{i}\in\mathcal{I}^{k}}\sum_{\mathrm{j}\in\mathcal{I}^{l}} \psi_{\mathbf{a}}^{s,q}(\mathrm{i})\psi_{\mathbf{a}}^{s,q}(\mathrm{j}).$$

Applying part (a) to the double sums completes the proof. □

We call a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n > 0$ (such as those in Lemma 3.5) for which there exists absolute constants $0 < K_1 \leq K_2$ such that

$$K_1 a_n a_m \leq a_{n+m} \leq K_2 a_n a_m \tag{3.22}$$

for all $n, m \in \mathbb{N}$, *almost multiplicative*. For such sequences the limit $\lim_{n\to\infty} a_n^{1/n}$ exists; see, for example, [**5**, Corollary 1.2].

It follows from Lemma 3.5 that, for each $\mathbf{a} \in [0, 1]^2$, we may define a function $P_{\mathbf{a}} : \mathbb{R} \times [0, \infty) \to [0, \infty)$ by

$$P_{\mathbf{a}}(s, q) = \lim_{k\to\infty} (\Psi_{\mathbf{a},k}^{s,q})^{1/k}.$$

Note that the value of $P_{\mathbf{a}}(s, q)$ is unchanged if we replace the right-hand side of (3.17) by the right-hand side of (3.18) in the definition of $\psi_{\mathbf{a}}^{s,q}(\mathrm{i})$ and thus of $\Psi_{\mathbf{a},k}^{s,q}$. Moreover, as $\psi_{\mathbf{a}}^{s,q}(\mathrm{i}) \asymp_{s,q} \psi_{\mathbf{b}}^{s,q}(\mathrm{i})$ and thus $\Psi_{\mathbf{a},k}^{s,q} \asymp_{s,q} \Psi_{\mathbf{b},k}^{s,q}$ for all $\mathbf{a}, \mathbf{b} \in [0, 1]^2$, it is easy to see that $P_{\mathbf{a}}$ is independent of the choice of $\mathbf{a}$. Thus we shall just write P instead of $P_{\mathbf{a}}$. For a fixed $q \geq 0$, we think of the function $s \mapsto \log P(s, q)$ as the *topological pressure* of the system.

We also write

$$\begin{aligned}\alpha_{\min} &= \inf\{\alpha_2(D_{\mathbf{a}}S_i) : \mathbf{a} \in [0, 1]^2,\ i \in \mathcal{I}\},\\ \alpha_{\max} &= \sup\{\alpha_1(D_{\mathbf{a}}S_i) : \mathbf{a} \in [0, 1]^2,\ i \in \mathcal{I}\},\\ p_{\min} &= \min\{p_i : i \in \mathcal{I}\},\\ p_{\max} &= \max\{p_i : i \in \mathcal{I}\}\end{aligned}$$

and note that $0 < \alpha_{\min}, \alpha_{\max}, p_{\min}, p_{\max} < 1$.

Recall that the L^q-spectrum of a given measure is Lipschitz continuous (as it is concave and decreasing) on $[\lambda, \infty)$ for all $\lambda > 0$. Let L_λ denote the Lipschitz constant of β on $[\lambda, \infty)$. We can now state some basic properties of P.

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LEMMA 3.6.

(1) *For* $s, r \in \mathbb{R}$ *and* $\lambda > 0$, *define*

$$U(s, r, \lambda) = min\{\alpha_{min}^s p_{min}^r, \alpha_{min}^s p_{max}^r, \alpha_{max}^s p_{min}^r, \alpha_{max}^s p_{max}^r\}(\alpha_{max}/\alpha_{min})^{min\{-L_\lambda r, 0\}}$$

and

$$V(s, r, \lambda) = max\{\alpha_{min}^s p_{min}^r, \alpha_{min}^s p_{max}^r, \alpha_{max}^s p_{min}^r, \alpha_{max}^s p_{max}^r\}(\alpha_{max}/\alpha_{min})^{max\{-L_\lambda r, 0\}}.$$

Then, for all $s, t \in \mathbb{R}$, $\lambda > 0$, $q \geq \lambda$ *and* $r \geq \lambda - q$,

$$U(s, r, \lambda)P(t, q) \leq P(s + t, q + r) \leq V(s, r, \lambda)P(t, q),$$

and for all $s, t \in \mathbb{R}$,

$$min\{\alpha_{min}^s, \alpha_{max}^s\}P(t, 0) \leq P(s + t, 0) \leq max\{\alpha_{min}^s, \alpha_{max}^s\}P(t, 0).$$

Also, for all $s \in \mathbb{R}$ *and* $q \geq 0$,

$$P(s, q) \leq p_{max}^q P(s, 0).$$

(2) P *is continuous on* $\mathbb{R} \times (0, \infty)$ *and on* $\mathbb{R} \times \{0\}$.

(3) P *is strictly decreasing in* $s \in \mathbb{R}$ *and* $q \in (0, \infty)$.

(4) *For each* $q \geq 0$, *there exists a unique* $s \geq 0$ *such that* $P(s, q) = 1$.

Proof. This is essentially the same as the proof of the analogous result of Fraser [**12**, Lemma 2.3] and as such is omitted. □

It follows from Lemma 3.6 that we may define a function $\gamma : [0, \infty) \to \mathbb{R}$ by $P(\gamma(q), q) = 1$, which we shall refer to as a *moment scaling function*. The moment scaling function satisfies the following useful properties.

LEMMA 3.7.

(1) γ *is strictly decreasing on* $[0, \infty)$.

(2) γ *is continuous on* $(0, \infty)$.

(3) $\gamma(1) = 0$ *and* $\lim_{q\to\infty} \gamma(q) = -\infty$.

(4) γ *is convex on* $(0, \infty)$.

Proof. This follows by the same reasoning as in the proof of [**12**, Lemma 2.5]. □

We can now state our main theorem which relates γ to the L^q-spectrum $\tau_\mu(q)$ of μ.

THEOREM 3.8. *Let* μ *be a nonlinear measure which satisfies the domination condition (3.1).*

(1) *For* $q \in [0, 1]$,

$$\overline{\tau}_\mu(q) \leq \gamma(q).$$

(2) *For* $q \geq 1$,

$$\underline{\tau}_\mu(q) \geq \gamma(q).$$

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(3) *If μ also satisfies the ROSC then, for all $q \geq 0$,*

$$\tau_\mu(q) = \gamma(q).$$

We shall prove this theorem in §4.

As a corollary we are able to calculate the box dimension of the support of these measures. We recall the definition of the box dimension.

Definition 3.9. Let $X \subset \mathbb{R}^n$ be bounded and non-empty and let $N_\delta(X)$ denote the minimal number of sets of diameter at most δ needed to cover X. The upper and lower box dimension are defined to be

$$\overline{\dim}_B X = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(X)}{-\log \delta}$$

and

$$\underline{\dim}_B X = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(X)}{-\log \delta},$$

respectively. If these numbers coincide then we define the box dimension of X, denoted $\dim_B X$, to be their common value.

COROLLARY 3.10. *Let F be a nonlinear attractor which satisfies the domination condition (3.1). Then:*

(1)
$$\overline{\dim}_B F \leq \gamma(0);$$

(2) *if F also satisfies the ROSC then*

$$\dim_B F = \gamma(0).$$

Proof. It is well known that the upper and lower box dimension of the support of a measure is given by the upper and lower L^q-spectrum at 0. The result is then immediate from Theorem 3.8. □

Note that $\gamma(0)$ depends on $\beta(0) = \dim_B \pi F$, the box dimension of the projection of F onto the x-axis. Also note that by standard results (e.g. [8, Corollary 3.10]), the *packing* dimension of a nonlinear attractor coincides with the upper box dimension and so Corollary 3.10 also yields the packing dimension.

4. *Calculating the L^q-spectrum*

We begin this section by introducing some notation. For $\mathtt{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}^*$, let $\widehat{\mathtt{i}} \in \mathcal{I}^* \cup \{\omega\}$ be given by

$$\hat{\mathtt{i}} = (i_1, i_2, \ldots, i_{k-1})$$

where ω is the empty word. For $\delta \in (0, 1]$ and $\mathbf{a} \in [0, 1]^2$, we define the δ-*stopping* by

$$\mathcal{I}_{\mathbf{a},\delta} = \{\mathtt{i} \in \mathcal{I}^* : \alpha_2(D_{\mathbf{a}} S_{\mathtt{i}}) < \delta \leq \alpha_2(D_{\mathbf{a}} S_{\hat{\mathtt{i}}})\}$$

where S_ω is the identity map. Note that if $\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}$ then

$$\alpha_{\min}\delta \leq \alpha_2(D_{\mathbf{a}} S_{\mathtt{i}}) < \delta. \tag{4.1}$$

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For $\mathtt{i} \in \mathcal{I}^*$, let $\mu_{\mathtt{i}} = p(\mathtt{i})\mu \circ S_{\mathtt{i}}^{-1}$ and $F_{\mathtt{i}} = S_{\mathtt{i}}(F) = \operatorname{supp}\mu_{\mathtt{i}}$. Note that, for all $\mathbf{a} \in [0, 1]^2$ and $\delta \in (0, 1]$,

$$\mu = \sum_{\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}} \mu_{\mathtt{i}}.$$

LEMMA 4.1. *Let* $\mathbf{a} \in [0, 1]^2$, $t \in \mathbb{R}$ *and* $q \geq 0$.

(1) *If* $t > \gamma(q)$ *then*

$$\sum_{\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}} \psi_{\mathbf{a}}^{t,q}(\mathtt{i}) \lesssim_{t,q} 1$$

for all $\delta \in (0, 1]$.

(2) *If* $t < \gamma(q)$ *then*

$$\sum_{\mathtt{i} \in \mathcal{I}_{\mathbf{a},\delta}} \psi_{\mathbf{a}}^{t,q}(\mathtt{i}) \gtrsim_{t,q} 1$$

for all $\delta \in (0, 1]$.

Proof. The proof follows that of [12, Lemma 7.1] which only depends on the multiplicative properties of Ψ (which we have established here) so is omitted. □

Our next lemma allows us to control the length of the side of $S_{\mathtt{i}}([0, 1]^2)$ in terms of the length of its base.

LEMMA 4.2. *There exists* $L > 0$ *such that, for all* $\mathtt{i} \in \mathcal{I}^*$ *and all* $0 \leq a < b \leq 1$,

$$\frac{|g_{\mathtt{i}}(b, 0) - g_{\mathtt{i}}(a, 0)|}{|f_{\mathtt{i}}(b) - f_{\mathtt{i}}(a)|} \leq L, \quad \frac{|g_{\mathtt{i}}(b, 1) - g_{\mathtt{i}}(a, 1)|}{|f_{\mathtt{i}}(b) - f_{\mathtt{i}}(a)|} \leq L,$$

noting that $f_{\mathtt{i}}$ *depends only on the first coordinate of its argument.*

Proof. By the mean value theorem there exist $c_1, c_2 \in (a, b)$ such that

$$\frac{|g_{\mathtt{i}}(b, 0) - g_{\mathtt{i}}(a, 1)|}{|f_{\mathtt{i}}(b) - f_{\mathtt{i}}(a)|} = \frac{|g_{\mathtt{i},x}(c_1, 0)||b - a|}{|f_{\mathtt{i},x}(c_2)||b - a|} = \frac{|g_{\mathtt{i},x}(c_1, 0)|}{|f_{\mathtt{i},x}(c_2)|} \lesssim 1$$

by Lemma 3.3. The same reasoning holds when replacing 0 with 1. Taking L to be the maximum of the two implied constants completes the result. □

The standard inequalities, that if $k \in \mathbb{N}$, $a_1, \ldots, a_k \geq 0$ and $q \geq 0$ then

$$\left(\sum_{i=1}^{k} a_i\right)^q \asymp_{k,q} \sum_{i=1}^{k} a_i^q, \tag{4.2}$$

will be helpful when manipulating moment sums.

Recall from (1.1) that $\mathcal{D}_\delta^q$ denotes the qth power moment sum of a measure over the δ-mesh cubes $\mathcal{D}_\delta$. Our next result compares the moment sums of $\mu_{\mathtt{i}} = p(\mathtt{i})\mu \circ S_{\mathtt{i}}^{-1}$ on $S_{\mathtt{i}}(F)$ with moment sums of the projection of μ onto the horizontal axis. This is analogous to [12, Lemma 7.2], but in the nonlinear case more care is needed.

LEMMA 4.3. *For each $q \geq 0$ and $\mathbf{a} \in [0,1]^2$, there exist numbers $\hat{A}, \hat{B} > 0$ such that if we write*

$$\hat{A}_{\mathrm{i},\delta} = \frac{\hat{A}\delta}{\alpha_1(D_{\mathbf{a}}S_{\mathrm{i}})} \quad \textit{and} \quad \hat{B}_{\mathrm{i},\delta} = \frac{\hat{B}\delta}{\alpha_1(D_{\mathbf{a}}S_{\mathrm{i}})} \tag{4.3}$$

for $\delta \in (0,1]$ and $\mathrm{i} \in \mathcal{I}_{\mathbf{a},\delta}$ then

$$\mathcal{D}^q_{\hat{B}_{\mathrm{i},\delta}}(p(\mathrm{i})\pi\mu) \lesssim \mathcal{D}^q_\delta(\mu_{\mathrm{i}}) \lesssim \mathcal{D}^q_{\hat{A}_{\mathrm{i},\delta}}(p(\mathrm{i})\pi\mu). \tag{4.4}$$

Proof. As $\mathrm{i} \in \mathcal{I}_{\mathbf{a},\delta}$ we have $\alpha_2(D_{\mathbf{a}}S_{\mathrm{i}}) < \delta$. We shall show that there are at most a constant number k squares of the δ-mesh that intersect $S_{\mathrm{i}}([0,1]^2) \supseteq \operatorname{supp}\mu_{\mathrm{i}}$ in any vertical column of mesh squares.

For this we estimate the height of the intersection of $S_{\mathrm{i}}([0,1]^2)$ with a given vertical strip of width δ. Note that for any such vertical strip there exists some $0 \leq a < b \leq 1$ such that

$$|f_{\mathrm{i}}(b) - f_{\mathrm{i}}(a)| = \delta \tag{4.5}$$

apart from at most two vertical strips (at the left and right ends of $S_{\mathrm{i}}([0,1]^2)$) for which

$$|f_{\mathrm{i}}(b) - f_{\mathrm{i}}(a)| \leq \delta \tag{4.6}$$

with one of a or b equal to either 0 or 1 (this is displayed in Figure 2). Then, for $a', b' \in [a,b]$,

$$\begin{aligned} |g_{\mathrm{i}}(b',1) - g_{\mathrm{i}}(a',0)| &\leq |g_{\mathrm{i}}(b',1) - g_{\mathrm{i}}(a',1)| + |g_{\mathrm{i}}(a',1) - g_{\mathrm{i}}(a',0)| \\ &\leq L|f_{\mathrm{i}}(b') - f_{\mathrm{i}}(a')| + |g_{\mathrm{i},y}(a',c)| \\ &\lesssim L\delta + \alpha_2(D_{(a',c)}S_{\mathrm{i}}) \\ &\lesssim \delta, \end{aligned} \tag{4.7}$$

where we have estimated the first term of (4.7) using Lemma 4.2 and (4.5)–(4.6) and the second term using the mean value theorem with $c \in (0,1)$ followed by (3.14), (3.15) and (4.1).

We have shown that the height of the intersection of $S_{\mathrm{i}}([0,1]^2)$ with every vertical strip with base length δ is at most $k'\delta$, where k' is independent of i. Thus at most $k = \lceil k' \rceil + 1$ squares in any column of the δ-mesh intersect $S_{\mathrm{i}}([0,1]^2)$ so, using (4.2), $\mathcal{D}^q_\delta(\mu_{\mathrm{i}}) \asymp \mathcal{D}^q_\delta(\pi\mu_{\mathrm{i}})$, where $\pi\mu_{\mathrm{i}}$ is the projection of μ_{i} onto the x-axis. In terms of the projection of pre-images of the intersection of δ-mesh cubes Q with $S_{\mathrm{i}}([0,1]^2)$,

$$\begin{aligned} \mathcal{D}^q_\delta(\mu_{\mathrm{i}}) &= \sum_{Q \in \mathcal{D}_\delta} \mu_{\mathrm{i}}(Q \cap S_{\mathrm{i}}([0,1]^2))^q \\ &= p(\mathrm{i})^q \sum_{Q \in \mathcal{D}_\delta} \mu(S_{\mathrm{i}}^{-1}(Q \cap S_{\mathrm{i}}([0,1]^2)))^q \\ &\asymp p(\mathrm{i})^q \sum_{Q \in \mathcal{D}_\delta} \pi\mu(\pi S_{\mathrm{i}}^{-1}(Q \cap S_{\mathrm{i}}([0,1]^2)))^q. \end{aligned} \tag{4.8}$$

If Q is a δ-mesh cube that intersects $S_{\mathrm{i}}([0,1]^2)$ other than one overlapping its left or right edge, then $\pi S_{\mathrm{i}}^{-1}(Q \cap S_{\mathrm{i}}([0,1]^2))$ is an interval in $\mathbb{R}$ of length δ. Writing $\hat{a}, \hat{b}$ for the

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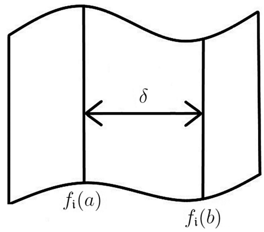

FIGURE 2. $S_{\mathtt{i}}([0,1]^2)$ together with two points $f_{\mathtt{i}}(a)$ and $f_{\mathtt{i}}(b)$ which together form a vertical strip of width δ.

endpoints of $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$, then using the mean value theorem and (3.13),

$$|\hat{a}-\hat{b}| = \frac{|f_{\mathtt{i}}(\hat{a})-f_{\mathtt{i}}(\hat{b})|}{|f_{\mathtt{i},x}(\hat{c})|} = \frac{\delta}{|f_{\mathtt{i},x}(\hat{c})|} \asymp \frac{\delta}{\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})},$$

where $\hat{c} \in [0,1]$.

If Q is one of the two cubes at the left or right end of $S_{\mathtt{i}}([0,1]^2)$ then we simply 'glue' $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ to the adjacent interval (which will be on the right or left, respectively). This will create a new interval which also has length comparable to $\delta/\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})$.

Every projection of a pre-image $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ can be covered by an interval of length $\hat{A}\delta/\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})$ and contains an interval of length $\hat{B}\delta/\alpha_1(D_{\mathbf{a}}S_{\mathtt{i}})$, for some constants $\hat{A} \geq \hat{B} > 0$. Recall the definitions (4.3) of $\hat{A}_{\mathtt{i},\delta}$ and $\hat{B}_{\mathtt{i},\delta}$ and write $\mathcal{J}_\delta$ for the δ-mesh on $\mathbb{R}$ centred at the origin. From (4.8), noting that each $J \in \mathcal{J}_{\hat{A}_{\mathtt{i},\delta}}$ can intersect $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ for at most $k\lceil \hat{A}/\hat{B}+1\rceil$ many $Q \in \mathcal{D}_\delta$, and using (4.2),

$$\begin{aligned}
\mathcal{D}_\delta^q(\mu_{\mathtt{i}}) &\asymp p(\mathtt{i})^q \sum_{Q\in\mathcal{D}_\delta} \pi\mu(\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2)))^q \\
&\lesssim p(\mathtt{i})^q \sum_{J\in\mathcal{J}_{\hat{A}_{\mathtt{i},\delta}}} \pi\mu(J)^q \\
&= \sum_{J\in\mathcal{J}_{\hat{A}_{\mathtt{i},\delta}}} (p(\mathtt{i})\pi\mu(J))^q \\
&= \mathcal{D}^q_{\hat{A}_{\mathtt{i},\delta}}(p(\mathtt{i})\pi\mu).
\end{aligned}$$

Similarly, each $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ intersects at most $\lceil \hat{A}/\hat{B}+1\rceil$ intervals $J \in \mathcal{J}_{\hat{B}_{\mathtt{i},\delta}}$, and each interval $J \in \mathcal{J}_{\hat{B}_{\mathtt{i},\delta}}$ intersects $\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2))$ for at most $2k$ sets $Q \in \mathcal{D}_\delta$, so

$$\begin{aligned}
\mathcal{D}_\delta^q(\mu_{\mathtt{i}}) &\asymp p(\mathtt{i})^q \sum_{Q\in\mathcal{D}_\delta} \pi\mu(\pi S_{\mathtt{i}}^{-1}(Q \cap S_{\mathtt{i}}([0,1]^2)))^q \\
&\gtrsim p(\mathtt{i})^q \sum_{J\in\mathcal{J}_{\hat{B}_{\mathtt{i},\delta}}} \pi\mu(J)^q
\end{aligned}$$

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$$= \sum_{J \in \mathcal{J}_{\hat{B}_{\mathrm{i},\delta}}} (p(\mathrm{i})\pi\mu(J))^q$$
$$= \mathcal{D}^q_{\hat{B}_{\mathrm{i},\delta}}(p(\mathrm{i})\pi\mu),$$

giving the result. □

Notice that a simple consequence of the Definition 1.3 of the L^q-spectrum is that, for all $\varepsilon > 0$, $q \geq 0$, $p > 0$ and $0 < \delta \leq 1$,

$$p^q \delta^{-\beta(q)+\varepsilon/2} \lesssim_{\varepsilon,q} \mathcal{D}^q_\delta(p\pi\mu) \lesssim_{\varepsilon,q} p^q \delta^{-\beta(q)-\varepsilon/2}. \tag{4.9}$$

We now turn to proving our main result, Theorem 3.8.

Proof of Theorem 3.8. The first two parts of this proof follow Fraser's proof of [12, Theorem 2.6], but we reproduce it here due to its centrality to our result.

Part (1). Let $q \in [0, 1]$ and let $\delta \in (0, 1]$ and $\mathbf{a} \in [0, 1]^2$. It is sufficient to show that $\overline{\tau}_\mu(q) \leq \gamma(q)$. As $q \in [0, 1]$,

$$\mathcal{D}^q_\delta(\mu) = \sum_{Q \in \mathcal{D}_\delta} \mu(Q)^q = \sum_{Q \in \mathcal{D}_\delta} \bigg(\sum_{\mathrm{i} \in \mathcal{I}_\delta} \mu_\mathrm{i}(Q)\bigg)^q$$
$$\leq \sum_{Q \in \mathcal{D}_\delta} \sum_{\mathrm{i} \in \mathcal{I}_\delta} \mu_\mathrm{i}(Q)^q = \sum_{\mathrm{i} \in \mathcal{I}_\delta} \sum_{Q \in \mathcal{D}_\delta} \mu_\mathrm{i}(Q)^q = \sum_{\mathrm{i} \in \mathcal{I}_\delta} \mathcal{D}^q_\delta(\mu_\mathrm{i}).$$

Thus for all $\varepsilon > 0$,

$$\begin{aligned}
\delta^{\gamma(q)+\varepsilon}\mathcal{D}^q_\delta(\mu) &\leq \delta^{\gamma(q)+\varepsilon} \sum_{\mathrm{i} \in \mathcal{I}_\delta} \mathcal{D}^q_\delta(\mu_\mathrm{i}) \\
&\lesssim \delta^{\gamma(q)+\varepsilon} \sum_{\mathrm{i} \in \mathcal{I}_\delta} \mathcal{D}^q_{\hat{A}_{\mathrm{i},\delta}}(p(\mathrm{i})\pi_\mathrm{i}\mu) && \text{by (4.4)} \\
&\lesssim_{\varepsilon,q} \delta^{\gamma(q)+\varepsilon} \sum_{\mathrm{i} \in \mathcal{I}_\delta} p(\mathrm{i})^q \bigg(\frac{\hat{A}\delta}{\alpha_1(D_\mathbf{a}S_\mathrm{i})}\bigg)^{-\beta(q)-\varepsilon/2} && \text{by (4.9)} \\
&\lesssim_{\varepsilon,q} \sum_{\mathrm{i} \in \mathcal{I}_\delta} p(\mathrm{i})^q \alpha_1(D_\mathbf{a}S_\mathrm{i})^{\beta(q)+\varepsilon/2}\alpha_2(D_\mathbf{a}S_\mathrm{i})^{\gamma(q)+\varepsilon-\beta(q)-\varepsilon/2} && \text{by (4.1)} \\
&= \sum_{\mathrm{i} \in \mathcal{I}_\delta} \psi_\mathbf{a}^{\gamma(q)+\varepsilon,q}(\mathrm{i}) && \text{by (3.17)} \\
&\lesssim_{\varepsilon,q} 1. && \text{by Lemma 4.1}
\end{aligned}$$

So $\overline{\tau}_\mu(q) \leq \gamma(q) + \varepsilon$ by (1.2), giving (1) on letting $\varepsilon \to 0$.

Part (2). We suppose $q \geq 1$ and as before let $\delta \in (0, 1]$ and $\mathbf{a} \in [0, 1]^2$. It is sufficient to show that $\underline{\tau}_\mu(q) \geq \gamma(q)$. As $q \geq 1$,

$$\mathcal{D}^q_\delta(\mu) = \sum_{Q \in \mathcal{D}_\delta} \mu(Q)^q = \sum_{Q \in \mathcal{D}_\delta} \bigg(\sum_{\mathrm{i} \in \mathcal{I}_\delta} \mu_\mathrm{i}(Q)\bigg)^q$$
$$\geq \sum_{Q \in \mathcal{D}_\delta} \sum_{\mathrm{i} \in \mathcal{I}_\delta} \mu_\mathrm{i}(Q)^q = \sum_{\mathrm{i} \in \mathcal{I}_\delta} \sum_{Q \in \mathcal{D}_\delta} \mu_\mathrm{i}(Q)^q = \sum_{\mathrm{i} \in \mathcal{I}_\delta} \mathcal{D}^q_\delta(\mu_\mathrm{i}).$$

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Thus for all $\varepsilon > 0$,

$$\begin{aligned}
\delta^{\gamma(q)-\varepsilon}\mathcal{D}_\delta^q(\mu) &\geq \delta^{\gamma(q)-\varepsilon}\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mathcal{D}_\delta^q(\mu_\mathtt{i}) \\
&\gtrsim \delta^{\gamma(q)-\varepsilon}\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mathcal{D}_{\hat{B}_{\mathtt{i},\delta}}^q(p(\mathtt{i})\pi_\mathtt{i}\mu) && \text{by (4.4)} \\
&\gtrsim_{\varepsilon,q} \delta^{\gamma(q)-\varepsilon}\sum_{\mathtt{i}\in\mathcal{I}_\delta}p(\mathtt{i})^q\left(\frac{\hat{B}\delta}{\alpha_1(D_\mathbf{a}S_\mathtt{i})}\right)^{-\beta(q)+\varepsilon/2} && \text{by (4.9)} \\
&\gtrsim_{\varepsilon,q} \sum_{\mathtt{i}\in\mathcal{I}_\delta}p(\mathtt{i})^q\alpha_1(D_\mathbf{a}S_\mathtt{i})^{\beta(q)-\varepsilon/2}\alpha_2(D_\mathbf{a}S_\mathtt{i})^{\gamma(q)-\varepsilon-\beta(q)+\varepsilon/2} && \text{by (4.1)} \\
&= \sum_{\mathtt{i}\in\mathcal{I}_\delta}\psi_\mathbf{a}^{\gamma(q)-\varepsilon,q}(\mathtt{i}) && \text{by (3.17)} \\
&\gtrsim_{\varepsilon,q} 1. && \text{by Lemma 4.1}
\end{aligned}$$

So $\underline{\tau}_\mu(q) \geq \gamma(q) - \varepsilon$, giving (2) on letting $\varepsilon \to 0$.

Part (3). We now assume μ satisfies the ROSC. Due to parts (1) and (2) we only need to provide an upper bound when $q > 1$ and a lower bound when $q < 1$.

We begin by considering the case when $q > 1$. For (1) we obtained an upper bound when $q \in [0, 1]$, but the only place in the proof where we used the assumption $q \leq 1$ was

$$\mathcal{D}_\delta^q(\mu) \leq \sum_{\mathtt{i}\in\mathcal{I}_\delta}\mathcal{D}_\delta^q(\mu_\mathtt{i}).$$

Thus for $q > 1$ we shall use the ROSC to show that

$$\mathcal{D}_\delta^q(\mu) \lesssim \sum_{\mathtt{i}\in\mathcal{I}_\delta}\mathcal{D}_\delta^q(\mu_\mathtt{i}).$$

It follows from Hölder's inequality that, for $Q \in \mathcal{D}_\delta$,

$$\left(\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mu_\mathtt{i}(Q)\right)^q \leq k^{q-1}\sum_{\mathtt{i}\in\mathcal{I}_\delta}\mu_\mathtt{i}(Q)^q,$$

where

$$k := |\{\mathtt{i} \in \mathcal{I}_\delta : \mu_\mathtt{i}(Q) > 0\}|. \tag{4.10}$$

To complete the proof we need to bound k uniformly for all δ and $Q \in \mathcal{D}_\delta$. Fix $\delta \in (0, 1]$ and $Q \in \mathcal{D}_\delta$ such that $\mu(Q) > 0$. For convenience, if $A > 0$ then we write AQ to denote the cube with the same centre as Q but with sidelength $A\delta$.

Let $\mathtt{i} \in \mathcal{I}_\delta$ be such that $S_\mathtt{i}((0, 1)^2) \cap Q$ is non-empty (such an $\mathtt{i}$ must exist as by assumption $\mu(Q) > 0$). Let $\mathbf{a} \in S_\mathtt{i}((0, 1)^2) \cap Q$ and consider the vertical 'slice' of $S_\mathtt{i}((0, 1)^2)$ that contains $\mathbf{a}$. By (4.1) and Lemma 3.4, $g_{\mathtt{i},y}(\mathbf{a}) \asymp \alpha_2(D_\mathbf{a}S_\mathtt{i}) \asymp \delta$. Together with the mean value theorem, this implies that the height of this vertical slice is comparable to δ, say it is bounded above by $M\delta$ for some $M > 1$ which is independent of δ.

Lemma 4.2 implies that if we draw a line of slope L (where we can assume $L > 1$) from the base of the vertical slice in both directions, and a line of slope $-L$ from the top of the vertical slice in both directions, then of the two isosceles triangles formed by these lines

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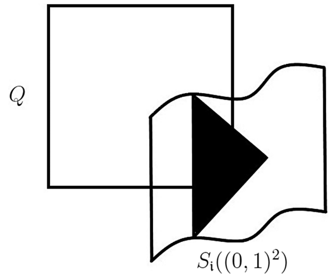

FIGURE 3. $S_{\mathfrak{i}}((0, 1)^2)$ and Q, together with the triangle $\Delta_{\mathfrak{i}}$ contained in $S_{\mathfrak{i}}((0, 1)^2)$.

and the vertical slice at least one must lie within $S_{\mathfrak{i}}((0, 1)^2)$. As the length of the vertical slice is comparable to δ the area of this triangle is comparable to δ^2. We write $\Delta_{\mathfrak{i}}$ for the triangle which is contained in $S_{\mathfrak{i}}((0, 1)^2)$, see Figure 3.

Each triangle $\Delta_{\mathfrak{i}}$ (associated with $\mathfrak{i} \in \mathcal{I}_\delta$ such that $S_{\mathfrak{i}}((0, 1)^2) \cap Q \neq \emptyset$) is contained in the square which has the same centre as Q and sidelength $3M\delta$ (i.e. the square $3MQ$). Let $\mathcal{L}$ denote two-dimensional Lebesgue measure.

As the area of each $\Delta_{\mathfrak{i}}$ is comparable to δ^2 and the ROSC guarantees that the interiors of the $\Delta_{\mathfrak{i}}$ are pairwise disjoint, it follows from (4.10) that

$$k\delta^2 \lesssim \sum_{\substack{\mathfrak{i}\in\mathcal{I}_\delta: \\ S_{\mathfrak{i}}((0,1)^2)\cap Q\neq\emptyset}} \mathcal{L}(\Delta_{\mathfrak{i}}) \leq (3M\delta)^2 = 9M^2\delta^2.$$

Hence $k \lesssim 1$, completing the proof of the upper bound for $q > 1$.

When $0 \leq q < 1$ a similar approach to the $q > 1$ case above establishes that

$$\mathcal{D}_\delta^q(\mu) \gtrsim \sum_{\mathfrak{i}\in\mathcal{I}_\delta} \mathcal{D}_\delta^q(\mu_{\mathfrak{i}}).$$

We omit the proof which is very similar. □

Acknowledgements. KJF and JMF were supported by an EPSRC Standard Grant (EP/R015104/1). JMF was also supported by a Leverhulme Trust Research Project Grant (RPG-2019-034). LDL was supported by an EPSRC Doctoral Training Grant (EP/N509759/1).

REFERENCES

[1] B. Bárány. Subadditive pressure for IFS with triangular maps. *Bull. Pol. Acad. Sci. Math.* **57** (2009), 263–278.
[2] J. Barral and D.-J. Feng. Multifractal formalism for almost all self-affine measures. *Comm. Math. Phys.* **318** (2013), 473–504.
[3] Y. Cao, Y. Pesin and Y. Zhao. Dimension estimates for non-conformal repellers and continuity of sub-additive topological pressure. *Geom. Funct. Anal.* **29** (2019), 1325–1368.
[4] K. J. Falconer. Bounded distortion and dimension for non-conformal repellers. *Math. Proc. Cambridge Philos. Soc.* **115** (1994), 315–334.
[5] K. J. Falconer. *Techniques in Fractal Geometry*. Wiley, Chichester, 1997.

[6] K. J. Falconer. Generalised dimensions of measures on self-affine sets. *Nonlinearity* **12** (1999), 877–891.

[7] K. J. Falconer. Generalised dimensions of measures on almost self-affine sets. *Nonlinearity* **23** (2010), 1047–1069.

[8] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*, 3rd edn. Wiley, Hoboken, NJ, 2014.

[9] K. J. Falconer and J. Miao. Dimensions of self-affine fractals and multifractals generated by upper-triangular matrices. *Fractals* **15** (2007), 289–299.

[10] D.-J. Feng and K. Simon. Dimension estimates for C^1iterated function systems and repellers. Part I. *Preprint*, 2020, arXiv:2007.15320.

[11] D.-J. Feng and Y. Wang. A class of self-affine sets and self-affine measures. *J. Fourier Anal. Appl.* **11** (2005), 107–124.

[12] J. M. Fraser. On the L^q-spectrum of planar self-affine measures. *Trans. Amer. Math. Soc.* **368** (2016), 5579–5620.

[13] H. Hu. Dimensions of invariant sets of expanding maps. *Comm. Math. Phys.* **176** (1996), 307–320.

[14] J. E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747.

[15] I. Kolossváry and K. Simon. Triangular Gatzouras-Lalley-type planar carpets with overlaps. *Nonlinearity* **32** (2019), 3294–3341.

[16] A. Manning and K. Simon. Subadditive pressure for triangular maps. *Nonlinearity* **20** (2007), 133–149.

[17] S.-M. Ngai. A dimension result arising from the L^q-spectrum of a measure. *Proc. Amer. Math. Soc.* **125** (1997), 2943–2951.

[18] L. Olsen. A multifractal formalism. *Adv. Math.* **116** (1995), 82–196.

[19] Y. Peres and B. Solomyak. Existence of L^q-dimensions and entropy dimension for self-conformal measures. *Indiana Univ. Math. J.* **49** (2000), 1603–1621.